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► **To cite this version:**

Claude Ponsard. Fuzzy economic spaces. [Research Report] Institut de mathématiques économiques (IME). 1980, 44 p., bibliographie. hal-01527230

**HAL Id: hal-01527230**

**<https://hal.science/hal-01527230>**

Submitted on 24 May 2017

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RP 210 (15)

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N° 43

FUZZY ECONOMIC SPACES

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April 1980

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4, BOULEVARD GABRIEL — 21000 DIJON

27 OCT. 1980



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This paper is to be presented at the First World Regional Science Congress, organized by the Regional Science Association in conjunction with the World University of the World Academy of Art and Science and Harvard University, to be held June 15-25, 1980, at the Roscoe Pound Building of the Harvard Law School, Cambridge, Massachusetts (U.S.A.).

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## SUMMARY

- 0 - Introduction
- 1 - Fuzzy Economic Universes
- 2 - Fuzzy Spatial Behaviours
- 3 - Conclusion
- 4 - Bibliography

## 0 - INTRODUCTION

0.1. Traditional spatial economic analysis is limited to the description of precise spaces.

To say that an economic space is precise means: (1) that this space has, or else, has not given constituent characteristics and (2) that the agents located there prefer, or else, do not prefer one possible action to another.

Proposition (1) implies that an economic space is perfectly delimited and that it can be clearly partitioned into homogeneous subspaces. Proposition (2) implies that economic agents undertake exact economic calculations and optimize, under rigid constraints of resource limitation, objective functions whose arguments are clearly defined.

Thus, traditional spatial economic analysis is based on a binary logic: presence or absence of the space's characteristics, preference or non-preference of agents with respect to possible actions. This logic supposes the principle of the excluded middle.

0.2. However, the real world is imprecise. The observed economic spaces (areas of influence, regions, attraction zones, market areas, etc.) have "more or less" the given characteristics; instead of having frontiers, they have ill-chiselled limits; they partially overlap one another and they do not allow themselves to be subdivided without ambiguity. Likewise, economic agents pursue vague objectives, sometimes incompatible or contradictory, and they appraise imperfectly the constraints which limit their resources.

The analyst who admits that the lights and shades of expression "modify everything" and are, at the same time essential, intends to retain them in full. But he must go beyond the usual literary comments which are often juxtaposed

The author thanks M. Emmanuel ANIZOBA, Docteur ès Sciences économiques of the University of Fribourg (Switzerland) for translating this paper into english. Of course, the author alone is responsible for the text (C.P.)

to scientific analysis and whose purpose is to relativize the conclusions, in other words, to contest implicitly the results. He is bound to give a formalized expression of these nuances and gradations of the real world and he ought to reconcile the imprecision inherent in the latter with the precision of the mathematical model being used.

0.3. It is true that n-ary logics have been in use for some time now: POST (1921), LUKASIEWICZ (1937), MOISIL (1940). But, it is with the recent development of the theory of fuzzy subsets that the elaboration of a spatial economic study, perfectly rigorous and fully formalized, has become possible.

This theory, presented for the first time by ZADEH (1965) [50] is making great strides and penetrates every branch of mathematics. A few primers are now available [18][21][23][39][51] .

0.4. Since 1974, the Institut de Mathématiques Economiques of the University of Dijon, associated with the Centre National de la Recherche Scientifique (France), devotes an important part of its researches to the theory of fuzzy subsets and its applications, especially its applications to spatial economic analysis.

In the Institute, research has followed four directions:

0.4.1. : Firstly, it was absolutely necessary to rigorously formulate the axiomatic framework of the theory of fuzzy subsets [29] , to clearly distinguish between the latter and probability calculus [30][38][41] , to assemble the principal mathematical results [39] , to present the concepts and theorems which are useful to economics [27] and more importantly to spatial economic analysis [28] and to resolve a number of algorithmic problems [40] .

0.4.2. : Next, many types of fuzzy economic spaces were studied: attraction zones for sale-points [8][11] , areas of fuzzy spatial interactions [12][13] , fuzzy regions [31] , french fuzzy regions defined by a fuzzy numeric taxonomy [3][45][46],

fuzzy regional dynamic systems [33] , fuzzy interregional relations [4] , fuzzy hierarchy of a system of central places [32] and fuzzy urban spaces [42] .

0.4.3. : Then, analyses of fuzzy spatial behaviours of the consumer [34][35] and of the producer [36] led to a reformulation of the theories of partial equilibria which prepares the way for that of the theory of general spatial equilibrium and of the optimum [under study] .

0.4.4. : Finally, various contributions have been made to general economics: fuzzy multicriterion analysis [22] , fuzzy decision theory [19][20] and fuzzy econometrics [14] .

0.5. The aim of the present study is not to summarize the totality of these works.

The time has come for the presentation, with all the rigour called for in this new and, for some people, unwonted field, of the scientific foundations of the theory of fuzzy economic spaces in the course of elaboration.

This reconsideration of the foundations of the theory should answer two series of questions:

0.5.1. : On what axiomatic framework is the description of economic universes based? Has it at its disposal specific and novel mathematical instruments, sufficiently pertinent and sophisticated?

0.5.2. : Can the description of fuzzy spatial behaviours of economic agents rely on a coherent and an appropriate type of economic calculation? On what theory of value is a fuzzy economic calculation based?

0.6. The above set of questions command the plan which will be followed: 1 - Fuzzy economic universes; 2 - Fuzzy spatial behaviours.

0.7. Remark: In order to avoid any ambiguity in the notation of mathematical terms, ordinary concepts (non-fuzzy) are underlined, whereas fuzzy concepts are not.

For instance,  $A \subset \underline{E}$  is read: A is a fuzzy subset of the ordinary reference set  $\underline{E}$ . Furthermore:  $\underline{g}(x)$  designates an ordinary function, whereas  $f(x)$  defines a fuzzy function. Likewise:  $[\underline{a}_1, \underline{a}_2]$  designates a non-fuzzy interval, whereas  $[t_1, t_2]$  represents a fuzzy interval.

For lack of space, the results of numerous theorems are cited without demonstrations, but the complete references indicate in what books and articles these demonstrations can be found.

## 1 - FUZZY ECONOMIC UNIVERSES

1.1. Every space is defined by a given set of localities equipped with characteristics and a configuration. A space,  $\underline{S}$ , is thus represented by a triplet as follows:  $\underline{S} = (\text{set of localities, set of characteristics of the localities, configuration of the localities})$ .

1.2. The word "locality" is an undefined term in the axiomatics of spatial analysis [1]. One can replace it by equivalent words such as site, spot, place. It designates, intuitively, a part of the space considered as an elementary unit.

Formally, localities are the points of the space under consideration. A set of localities is denoted by  $\underline{E} = \{e_p\}_{p \in \underline{P}}$  where  $\underline{P}$  is a finite or infinite set of indices.

The non-triviality of  $\underline{E}$  implies that  $\text{Card}(\underline{E}) > 1$ . The cardinality of  $\underline{E}$  occupies an important place in the abstract description of a given space. If the cardinal of  $\underline{E}$  is finite or countable infinite, the abstract space is discrete. If the cardinal of  $\underline{E}$  is non-countable infinite the abstract space is continuous. We shall verify, in paragraph 1.4., the existence of close relations between the cardinality of  $\underline{E}$  and the formal properties of the configurations of localities and, with these properties, the concrete interpretation associated with them.

1.3. The set of localities  $\underline{E}$  is the reference set which represents the support for physical, biological and human processes. Its complete description requires that it be equipped with a set of characteristics which particularize the localities.

1.3.1. The geographical space is equipped with material and immaterial characteristics. The first designate more particularly the elements of climate, relief, air, water, fauna and flora, soil and subsoil, peopling, rural and urban habitat, transportation networks, etc. The second indicate such characteristics as accessibility, viability, aesthetic and sanitary qualities, etc. In a wider sense, the characteristics of manking

(ethnic, languages, modes of life,...) and those of objects (physical, chemical, biological, cultural,..., properties) are likened to the characteristics inherent in the localities where they are durably implanted.

1.3.2. Each of these characteristics has at least one dimension owing to the fact that it imposes conditions on economic activity. One then proceeds from concrete characteristics of the geographical space to those of the economic space which is, as a set equipped with properties, an abstract space. In a wider sense, we assign to the economic space the characteristics of agents (elasticity of demand or of supply over a market segment or surface, propensity of a region to export and to import,...) and of economic goods (agricultural, industrial and commercial characteristics of a zone, touristic character of a town,...).

Precisely, every subset of localities equipped with appropriate characteristics is a particular economic universe. By universe, we mean a spot where an activity is taking place. For instance, a market area is a subset of localities which has the power of continuum and which is equipped with all the characteristics specifying the number, the size and the behaviours of the co-traders, as well as the nature and the properties of the exchanged goods and the conditions of their transportation. In this context, a market area is an economic universe. Likewise, an economic region, a zone of influence of an agglomeration, a zone of attraction of a commercial unit, of a bank or an hospital, etc., are economic universes.

