Joint distribution of a Lévy process and its running supremum

Laure Coutin, Waly Ngom, Monique Pontier

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Abstract

Let X be a jump-diffusion process and X^* its running supremum. In this paper, we first show that for any $t > 0$, the law of the pair (X_t^*, X_t) has a density with respect to Lebesgue measure and compute this one. This allows us to show that for any $t > 0$, the pair formed by the random variable X_t and the running supremum X_t^* of X at time t can be characterized as a solution of a weakly valued-measure partial differential equation. Then we compute the marginal density of X_t^* for all $t > 0$.

Keywords: Lévy process, partial differential equation, running supremum process, first hitting time.

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1 Introduction

Consider a Lévy process $(X_t, t \geq 0)$, starting from zero, which is right continuous left limited. If moreover X is the sum of a drifted Brownian motion and a compound Poisson process, it is called a mixed diffusive-jump process. As any Lévy process, X has stationary and independent increments and is characterized by its Laplace transform. The mixed diffusive-jump processes and the notion of first passage time (behavior of certain processes at first passage time) are very useful and widely studied.
Introducing the running supremum at time t, X_t^* and the first passage τ_b of X at level b, the probability $\mathbb{P}(X_t \geq a, X_t^* \geq b) = \mathbb{P}(X_t \geq a, \tau_b \leq t)$ for some fixed real numbers (a, b), $a \leq b$ and $b > 0$, is of great importance, for example, in pricing barrier options while the logarithm of the underlying asset price is modeled by a jump-diffusion process. In this idea, Kou and Wang [5] give the explicit expression of the Laplace transform of the joint distribution of the double exponential mixed diffusive-jump process and its running supremum.
In [4], Jeanblanc et al. consider the first passage time by a diffusion at a deterministic function h that depends on time and they define a function of τ_h and X which satisfies the Fokker-Planck Equation.

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In [1], it is well noted (Theorem 2.2.9 and Exercise 2.2.10) that the $\frac{1}{2}-$stable subordinator is the first passage time of a standard Brownian motion and the inverse Gaussian subordinator is the first passage time of standard Brownian motion with a drift.
Mark Veillette and Murad S. Taqqu study in [9] the first passage time of a subordinator D. Since D is in general non-Markovian with non-stationary and non-independent increments, they derive a partial differential equation for the Laplace transform of the $n-$ time tail distribution $\mathbb{P}(\tau_{t_1} > s_1, \cdots, \tau_{t_n} > s_n)$ where $\tau_{t_k} = \inf\{s : D_s > t_k\}$ for a subordinator $(D_s, s \geq 0)$. With this result, they give a recursive formula for multiple-time moments of the local time of a Markov process in terms of its transition density.
The authors of [2] use a partial differential equation (PDE) approach to show that the calibration of an implied volatility surface and the pricing of contingent claims can be as simple in mixed diffusive-jump framework as it is in a diffusion framework.
This work characterizes the law of the pair U_t formed by the random variable X_t and the running supremum X_t^* of X at time t, with a valued-measure partial differential equation and gives an explicit expression for the density function of this pair. Then the marginal density of X_t^* is given.
The paper is organized as follows: Section 2 provides the main result. Section 3 gives the density function of the pair formed by the random variable X_t and its running supremum X_t^* and Section 4 is devoted to the proof of the main result of Section 2. To finish, one concludes and gives some auxiliary results in Appendix.

2 Valued measure differential equation for the joint law

We introduce some preliminary concepts for the diffusion part: for a standard Brownian motion W and a real number m, let be

$$\tilde{X}_t = mt + W_t, \quad \tilde{X}_t^* = \sup_{s \leq t} \tilde{X}_s. \tag{1}$$

In [4] page 147, Jeanblanc et al. show that the pair $(\tilde{X}_t^*, \tilde{X}_t)$ has a density with respect to Lebesgue measure on $\mathbb{R}^2$ noted $\tilde{p}(.,.;t)$ where

$$\tilde{p}(b, a; t) = \frac{2(2b - a)}{\sqrt{2\pi t^3}} \exp\left[-\frac{(2b - a)^2}{2t} + ma - m^2\frac{t}{2}\right] \mathbf{1}_{\{\max(0,a)<b\}}. \tag{2}$$

In all the following, Φ_G means the standard normal Gaussian distribution and one often uses the following:

$$1 - \Phi_G(x) = \Phi_G(-x) \leq \frac{1}{x\sqrt{2\pi}} \exp -\frac{x^2}{2}, \quad \forall x > 0. \tag{3}$$

In order to have a Lévy process with non zero jump part, let us introduce

$$X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i, \quad X_t^* = \sup_{s \leq t} X_t,$$

where N is a Poisson process with constant positive intensity λ, $(Y_i, i \in \mathbb{N}^*)$ is a sequence of independent and identically distributed random variables with the distribution function F_Y and the sequence of jump times of N is denoted by $(T_i), i \geq 1$. Let θ be the shift operator and $(U_t;\ \ t \geq 0)$ be the $\mathbb{R}^2-$value process defined by

$$U_t = (X_t^*, X_t), \quad t \geq 0. \tag{4}$$

The aim is to prove the theorem:

Theorem 2.1. *(i) For all $t > 0$, the law of the pair (X_t^*, X_t) has a density with respect to the Lebesgue measure, denoted by $p(b, a; t)$.*

(ii) For all $t > 0$, $a \in \mathbb{R}$ the map $h \mapsto p(a + h, a; t)$ has a limit when h goes to 0 denoted by $p(a+, a; t)$.

(iii) Let be $\varphi\ : \mathbb{R}^2 \to \mathbb{R}$ a C_b^3- bounded function with a support in $\{(b, a),\quad b > 0, b \geq a\}$ such that there exists $\delta > 1$ satisfying $\int_{\mathbb{R}} |\partial_1 \varphi(a, a)|^\delta da < \infty$.
For any $t > 0$,

$$\mathbb{E}\left(\varphi(U_t)\right) = \varphi(0, 0) + \int_0^t \mathbb{E}\left[m\partial_2\varphi(U_s) + \frac{1}{2}\partial_{22}^2\varphi(U_s)\right] ds \tag{5}$$
$$+ \int_0^t \frac{1}{2}\mathbb{E}\left[\partial_1\varphi(X_s, X_s)\frac{p(X_s+, X_s; s}{g(X_s; s)}\right] ds + \lambda \int_0^t \mathbb{E}\left(\int_{\mathbb{R}} \left[\varphi(U_s(y)) - \varphi(U_s)\right] F_Y(dy)\right) ds.$$

where $g(.; s)$ is the density of the random variable X_s and

$$U_s = (X_s^*, X_s), \quad U_s(y) = (\max(X_s^*, X_s + y), X_s + y), \quad s \geq 0. \tag{6}$$

In the next sections, details of the proof of Theorem 2.1 are given.

3 Existence of the density of the law of (X_t^*, X_t) and its properties

We note that

$$X_t^* = \max\{(\sup_{u \in [T_i, \inf(T_{i+1}, t)[} X_u, i = 0, ..., N_t), X_t\}$$

and use the joint density of $(\tilde{X}_t^*, \tilde{X}_t)$ given by (2) to show that the pair (X_t^*, X_t) law has a density which is right continuous on the diagonal, see Proposition 3.1 below which actually is the proof of Theorem 2.1 (i) and (ii).

Proposition 3.1. *(i) For all $t > 0$, the law of the random vector (X_t^*, X_t) admits a density with respect to the Lebesgue measure given by*

$$p(b, a, t) = \mathbb{E}\left(\sum_{k=0}^{N_t} \tilde{p}\left(b - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1} \leq t\}} - (X_t - X_{T_{k+1} \wedge t}), t \wedge T_{k+1} - T_k\right)\mathbf{1}_{\Delta_{k,t}}(b, a)\right)$$

where $\tilde{p}$ is given by (2) and

$$\Delta_{k,t} = \left\{(b, a) \mid b > \max\left(X_{T_k}^*, a + [X_{t \wedge T_{k+1}} - \sup_{u \in [T_{k+1}, t]} X_u]\mathbf{1}_{\{T_{k+1} < t\}}\right)\right\}. \tag{7}$$

(ii) Moreover, for all $a \in \mathbb{R}$, $t > 0$ the map $h \mapsto p(a+h, a; t)$ has a limit when h goes to 0 denoted by $p(a+, a; t)$ and

$$p(a+, a; t) =$$
$$\mathbb{E}\left(\sum_{k=0}^{N_t} \tilde{p}\left((a - X_{T_k})^+, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1} \le t\}} - (X_t - X_{T_{k+1}\wedge t}), t \wedge T_{k+1} - T_k\right) \mathbf{1}_{a \in D_{k,t}}\right)$$

where $D_{k,t} := \{a : a \ge \max\left(X^*_{T_k}, a + [X_{t\wedge T_{k+1}} - \sup_{u\in[T_{k+1},t]} X_u]\mathbf{1}_{\{T_{k+1}<t\}}\right)\}$

The proof of (i) relies on the following lemma:

Lemma 3.2. *Almost surely, for all t,*

$$X^*_t = \max\left(X_{T_k} + \sup_{u\in[T_k, T_{k+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_k}\right), \quad k = 0, ..., N_t\right). \tag{8}$$

Moreover, almost surely, for all t, there exists a unique k denoted as N^*_t such that

$$X^*_t = X_{T_k} + \sup_{u\in[T_k, T_{k+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_k}\right). \tag{9}$$

Proof. Let t be fixed.
(a) Note that

$$X^*_t = \max\left\{\max\left(X_{T_k} + \sup_{u\in[T_k, T_{k+1}\wedge t[} (X_u - X_{T_k}), \quad k = 0, ..., N_t\right), X_t\right\}. \tag{10}$$

For $k \in \mathbb{N}$, for all $u \in [T_k, T_{k+1}[$, $X_u - X_{T_k} = \tilde{X}_u - \tilde{X}_{T_k}$ where $\tilde{X}$ is the continuous process defined in (1), thus for $k \le N_t$,

$$\sup_{u\in[T_k, T_{k+1}\wedge t[} (X_u - X_{T_k}) = \sup_{u\in[T_k, T_{k+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_k}\right) \tag{11}$$

and

$$\max\left(X_{T_{N_t}} + \sup_{u\in[T_{N_t}, T_{N_t+1}\wedge t[} \left(X_u - X_{T_{N_t}}\right), X_t\right) = X_{T_{N_t}} + \sup_{u\in[T_{N_t}, T_{N_t+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_{N_t}}\right). \tag{12}$$

Plugging identities (11) and (12) in equality (10) yields (8).