Current language, in spatial analysis, makes use of the term "space" and is obliged to employ the plural and/or unite it with qualificatives in order strip it of its natural ambiguity and to precise what we propose to call economic universes: regional spaces, spaces pertaining to hospitals, etc. Whatever the vocabulary, what is of import is the utilization of rigorously defined concepts.

1.3.3. Usually, one supposes that a universe has or has not a given characteristic. For instance, a region is known to be agricultural or industrial, rich or poor. However, such

affirmations turn out to be too simplifying. They are based on the implicit use of arbitrary thresholds on and after which the presence or the absence of a characteristic is accepted. They lead to partitions of the space which violate the overlappings observed in reality and which correspond to zones of gradual transition from one subset to another. In other words, they lead to a tracing of clear-cut frontiers where only limitrophe and hybrid belts exist.

The dichotomy presence-absence causes the loss of an important amount of information, that which contains the nuances and which bestows it with richness. It is preferable to preserve all this information and to suitably express the fuzziness which envelops the observed universes. Referring back to the above-mentioned example, it is generally more exact to say that a region is simultaneously "not very" agricultural and "very" industrialized, or that it is "rather" rich. The presence, "more or less" clear, of a given characteristic is then fully recognized without loss of information. We thus avoid the construction of arbitrary partitions and the tracing of artificial frontiers.

Let  $\underline{A} = \{ \underline{A}_s \}$ , be the set of characteristics of the localities, with  $s \in \underline{S}$ ,  $\underline{S}$  being a set of indices, finite or not.

Formally, one supposes that a locality (elementary spatial unit) has more or less a given characteristic with which the entire space to which it belongs is equipped.

We define a set of  $v$  mappings, denoted by  $\mu^s$ , with  $s = 1, \dots, v, \dots$  from  $\underline{E}$  into  $\underline{M}$ , where  $\underline{M}$  is a preordered set and, such that:

$$\begin{aligned} \forall s \in \underline{S}, \mu^s : \underline{E} &\longrightarrow \underline{M} \\ \forall p \in \underline{P}, e_p &\longrightarrow \mu^s(e_p) \in \underline{M}. \end{aligned}$$

In this way, we have  $v$  fuzzy subsets,  $A_s$ , of the referential  $\underline{E}$ , such that:

$$\forall s \in \underline{S}, A_s = \left\{ e_p, \mu_{A_s} ; \forall e_p \in \underline{E} : \mu_{A_s}(e_p) \in \underline{M} \right\}.$$

We have,

$$\forall s \in \underline{S} : \text{Card}(\underline{M})^{\text{Card}(\underline{E})} = \text{Card} \mathcal{P}(\underline{E}),$$

where  $\mathcal{P}(\underline{E})$  is the fuzzy power-set of  $\underline{E}$ .

From  $\{0,1\}^{\underline{E}} \subset \underline{M}^{\underline{E}}$ , it follows that:  $2^{\text{Card}(\underline{E})} \subset \text{Card}(\underline{M})^{\text{Card}(\underline{E})}$

We note that the fuzzy power-set of  $\underline{E}$  has the structure of a vectorial lattice, whereas the non-fuzzy power-set (called clear power-set) of  $\underline{E}$ ,  $\mathcal{P}(\underline{E})$ , has the structure of a Boolean lattice.

We have, in this manner, expressed the idea that a locality has more or less a given characteristic.

For example, if  $s=1$  denotes agricultural character and if  $s=2$  denotes industrial character, then  $\underline{E}$  as a whole has the agricultural character and the industrial character. However  $A_1$  is the fuzzy subset of the localities having "more or less" the agricultural character. For a given  $p$ , the function  $\#_{A_1}(e_p)$  takes a high value if the locality  $e_p$  is very agricultural or a low value if the locality  $e_p$  is not very agricultural. Likewise,  $A_2$  is the fuzzy subset of the localities having more or less the industrial character. Etc.

We emphasize that fuzziness characterizes the subsets of the referential  $\underline{E}$ , but the referential itself is an ordinary set.

1.3.4. The theory of fuzzy subsets furnishes the mathematical model appropriate for the elaboration of a formalized theory of fuzzy economic universes. The scrutiny of the axiomatics of the theory of fuzzy subsets indicates that we can: (1) show that the theory of fuzzy economic universes has a rigorous scientific foundation and (2) state the relations of this theory with ordinary spatial economic theories (sure and random cases) and show that this new theory includes the previous ones as particular cases.

1.3.4.1. Every family,  $\mathcal{F}$ , of fuzzy subsets of  $\underline{E}$  forms what we propose to call a  $\mathfrak{Z}$ -algebra over  $\underline{E}$  [29] .

Let  $\{A_s / 1 \leq s \leq \infty\}$  be a family,  $\mathcal{F}$ , of fuzzy subsets of  $\underline{E}$ . We have:

$$A_1 \subset A_2 \subset \dots \subset A_s \subset \dots \quad \text{or} \quad A_1 \supset A_2 \supset \dots \supset A_s \supset \dots$$

and  $\{A_s\}$  is a monotone sequence.

We call  $\mathfrak{Z}$ -algebra any family  $\mathcal{F}$  of fuzzy subsets of  $\mathcal{P}(\underline{E})$ , the fuzzy power-set of  $\underline{E}$ , such that:

$$(1.3.4.1.1.): \underline{E} \in \mathcal{F}; \emptyset \in \mathcal{F}$$

$$(1.3.4.1.2.): \text{If } A_s \in \mathcal{F} \text{ and } \{A_s\} \text{ is monotone, then } \lim_{s \rightarrow \infty} A_s \in \mathcal{F}$$

(1.3.4.1.1.) and (1.3.4.1.2.) are the properties of a monotone family of fuzzy subsets of  $\underline{E}$ .

The pair  $(\underline{E}, \mathcal{F})$  defines a fuzzifiable set.

A fuzzy measure on a fuzzifiable set is a mapping,  $\mu$ , with domain the  $\mathfrak{Z}$ -algebra  $\mathcal{F}$  and co-domain a preordered or ordered and bounded set,  $\underline{M}$ , satisfying the following axioms:

$$(1.3.4.1.3.): \text{Axiom 1 : } \mu(\underline{E}) = 1; \mu(\emptyset) = 0$$

$$(1.3.4.1.4.): \text{Axiom 2 : } \forall s \in \underline{S}, \forall A_s \in \mathcal{F} : \underline{\Lambda} \leq \mu(A_s) \leq \underline{V}$$

where  $\underline{\Lambda}$  and  $\underline{V}$  are, respectively, the lower and upper bounds of  $\underline{M}$ .

$$1.3.4.1.5. : \text{Axiom 3 : } (\forall s \in \underline{S}, \forall A_s \in \mathcal{F}, \forall r, t \in \underline{S}, A_r \subset A_t)$$

$$\Rightarrow \mu(A_r) \leq \mu(A_t) .$$

$$1.3.4.1.6. : \text{Axiom 4 : If } A_s \in \mathcal{F} \text{ and if } \{A_s\} \text{ is monotone, then:}$$

$$\lim_{s \rightarrow \infty} \mu(A_s) = \mu(\lim_{s \rightarrow \infty} A_s) .$$

We note that axiom 4 should not be used unless the set  $\underline{E}$  is infinite. Also, the set  $\underline{M}$  can be non-numeric.

The triplet  $(\underline{E}, \mathcal{F}, \mu)$  defines a fuzzified set.

It is easy to check that numerous particular fuzzified sets can be defined according to the characterization of  $\underline{M}$ :

- Fuzzified sets of ZADEH [50] :  $\underline{M} = [0,1]$  . The interval  $[0,1]$  of  $\mathbb{R}$  is a totally ordered set. It has the structure of a distributive vectorial lattice with respect to the operations  $\wedge$  and  $\vee$  (respectively, the minimum and the maximum of a membership function with respect to an element). In empirical studies, fuzzified ZADEH sets are very much in use.

- Fuzzified sets of GOGUEN [16] :  $\underline{M} = \underline{L}$  , where  $\underline{L}$  is a lattice, partially or totally ordered. According to the properties of lattices, fuzzified sets which are more or less general can be defined distributive lattices, modular lattices (weak distributivity), Boolean lattices, etc.

- Finally, more general fuzzified sets can be constructed by bestowing on  $\underline{M}$  more general structures [21] .

For instance:  $\underline{M} = \{\underline{a}_1, \underline{a}_2\} \subseteq [0,1]$  : theory of phi-fuzzy subsets [32][45] .

$\underline{M} = \mathcal{P}(\underline{Y})$ , where  $\mathcal{P}(\underline{Y})$  is the power-set of the set  $\underline{Y}$ : theory of  $\mathcal{P}$ -fuzzy subsets.

$\underline{M}$  is a preorder,

etc.

It is evident that if  $\underline{M} = \{0,1\}$  , we have the ordinary set theory as a particular case.

1.5.4.2. The axiomatic framework of the theory of fuzzy subsets includes that of the theory of measure sets.

In effect, let  $\{\underline{A}_s / 1 \leq s \leq \infty\}$  be a family,  $\underline{Q}$  , of ordinary subsets of  $\underline{E}$ .

We know that a  $\sigma$ -algebra is any family,  $\underline{Q}$  , of subsets of  $\mathcal{P}(\underline{E})$ , the power-set of  $\underline{E}$ , such that:

$$(1.5.4.2.1.) : \underline{E} \in \underline{Q}$$

$$(1.5.4.2.2.) : (\forall s \in \underline{S}, \forall \underline{A}_s \in \underline{Q} : \underline{A}_s \in \underline{Q}) \Rightarrow \bar{\underline{A}}_s \in \underline{Q}$$

where  $\bar{\underline{A}}_s$  is the complement of  $\underline{A}_s$  in  $\underline{E}$ .

$$(1.5.4.2.3.) : (\forall s \in \underline{S}, \forall \underline{A}_s \in \underline{Q}, \forall r, t \in \underline{S}, \underline{A}_r \cap \underline{A}_t = \emptyset)$$

$$\Rightarrow \bigcup_{s=1}^{\infty} \underline{A}_s \in \underline{Q} .$$

The pair  $(\underline{E}, \underline{Q})$  defines a measurable set.

A positive measure on a measurable set is a mapping,  $m$ , with domain the  $\sigma$ -algebra  $\underline{Q}$  and with co-domain  $\mathbb{R}^+$ , such that:

(1.3.4.2.4.): Axiom 1 :  $m(\emptyset) = 0$

(1.3.4.2.5.): Axiom 2 :  $m(\bigcup_{s=1}^{\infty} \underline{A}_s) = \sum_{s=1}^{\infty} m(\underline{A}_s)$

The triplet  $(\underline{E}, \underline{Q}, m)$  defines a measure set on  $\underline{E}$ .

We note especially that a measure has the additive property given by (1.3.4.2.5), whereas the fuzzy measure has only the monotony property given by (1.3.4.1.5.) However, any family having the additive property has also the property of monotony, but the converse is false. We also observe that a  $\mathfrak{z}$ -algebra over  $\underline{E}$  has not the property of complementation, whereas a  $\sigma$ -algebra has (1.3.4.2.2.)

1.3.4.3. In the same manner, we show that the axiomatics of the theory of fuzzy subsets include those of probability theory as a particular case [30] [38] [41] .

It will suffice to recall that a probability is a particular case of a measure.

Let  $(\underline{E}, \underline{Q})$  be a probabilizable set defined as at points (1.3.4.2.1.) to (1.3.4.2.3.)

A probability on a probabilizable set is a mapping,  $p$ , from  $\underline{Q}$  into  $\mathbb{R}^+$ , such that:

(1.3.4.3.1.): Axiom 1 :  $p(\underline{E}) = 1$

(1.3.4.3.2.): Axiom 2 :  $\forall s \in \underline{S}, \forall \underline{A}_s \in \underline{Q} : p(\underline{A}_s) \geq 0$

(1.3.4.3.3.): Axiom 3 :  $(\forall s \in \underline{S}, \forall \underline{A}_s \in \underline{Q}, \forall r, t \in \underline{S}, \underline{A}_r \cap \underline{A}_t = \emptyset, r \neq t) \Rightarrow p(\bigcup_{s \in \underline{S}} \underline{A}_s) = \sum_{s \in \underline{S}} p(\underline{A}_s)$  .

The triplet  $(\underline{E}, \underline{Q}, p)$  defines a probabilized set. Again, we observe that the properties of a probability measure are particular cases of those of a fuzzy measure.

In addition, after a very careful distinction between the theory of fuzzy subsets and the theory of probability, we can construct a theory of fuzzy random sets which handles the probabilities of fuzzy events [9] [10] [21] [30] [38] [41] [52] . The latter turns out to be fundamental in the elaboration of fuzzy statistical theory.

1.4. Finally, an economic universe is not just any whatever. It has a configuration. In effect, it obeys the laws which induce its development. For example, the locations of traders in a market area are differently arranged according to the structure of the market and according to the mechanism of price formation associated with it: competition, monopoly, oligopoly, etc.

Firstly, an economic universe has a form. Generally, in the case of precise spaces, one uses representations based on elementary geometry or on analytic geometry: Von THUNEN's or ALONSO's concentric circles, WEBER's "triangle of localization", LOESCH's nested hexagons , etc. More generally, one can have recourse to general topology in order to formalize the representation of universes having any forms whatever [26] . In the case of imprecise spaces, one should, likewise, describe the fuzzy topological properties.

Next, if an economic universe is a surface, it has dimension too. If it exists, the measure of the surface leads then to the calculation of its area. In the case of imprecise spaces, one should resolve the problem of estimating the area of a surface having imprecise contours.

Then, the localities belonging to an economic universe occupy positions. They have a positional relation among them. We are thus led to a geometry of position, that is, to the theory of networks [25] . In the case of imprecise spaces, we have to define fuzzy networks.

Finally, having a positional relation among them, the points of an economic universe are separated by a distance or by many distances. Equipped with a distance, a space then becomes a metric space. Under this hypothesis, the distance induces a topology in the space under consideration. In the case of imprecise spaces, one should define fuzzy metric spaces.

In short, a configuration, in this context, is defined by a quadruplet: (form, dimension, position, distance).

Remark: This formulation is close to the one presented, in the case of precise spaces, in reference [17] . However, it concedes neither the same place nor the same role to distance and to the set of characteristics of the localities.

1.4.1. The description of the fuzzy morphology of a given imprecise universe is based on the generalization of the notion of open sets and the concepts derived thereof, from which one defines a fuzzy topology on  $\underline{E}$ .

1.4.1.1. A fuzzy subset,  $A$ , of  $\underline{E}$ , being defined as an element of the set of mappings,  $\underline{M}^{\underline{E}}$ , of  $\underline{E}$ , into a preordered set  $\underline{M}$ , the structure of  $\underline{M}$  induces a topological structure on  $\mathcal{P}(\underline{E})$ , the fuzzy power-set of  $\underline{E}$  [39].

A fuzzy topology on the referential  $\underline{E}$  is a family,  $\mathcal{V}(\underline{E})$ , of fuzzy subsets of  $\underline{M}^{\underline{E}}$ ;  $\mathcal{V}(\underline{E}) \subset \underline{M}^{\underline{E}}$  and its elements satisfy the following axioms:

$$(1.4.1.1.1.) : \emptyset \in \mathcal{V}(\underline{E})$$

$$(1.4.1.1.2.) : \underline{E} \in \mathcal{V}(\underline{E})$$

$$(1.4.1.1.3.) : [\forall r, t \in \underline{S}, A_r \in \mathcal{V}(\underline{E}) \text{ and } A_t \in \mathcal{V}(\underline{E})]$$

$$(1.4.1.1.4.) : \left[ \begin{array}{l} \Rightarrow (A_r \cap A_t) \in \mathcal{V}(\underline{E}) \\ A_s \in \mathcal{V}(\underline{E}), s \in \underline{S}, \underline{S} \text{ countable} \end{array} \right]$$

$$\Rightarrow \bigcup_{s \in \underline{S}} A_s \in \mathcal{V}(\underline{E}) .$$

The pair  $(\underline{E}, \mathcal{V})$  defines a fuzzy topological space. The elements of  $\mathcal{V}(\underline{E})$  are called open fuzzy subsets with respect to  $\underline{E}$ .

1.4.1.2. The union of all the open sets contained in  $A \in \mathcal{P}(\underline{E})$  is the interior of  $A$  denoted by  $A^\circ$ . If  $A$  is an open set, it is its own interior.

1.4.1.3. Provided that the set  $\underline{M}$  has the complement property, we can define the fuzzy subsets which are closed in  $\underline{E}$ . If  $A \in \mathcal{V}(\underline{E})$ , the complement of  $A$ , written  $\bar{A}$ , is called a closed fuzzy subset with respect to  $\underline{E}$ .

However, we saw in 1.3.4.1. that the definition of a  $\mathfrak{Z}$ -algebra on  $\underline{E}$  does not assume the existence of the complement property. In this case and with rigour in mind, it is impossible to construct closed sets. But, we can define a pseudo-complementation. For instance, if  $\underline{M} = [0, 1]$ , the pseudo-complement of  $A$ ,  $\bar{A}$ , is such that:

$$\forall e_p \in \underline{E}, p \in \underline{P}, A \subset \underline{E} : \mu_{\bar{A}}(e_p) = 1 - \mu_A(e_p) .$$

We easily verify that:  $A \cap \bar{A} \neq \emptyset$  and  $A \cup \bar{A} \neq \underline{E}$ . Under these conditions, if  $A \in \mathcal{C}(\underline{E})$ , the pseudo-complement of  $A$  is called a pseudo-closed fuzzy subset with respect to  $\underline{E}$ .