(b) Let two integers $i < j$ then,

$$X_{T_j} + \sup_{u\in[T_j, T_{j+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_j}\right) = X_{T_i} + \left(\tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}\right) + Y_{i+1} + \left(X_{T_j} - X_{T_{i+1}}\right) + \sup_{u\in[T_j, T_{j+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_j}\right)$$

and

$$X_{T_j} + \sup_{u\in[T_j, T_{j+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_j}\right) - X_{T_i} - \sup_{u\in[T_i, T_{i+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i}\right) =$$
$$\left\{Y_{i+1} + (X_{T_j} - X_{T_{i+1}}) + \sup_{u\in[T_j, T_{j+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_j})\right\} + \left\{(\tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}) - \sup_{u\in[T_i, T_{i+1}\wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i}\right)\right\}.$$

The two following random vectors are independent:

$$\left(\sup_{u \in [T_i, T_{i+1} \wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i} \right); \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i} \right), Y_{i+1} + \left(X_{T_j} - X_{T_{i+1}} \right) + \sup_{u \in [T_j, T_{j+1} \wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_j} \right)$$

and the law of the vector $\left(\sup_{u \in [T_i, T_{i+1} \wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i} \right); \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i} \right)$ admits a density with respect to the Lebesgue measure, hence the law of the random variable

$$\sup_{u \in [T_i, T_{i+1} \wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i} \right) + \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}$$

has a density with respect to the Lebesgue measure and is independent of

$$Y_{i+1} + \left(X_{T_j} - X_{T_{i+1}} \right) + \sup_{u \in [T_j, T_{j+1} \wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_j} \right).$$

Therefore, $X_{T_j} + \sup_{u \in [T_j, T_{j+1} \wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_j} \right) - X_{T_i} - \sup_{u \in [T_i, T_{i+1} \wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i} \right)$ is the sum of two independent random variables, one having a density, then also has a density. So for all t, almost surely, whenever $i \neq j$

$$X_{T_j} + \sup_{u \in [T_j, T_{j+1} \wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_j} \right) \neq X_{T_i} + \sup_{u \in [T_i, T_{i+1} \wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_i} \right)$$

(c) Above, we can exchange $\forall t > 0$ and almost surely, since the processes $(N_t,\ t \geq 0)$ and $\left(\left(\max \left(X_{T_k} + \sup_{u \in [T_k, T_{k+1} \wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_k} \right) \right),\ \ k \leq N_t \right),\ t \geq 0 \right)$ are right continuous. $\square$

Proof. of Proposition 3.1 (i): According to Lemma 3.2, let N_t^* denoting the index k where the maximum below is reached,

$$X_t^* = \max \left(X_{T_k} + \sup_{u \in [T_k, T_{k+1} \wedge t]} \left(\tilde{X}_u - \tilde{X}_{T_k} \right),\ \ k = 0, ..., N_t \right).$$

The fact $N_t^* = k$ is equivalent to: the supremum is reached on the interval $[T_k, T_{k+1} \wedge t]$, actually meaning $X_t^* = \sup_{[T_k, T_{k+1} \wedge t]} X_u$ and remark that $\sup_{[T_k, T_{k+1} \wedge t]} X_u \geq X_{T_k}^* \vee \sup_{[T_{k+1} \wedge t, t]} X_u$.
On the interval $[T_k, T_{k+1} \wedge t]$, $X_u = X_{T_k} + \tilde{X}_u - \tilde{X}_{T_k}$. Thus the following inequalities are equivalent to $N_t^* = k$:

(a) $\ X_{T_k} + \sup_{[T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq X_{T_k}^*$,

(b) $\ X_{T_k} + \sup_{[T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq \sup_{[T_{k+1} \wedge t, t]} X_u = [X_{T_{k+1} \wedge t} + \sup_{[T_{k+1}, t]} (X_u - X_{T_{k+1} \wedge t})] \mathbf{1}_{\{T_{k+1} < t\}} + X_t \mathbf{1}_{\{T_{k+1} \geq t\}}$.

Using $X_{T_{k+1}} = X_{T_k} + \tilde{X}_{T_{k+1}} - \tilde{X}_{T_k} + Y_{k+1}$, (b) is equivalent to

$$\sup_{[T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq [\tilde{X}_{T_{k+1}} - \tilde{X}_{T_k} + Y_{k+1} + \sup_{[T_{k+1}, t]} (X_u - X_{T_{k+1}})] \mathbf{1}_{\{T_{k+1} < t\}}) + (\tilde{X}_t - \tilde{X}_{T_k}) \mathbf{1}_{\{T_{k+1} \geq t\}}.$$

As a conclusion we get $\{N_t^* = k\} =$

$$\{ \sup_{[T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq X_{T_k}^* - X_{T_k} \} \cap \{ \sup_{[T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq \tilde{X}_{t \wedge T_{k+1}} - \tilde{X}_{T_k} + [Y_{k+1} + \sup_{[T_{k+1}, t]} (X_u - X_{T_{k+1}})] \mathbf{1}_{\{T_{k+1} \leq t\}})\}.$$

Thus

$$\{N_t^* = k\} = \left\{ \left(\sup_{u\in[T_k,T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}), \tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k} \right) \in \bar{\Delta}_{k,t} \right\}$$

where

$$\bar{\Delta}_{k,t} = \left\{ (b,a): \ | \ b > \max\left(X^*_{T_k} - X_{T_k}, a + [Y_{k+1} + \sup_{u\in[T_{k+1},t]} (X_u - X_{T_{k+1}})] \mathbf{1}_{\{T_{k+1}\le t\}} \right) \right\}. \tag{13}$$

Moreover on $\{k \le N_t\}$ so on $\{N_t^* = k\} \subset \{k \le N_t\}$

$$X_t = X_{T_k} + (\tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k}) + Y_{k+1}\mathbf{1}_{\{\mathbf{t}\ge \mathbf{T_{k+1}}\}} + (X_t - X_{t\wedge T_{k+1}}). \tag{14}$$

Let Φ be a bounded Borel function, hence

$$\mathbb{E}\left[\Phi(X_t^*, X_t)\right] = \mathbb{E}\left[\sum_{k=0}^{N_t} \Phi(X_t^*, X_t)\mathbf{1}_{\{N_t^*=k\}}\right] =$$

$$\mathbb{E}\left[\sum_{k=0}^{N_t} \mathbf{1}_{\{N_t^*=k\}} \Phi\left(X_{T_k} + \sup_{u\in[T_k,T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}), X_{T_k} + (\tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k}) + Y_{k+1}\mathbf{1}_{\{t\ge T_{k+1}\}} + (X_t - X_{t\wedge T_{k+1}}) \right)\right].$$

The four following random vectors are independent:

$$(X_{T_k}, X^*_{T_k}), \quad Y_{k+1}, \left(X_t - X_{t\wedge T_{k+1}}, \sup_{u\in[T_{k+1}\wedge t, t[} (X_u - X_{T_{k+1}\wedge t}) \right), \left(\sup_{u\in[T_k,T_{k+1}\wedge t]} \tilde{X}_u - \tilde{X}_{T_k}, \tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k} \right)$$

and conditionally to $\sigma\left(\mathcal{F}_{T_k}, Y_{k+1}, (X_u - X_{T_{k+1}}, \quad u \ge T_{k+1}\wedge t), \ T_k, T_{k+1}\right)$, the law of the random vector

$$\left(\sup_{u\in[T_k,T_{k+1}\wedge t]} \tilde{X}_u - \tilde{X}_{T_k}, \tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k} \right)$$

has a density with respect to the Lebesgue measure given by $\tilde{p}(b, a, T_{k+1}\wedge t - T_k)$ where $\tilde{p}$ is defined by (2). We obtain that $\mathbb{E}\left(\Phi(X_t^*, X_t)\right) =$

$$\int \mathbb{E}\left[\sum_{k=0}^{N_t} \Phi(X_{T_k} + b, X_{T_k} + a + Y_{k+1}\mathbf{1}_{\{\mathbf{t}\ge \mathbf{T_{k+1}}\}} + (X_t - X_{T_{k+1\wedge t}}))\tilde{p}(b, a, T_{k+1}\wedge t - T_k)\mathbf{1}_{\bar{\Delta}_{k,t}}(b,a) \right] dadb.$$

The change of variable formula $v = b + X_{T_k}$ and $u = X_{T_k} + a + Y_{k+1}\mathbf{1}_{\{\mathbf{t}\ge \mathbf{T_{k+1}}\}} + (X_t - X_{T_{k+1\wedge t}})$ concludes the proof.

Proof of Proposition 3.1 (ii): Let $a \in \mathbb{R}$, $t > 0$, the map

$$h \mapsto \sum_{k=0}^{N_t} \tilde{p}\left(a + h - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1}\le t\}} - (X_t - X_{T_{k+1}\wedge t}), t\wedge T_{k+1} - T_k\right) \mathbf{1}_{\Delta_{k,t}}(b,a)$$

has a right limit when h goes to 0 since both functions $h \mapsto \tilde{p}(a+h, a; t)$ and $h \to \mathbf{1}_{\Delta_k,t}(a+h, a)$ admit a limit when h decreases to 0. According to Proposition 6.2 in Appendix the family

$$\left(\sum_{k=0}^{N_t} \tilde{p}\left(a + h - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1}\le t\}} - (X_t - X_{T_{k+1}\wedge t}), t\wedge T_{k+1} - T_k\right) \mathbf{1}_{\Delta_{k,t}}(b,a) \right)_{h\in[0,1]}$$

is uniformly integrable.
Then, we can exchange the limit and the expectation and $h \mapsto p(a+h, a; t)$ has a limit when h decreases to 0 and

$$p(a+, a; t) =$$
$$\mathbb{E}\left(\sum_{k=0}^{N_t} \tilde{p}\left((a - X_{T_k})_+, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1}\leq t\}} - (X_t - X_{T_{k+1}\wedge t}), t\wedge T_{k+1} - T_k\right)\mathbf{1}_{a\in D_{k,t}}\right)$$

where $D_{k,t} := \{a : a \geq \max\left(X^*_{T_k}, a + [X_{t\wedge T_{k+1}} - \sup_{u\in[T_{k+1},t]} X_u]\mathbf{1}_{\{T_{k+1}<t\}}\right)\}$

□

As a corollary the law of X^*_t is deduced:

Corollary 3.3. *For any $t > 0$, the law of the random variable X^*_t has a density $p^*(., t)$ given by*

$$p^*(b,t) = 2\mathbb{E}\left(\sum_{k=0}^{N_t} 2e^{2(b-X_{T_k})m} H_m\left[(C_{t,k})^+\mathbf{1}_{T_{k+1}<t} + (b - X_{T_k}) + m(t\wedge T_{k+1} - T_k), t\wedge T_{k+1} - T_k\right]\mathbf{1}_{\{b>X^*_{T_k}\}}\right) \tag{15}$$

where $H_m : (x,t) \to \frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{x^2}{2t}\right] - m\Phi_G(-\frac{x}{\sqrt{t}})$ *and*
$C_{t,k} = (Y_{k+1} + \sup_{u\in[T_{k+1},t]}(X_u - X_{T_{k+1}}))\mathbf{1}_{\{T_{k+1}\leq t\}}$.