1.4.1.4. The intersection of all the closed (pseudo-closed) sets containing  $A \in \mathcal{P}(\underline{E})$  is the closure of  $A$ , denoted by  $\bar{A}$ . If  $A$  is closed (pseudo-closed), it is its own closure.

1.4.1.5. The exterior of a fuzzy subset  $A \in \mathcal{P}(\underline{E})$ , written  $\text{ext}(A)$ , is the interior of its complement (pseudo-complement).

We have:

$$\text{ext}(A) = \overset{\circ}{\bar{A}} = \bar{\overset{\circ}{A}} .$$

1.4.1.6. The fuzzy frontier of a fuzzy subset  $A$  is a fuzzy subset  $\overset{\star}{A} \in \mathcal{P}(\underline{E})$ , such that:

$$\overset{\star}{A} = \overline{\overset{\circ}{A} \cup \text{ext}(A)} = \overline{\overset{\circ}{A} \cup \overset{\circ}{\bar{A}}} = \overline{\overset{\circ}{A} \cup \bar{\overset{\circ}{A}}}$$

1.4.1.7. We can also define a fuzzy topology on  $\underline{E}$  using the notion of neighbourhood. With respect to  $\underline{E}$ , a fuzzy subset  $A_r$  is a neighbourhood of a fuzzy subset  $A_t$ , written  $V(A_t)$ , if there exists an open fuzzy subset,  $A_q$ , of  $\underline{E}$  such that:

$$A_t \subset A_q \subset A_r .$$

We notice that this definition is different from that of a neighbourhood in ordinary topology. In effect, in the case of non-fuzzy subsets we consider the neighbourhood of a point, whereas in the case of fuzzy subsets we consider the neighbourhood of a set. This difference is explained by the fact that  $A_r$  is an element of  $\underline{M}^E$ .

The family of all the neighbourhoods of  $A_t \in \mathcal{P}(\underline{E})$  is called a system of neighbourhoods of  $A_t$  and denoted by  $\mathcal{V}(A_t)$ .

1.4.1.8. From the above notions, we can generalize the usual properties of general topology and apply them to the fuzzy subsets of a given reference set.

We can then define the conditions for the convergence of sequences of fuzzy subsets and for the continuity of mappings

from one fuzzy topological space into another. It is especially possible to generalize the concept of homomorphism. Any bijection  $f$  of a fuzzy topological space into another, with  $f$  and  $f^{-1}$  continuous, is a fuzzy homomorphism. This notion is fundamental in topology because, a homomorphism is an isomorphism of topological structures. If two fuzzy topological spaces are homomorphic, any property which is true for one is also true for the other. In other words, fuzzy homomorphism preserves the properties of the fuzzy morphologies of the localities. These properties constitute the topological invariants of the fuzzy universe under description [2][47][48][49] .

1.4.2. In the case of precise spaces, if an economic universe has a surface and if, in addition, a measure of this surface (area) exists, we say that it is integrable. The case of imprecise spaces presents the difficulty of estimating areas of surfaces having fuzzy contours. Its resolution is furnished by the concept of fuzzy integral which generalizes the usual concept of the integral.

1.4.2.1. The area of a surface bounded by an ill-defined frontier is obtained by the integration of fuzzy functions of real variables [7] .

A fuzzy valued function,  $f$ , of a variable in a real interval is a function from  $\mathbb{R}$  into  $\mathcal{F}(\mathbb{R})$ , the fuzzy power-set of the real line:

$$\begin{array}{ccc} f : \mathbb{R} & \longrightarrow & \mathcal{F}(\mathbb{R}) \\ x \in [\underline{a}, \underline{b}] \subset \mathbb{R} & \longrightarrow & f(x) = y \end{array}$$

and whose membership function, written  $\mu_{f(x)}$ , is such that:

$$\mu_{f(x)} : \underline{t} \in \mathbb{R} \longrightarrow \mu_{f(x)}(\underline{t}) \in [0, 1]$$

By virtue of this definition,  $y$  is a fuzzy subset.

1.4.2.2. In addition, if we suppose that  $y$  is a normal, convex and piecewise continuous fuzzy subset of  $\mathbb{R}$ , then  $y$  is a fuzzy number [5][6] .

Let  $g_\lambda$  be a function from  $\mathbb{R}$  into  $\mathbb{R}$ :

$$\begin{array}{ccc} g_\lambda : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longrightarrow & g_\lambda(x) \end{array}$$

such that  $\forall x \in \mathbb{R}, \mu_{f(x)} [g_\lambda(x)] = \lambda$

The graph of this function is called a level-curve  $\lambda$ . We assume that the integral  $\int_a^b g_\lambda(x) dx$  exists and that the function  $g_\lambda(x)$  is continuous.

Let  $\underline{L}$  be the set of functions,  $\underline{l}$ , from  $\mathbb{R}$  into  $\mathbb{R}$  such that:

$$(1.4.2.2.1.) : \int_a^b \underline{l}(x) dx \text{ exists}$$

(1.4.2.2.2.):  $\underline{l}$  is the countable union of pieces of level-curves, that is,  $\underline{l} = \bigcup_{i \in \mathbb{N}_0} \underline{l}_i$ , where  $\underline{l}_i$  is continuous. Each curve,  $\underline{l}_i$ , delimits a surface,  $\underline{T}_i$ , whose area is designated by  $\underline{A}_i$ .

The fuzzy integral, written  $I(\underline{a}, \underline{b})$ , of  $f(x)$  over the non-fuzzy interval  $[\underline{a}, \underline{b}]$  is defined by the membership function:

$$\mu_{I(\underline{a}, \underline{b})}(\underline{A}) = \sup_{\substack{\underline{l} \in \underline{L} \\ \underline{A} = \sum_i \underline{A}_i}} \min_i \mu(\underline{A}_i)$$

where  $\mu(\underline{A}_i)$  is the level of the curve  $\underline{l}_i$  bounding  $\underline{T}_i$ .

1.4.2.3. We have the following result [7]:

(1.4.2.3.1.):  $\forall \underline{A} \in \mathbb{R}$ , there exists a curve  $g_\lambda$  of level  $\lambda$  over the interval  $[\underline{a}, \underline{b}]$ , bounding a surface  $\underline{T}$  with an area  $\underline{A}$  such that:

$$\mu_{I(\underline{a}, \underline{b})}(\underline{A}) = \mu_{I(\underline{a}, \underline{b})} \left[ \int_a^b g_\lambda(x) dx \right] = \lambda$$

This proposition guarantees the existence and the uniqueness of the solution.

(1.4.2.3.2.): Remark: This theorem leans on the condition that  $\mu_{f(x)}(t)$  is continuous at  $t$  and at  $x$  and that  $\mu_{f(x)}(t)$  is not locally constant at  $t$  over the intervals  $[\underline{t}_1, \underline{t}_2]$ .

If the continuity at  $t$  of the fuzzy valued function is not satisfied, many level curves  $\lambda$  can coincide. If  $\mu_{f(x)}(t)$  is constant over intervals  $[\underline{t}_1, \underline{t}_2]$ , some level curves  $\lambda$  can degenerate into zones of level  $\lambda$ .

The result given in (1.4.2.3.1.) remains valid for the more general definition of the fuzzy integral  $I(\underline{a}, \underline{b})$ , by putting:

$$\mu I(\underline{a}, \underline{b})(\underline{A}) = \sup_{g: \int_a^b g(x) dx = \underline{T}} \mu(g)$$

where  $\mu(g)$  is the true value of the surface  $\underline{T}$ , of area  $\underline{A}$ , bounded by the axes,  $y = 0$ ,  $x = a$ ,  $x = b$  and the curve  $\underline{g}$  such that:

$$\mu(g) = \min_{x \in [\underline{a}, \underline{b}]} \mu_{f(x)} [g(x)]$$

in other words, the smallest true value of the points on the frontier of the given surface, which is the curve  $\underline{y} = \underline{g}(x)$ .

1.4.2.4. From the afore said the theory of fuzzy integration generalizes the Riemann integral and handles integral calculus in the framework of fuzzy intervals as well as the integrals of functions of a fuzzy variable [7] .

1.4.2.5. Remark: In spite of this, there exists another theory of fuzzy integration which generalizes the Lebesgue integral [44] .

1.4.3. The points of an economic universe are in a positional relation. In the case of precise spaces, we can use a network to formally describe these positional relations and study their properties. As for imprecise spaces, it is easy to construct fuzzy networks. However, their interpretation calls for comment.

1.4.3.1. Let  $\underline{E}$  be a referential and  $\Gamma$  be a correspondence in  $\underline{E}$ . A fuzzy network,  $G$ , is a pair  $G = (\underline{E}, \Gamma)$  such that:

$$\forall e_p \in \underline{E}, \forall e_q \in \Gamma(e_p), p, q \in \underline{P} : \mu_{\underline{E}^2}(e_p, e_q) \in \underline{M}$$

where  $\underline{M}$  is a preordered set. More often than not, one chooses  $\underline{M} = [0, 1]$  .