Proof. Let $\tilde{q}$ be the function such that $\tilde{p}(b,a,.) = \tilde{q}(b,a,.)\mathbf{1}_{b>a\vee 0}$ where

$$\tilde{p}(b,a;t) = \frac{2(2b-a)}{\sqrt{2\pi t^3}}\exp\left[-\frac{(2b-a)^2}{2t} + ma - m^2\frac{t}{2}\right]\mathbf{1}_{\{\max(0,a)<b\}}.$$

Remark that

$$(2b-a)^2 - 2mta + m^2t^2 = [a - (2b+mt)]^2 - 4bmt, \tag{16}$$

thus we obtain

$$\tilde{q}(b,a,t) = \frac{2e^{2bm}}{\sqrt{2\pi t}}\left(\frac{2b+mt-a}{t}\exp\left[-\frac{[a-(2b+mt)]^2}{2t}\right] - m\exp\left[-\frac{[a-(2b+mt)]^2}{2t}\right]\right).$$

Hence, for any A,

$$0 \leq \int_{-\infty}^{A} \tilde{p}(b,a,t)da = 2e^{2bm}\mathbf{1}_{b>0}H_m(x,t),\ x = b\wedge A - 2b - mt. \tag{17}$$

Let k be fixed and $P^*_k(b,t)$ be given by

$$P^*_k(b,t) := \int_{\mathbb{R}} \tilde{p}\left(b - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{T_{k+1}\leq t} - (X_t - X_{T_{k+1}\wedge t}), t\wedge T_{k+1} - T_k\right)\mathbf{1}_{\Delta_{k,t}}(b,a)da$$

then the density of X^*_t is given by

$$p^*(b,t) = \mathbb{E}\left(\sum_{k=0}^{N_t} P^*_k(b,t)\right).$$

With the change of variables $u = a - X_{T_k} - Y_{k+1}\mathbf{1}_{T_{k+1}\le t} - (X_t - X_{t\wedge T_{k+1}})$, it follows

$$P_k^*(b,t) := \int_{\mathbb{R}} \tilde{p}\left(b - X_{T_k}, u, t\wedge T_{k+1} - T_k\right)\mathbf{1}_{\Delta_{k,t}}(b, u + X_{T_k} + Y_{k+1}\mathbf{1}_{T_{k+1}\le t} + (X_t - X_{t\wedge T_{k+1}}))du.$$

According to the definition of $\Delta_{k,t}$ (7)

$$\begin{aligned}&\mathbf{1}_{\Delta_{k,t}}(b, u + X_{T_k} + Y_{k+1}\mathbf{1}_{\{T_{k+1}\le t\}} + (X_t - X_{t\wedge T_{k+1}})) = \\ &\mathbf{1}_{\{b > X^*_{T_k}\}}\mathbf{1}_{\{b > u + X_{T_k} + [Y_{k+1} + \sup_{u\in[T_{k+1},t]}(X_u - X_t)]\mathbf{1}_{\{T_{k+1}\le t\}}\}}\end{aligned}$$

On the event $T_{k+1}\le t$ (id est $k < N_t$)

$$P_k^*(b,t) = \int_{\mathbb{R}} \tilde{q}(b - X_{T_k}, u, T_{k+1} - T_k)\mathbf{1}_{b > X^*_{T_k}}\mathbf{1}_{]-\infty, \min(b - X_{T_k}, b - X_{T_k} - C_{t,k}[}(u)du$$

since $C_{t,k} = (Y_{k+1} + \sup_{u\in[T_{k+1},t]}(X_u - X_t))\mathbf{1}_{\{T_{k+1}\le t\}}$.
And on the event $T_{k+1} > t$

$$P_k^*(b,t) = \int_{\mathbb{R}} \tilde{q}(b - X_{T_k}, u, t - T_k)\mathbf{1}_{b > X^*_{T_k}}\mathbf{1}_{]-\infty, b - X_{T_k}[}(u)du.$$

(a) On the event $T_{k+1}\le t$ applying (17) to $A = b - X_{T_k} - C_{t,k}$, with $T_{k+1} - T_k$ and $b - X_{T_k}$ instead of t and b, $(b - X_{T_k})\wedge A = b - X_{T_k} - (C_{t,k})^+$ (since $0\wedge(-x) = -x^+$) so on this event

$$P_k^*(b,t) = 2e^{2m(b - X_{T_k})}H_m\left[(C_{t,k})^+ + (b - X_{T_k}) + m(T_{k+1} - T_k), T_{k+1} - T_k\right].$$

(b) On the event $T_{k+1} > t$, applying (17) to $A = b - X_{T_k}$ and taking $t - T_k$ and $b - X_{T_k}$ instead of t and b, so on this event

$$P_k^*(b,t) = 2e^{2(b - X_{T_k})m}H_m\left[-(b - X_{T_k}) - m(t - T_k), t - T_k\right].$$

To summarize both cases

$$P_k^*(b,t) = 2e^{2(b - X_{T_k})m}H_m\left[(C_{t,k})^+\mathbf{1}_{\{T_{k+1}\le t\}} + (b - X_{T_k}) + m(t\wedge T_{k+1} - T_k), t\wedge T_{k+1} - T_k\right]$$

and the proof is achieved. □

4 Proof of Theorem 2.1 (iii)

To prove the end of this theorem, we proceed as follows: we will compute $\lim_{h\to 0} h^{-1}A(t,h) = a(t)$ where

$$A(t,h) := \mathbb{E}\left[\varphi(U_{t+h}) - \varphi(U_t)\right]. \tag{18}$$

After that, we will use [8] 11.82 p. 368: If f is a function such that f' is finite everywhere and integrable, then for all $a\le b$, $f(b) - f(a) = \int_a^b f'(s)ds$. The study of $a(t) := \lim_{h\to 0} h^{-1}A(t,h)$ could prove that for all $t > t_0 > 0$,

$$\mathbb{E}\left(\varphi(U_t)\right) = \int_{t_0}^t a(s)ds + \mathbb{E}\left(\varphi(X^*_{t_0}, X_{t_0})\right),\quad \forall t > t_0.$$

A last step will be to prove the convenient properties of the function a, for all $T > 0$, there exist $v \in]0, 1[$, $\xi \geq 0$ and a constant C such that

$$|a(t)| \leq C \left[\frac{1}{t^v} + 1 + t^\xi \right], \quad \forall t \in]0, T]. \tag{19}$$

Then letting t_0 going to 0 and using the fact that φ is continuous bounded, X and X^* are right continuous, estimation (19) and Lebesgue dominated theorem, for all $t > 0$

$$\mathbb{E}\left(\varphi(U_t)\right) = \int_0^t a(s)ds + \varphi(0, 0).$$

Proof of the function $a : t \mapsto \mathbb{E}(\varphi(U_t))$ admits a derivative on $]0, +\infty[$ denoted as a and satisfying (19). The idea is to split $A(t, h)$ in three parts according to the values of $N_{t+h} - N_t$:

$$A(t, h) = \sum_{i=0}^{2} A_i(t, h) \tag{20}$$

where

$$\begin{aligned} A_i(t, h) &:= \mathbb{E}\left([\varphi(U_{t+h}) - \varphi(U_t)]\, \mathbf{1}_{\{N_{t+h}-N_t=i\}}\right), \quad i = 0, 1 \\ A_2(t, h) &:= \mathbb{E}\left([\varphi(U_{t+h}) - \varphi(U_t))]\, \mathbf{1}_{\{N_{t+h}-N_t\geq 2\}}\right). \end{aligned} \tag{21}$$

Lemma 4.1. *Under the hypothesis of Theorem 2.1*

$$\lim_{h\to 0} h^{-1} A_2(t, h) = 0. \tag{22}$$

Proof. By hypothesis φ is bounded and we get

$$|A_2(t, h)| \leq 2\|\varphi\|_\infty \mathbb{P}(N_{t+h} - N_t \geq 2) \leq 2\|\varphi\|_\infty (1 - e^{-\lambda h} - \lambda h e^{-\lambda h}).$$

Thus, $\lim_{h\to 0} h^{-1} A_2(t, h) = 0$. □

This lemma added to the three next propositions proves Theorem 2.1 (iii): this lemma treats the term $A_2(t, h)$ while Proposition 4.2 treats the term $A_1(t, h)$. Propositions 4.3 and 4.4 treat the term $A_0(t, h)$.

Proposition 4.2. *Let be* $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *a* C_b^3- *bounded function. Then*

$$\lim_{h\to 0} h^{-1} A_1(t, h) = \lambda \mathbb{E} \int_{\mathbb{R}} [\varphi(U_t(y)) - \varphi(U_t)] F_Y(dy).$$

where U_t *is defined by (4) and* $U_t(y)$ *by (6).*

This proposition gets the last term on the right hand in (5).

Proof. Introducing the term $\varphi(U_t(Y_{N_{t+h}}))$, let be $A_1(t, h) := A_{1,1}(t, h) + A_{1,2}(t, h)$ where

$$\begin{aligned} A_{1,1}(t, h) &= \mathbb{E}\left(\left\{\varphi(U_t(Y_{N_{t+h}})) - \varphi(U_t)\right\} \mathbf{1}_{\{N_{t+h}=N_t+1\}}\right), \\ A_{1,2}(t, h) &= \mathbb{E}\left(\left\{\varphi(U_{t+h}) - \varphi(U_t(Y_{N_{t+h}}))\right\} \mathbf{1}_{\{N_{t+h}=N_t+1\}}\right). \end{aligned}$$

• Since φ is $\mathcal{C}^1$ class with bounded derivative, Lemma 6.1 (Appendix) implies that on the event $\{N_{t+h} - N_t = 1\}$ we have
$|\varphi(U_{t+h}) - \varphi(U_t(Y_{N_{t+h}}))| \leq \|\nabla\varphi\|_\infty (2\sup_{0\leq u\leq h}|\tilde{X}_{t+u} - \tilde{X}_t| + h)$. Thus

$$h^{-1}A_{1,2}(t,h) \leq \lambda e^{-\lambda h}\|\nabla\varphi\|_\infty (2\sup_{0\leq u\leq h}|\tilde{X}_{t+u} - \tilde{X}_t| + h) \to 0$$

when $h \to 0$: Indeed, the process $\tilde{X}$ satisfies $0 \leq \tilde{X}_h \leq |m|h + W_h^*$ and Burkholder Davis Gundy's inequality for all $i \geq 1$ implies there exists a constant $C_i > 0$ such that $E[(W_h^*)^i] \leq C_i h^{i/2}$, hence for $h \leq 1$,

$$\mathbb{E}\left((\tilde{X}_h^*)^i\right) \leq C_i h^{i/2}, \tag{23}$$

thus $E(\tilde{X}_h^*) \leq C\sqrt{h}$.
• Let us deal with $A_{1,1}(t,h)$ to show

$$\lim_{h\to 0} h^{-1}A_{1,1}(t,h) = \lambda\mathbb{E}\int_{\mathbb{R}}[\varphi(U_t(y)) - \varphi(U_t)]F_Y(dy). \tag{24}$$

On the event $\{N_t = n, N_{t+h} = n+1\}$, the equality $U_t(Y_{N_{t+h}}) = U_t(Y_{n+1})$ holds. The independence properties arising from the structure of the process X, the use of the laws of Y_i, T_n, the decomposition of T_{n+1}, $T_{n+1} = T_n + S_{n+1}$, and the conditioning to $\mathcal{F}_{T_n}$, yield:

$$A_{1,1}(t,h) = \sum_{n=0}^{+\infty}\mathbb{E}\left[1_{\{T_n\leq t<T_{n+1}\leq t+h<T_{n+2}\}}(\varphi(U_t(Y_{n+1})) - \varphi(U_t))\right] =$$
$$\sum_{n=0}^{+\infty}\mathbb{E}\left[1_{\{T_n\leq t\}}\int_{t-T_n}^{t+h-T_n} ds\int_{\mathbb{R}}(\varphi(U_t(y)) - \varphi(U_t))\,F_Y(dy)\lambda e^{-\lambda(t+h-T_n)}\right].$$

By hypothesis, the function φ is bounded and when h goes to 0, Lebesgue's dominated convergence theorem yields

$$\lim_{h\to 0} h^{-1}A_{1,1}(t,h) = \sum_{n=0}^{+\infty}\mathbb{E}\left[\lambda e^{-\lambda(t-T_n)}1_{\{T_n\leq t\}}\int_{\mathbb{R}}(\varphi(U_t(y)) - \varphi(U_t))\,F_Y(dy)\right]. \tag{25}$$

Since $e^{-\lambda(t-T_n)}\mathbf{1}_{\{T_n\leq t\}} = \mathbb{E}[\mathbf{1}_{\{T_n\leq t<T_{n+1}\}}/\mathcal{F}_t]$, it follows

$$\lim_{h\to 0} h^{-1}A_{1,1}(t,h) = \sum_{n=0}^{+\infty}\mathbb{E}\left[\lambda 1_{\{T_n\leq t<T_{n+1}\}}\int_{\mathbb{R}}(\varphi(U_t(y)) - \varphi(U_t))\,F_Y(dy)\right] =$$
$$\mathbb{E}\left[\lambda\int_{\mathbb{R}}(\varphi(U_t(y)) - \varphi(U_t))\,F_Y(dy)\right]. \tag{26}$$

□

We now turn to the study of $h^{-1}A_0(t,h)$ when h goes to 0. On the event $\{N_{t+h} - N_t = 0\}$, $T_{N_t} = T_{N_{t+h}}$, hence $X^*_{T_{N_{t+h}}} = X^*_{T_{N_t}}$ and $X_{T_{N_{t+h}}} = X_{T_{N_t}}$,

$$X_{t+h} = X_t + \tilde{X}_h\circ\theta_t,\ X^*_{t+h} = \max(X^*_t, X_t + \tilde{X}^*_h\circ\theta_t).$$

Using Markov property at t and the fact that the processes N and $\tilde{X}$ are independent

$$A_0(t,h) = e^{-\lambda h}\mathbb{E}\left(\mathbb{E}\left(\varphi(\max(x^*, x+\tilde{X}_h^*), x+\tilde{X}_h) - \varphi(x^*,x)\right)_{|x^*=X_t^*, x=X_t}\right).$$

Let us introduce

$$a_0(h,x^*,x) := \mathbb{E}\left(\varphi(\max(x^*, x+\tilde{X}_h^*), x+\tilde{X}_h) - \varphi(x^*,x)\right).$$

To study the term $a_0(h,x^*,x)$, we make a Taylor expansion at a neighborhood of (x^*,x):

$$\begin{aligned}
a_0(h,x^*,x) : &= \partial_2\varphi(x^*,x)mh + \frac{1}{2}\partial^2_{22}\varphi(x^*,x)[m^2h^2+h] + \\
&+ \partial^2_{1,2}\varphi(x^*,x)\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]\tilde{X}_h\right) \\
&+ \frac{1}{2}\partial^2_{1,1}\varphi(x^*,x)\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]^2\right) + R_0(h,x^*,x), \\
&+ \partial_1\varphi(x^*,x)\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]\right)
\end{aligned}$$

where, using ∇^i the tensor of order i,

$$|R_0(h,x^*,x)| \leq 4\|\nabla^3\varphi\|_\infty\left[\mathbb{E}\left(\left|\max(x^*, x+\tilde{X}_h^*) - x^*\right|^3\right) + \mathbb{E}\left(\left|\tilde{X}_h\right|^3\right)\right].$$

This allows us to write:

$$A_0(t,h) = \sum_{i=1}^{3} A_{0,i}(t,h), \quad A_{0,i}(t,h) := \mathbb{E}\left(a_{0,i}(h,,x^*,x)_{|x^*=X_t^*, x=X_t}\right) \tag{27}$$

where

$$\begin{aligned}
a_{0,1}(h,x^*,x) : &= \partial_2\varphi(x^*,x)mh + \frac{1}{2}\partial^2_{22}\varphi(x^*,x)[m^2h^2+h] \\
a_{0,2}(h,x^*,x) : &= \partial^2_{1,2}\varphi(x^*,x)\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]\tilde{X}_h\right) \\
&\quad + \frac{1}{2}\partial^2_{1,1}\varphi(x^*,x)\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]^2\right) + R_0(h,x^*,x), \\
a_{0,3}(h,x^*,x) : &= \partial_1\varphi(x^*,x)\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]\right).
\end{aligned}$$

Proposition 4.3. *Let be $\varphi : \mathbb{R}^2 \to \mathbb{R}$ a C_b^3- bounded function. Then for any $t > 0$*

$$\lim_{h\to 0} h^{-1}(A_{0,1} + A_{0,2})(t,h) = \mathbb{E}\left(\partial_2\varphi(U_t)m + \frac{1}{2}\partial^2_{22}\varphi(U_t)\right) \tag{28}$$

Proof. (a) Since $\tilde{X}$ and $\tilde{X}^*$ are continuous processes and φ a three times differentiable function with bounded differential, it follows

$$\lim_{h\to 0} h^{-1}A_{0,1}(t,h) = \mathbb{E}\left(\partial_2\varphi(U_t)m + \frac{1}{2}\partial^2_{22}\varphi(U_t)\right). \tag{29}$$

(b) The second term satisfies: Under hypothesis of Theorem 2.1,

$$\lim_{h\to 0} h^{-1}A_{0,2}(t,h) = 0.$$

Indeed, we first note that $\max(x^*, x+\tilde{X}^*_h) - x^* = (\tilde{X}^*_h - (x^*-x))^+ \leq \tilde{X}^*_h \mathbf{1}_{\{\tilde{X}^*_h > x^*-x\}}$. Using Cauchy-Schwarz inequality and (23) there exists a constant D_i such that

$$\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}^*_h) - x^*\right]^i\right) \leq D_i h^{i/2}\sqrt{\mathbb{P}(\tilde{X}^*_h > x^*-x)}.$$

The function φ is three times differentiable with bounded differential, we deduce from the expression of $a_{0,2}$ that there exists a constant $C>0$ such that

$$A_{0,2}(t,h) \leq \left[\sum_{i=2}^{3} \|\nabla^i\varphi\|\right] C\sum_{i=2}^{3} h^{i/2}\mathbb{E}\left(\sqrt{\mathbb{P}(\tilde{X}^*_h > x^*-x)}_{|x^*=X^*_t, x=X_t}\right).$$

The law of the pair (X^*_t, X_t) has a density with respect to Lebesgue measure on $\mathbb{R}^2$, (cf. Proposition 3.1) almost surely $X^*_t > X_t$, it follows with Lebesgue dominated convergence Theorem

$$\lim_{h\to 0} h^{-1}A_{0,2}(t,h) = 0.$$

□

We now deal with the term $h^{-1}A_{0,3}(t,h)$.

Proposition 4.4. *Let be $\varphi : \mathbb{R}^2 \to \mathbb{R}$ a C^3_b- bounded function such that there exists $\delta > 1$ satisfying $\int_{\mathbb{R}} |\partial_1\varphi(a,a)|^\delta da < \infty$. Then for any $t>0$*

$$\lim_{h\to 0}\frac{1}{h}\mathbb{E}\left(\partial_1\varphi(U_t)\mathbb{E}\left(\max(x^*, x+\tilde{X}^*_h) - x^*\right)_{x^*=X^*_t, x=X_t}\right) = \frac{1}{2}\mathbb{E}\left[\partial_1\varphi(X_t, X_t)\frac{p(X_t+, X_t, t)}{g(X_t,t)}\right].$$

Proof. We first need the following lemmas:

Lemma 4.5. *For any $t>0$, the law of $\tilde{X}^*_t$ has the density with respect to Lebesgue measure on $\mathbb{R}$,*

$$\tilde{p}^*(b,t) := 2\left[\frac{1}{\sqrt{2\pi t}}\exp -\frac{(b-mt)^2}{2t} - me^{2bm}\Phi_G(\frac{-b-mt}{\sqrt{t}})\right]\mathbf{1}_{]0,+\infty[}(b). \tag{30}$$

Remark 4.6. *This result is consistent with the fact that when $m=0$, $\tilde{X}^*_t$ and $|\tilde{X}_t|$ have the same law (cf. Proposition 3.7, Revuz-Yor [7]).*

Proof. This is obviously the derivative with respect to b of the law provided in [4] page 147. □

Lemma 4.7. *Let be $h>0$ and $H(x) := \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} - x\Phi_G(-x)$:*

$$\frac{1}{2}\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}^*_h) - x^*\right]\right) =$$
$$-mh\int_0^\infty e^{2bm\sqrt{h}}(b - \frac{(x^*-x)}{\sqrt{h}})_+\Phi_G(-b-m\sqrt{h})db + \sqrt{h}H\left(\frac{(x^*-x-mh)}{\sqrt{h}}\right).$$

Proof. Recall that

$$\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]\right) = \mathbb{E}\left(\left[\tilde{X}_h^* - (x^*-x)\right]\mathbf{1}_{\{\tilde{X}_h^* > x^*-x\}}\right).$$