In a fuzzy network, the value given to an arc or a line expresses the degree of membership of this arc or line to the network. One should thus not mistake a fuzzy network for an ordinary valued network. In the latter case, an arc (or a line) exists or does not exist: we choose  $\underline{M} = \{0, 1\}$  . If the arcs (or the lines) have given values, the latter express other properties.

In a fuzzy network, the existence of all the arcs does not offer the same precision. Between any pair of points, the value given to an arc or a line is a fuzzy measure. The latter indicates the existence of a hierarchy between the arcs or lines which is either given objectively and independently of any human appreciation, or given subjectively and expressing a personal opinion.

We can also state clearly the fuzzy positional relations, such as, for instance: a given locality is "strongly" connected with another situated to the North, "very weakly" with a locality situated to the South, "moderately" with a locality to the East and "weakly" with a locality to the West.

1.4.3.2. We know that one of the principal applications of network theory is the treatment of path problems.

Let  $G \subset \underline{E}^2$  be a finite fuzzy network. A path in  $G$ , denoted by  $c$ , is an ordered  $r$ -tuple, with or without repetition of the vertices of the network, such that:

$$c = (e_{p_1}, e_{p_2}, \dots, e_{p_r})$$

with  $e_{p_k} \in \underline{E}$ ,  $k=1,2,\dots,r$ , and the condition:

$$\forall (e_{p_k}, e_{p_{k+1}}) \in \underline{E}^2 : \mu_G(e_{p_k}, e_{p_{k+1}}) > 0, \quad k=1,2,\dots,r-1.$$

To each path  $c$ , we associate a value,  $v(c)$ , defined by:

$$v(c) = \mu_G(e_{p_1}, e_{p_2}) \wedge \dots \wedge \mu_G(e_{p_{r-1}}, e_{p_r}),$$

which expresses the degree of membership of the path  $c$  to the fuzzy network and which is equal to the minimal value of the membership functions of the arcs at this path.

Let  $C(e_p, e_q)$  be the set of all the paths existing between any two points. We have by definition:

$$C(e_p, e_q) = (e_{p_1} = e_p, e_{p_2}, \dots, e_{p_{r-1}}, e_{p_r} = e_q).$$

We can thus construct "the strongest path", denoted by  $C^*(e_p, e_q)$ , from  $e_p$  to  $e_q$ , by putting:

$$v^*(e_p, e_q) = \bigvee_{C(e_p, e_q)} v(e_{p_1} = e_p, \dots, e_{p_{r-1}}, e_{p_r} = e_q).$$

The strongest path of a fuzzy network is not necessarily unique. In addition, it is evident that it not expresses a flow between two points; it indicates the sequence of arcs which have the strongest degree of membership to the network [32] .

1.4.3.3. Remark: This definition of the strongest path uses the Max-Min composition. More generally, one replaces the operation ( $\wedge$ ) in the expression  $v(c)$  with an operation ( $\star$ ) provided that the latter has the properties of associativity and of monotony. For instance, if ( $\star$ ) is multiplication ( $\cdot$ ), we obtain:

$$v(e_{p_1}, e_{p_r}) = \mu_G(e_{p_1}, e_{p_2}) \cdot \dots \cdot \mu_G(e_{p_{r-1}}, e_{p_r}) .$$

1.4.3.4. From the afore said we can proceed to study the properties of fuzzy networks. One of the most interesting for spatial analysis is the property of connexity which permits us to express the more or less great accessibility of the points of the given fuzzy universe. We generalize the usual concepts of connexity found in network theory [42] .

A fuzzy network is strongly connected at the level  $\alpha$  if and only if, between any pair  $(e_p, e_q) \in \underline{E}^2$ , there exists at least a strongest path with value  $\alpha$ . It is clear that if a network is strongly connected at the level  $\alpha$ , then the same holds true at the level  $\alpha'$  if  $\alpha' \leq \alpha$  .

In order to measure the degree of strong connexity of a fuzzy network  $G$ , at the level  $\alpha$ , we define the connexity number, written  $w_\alpha(G)$ . Let  $w_\alpha(e_p, e_q)$  be the maximum number of disjoint paths between the vertices  $e_p$  and  $e_q$  having the value  $\alpha$ . The connexity number of the fuzzy network,  $G$ , at the level  $\alpha$  is given by:

$$w_\alpha(G) = \bigwedge_{(e_p, e_q)} w_\alpha(e_p, e_q) .$$

It characterizes the diversity of the paths valued  $\alpha$  and joining the two points which are the least well-linked at this level.

If  $w_\alpha(G) = 0$ , the fuzzy network is not strongly connected at the level  $\alpha$ . It could evidently be at the level of  $\alpha'$ , with  $\alpha' < \alpha$  .

If  $w_\alpha(G) = s - 1$ , any two vertices are joined by  $(s - 1)$  distinct paths having a value greater than or equal to  $\alpha$ . This number is especially useful in the study of the articulation of a fuzzy network. In effect,  $w_\alpha(G)$  is the minimum number of elements of a set of the vertices of a fuzzy network whose suppression implies that the latter ceases to be strongly connected at the level  $\alpha$ . In the particular case where  $w_\alpha(G) = 1$ , there exists a vertex, called  $\alpha$ -point of articulation, whose suppression brings about the appearance of a sub-network not strongly connected at the level  $\alpha$ .

Finally, a fuzzy network is semi-strongly connected at the level  $\alpha$  if and only if, for every pair  $\{e_p, e_q\} \in \underline{E}^2$  there exists at least one strongest path valued  $\alpha$  in one of the two directions. It is simply connected at the level  $\alpha$  if and only if, for every pair  $\{e_p, e_q\} \in \underline{E}^2$ , there exists at least one strongest chain with value  $\alpha$ .

1.4.4. Being in a positional relation among themselves, the points of an economic universe are separated by distances.

1.4.4.1. Considering the case of precise spaces, the distances are mappings of  $\underline{E}^2$  into  $\mathbb{R}$  which associate with every pair of the points of  $\underline{E}^2$  a real number and which satisfy the well-known conditions of non-negativity, regularity, symmetry and triangular inequality. A space equipped with a distance,  $\underline{d}$ , is a metric space, written  $(\underline{E}, \underline{d})$ . The problem consists then in choosing a distance compatible with the disposition of the localities (euclidean, rectilinear, circum-radial distances, network-distance, etc.). However, the stringent character of the conditions that a distance must satisfy has sometimes led to the questioning of the expediency to resort to weak metrics in the definition of which the conditions of regularity and/or of symmetry are weakened [37]. But in non-regular and/or non-symmetric spaces, weak distances (deviation, quasi-metric, weak metric) remain precise.

The approximation of a geographical distance by a topological distance or even by a weak distance does not lead to the explanation of the fuzzy character often depicted by the

relation of remoteness between two localities. An economic agent can estimate that two points are "very close", "rather remote", "very distant" with respect to each other, either because he lacks complete information on the objective measure of the geographical distance, or because he indulges in an appreciation into which enter more or less explicit considerations such as the amount of time needed to ply between the points, the characteristics of the means of transport, the negative utility resulting from the journey, etc. Instead of weighting a precise distance in a more or less arbitrary manner by coefficients representing these factors approximately, it is preferable to define a fuzzy distance, denoted by  $d$ .

1.4.4.2. A fuzzy metric space,  $(\underline{E}, d)$ , can firstly be defined in the following way [42] :

Let  $\mathcal{R}$  be a fuzzy binary relation in  $\underline{E}^2$ , that is, a binary relation such that:

$$\forall (e_p, e_q) \in \underline{E}^2, e_p \mathcal{R} e_q \iff \mu_{\mathcal{R}}(e_p, e_q) \in \underline{M}$$

where  $\underline{M}$  is a membership set (generally,  $\underline{M} = [0, 1]$ ) and  $\mu_{\mathcal{R}}(e_p, e_q)$  is the degree of membership of the ordered pair  $(e_p, e_q)$  to the binary relation  $\mathcal{R}$ .

A fuzzy distance is a fuzzy binary relation in  $\underline{E}^2$  satisfying the following axioms:

(1.4.4.2.1.) : Positivity:  $\forall (e_p, e_q) \in \underline{E}^2, \mu_d(e_p, e_q) > 0 \iff e_p \neq e_q$

(1.4.4.2.2.) : Regularity:  $\forall (e_p, e_q) \in \underline{E}^2, \mu_d(e_p, e_q) = 0 \iff e_p = e_q$

(1.4.4.2.3.) : Symmetry:  $\forall (e_p, e_q) \in \underline{E}^2, \mu_d(e_p, e_q) = \mu_d(e_q, e_p)$

(1.4.4.2.4.) : Triangular inequality:  $\forall (e_p, e_q, e_r) \in \underline{E}^3,$

$$\mu_d(e_p, e_r) \leq \mu_d(e_p, e_q) \star \mu_d(e_q, e_r)$$

where  $(\star)$  is an operation to be chosen according to the properties of the fuzzy economic universe under consideration.