Lemma 4.5 gives the density of $\tilde{X}_h^*$ and the change of variable $b \to \sqrt{h}b$ yields

$$\frac{1}{2}\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]\right) =$$
$$\int_{\frac{x^*-x}{\sqrt{h}}}^{\infty} \sqrt{h}\left[b - \frac{(x^*-x)}{\sqrt{h}}]\right]_+ \left[\frac{1}{\sqrt{2\pi}}e^{-\frac{(b-m\sqrt{h})^2}{2}} - m\sqrt{h}e^{2bm\sqrt{h}}\Phi_G(-b-m\sqrt{h})\right] db.$$

This can be written again as

$$\frac{1}{2}\mathbb{E}\left(\left[\max(x^*, x+\tilde{X}_h^*) - x^*\right]\right) = \int_{\frac{x^*-x}{\sqrt{h}}}^{\infty} \sqrt{h}[b - m\sqrt{h} - \frac{(x^*-x)-mh}{\sqrt{h}}]\frac{1}{\sqrt{2\pi}}e^{-\frac{(b-m\sqrt{h})^2}{2}} db$$
$$- mh\int_{\frac{x^*-x}{\sqrt{h}}}^{\infty} e^{2bm\sqrt{h}}(b - \frac{x^*-x}{\sqrt{h}})\Phi_G(-b-m\sqrt{h})db.$$

The lemma is proved using the integration by parts formula and the definition of H. $\square$

Lemma 4.7 allows to compute $h^{-1}A_{0,3}(t,h)$ including $\mathcal{F}_t-$conditional expectation under the expectation:

$$\frac{1}{h}\mathbb{E}\left(\partial_1\varphi(U_t)\mathbb{E}\left(\max(x^*, x+\tilde{X}_h^*) - x^*\right)_{x^*=X_t^*, x=X_t}\right) = \frac{2}{\sqrt{h}}\mathbb{E}\left(\partial_1\varphi(U_t)H(\frac{(X_t^*-X_t-mh)}{\sqrt{h}})\right)$$
$$-2m\mathbb{E}\left(\partial_1\varphi(U_t)\int_0^{\infty} e^{2bm\sqrt{h}}\Phi_G(-b-m\sqrt{h})(b - \frac{X_t^*-X_t}{\sqrt{h}})_+ db\right).$$

(a) Firstly, we show that

$$\lim_{h\to 0} -m\mathbb{E}\left[\partial\varphi_1(U_t)\int_0^{+\infty} e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})(b - \frac{X_t^*-X_t}{\sqrt{h}})_+ db\right] = 0.$$

The term $\partial\varphi_1(U_t)\int_0^{+\infty} e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})(b - \frac{X_t^*-X_t}{\sqrt{h}})_+$ is uniformly bounded with respect to h : Indeed, $b>0$ and let $0 < h \leq 1$,

$$0 \leq e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})(b - \frac{X_t^*-X_t}{\sqrt{h}})_+ \leq e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})b.$$

The function $(h,b) \mapsto e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})b$ is continuous on the compact interval $[0 \;\; 1] \times [0 \;\; 2|m|]$, then it is bounded on this interval.
Now, consider $b > 2|m|$. Therefore $b + m\sqrt{h} > |m| > 0$, $\;\; b - m\sqrt{h} > \frac{b}{2}$ and $\frac{b}{(b+m\sqrt{h})} \leq 2$. We use the inequality (3): $1 - \Phi_G(x) = \Phi_G(-x) \leq \frac{1}{x\sqrt{2\pi}}\exp -\frac{x^2}{2}$, $\;\; \forall x > 0$, to obtain for $b > 2|m|$, $h \in [0,1]$

$$e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})b \leq \frac{b}{(b+m\sqrt{h})\sqrt{2\pi}}e^{2mb\sqrt{h}}e^{-\frac{(b+m\sqrt{h})^2}{2}} \leq \frac{2}{\sqrt{2\pi}}e^{-\frac{b^2}{8}}.$$

This implies that the term $\partial\varphi_1(U_t)\int_0^{+\infty} e^{2mb\sqrt{h}}\Phi_G(-b-m\sqrt{h})(b-\frac{X_t^*-X_t}{\sqrt{h}})_+db$ is uniformly bounded by a constant. The result follows by Lebesgue dominated convergence Theorem: Indeed, almost surely $X_t^*-X_t>0$ and on this set, the integrand goes almost surely to 0. .

(b) Secondly our goal is to compute the limit when h goes to 0 of the term

$$B_1^*(t,h)=E[\partial_1\varphi(U_t)\frac{1}{\sqrt{h}}H(\frac{X_t^*-X_t-mh}{\sqrt{h}})].$$

The proof is divided into four steps.

1. Firstly, we prove that
$$\lim_{h\to 0}|B_1^*(t,h)-E[\partial_1\varphi(U_t)\frac{1}{\sqrt{h}}H(\frac{X_t^*-X_t}{\sqrt{h}})]|=0. \tag{31}$$

2. Secondly, we prove that
$$\lim_{h\to 0}\left|B_1^*(t,h)-E[\partial_1\varphi(X_t,X_t)\frac{1}{\sqrt{h}}H(\frac{X_t^*-X_t}{\sqrt{h}})]\right|=0. \tag{32}$$

3. Thirdly, we prove that
$$\lim_{h\mapsto 0}B_1^*(t,h)=\frac{1}{2}\int_{\mathbb{R}}\partial_1\varphi(a,a;t)p(a+,a;t)da.$$

4. Finally we observe that
$$\lim_{h\mapsto 0}B_1^*(t,h)=\frac{1}{2}E[\partial_1\varphi(X_t,X_t)\frac{p(X_t+,X_t;t)}{g(X_t,t)}].$$

Step 1: The function H defined by $H(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}-x\Phi_G(-x)$ is differentiable with differential given by $x\mapsto\Phi_G(-x)$, which is positive and bounded by 1. Hence for all $(x,y)\in\mathbb{R}$, there exists $\lambda\in[0,1]$ such that $H(y)-H(x)=(y-x)H'(\lambda x+(1-\lambda)y)$. Thus there exists $\lambda\in[0,1]$ such that

$$\frac{1}{\sqrt{h}}\left[H(\frac{X_t^*-X_t}{\sqrt{h}})-H(\frac{X_t^*-X_t-mh}{\sqrt{h}})\right]=mH'(\frac{X_t^*-X_t}{\sqrt{h}}+\lambda m\sqrt{h})$$

Then for $h>0$,

$$\left|B_1^*(t,h)-E[\partial_1\varphi(U_t)\frac{1}{\sqrt{h}}H(\frac{X_t^*-X_t}{\sqrt{h}})]\right|\leq|m|\|\partial_1\varphi\|_\infty E[\sup_{x\geq\frac{X_t^*-X_t}{\sqrt{h}}-|m|\sqrt{h}}H'(x)].$$

The fact that almost surely $X_t^*-X_t>0$ proves that the almost sure limit of $\frac{X_t^*-X_t}{\sqrt{h}}-|m|\sqrt{h}$ is $+\infty$. Moreover, $H'=\Phi_G$ is positive bounded and satisfies $\lim_{x\to\infty}H'(x)=0$, so Lebesgue theorem achieves the proof of (31).

Step 2: Using regularity assumption on φ

$$\left|B_1^*(t,h)-E[\partial_1\varphi(X_t,X_t)\frac{1}{\sqrt{h}}H(\frac{X_t^*-X_t}{\sqrt{h}})]\right|\leq$$
$$|m|\sqrt{h}\|\partial_1\varphi\|_\infty+\|\partial_{11}^2\varphi\|_\infty E\left[\frac{X_t^*-X_t}{\sqrt{h}}H(\frac{X_t^*-X_t}{\sqrt{h}})\right]$$

The function $x \mapsto xH(x)$ is bounded on $\mathbb{R}^+$, and $\lim_{x\to\infty} xH(x) = 0$. Then, since $X_t^* - X_t$ law has a density and

$$\text{almost sure } \lim_{h\to 0} \frac{X_t^* - X_t}{\sqrt{h}} H(\frac{X_t^* - X_t}{\sqrt{h}}) = 0,$$

the dominated Lebesgue Theorem yields:

$$\lim_{h\to 0} E\left(\frac{X_t^* - X_t}{\sqrt{h}} H(\frac{X_t^* - X_t}{\sqrt{h}})\right) = 0.$$

The proof of (32) is achieved.

Step 3: Introducing the density of the law of (X_t^*, X_t) according to Proposition 3.1 (i):

$$B_1^*(t,h) = \int_{\mathbb{R}^2} \partial\varphi_1(a,a) \frac{1}{\sqrt{h}} H(\frac{b-a}{\sqrt{h}}) p(b,a;t) da db + o(1).$$

We perform the change of variable $b = a + u\sqrt{h}$ and

$$B_1^*(t,h) = \int_{\mathbb{R}\times\mathbb{R}^+} \partial_1\varphi(a,a) H(u) p(a+u\sqrt{h}, a, t) da du + o(1)$$

Note that, for all $a > 0, u > 0$, according to the Proposition 3.1 (ii),

$$\lim_{h\to 0} \partial_1\varphi(a,a) H(u) p(a+u\sqrt{h}, a, u) = \partial_1\varphi(a,a) H(u) p(a+, a, t).$$

Jensen inequality induces

$$(B_1^*(t,h))^\delta \le \int_{\mathbb{R}\times\mathbb{R}^+} (\partial_1\varphi)^\delta(a,a) H^\delta(u) p^\delta(a+u\sqrt{h}, a, t) da du + o(1).$$

According to Proposition 6.2 and (39)

$$p^\delta(a+u\sqrt{h}, a, t) \le C(\delta, T, m)\left[\frac{1}{t^v}\frac{1}{|a+2u\sqrt{h}+mt|^\gamma} + \frac{1}{t^\beta} + 1 + t^\xi\right] e^{6(m_+)^2\delta t}$$

Integrability and boundedness assumptions on $\partial_1\varphi$ induces that for all h there exists D_i such that

$$\int_{\mathbb{R}\times\mathbb{R}^+} (\partial_1\varphi)^\delta(a,a) H^\delta(u) p^\delta(a+u\sqrt{h}, a, t) da du \le D_1 + D_2 \int_{\mathbb{R}\times\mathbb{R}^+} |\partial_1\varphi|^\delta(a,a) \frac{1}{|a+2u\sqrt{h}+mt|^\gamma} H^\delta(u) da du.$$

The integral with respect to da is shared in two parts factor, of $\int_{\mathbb{R}^+} H^\delta(u) du$:

$$\int_{|a+2u\sqrt{h}+mt|\le 1} |\partial_1\varphi|^\delta(a,a) \frac{1}{|a+2u\sqrt{h}+mt|^\gamma} da + \int_{|a+2u\sqrt{h}+mt|>1} |\partial_1\varphi|^\delta(a,a) \frac{1}{|a+2u\sqrt{h}+mt|^\gamma} da.$$

The second term is bounded by $\int_{\mathbb{R}} |\partial_1\varphi|^\delta(a,a) da < \infty$, the first one is bounded by

$$\|\partial_1\varphi\|_\infty \int_{|a+2u\sqrt{h}+mt|\le 1} \frac{1}{|a+2u\sqrt{h}+mt|^\gamma} da = \|\partial_1\varphi\|_\infty \int_{|x|\le 1} |x|^{-\gamma} dx = \frac{2}{1-\gamma}\|\partial_1\varphi\|_\infty.$$

Thus for all $T > 0$ there exist $v \in]0,1[$, $\xi \geq 0$ and a constant C such that

$$sup_{h\in]0,1]} \int_{\mathbb{R}\times\mathbb{R}^+} |\partial_1\varphi(a,a)|^\delta H^\delta(u) p^\delta(a+u\sqrt{h},a,t)dadu \leq C\left[\frac{1}{t^v}+1+t^\xi\right], \quad \forall t \in]0,T]. \quad (33)$$

The family $\{(a,u) \to \partial_1\varphi(a,a)H(u)p(a+2u\sqrt{h},a,t), h \in [0,1]\}$ is uniformly integrable with respect to Lebesgue measure $dadu$, so we can exchange the limit and the integral:

$$\lim_{h\to 0} B_1^*(t,h) = \int_{\mathbb{R}\times\mathbb{R}^+} \partial\varphi_1(a,a)H(u)p(a+,a;t)dadu.$$

Note that $\int_{\mathbb{R}^+} H(u)du = \frac{1}{2}$ ends the proof of this step.

Step 4: Propositions 4.2 4.3 and 4.4, it is proved that

$$a(t) = \lambda\mathbb{E}\int_{\mathbb{R}}[\varphi(U_t(y)) - \varphi(U_t)]F_Y(dy) + \mathbb{E}\left(\partial_2\varphi(U_t)m + \frac{1}{2}\partial_{22}^2\varphi(U_t)\right) + \frac{1}{2}\mathbb{E}\left[\partial_1\varphi(X_t,X_t)\frac{p(X_t+,X_t,t)}{g(X_t,t)}\right].$$

The two first terms are bounded, so we have only to check (19) on the third term.
Similarly to Step 3, we get

$$\lim_{h\to 0} E[|\partial_1\varphi(U_t)|\frac{1}{\sqrt{h}}H(\frac{X_t^* - X_t - mh}{\sqrt{h}})] = \int_{\mathbb{R}\times\mathbb{R}^+} |\partial\varphi_1(a,a)|H(u)p(a+,a;t)dadu.$$

so $a \mapsto \partial\varphi_1(a,a)p(a+,a;t)$ belongs to $L^1(\mathbb{R},da)$. Then, since $g(.,t)$ denotes the density of the law of X_t $a \mapsto \partial\varphi_1(a,a)\frac{p(a+,a;t)}{g(a;t)}$ belongs to $L^1(\mathbb{R},P_{X_t})$.
Note that from estimation (33) for all $T > 0$ there exist $v \in]0,1[$, $\xi \geq 0$ and a constant C such that

$$\left|\frac{1}{2}\mathbb{E}\left[\partial_1\varphi(X_t,X_t)\frac{p(X_t+,X_t,t)}{g(X_t,t)}\right]\right| \leq C\left[\frac{1}{t^v}+1+t^\xi\right], \quad \forall t \in]0,T],$$

and the function a satisfies (19).

□

5 Conclusion

The aim of this paper is to have a complete study of the law of one Lévy process X and its running supremum. Recall that X^* is not Markovian, but the pair $U := (X^*, X)$ is. In the second section, we give the main result (Theorem 2.1): the density of the law of U_t, its right continuity on the diagonal and a weakly valued-measure differential equation which characterizes the law of U_t.
To complete the study of the survival probability initiated by Coutin and Dorobantu [3], as a consequence, one gives the marginal density of the law of X_t^* (Corollary 3.3). A perspective could be the proof of regularity of the survival probability on $\mathbb{R}^+ \times \mathbb{R}$, meaning
$(b,t) \to \mathbb{P}(\tau_b > t) = \mathbb{P}(X_t^* \leq b) = \int_0^b p^*(x,t)dx$.
Another perspective is to study filtering, for instance to generalize W. Ngom [6] to incomplete observation.

6 Appendix

Lemma 6.1. *On the event* $\{N_{t+h} = N_t + 1\}$, $\left|U_t(Y_{N_{t+h}}) - U_{t+h}\right| \leq 2\sup_{0\leq u\leq h} |\tilde{X}_{t+u} - \tilde{X}_t|$.

Proof. : (a) On the event $\{N_{t+h} - N_t = 1\} = \cup_n\{N_t = n, N_{t+h} - N_t = 1\}$, we compute $U_t(Y_{N_{t+h}}) = (\max(X_t^*, X_t + Y_{N_{t+h}}); X_t + Y_{N_{t+h}})$ and $U_{t+h} = (X^*_{t+h}, X_{t+h})$.
(b) On the event $\{N_t = n, N_{t+h} = n+1\}$,

$$U_t(Y_{N_{t+h}}) = (\max(X_t^*, X_t + Y_{n+1}); X_t + Y_{n+1}), \quad U_{t+h} = (X^*_{t+h}, X_{t+h}).$$

(c) We bound up $|U_t(Y_{N_{t+h}}) - U_{t+h}|$ component by component:

• Concerning the second component, on the event $\{N_t = n, N_{t+h} = n+1\}$, we have

$$X_{t+h} = X_t + Y_{n+1} + (X_{t+h} - X_t - Y_{n+1}).$$

Since there is one only jump at time T_{n+1} for the process X between t and $t+h$, hence $X_{t+h} - X_t - Y_{n+1} = \tilde{X}_{t+h} - \tilde{X}_t$ and

$$|X_{t+h} - X_t - Y_{n+1}|\mathbf{1}_{\{N_t=n,N_{t+h}=n+1\}} \leq \sup_{0\leq u\leq h} |\tilde{X}_{t+u} - \tilde{X}_t|\mathbf{1}_{\{N_t=n,N_{t+h}=n+1\}}. \tag{34}$$

• The first component is

$$X^*_{t+h} - \max(X_t^*, X_t + Y_{n+1})$$

with

$$\begin{aligned} X^*_{t+h} &= \max(X_t^*, X_t + \sup_{t\leq u<T_{n+1}} (\tilde{X}_u - \tilde{X}_t), X_t + Y_{n+1} + (\tilde{X}_{T_{n+1}} - \tilde{X}_t) + \sup_{T_{n+1}\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}})) \\ &= \max(X_t^*, X_t + \sup_{t\leq u<T_{n+1}} (\tilde{X}_u - \tilde{X}_t), X_t + Y_{n+1} + \sup_{T_{n+1}\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_t)). \end{aligned}$$

(a) On the event $\{X_t^* \geq X_t + Y_{n+1}\}$,

$$\begin{aligned} &X^*_{t+h} - \max(X_t^*, X_t + Y_{n+1}) = \\ &0 \vee (X_t + \sup_{t\leq u<T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - X_t^*) \vee (X_t + Y_{n+1} + \sup_{T_{n+1}\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_t) - X_t^*). \end{aligned}$$

Since $X_t \leq X_t^*$:

$$X_t + \sup_{t\leq u\leq T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - X_t^* \leq \sup_{t\leq u\leq T_{n+1}} (\tilde{X}_u - \tilde{X}_t) \leq \sup_{t\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_t)$$

and on the event $\{X_t^* \geq X_t + Y_{n+1}\}$

$$X_t + Y_{n+1} + \sup_{T_{n+1}\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_t) - X_t^* \leq \sup_{T_{n+1}\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_t) \leq \sup_{t\leq u\leq t+h} (\tilde{X}_u - \tilde{X}_t).$$

On this event, globally

$$0 \leq X^*_{t+h} - \max(X_t^*, X_t + Y_{n+1}) \leq \sup_{0\leq u\leq h} |\tilde{X}_{t+u} - \tilde{X}_t|. \tag{35}$$

(b) On the event $\{X_t^* < X_t + Y_{n+1}\}$, the first component is equal to

$$(X_t^* - X_t - Y_{n+1}) \vee (\sup_{t \le u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - Y_{n+1}) \vee (\sup_{T_{n+1} \le u \le t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}})). \tag{36}$$

On this event, the first element in (36) $(X_t^* - X_t - Y_{n+1}) \le 0$ and the third one being non negative, thus the first component is $(\sup_{t \le u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - Y_{n+1}) \vee (\sup_{T_{n+1} \le u \le t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}}))$.

As a conclusion, globally:

$$|X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1})| \mathbf{1}_{\{N_t = n, N_{t+h} = n+1\}} \le \sup_{t \le u \le t+h} |\tilde{X}_u - \tilde{X}_t|. \tag{37}$$

Inequalities (34), (35) and (37) lead to the result. $\square$

6.1 Integrability properties

In the sequel, P is the random field defined by

$$P(b, a; t) := \sum_{k=0}^{N_t} \tilde{p}\left(b - X_{T_k}, a - X_{T_k} - Y_{k+1} \mathbf{1}_{\{t > T_{k+1}\}} - (X_t - X_{\min(t, T_{k+1})}); \min(t, T_{k+1}) - T_k\right) \tag{38}$$

which satisfies

$$p(b, a, t) \le \mathbb{E}[P(b, a; t)]. \tag{39}$$

Proposition 6.2. *Let $\delta \in [1, \frac{3}{2}[$, there exists $(v, \gamma, \beta) \in [0, 1[^3$ and $\xi \ge 0$, such that for all $T > 0$ there exists a constant $C(\delta, T, m)$ satisfying for all $t \in]0, T]$, $b > \max(a, 0)$*

$$\mathbb{E}\left(P(b, a; t)^\delta\right) \le C(\delta, T, m) \left[\frac{1}{t^v} \frac{1}{(2b - a + mt)^\gamma} + \frac{1}{t^\beta} + 1 + t^\xi\right] e^{6(m_-)^2 \delta t} \tag{40}$$

where $x_+ = \max(x, 0)$ and $x_- = \max(-x, 0)$ for any $x \in \mathbb{R}$.

The proof is based on the following three lemmas.

Recall that for all $t > 0$ the law of the pair $(\tilde{X}_t^*, \tilde{X}_t)$ has a density with respect to the Lebesgue measure given by

$$\tilde{p}(b, a; t) = \frac{2(2b - a)}{\sqrt{2\pi t^3}} e^{-\frac{(2b-a)^2}{2t} + ma - \frac{m^2 t}{2}} \mathbf{1}_{\{b > \max(0, a)\}}.$$

We have the following estimations on $\tilde{p}$.