If  $(\star) = (V)$  : the relation  $d$  is a fuzzy ultra-metric. In addition, every Max-Min transitive fuzzy binary relation, that is, such that:

$$\forall (e_p, e_q, e_r) \in \underline{E}^3, \mu_{\mathcal{R}}(e_p, e_r) \geq V_{eq} [\mu_{\mathcal{R}}(e_p, e_q) \wedge \mu_{\mathcal{R}}(e_q, e_r)]$$

and which is anti-reflexive and symmetric, is a fuzzy distance. We have in effect:

$$[\mu_{\mathcal{R}}(e_p, e_q) \wedge \mu_{\mathcal{R}}(e_q, e_r)] = 1 - [\mu_{\mathcal{R}}(e_p, e_q) \vee \mu_{\mathcal{R}}(e_q, e_r)]$$

From whence, we have:

$$\mu_{\mathcal{R}}(e_p, e_r) \leq \vee [\mu_{\mathcal{R}}(e_p, e_q), \mu_{\mathcal{R}}(e_q, e_r)]$$

and conversely.

Thus defined, fuzzy distance expresses the imprecise evaluation of the remoteness of the points of a universe, since the membership function,  $\mu_d(e_p, e_q)$ , of the ordered pair  $(e_p, e_q)$ , of localities having the fuzzy distance  $d$ , indicates the more or less great importance of the remoteness of  $e_p$  and of  $e_q$ .

From thence, the concepts of weak distances (deviation, quasi-metric, etc.) can be extended to the cases of fuzzy relations of remoteness. The same holds true in respect of concepts of distance from a point to a subset of points and of distance between two subsets of a metric space.

1.4.4.3. The appreciation of the remoteness of a point from another is formulated in a second manner, different from the previous one, if one supposes that the distance from  $e_p$  to  $e_q$  is "near  $k$ ", where  $k$  is a constant. In spite of its apparent exactitude, such a judgment is in fact characterized by an imprecision concerning the evaluation of the distance [42].

We thus define a precise distance by a mapping,  $\underline{d}$ , from  $\underline{E}^2$  into  $\mathbb{R}^+$  and a fuzzy relation  $\mathcal{R}$  in  $\underline{E}^2$  to which corresponds, by the mapping  $\underline{d}$ , a fuzzy subset  $D \subset \mathbb{R}^+$ . We say that  $D$  is induced by  $\underline{d}$  in  $\mathbb{R}^+$ .

The membership function of  $D$  is given by:

$$\forall z \in D : \mu_D(z) = \vee_{(e_p, e_q) \in \underline{d}^{-1}(z)} [\mu_{\mathcal{R}}(e_p, e_q)] .$$

This membership function expresses a degree of likelihood or of credibility of the appreciation placed on the distance  $\underline{d}$ .

One verifies immediately that, in the particular case where  $\mu_D(z) = 1$ , no indetermination characterizes this appreciation. This definition generalizes the usual notion of distance, which

was not the case in the formulation presented in paragraph 1.4.4.3.

#### 1.4.4.4. Remarks :

(1.4.4.4.1): There are other possible definitions of fuzzy metaspaces which could be obtained from the membership functions and fuzzy binary relations, such as, for instance, the relations of similarity and of dissimilarity [21] .

(1.4.4.4.2): One should be attentive to the fact that some generalizations of the definition of distance have no connection with geographical distance and its approximation by a topological distance. The same holds true, for example, for the generalized euclidean distance and the generalized distance of Hamming between two fuzzy subsets [21][24] . These concepts express the difference which characterizes the elements of a referential with respect to their respective degrees of membership of the two fuzzy subsets.

## 2 - FUZZY SPATIAL BEHAVIOURS

2.1. A spatial economy, considered at any given moment, is completely described by the set of agents implanted in it and by the set of actions which are, a priori, within their reach.

Let  $\underline{G} = \{ \underline{G}_t^{p'} \}$  denote the set of agents indexed by  $t$ , with  $t \in \underline{T}$ ,  $\underline{T}$  being a finite set of indices; these agents are located in localities indexed by  $p'$ , with  $p' \in \underline{P}$ ,  $\underline{P}$  being a set, finite or not, of indices.

To simplify, we denote an agent by the index which identifies him and by the index which refers to his location.

Let  $\underline{H} = \{ \underline{h}_k^{p'} \}$  be the set of possible actions,  $h$ , indexed by  $k$ ,  $k \in \underline{K}$ , where  $\underline{K}$  is a set, finite or not, of indices; these actions are feasible in localities indexed by  $p \in \underline{P}$ , and this, by an agent  $t$  implanted in locality  $p'$ .

Note that the set of indices which refers to the locations of the agents is the same as that which indexes the localities of their possible actions, since the former covers the description of the same space. Generally, however, we have  $p \neq p'$  for an agent  $t$  and a possible action  $h$ .

With the above notations, the framework for the analysis of a spatial economy is specified by the set  $\underline{G}$  and the family  $\{ \underline{h}_k^{p'} \}$ ,

Writting  $m = k.p.p'.t$ , the social state of a spatial economy is an  $m$ -tuple

$$h = ( {}^1_1 h_1^1, \dots, {}^p_k h_t^{p'}, \dots )$$

of the set of all possible social states, denoted by

$$( \prod_t h_t \times \prod_{p'} h^{p'} ) \times ( \prod_k h \times \prod_p P_h ) .$$

2.1.1. In order to describe the theoretic foundations of the analysis of a spatial economy, attention must be focused on the description of individual behaviour with respect to the set of a priori possible actions, for, the study of social behaviours refers to the problem of aggregation of individual behaviours and collective choices.

Under these conditions, one reasons with  $t$  and  $p'$  fixed and it suffices to write  $\underline{H} = \left\{ \begin{matrix} p_h \\ k \end{matrix} \right\}$  to designate the set of possible actions.

We admit, to begin with, that the choice made by the agent, within the set of possible actions, is guided by the preference of this agent with respect to the different actions between which he must opt. In 2.2., we shall consider the introduction of a limitation constraint affecting the set of possible actions.

2.1.2. We recognize fully that, placed in a fuzzy univers, the economic agent does not generally manifest a perfect aptitude to clearly discriminate, among the a priori possible actions, between those he prefers and those he does not prefer. His knowledge about the space in which he lives is incomplete and imperfect, and his information on the actions is insufficient. In addition, even if he is well informed, it does not follow that his behaviour obeys a binary logic of the type: preference-non preference. Human nature is such that the spatial behaviour of the individual is generally imprecise.

Behaviour is thus characterized by a structure  $(\underline{H}, \mathcal{R})$ , where  $\mathcal{R}$  is a fuzzy binary relation between the elements of  $\underline{H}^2$ . This relation is such that:

$$\begin{aligned} p_h \mathcal{R} q_h = \left\{ \begin{matrix} (p_h, q_h), \mu_{\mathcal{R}} ; \forall p_h \in \underline{H}, q_h \in \underline{H} : \\ \mu_{\mathcal{R}} (p_h, q_h) \in \underline{M} \end{matrix} \right\} \end{aligned}$$

where  $\underline{M}$  is a preordered and bounded membership set (a non-

numeric set will do) and  $\mu_{\mathcal{R}}(p_k^h, q_l^h)$  designates the level of preference between the two actions under consideration.

The relation  $\mathcal{R}$  indicates that the agent puts into correspondence two possible actions, and this, at a "more or less" high level of preference. The level expresses the degree of imprecision which affects this correspondence.

This structure of imprecise individual preference  $(\underline{H}, \mathcal{R})$  can also be written as  $(\underline{H}, \succ, \sim)$  in order to properly distinguish between strict preference, denoted by  $\succ$ , and indifference symbolized by  $\sim$ .

2.1.2.1. In the particular case where individual behaviour obeys a binary logic, strict preference is asymmetric and irreflexive.

Generally, an economic agent has, more or less, a preference between two possible actions. If

$\mu_{\mathcal{R}}(p_k^h, q_l^h) > \mu_{\mathcal{R}}(q_l^h, p_k^h)$ , we propose to call  $\mu_{\mathcal{R}}(p_k^h, q_l^h)$  the strong degree of preference of  $p_k^h$  with respect to  $q_l^h$  and to call  $\mu_{\mathcal{R}}(q_l^h, p_k^h)$  the weak degree of preference of  $q_l^h$  with respect to  $p_k^h$ .

In this manner, the structure  $(\underline{H}, \succ)$  is equipped with the fuzzy antisymmetric property à la KAUFMANN [21], because:

$$\forall (p_k^h, q_l^h) \in \underline{H}^2, p_k^h \neq q_l^h, \mu_{\mathcal{R}}(p_k^h, q_l^h) \neq \mu_{\mathcal{R}}(q_l^h, p_k^h)$$

$$\text{or } \mu_{\mathcal{R}}(p_k^h, q_l^h) = \mu_{\mathcal{R}}(q_l^h, p_k^h) = 0.$$

However, the hypothesis according to which

$\mu_{\mathcal{R}}(p_k^h, q_l^h) = \mu_{\mathcal{R}}(q_l^h, p_k^h) = 0$  will not be taken up until

paragraph 2.1.2.2. in the framework of the definition of the structure of indifference and where this hypothesis will be shown to be a particular case of fuzzy symmetry.

For the structure  $(\underline{H}, \succ)$ , what is of import here is that:

$$\forall (p_k^h, q_l^h) \in \underline{H}^2, p_k^h \neq q_l^h, \not\exists (p_k^h, q_l^h)$$

such that

$$\mu_{\mathcal{R}}(p_k^h, q_l^h) = \mu_{\mathcal{R}}(q_l^h, p_k^h) \neq 0.$$

In addition, we admit that imprecise preference is Max-Min transitive in order to subsequently ensure that the set of imprecise choices is not empty (see 2.1.4.).

We have thus:

$$\forall (p_k^h, q_l^h), (q_l^h, r_m^h), (p_k^h, r_m^h) \in (\underline{H}^2)^3 :$$

$$\mu_{\mathcal{R}}(p_k^h, r_m^h) \geq \bigvee_{q_l^h} [\mu_{\mathcal{R}}(p_k^h, q_l^h) \wedge \mu_{\mathcal{R}}(q_l^h, r_m^h)]$$

which means that if the preference for  $p_k^h$  with respect to  $r_m^h$  is at least equal to the maximum of  $q_l^h$  for the smallest preference values of  $p_k^h$  with respect to  $q_l^h$  and of  $q_l^h$  with respect to  $r_m^h$ , then the fuzzy relation  $\mathcal{R}$  is Max-Min transitive. In short, the direct preference between two possible actions is at least as strong as the indirect preferences which require the intervention of a third possible action.

We note that this formulation expresses a transitive property which is weaker than the classical formulation. The latter can be obtained by replacing the Max operator by the boolean sum and the Min operator by the boolean product (which are equivalent in the context of boolean operators).

2.1.2.2. In the particular case where individual behaviour obeys a binary logic, indifference is symmetric and reflexive. Generally, an economic agent can be more or less indifferent between two possible actions. Thus, the structure  $(\underline{H}, \sim)$  is equipped with the property of fuzzy symmetry. We have:

$$\forall (p_k^h, q_l^h) \in \underline{H}^2 : \mu_{\mathcal{R}}(p_k^h, q_l^h) = \mu_{\mathcal{R}}(q_l^h, p_k^h).$$

The property of fuzzy symmetry implies that of fuzzy reflexivity when we have  $l = k$  and  $q = p$ . We have:

$$\forall (p_k^h, p_k^h) \in \underline{H}^2 : \mu_{\mathcal{R}}(p_k^h, p_k^h) \geq 0.$$

The degree of indifference can be zero. In fact, it is possible for the agent to discriminate between any two actions in which he has absolutely no interest, or else, to consider himself in the impossibility of comparing any two actions. In the two cases, the set of choices is empty and the membership function to the relation  $\mathcal{R}$ , of the two considered actions is equal to zero.

An important consequence is deduced from the above. The property of totality of the indifference relation and of the preference relation is guaranteed. It follows that the axiom of comparability need not be stated, for if two actions  $p_k^h$  and  $q_l^h$  are considered to be non-comparable, then  $\mu_{\mathcal{R}}(p_k^h, q_l^h) = 0$ . The situation of non-comparability does not raise specific problems as in the framework of classical theory.

On the other hand, the fact that indifference of zero degree expresses the case of the impossibility to compare two actions, this does not imply that indifference is, a priori, a transitive relation. The same condition prevails under the hypothesis that indifference is of degree greater than zero, but different from one. Imprecise indifference does not necessarily have a structure of equivalence.

There is thus good reason to lay down the condition of Max-Min transitivity of indifference in order to ensure the coherence of individual fuzzy behaviour.

2.1.3. It is easy to verify that fuzzy preference defined as follows:

$$\begin{aligned} \forall (p_k^h, q_l^h) \in \underline{H}^2, \text{ we have : } & \mu_{\mathcal{R}}(p_k^h, q_l^h) > \mu_{\mathcal{R}}(q_l^h, p_k^h) \\ \text{or} & \mu_{\mathcal{R}}(q_l^h, p_k^h) > \mu_{\mathcal{R}}(p_k^h, q_l^h) \\ \text{or} & \mu_{\mathcal{R}}(p_k^h, q_l^h) = \mu_{\mathcal{R}}(q_l^h, p_k^h), \end{aligned}$$

defines a structure  $(\underline{H}, \succ)$  of fuzzy total preorder.

However, contrary to classical results [ 15 ] , if the structure of imprecise preference-indifference,  $(\underline{H}, \preceq)$  is really a fuzzy preorder, it does not follow that indifference is right congruent (respectively left congruent) with respect to strong preference and, hence, that  $(\underline{H}, \sim)$  is a structure of equivalence.

However, there is proof [ 34 ] that if the following two conditions are satisfied:

- (i) the strong preferences, for the indifferent actions  $p_k^h$  and  $q_l^h$  with respect to the indifferent actions  $r_m^h$  and  $s_m^h$  , are at least equal to the level of indifference between the actions  $p_k^h$  and  $q_l^h$
- (ii) the weak preferences, for the indifferent actions  $r_m^h$  and  $s_n^h$  with respect to the indifferent actions  $p_k^h$  and  $q_l^h$  , are strictly equal to the level of indifference between the actions  $r_m^h$  and  $s_n^h$
- then the structure  $(\underline{H}, \sim)$  is a structure of equivalence.

It can be shown that these two conditions respect the property of Max-Min transitivity [ 34 ] .

In other words, within the fuzzy preorder, all the non-fuzzy subsets, denoted by  $\underline{p}_k^H$  ,  $\underline{p}_k^H \subset \underline{H}$  , such that:

$$\forall (p_k^h, q_l^h) \in \underline{p}_k^H \quad , \quad \mu_{\mathcal{R}}(p_k^h, q_l^h) = \mu_{\mathcal{R}}(q_l^h, p_k^h)$$

form between them subrelations of similarity. But a subrelation of similarity is maximal if it is not a subrelation of similarity of any other of the same nature, and this, within the relation under consideration. The property of fuzzy symmetry, in conjunction with the properties of reflexivity and Max-Min transitivity, does not necessarily imply that the maximal subrelations of similarity are disjoint.

On the other hand, if conditions (i) and (ii) are satisfied, then the maximal subrelations of similarity form classes of similarity. We say then that the fuzzy preorder is reducible .

Finally, it can be shown that conditions (i) and (ii) bestow on the structure  $(\underline{H}, \succsim)$  the properties of right and left congruence of imprecise indifference with respect to imprecise strong preference [34] .

The fuzzy preorder being reducible , the relation formed between themselves by the classes of similarity is necessarily antisymmetric, and as such, these classes of similarity form between themselves a fuzzy order relation.

2.1.4. We are now in a position to define an imprecise individual behaviour for decidable choice.

Let  $(\underline{H}, \succsim)$  be a structure of fuzzy preference and  $J \subset \underline{H}$  be a non-empty fuzzy subset of  $\underline{H}$ .

An action  $\frac{p}{k}h \in J$  is said to be optimal with respect to  $J$  if and only if  $\frac{p}{k}h$  is a maximal element of the structure  $(J, \succsim)$  , that is, if and only if:

$$\nexists \frac{q}{l}h \in J \text{ such that } \mu_J(\frac{q}{l}h, \frac{p}{k}h) > \mu_J(\frac{p}{k}h, \frac{q}{l}h) .$$

We denote by  $N(J)$  the set of optimal actions in  $J$ .

An action  $\frac{p}{k}h \in J$  is a best element of  $J$ , if and only if  $\frac{p}{k}h$  is a maximum of the structure  $(J, \succsim)$ , that is, if and only if:

$$\forall \frac{q}{l}h \in J : \mu_J(\frac{p}{k}h, \frac{q}{l}h) \geq \mu_J(\frac{q}{l}h, \frac{p}{k}h) .$$

Let  $S(J)$  stand for the set of best elements of  $J$ ; we propose to call  $S(J)$  the set of imprecise choices.

With the set  $S(J)$  given, we can formulate the description of the imprecise behaviour, associated with the structure  $(\underline{H}, \succsim)$ , of the economic agent. We define the mapping  $S$  of the fuzzy power-set of  $\underline{H}$  into  $\underline{H}$ , such that:

$$\begin{aligned} S : \mathcal{P}(\underline{H}) &\longrightarrow \underline{H} \\ J &\longrightarrow S(J) \\ &= \left\{ \frac{p}{k}h \in J; \forall \frac{q}{l}h \in J; \mu_J(\frac{p}{k}h, \frac{q}{l}h) \geq \mu_J(\frac{q}{l}h, \frac{p}{k}h) \right\}. \end{aligned}$$

Choice is always possible and imprecise individual choice behaviour is decidable, if and only if the mapping  $S$  is defined everywhere.