Lemma 6.3. *For all $\alpha \in]0, 1[$, there exists a constant $C(\alpha, m)$ such that for all $t > 0$*

$$\tilde{p}(b, a; t) \le C(\alpha, m) \left[\frac{1}{t^{1 - \frac{\alpha}{2}} (2b - a + mt)^\alpha} + (m_-) \frac{1}{\sqrt{t}}\right] e^{-\frac{(2b - a + mt)^2}{8t} + 6(m_+)^2 t} \mathbf{1}_{\{b > \max(0, a)\}}. \tag{41}$$

Proof. We factorize

$$(2b - a)^2 - 2mat + m^2 t^2 = [a - (2b + mt)]^2 - 4bmt$$

and write

$$\tilde{p}(b, a; t) = \frac{2}{\sqrt{2\pi}} \left[\frac{(2b - a + mt)}{t\sqrt{t}} - \frac{m}{\sqrt{t}}\right] e^{-\frac{(2b - a + mt)^2}{2t} + 2bm} \mathbf{1}_{\{b > \max(0, a)\}}. \tag{42}$$

Let be $C_\beta := \sup_{x \in [0, +\infty[} x^\beta e^{-\frac{x^2}{4}} < +\infty$.

- For $m \leq 0$ and since $b > 0$ we obtain

$$\tilde{p}(b,a;t) \leq \frac{2}{\sqrt{2\pi}} \left[\frac{C_{1+\alpha}}{t^{1-\frac{\alpha}{2}} (2b-a+mt)^{\alpha}} + \frac{m_-}{\sqrt{t}} \right] e^{-\frac{(2b-a+mt)^2}{4t}} \mathbf{1}_{\{b > \max(0,a)\}},$$

and estimation (41) for $m \leq 0$.

- For $m > 0$, using (42)

$$\tilde{p}(b,a;t) \leq \frac{2}{\sqrt{2\pi}} \left[\frac{C_{1+\alpha}}{t^{1-\frac{\alpha}{2}} (2b-a+mt)^{\alpha}} \right] e^{-\frac{(2b-a+mt)^2}{4t} + 2bm} \mathbf{1}_{\{b > \max(0,a)\}}.$$

Since

$$\frac{(2b-a+mt)^2}{8t} - 2bm = \frac{1}{8t} \left((2b-a)^2 + 2(2b-a)mt + m^2t^2 - 16bmt \right),$$

using $2b - a \geq b$ and $m > 0$,

$$\frac{(2b-a+mt)^2}{8t} + 2bm \geq \frac{1}{8t} \left(b^2 + 2bmt + m^2t^2 - 16bmt \right) = \frac{1}{8t} \left((b-7mt)^2 - 48m^2t^2 \right) \geq -6m^2t^2.$$

Thus

$$\tilde{p}(b,a;t) \leq \frac{2}{\sqrt{2\pi}} \left[\frac{C_{1+\alpha}}{t^{1-\frac{\alpha}{2}} (2b-a+mt)^{\alpha}} \right] e^{-\frac{(2b-a+mt)^2}{8t} + 6m^2t} \mathbf{1}_{\{b > \max(0,a)\}}.$$

We obtain the estimation (41) for $m > 0$.

$\square$

Lemma 6.4. *Let $0 < \alpha < 1$ there exists a constant $C(\alpha)$ such that for all $T > 0, \sigma > 0$*

$$\sup_{c \in \mathbb{R}} \mathbb{E} \left(|c + \sigma G|^{-\alpha} e^{-\frac{(c+\sigma G)^2}{2T}} \right) \leq C(\alpha) \frac{T^{\frac{1-\alpha}{2}}}{\sigma}, \tag{43}$$

where G is a standard Gaussian variable.

Proof. First we prove inequality (43) for $\sigma = 1$. Let

$$I(c,T) := \mathbb{E} \left(|c+G|^{-\alpha} e^{-\frac{(c+G)^2}{2T}} \right) = I(c,T,+) + I(c,T,-) \tag{44}$$

where

$$I(c,T,\pm) := \mathbb{E} \left((c+G)_{\pm}^{-\alpha} e^{-\frac{(c+G)^2}{2T}} \right). \tag{45}$$

Using the density of G

$$I(c,T,+) = \int_{-c}^{\infty} (c+g)^{-\alpha} e^{-\frac{(c+g)^2}{2T}} \frac{e^{-\frac{g^2}{2}}}{\sqrt{2\pi}} dg$$

and an integration by part

$$I(c,T,+)=\int_{-c}^{\infty}\frac{1}{1-\alpha}\left[\frac{(c+g)^{2-\alpha}}{T}+g(c+g)^{1-\alpha}\right]e^{-\frac{(c+g)^2}{2T}}\frac{e^{-\frac{g^2}{2}}}{\sqrt{2\pi}}dg.$$

Note that $x\mapsto x^{\beta}e^{-\frac{x^2}{4}}$ is bounded on $\mathbb{R}^+$ by C_β then

$$I(c,T,+)\leq\int_{-c}^{\infty}\frac{1}{1-\alpha}\left[C_{2-\alpha}T^{-\frac{\alpha}{2}}\frac{e^{-\frac{(c+g)^2}{4T}}}{\sqrt{2\pi}}+C_1C_{1-\alpha}T^{\frac{1-\alpha}{2}}\frac{e^{-\frac{g^2}{4}}}{\sqrt{2\pi}}\right]dg.$$

Integrating with respect to g yields

$$I(c,T,+)\leq 2\left[C_{2-\alpha}T^{\frac{1-\alpha}{2}}+C_1C_{1-\alpha}T^{\frac{1-\alpha}{2}}\right]. \tag{46}$$

Using the same lines we obtain

$$I(c,T,-)\leq 2\left[C_{2-\alpha}T^{\frac{1-\alpha}{2}}+C_1C_{1-\alpha}T^{\frac{1-\alpha}{2}}\right]. \tag{47}$$

Plugging inequalities (46) and (47) into (44) yields (43) for $\sigma=1$.
Replace c by $\frac{c}{\sigma}$ and T by $\frac{T}{\sigma^2}$ yields (43). □

Lemma 6.5. *For all $\alpha<1,\beta<1,\delta>0$, $T>0$ there exists a constant $C(\alpha,\beta,\delta,T)$ such that for all $t\in]0,T]$,*

$$\mathbb{E}\left(\mathbf{1}_{\{N_t>0\}}(N_t+1)^{\delta}\left[T_{N_t}^{-\beta}(t-T_{N_t})^{-\alpha}+\sum_{k=0}^{N_t-1}(t-T_{k+1})^{-\alpha}(T_{k+1}-T_k)^{-\beta}\right]\right)\leq C(\alpha,\beta,\delta,T)t^{1-\alpha-\beta}. \tag{48}$$

Proof. Let $S(1,\alpha,\beta,\delta,t)$ be the random variable defined by

$$S(1,\alpha,\beta,\delta,t):=\mathbb{E}\left(\mathbf{1}_{\{N_t>0\}}(N_t+1)^{\delta}\left[T_{N_t}^{-\beta}(t-T_{N_t})^{-\alpha}\right]\right). \tag{49}$$

Note that $(T_{k+1}-T_k)_{k\geq 0}$ are independent variables, identically distributed and T_1 follows an exponential distribution with parameter λ.

Then, $S(1,\alpha,\beta,\delta,t)=\sum_{n=1}^{\infty}S(1,\alpha,\beta,\delta,t;n)$ where for $n\geq 1$:

$$\begin{aligned}&S(1,\alpha,\beta,\delta,t;n) &(50)\\ =&(n+1)^{\delta}\int_{[0,\infty[^{n+1}}(u_1+...+u_n)^{-\beta}(t-u_1-...-u_n)^{-\alpha}\lambda^{n+1}e^{-\lambda(u_1+...+u_{n+1})}\mathbf{1}_{\{u_1+..+u_n\leq t\leq u_1+...+u_{n+1}\}}du_1...du_{n+1}.\end{aligned}$$

We integrate with respect to u_{n+1} between $t-u_1-...-u_n$ and infinity

$$S(1,\alpha,\beta,\delta,t;n)=\lambda^n(n+1)^{\delta}e^{-\lambda t}\int_{[0,\infty[^n}(u_1+...+u_n)^{-\beta}(t-u_1-...-u_n)^{-\alpha}\mathbf{1}_{\{u_1+..+u_n\leq t\}}du_1...du_n.$$

We perform the change of variable $tv_i=u_1+...+u_i$, $i=1,...,n$

$$S(1,\alpha,\beta,\delta,t;n)=\lambda^n t^{n-\alpha-\beta}e^{-\lambda t}\int_{[0,1]^n}(v_n)^{-\beta}(1-v_n)^{-\alpha}\mathbf{1}_{\{v_1<..<v_n\leq 1\}}dv_1...dv_n.$$

We integrate with respect to $v_1, ..., v_{n-1}$

$$S(1,\alpha,\beta,\delta,t;n) = \frac{(n+1)^\delta \lambda^n t^{n-\alpha-\beta}}{(n-1)!} e^{-\lambda t} \int_0^1 (v_n)^{-\beta+n-1}(1-v_n)^{-\alpha} dv_n$$
$$= \frac{(n+1)^\delta \lambda^n t^{n-\alpha-\beta}}{(n-1)!} e^{-\lambda t} B(n-\beta, 1-\alpha) \tag{51}$$

The radius of convergence of entire series $(\sum_{n\geq 1} \frac{(n+1)^\delta \lambda^n t^{n-1}}{(n-1)!} B(n-\beta,1-\alpha))$ is infinite, so the series is continuous on $\mathbb{R}$ and then bounded on $[0,T]$. There exists a constant $C_1(\alpha,\beta,\delta,T)$ such that for all $t \in]0,T]$ adding inequalities (51) yields

$$S(1,\alpha,\beta,\delta,t) \leq C_1(\alpha,\beta,\delta,T) t^{1-\alpha-\beta}. \tag{52}$$

Let $S(2,\alpha,\beta,\delta,t)$ be the random variable defined by

$$S(2,\alpha,\beta,\delta,t) := \mathbb{E}\left(\mathbf{1}_{\{N_t>0\}} (N_t+1)^\delta \sum_{k=0}^{N_t-1} \left[(t-T_{k+1})^{-\beta} (T_{k+1}-T_k)^{-\alpha} \right] \right). \tag{53}$$

Note that $(T_{k+1}-T_k)_{k\geq 0}$ are independent variables, identically distributed and T_1 follows an exponential distribution with parameter λ.

Then,

$$S(2,\alpha,\beta,\delta,t) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} S(2,\alpha,\beta,\delta,t;n,k)$$

where for $n \geq 1$, $0 \leq k \leq n-1$

$$S(2,\alpha,\beta,\delta,t;n,k) \tag{54}$$
$$=(n+1)^\delta \int_{[0,\infty[^{n+1}} (t-u_1-...-u_{k+1})^{-\beta} (u_{k+1})^{-\alpha} \lambda^{n+1} e^{-\lambda(u_1+...+u_{n+1})} \mathbf{1}_{\{u_1+..+u_n \leq t \leq u_1+...+u_{n+1}\}} du_1..du_{n+1}.$$

We integrate with respect to u_{n+1} between $t-u_1-..u_n$ and infinity

$$S(2,\alpha,\beta,\delta,t;n,k) = (n+1)^\delta e^{-\lambda t} \lambda^n \int_{[0,\infty[^n} (t-u_1-...-u_{k+1})^{-\beta} (u_{k+1})^{-\alpha} \mathbf{1}_{\{u_1+..+u_n \leq t\}} du_1...du_n.$$

We perform the change of variable $tv_i = u_1+...+u_i$, $i=1,...,n$

$$S(2,\alpha,\beta,\delta,t;n,k) = (n+1)^\delta t^{n-\alpha-\beta} e^{-\lambda t} \lambda^n \int_{[0,1]^n} (1-v_{k+1})^{-\beta} (v_{k+1}-v_k)^{-\alpha} \mathbf{1}_{\{v_1<..<v_n\leq 1\}} dv_1...dv_n.$$

We integrate with respect to $v_1,...,v_k$ and $v_{k+2},...,v_n$

$$S(2,\alpha,\beta,\delta,t;n,k) = \frac{(n+1)^\delta t^{n-\alpha-\beta} e^{-\lambda t} \lambda^n}{k!(n-k)!} \int_{[0,1]^2} (1-v_{k+1})^{-\beta+n-k} (v_{k+1}-v_k)^{-\alpha+k} dv_k dv_{k+1}$$
$$\leq \frac{(n+1)^\delta t^{n-\alpha-\beta} e^{-\lambda t} \lambda^n}{n!} \frac{k!(n-k)!}{n!} \max\left(1, B(1-\beta, 1-\alpha)\right) \tag{55}$$

since

$$0 \leq \int_{[0,1]^2} (1-v_{k+1})^{-\beta+n-k} (v_{k+1}-v_k)^{-\alpha+k} dv_k dv_{k+1} \leq \max(1, \int_{[0,1]^2} (1-v_{k+1})^{-\beta} (v_{k+1}-v_k)^{-\alpha} dv_k dv_{k+1}).$$

Then, adding inequalities (55) from $k=0$ to $n-1$

$$\sum_{k=0}^{n} S(2,\alpha,\beta,\delta,t;n,k) \le \max\left(1, \frac{\Gamma(-\alpha+1)\Gamma(-\beta+1)}{\Gamma(-\alpha-\beta+1)},\right) \frac{(n+1)^\delta 2^n t^{n-\alpha-\beta} e^{-\lambda t}\lambda^n}{n!}. \tag{56}$$

The radius of convergence of entire series $(\sum_{n\geq 1} \frac{(n+1)^\delta \lambda^n 2^n t^{n-1}}{(n)!})$ is infinite, so the series is continuous on $\mathbb{R}$ and then bounded on $[0,T]$: there exists a constant $C_2(\alpha,\beta,\delta,T)$ such that for all $t\in]0,T]$ and adding inequalities (56) yields

$$S(2,\alpha,\beta,\delta,t) \le C_2(\alpha,\beta,\delta,T) t^{1-\alpha-\beta}. \tag{57}$$

Inequality (48) is a consequence of inequalities (52) and (57). □

Proof. of the proposition: recall

$$P(b,a;t) := \sum_{k=0}^{N_t} \tilde{p}\left(b - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{t>T_{k+1}\}} - (X_t - X_{\min(t,T_{k+1})}); \min(t,T_{k+1}) - T_k\right).$$

- Note that from Lemma 6.3, there exists a constant $C(\alpha,m)$ such that

$$P(b,a;t)\mathbf{1}_{\{N_t=0\}} \le C(\alpha,m)\left[\frac{1}{t^{1-\frac{\alpha}{2}}(2b-a+mt)^\alpha} + (m)_+\frac{1}{\sqrt{t}}\right] e^{6(m_+)^2 t}\mathbf{1}_{\{b>\max(0,a)\}}$$

  Taking $\xi=0$, $\beta=\frac{\delta}{2}$. Since $\delta\in[1,\frac{3}{2}[$ then $2(1-\frac{1}{\delta}) < \frac{1}{\delta}$. If $\alpha = 1-\frac{1}{2\delta}$ then $\gamma := \alpha\delta = \delta - \frac{1}{2} < 1$ and $v := \delta(1-\frac{\alpha}{2}) < 1$. Then, using $(x+y)^\delta \le 2^{\delta-1}[x^\delta + y^\delta]$ there exists a constant $C_0(\delta)$ such that for all $t>0$, $b>\max(a,0)$

$$\mathbb{E}\left(P(b,a;t)^\delta \mathbf{1}_{\{N_t=0\}}\right) \le C_0(\delta)\left[\frac{1}{t^v}\frac{1}{(2b-a+mt)^\gamma} + \frac{1}{t^\beta}\right] e^{6(m_+)^2 t\delta}. \tag{58}$$

- Using estimation (41) and development (38)

$$P(b,a;t) := \sum_{k=0}^{N_t} \tilde{p}\left(b - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{t>T_{k+1}\}} - (X_t - X_{\min(t,T_{k+1})}); \min(t,T_{k+1}) - T_k\right),$$

  $P(b,a;t)\mathbf{1}_{\{N_t>0\}} \le C(\alpha,m)\mathbf{1}_{\{N_t>0\}} \sum_{k=0}^{N_t} P_k(b,a,t)$ where

$$P_k(b,a;t) \le \left[\frac{1}{(t\wedge T_{k+1} - T_k)^{1-\alpha/2} Z_k(t)} + \frac{m_-}{\sqrt{t\wedge T_{k+1} - T_k}}\right] e^{-\frac{Z_k(t)^2}{8t}} e^{6(m_-)^2(\min(t,T_{k+1})-T_k)}$$

  and $Z_k(t) := (2b - a - X_{T_k} + Y_{k+1}\mathbf{1}_{\{t>T_{k+1}\}} + X_t - X_{t\wedge T_{k+1}} + m(T_{k+1}\wedge t - T_k)$.

We define a family $(C_k(t),\sigma_k(t),S_k(t),G_k(t))_{k\in\mathbb{N}\cup\{0\}}$ by
- $(C_k(t),\sigma_k(t),S_k(t),G_k(t)) = (0,0,0,0)$ for $k>N_t$,
- for $k<N_t$:

$$C_k(t) = 2b - a - X_{T_k} + \sum_{i=k+1}^{N_t} Y_i + m(t - T_{k+1}) + m(T_{k+1} - T_k);$$

$$\sigma_k(t) = \sqrt{t - T_{k+1}};\ S_k(t) = T_{k+1} - T_k;\ G_k(t) = \frac{B_t - B_{T_{k+1}}}{\sqrt{t - T_{k+1}}}$$

• for $k = N_t$

$$C_k(t) = 2b - a - \sum_{i=0}^{N_t} Y_i - mT_{N_t} + m(t - T_{N_t});\ \sigma_k(t) = \sqrt{T_{N_t}};\ S_k(t) = t - T_{N_t},;\ G_k(t) = \frac{B_{T_{N_t}}}{\sqrt{T_{N_t}}}.$$

Then, from estimation (41) and the definition of $(C_k(t), \sigma_k(t), S_k(t), G_k(t))$

$$P(b,a;t)\mathbf{1}_{\{N_t \neq 0\}} \leq \mathbf{1}_{\{N_t \neq 0\}} \sum_{k=0}^{N_t} \left[\frac{1}{S_k^{1-\frac{\alpha}{2}}(C_k + \sigma_k G_k)^\alpha} + (m)_- \frac{1}{\sqrt{S_k}} \right] e^{6(m_+)^2 t} \mathbf{1}_{\{b > \max(0,a)\}}$$

Note that $S_k \leq t$ and using Jensen inequality

$$P(b,a;t)^\delta \mathbf{1}_{\{N_t \neq 0\}} \leq \mathbf{1}_{\{N_t \neq 0\}} (N_t + 1)^\delta \sum_{k=0}^{N_t} \left[\frac{1}{S_k^{\delta(1-\frac{\alpha}{2})}(C_k + \sigma_k G_k)^{\alpha\delta}} + (m)_-^\delta \frac{1}{\sqrt{S_k}^\delta} \right] e^{6\delta(m_+)^2 t} \mathbf{1}_{\{b > \max(0,a)\}}$$

Conditionally to $\{N_t \neq 0\}$, the random variable G_k are independent of C_k, σ_k and S_k and their law is the standard Gaussian distribution. Using Lemma 6.5 for $\sigma = \sigma_k$ and $T = S_k$ yields there exists a constant $C(\alpha, \delta)$ such that

$$\mathbb{E}\left(P(b,a;t)^\delta \mathbf{1}_{\{N_t \neq 0\}}\right) \leq$$

$$C(\alpha,\delta)\mathbb{E}\left(\mathbf{1}_{\{N_t \neq 0\}}(N_t+1)^\delta \sum_{k=0}^{N_t - 1} \left[\frac{1}{(T_{k+1} - T_k)^{\delta - \frac{1}{2}} \sqrt{t - T_{k+1}}} + (m)_-^\delta \frac{1}{\sqrt{t - T_{k+1}}^\delta} \right]\right) e^{6\delta(m_+)^2 t} \mathbf{1}_{\{b > \max(0,a)\}}$$

$$+ C_(\alpha,\delta)\mathbb{E}\left(\mathbf{1}_{\{N_t \neq 0\}}(N_t+1)^\delta \left[\frac{1}{T_{N_t}^{\delta - \frac{1}{2}} \sqrt{t - T_{N_t}}} + (m)_-^\delta \frac{1}{\sqrt{T_{N_t}}^\delta} \right]\right) e^{6\delta(m_+)^2 t} \mathbf{1}_{\{b > \max(0,a)\}}.$$

According to Lemma 6.5, there exists a constant $C(\alpha, \delta, T)$ such that for all $t \in]0, T]$

$$\mathbb{E}\left(P(b,a;t)^\delta \mathbf{1}_{\{N_t \neq 0\}}\right) \leq C(\alpha,\delta,T)\left[t^{1-\delta} + t^{\frac{1-\delta}{2}}\right] e^{6\delta(m_+)^2 t} \mathbf{1}_{\{b > \max(0,a)\}}. \tag{59}$$

Adding inequalities (58) and (59) yields inequality (40).

□

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