2.1.5. The imprecise choice behaviour associated with any preference structure can be non-decidable. It is thus necessary to specify the conditions that a fuzzy structure,  $(\underline{H}, \succcurlyeq)$ , must satisfy in order that the associated imprecise choice behaviour be decidable.

2.1.5.1. Let us consider, firstly, the case where  $\text{card}(\underline{H})$  is finite. If  $(\underline{H}, \succcurlyeq)$  is an imprecise preference structure which is a finite structure of fuzzy total preorder, then we can associate it with a decidable imprecise choice behaviour.

In fact, let  $J$  be any fuzzy non-empty subset of  $\underline{H}$ . Since  $\text{card}(\underline{H})$  is finite, it suffices to make a pairwise comparison of its elements in order to obtain the greatest elements of  $J$ . Thus  $S(J)$  is non-empty.

2.1.5.2. We next consider the case where  $\text{card}(\underline{H})$  is infinite. If  $(\underline{H}, \succcurlyeq)$  is an imprecise preference structure which is an infinite structure of fuzzy total preorder, then we can associate it with a decidable imprecise choice behaviour, if and only if the structure satisfies the condition of an ascending chain, that is, if and only if every increasing sequence,  $(\frac{p}{k}h)_{(n)}$ , of elements of  $\underline{H}$  is finite.

Indeed, if by hypothesis the condition of ascending chain is not satisfied, then there exists an increasing sequence,  $(\frac{p}{k}h)_{(n)}$ , of elements of  $\underline{H}$  and the fuzzy subset,  $J$ , of  $\underline{H}$  defined by:

$$J = \left\{ \frac{1}{1}h_1, \dots, \frac{p}{k}h_n, \dots, \mu_J ; \right. \\ \left. \forall \frac{p}{k}h \in \underline{H}, \forall \frac{q}{1}h \in \underline{H} : \mu_J(\frac{p}{k}h, \frac{q}{1}h) \in \underline{M} \right\}$$

has no best element, and hence  $S(J) = \emptyset$ . Imprecise choice behaviour is not decidable. Whence the contradiction.

Conversely, if there exists by hypothesis  $J \in \mathcal{P}(\underline{H}) - \emptyset$  such that  $S(J) = \emptyset$  and if we choose  $\frac{p}{k}h \in J$ , then, as  $\frac{p}{k}h$  is not maximal since  $S(J) = N(J) = \emptyset$ , we can find  $\frac{q}{1}h \in J$  such that:

$$\mu_J(\frac{p}{k}h, \frac{q}{1}h) < \mu_J(\frac{q}{1}h, \frac{p}{k}h).$$

Since  $q_{1h} \in J$  is not maximal, we can find  $r_{mh} \in J$  such that

$$\mu_J(p_{kh}, q_{1h}) < \mu_J(q_{1h}, p_{kh}) < \mu_J(q_{1h}, r_{mh}) < \mu_J(r_{mh}, q_{1h}),$$

and so forth. We thus construct an infinite increasing sequence, which contradicts the condition of ascending chain. Hence, it follows that  $S(J) \neq \emptyset$  for each  $J \in \mathcal{P}(\underline{H}) - \emptyset$ .

2.1.6. It can be verified that the function  $\mu$  is a fuzzy measure on  $\underline{H}$ . To see this, let  $\mathcal{K}$  be a family of fuzzy subsets of  $\underline{H}$ . We have:  $H_1, \dots, H_k, \dots, H_n, \dots \in \mathcal{P}(\underline{H})$  where  $\mathcal{P}(\underline{H})$  is the fuzzy power set of  $\underline{H}$ .

2.1.6.1. Since  $\mu_{\mathcal{R}}(p_{kh}, q_{1h}) = 0$ , if  $p_{kh}$  and  $q_{1h}$  are two actions which are considered uninteresting or incomparable, we have:

$$\emptyset \in \mathcal{K}$$

As  $p_{kh}$  and  $q_{1h}$ ,  $\bigvee_k p_{kh} \in \underline{H}$ ,  $\bigvee_1 q_{1h} \in \underline{H}$ , are elements of the reference set  $\underline{H}$ , we also have:

$$\underline{H} \in \mathcal{K}$$

2.1.6.2. Any sequence,  $\{H_n\}$ , element of  $\mathcal{K}$ , is monotone, because conditions (i) and (ii) defined in section 2.1.3. guarantee that the similarity classes form, among themselves and by inclusion, a fuzzy order relation. It follows that:

$$\lim_{n \rightarrow \infty} H_n = \bigcap_{k=1}^{\infty} H_k \implies \lim_{n \rightarrow \infty} H_n \in \mathcal{K}$$

By 2.1.6.1. and 2.1.6.2., we verify that  $\mathcal{K}$  is a  $\xi$ -algebra over  $\underline{H}$ .

2.1.6.3. The function  $\mu$  is a fuzzy measure over the fuzzifiable set  $(\underline{H}, \mathcal{K})$ .

Indeed:

2.1.6.3.1. By the definition of  $\mu$ , we have:

$$\mu(\emptyset) = 0 \quad \text{and} \quad \mu(\underline{H}) = 1.$$

2.1.6.3.2. The function  $\mu$  taking its values in  $\underline{M}$  which is preordered and bounded, we have:

$$\forall H_k \in \mathcal{K}, \quad \Lambda \leq \mu(H_k) \leq V$$

where  $\Lambda$  and  $V$  are respectively the lower and upper bounds of  $\underline{M}$ .

2.1.6.3.3. The order relation, by inclusion, on  $\mathcal{H}$  implies that:

$$\forall H_j \in \mathcal{H}, \forall H_k \in \mathcal{H} \quad , \text{ if } H_j \subset H_k \quad , \text{ we have :}$$

$$\mu(H_j) \leq \mu(H_k) \quad .$$

2.1.6.3.4. If the set  $\underline{H}$  is infinite, and since each sequence  $\{H_n\}$ , element of  $\mathcal{H}$ , is monotone, then by the theorem of nested sets, we have:

$$\lim_{n \rightarrow \infty} \mu(H_n) = \mu(\lim_{n \rightarrow \infty} H_n)$$

By 2.1.6.3.1., 2.1.6.3.2., 2.1.6.3.3. and 2.1.6.3.4., we verify that  $\mu$  is a fuzzy measure and that the triplet  $(\underline{H}, \mathcal{H}, \mu)$  is a fuzzified set.

2.1.7. The economic interpretation of this result is important. The theory presented here, of fuzzy spatial preference (on possible actions), is neither ordinal nor cardinal.

We propose to call it a "valuation theory of preference", the term "valuation" being taken to mean "fuzzy measure".

Now, if  $\underline{M}$ , the membership set to which the function  $\mu$  belongs, is equal to  $\{0,1\}$ , we fall on the ordinal conception of preference as a particular case. The agent prefers or does not prefer one action to another. Therefore,  $\mu$  is a boolean characteristic function.

In the same manner, if the function  $\mu$  is a measure proper, that is, if the  $\mathcal{F}$ -algebra over  $\underline{H}$  is a  $\sigma$ -algebra over  $\underline{H}$ , and if  $\mu$  has the additive property, we find the cardinal conception of preference as a particular case. The agent measures by how much he prefers one action to another action and he adds up his preferences for two or more actions.

Following the same reasoning, if the function  $\mu$  is a probability, that is, if the  $\mathcal{F}$ -algebra over  $\underline{H}$  is a  $\sigma$ -algebra over  $\underline{H}$  and if  $\mu$  verifies the axioms of BOREL-KOLMOGOROV, we find the stochastic theory of preference as a particular case. A probability is a particular measure.

We thus verify the generality of the "valuation theory of preference".

2.2. The set of possible actions has, until now, been defined in purely physical terms. But owing the scarcity which fundamentally characterizes the means at the disposal of an economic agent, there always exists a limitation constraint affecting this set. Individual behaviour is then analysed in the framework of economic calculus which consists of the optimization of an objective function under resource limitation constraint.

In consumer theory, the set  $H$  becomes a set of possible consumptions and the resource limitation constraint is a budget constraint. The optimal demand is that which optimizes preference (which maximizes utility eventually) over the set of efficient consumptions, and this, while taking into account the budget constraint.

In producer theory the set  $H$  becomes a set of possible productions and the constraint becomes technological. The optimal supply is that which, respecting the technological constraint, maximizes profit over the set of efficient productions.

Now, it can be shown that if the preferences are fuzzy, the existence of a budget constraint implies that all the possible consumptions are more or less efficient. Formally, the fuzzy subset of more or less efficient consumptions plays the role of a fuzzy constraint. Consumer's equilibrium is attained for a fuzzy demand which optimizes imprecise preference (which maximizes fuzzy utility eventually) under this fuzzy constraint [ 34 ][ 35 ] .

In the same manner, we show that all the possible productions are more or less efficient and that the producer's equilibrium is attained for a fuzzy supply which, under this fuzzy constraint, raises the profit to a more or less high level [ 36 ] .

2.2.1. These models of fuzzy spatial equilibrium of the consumer and the producer are based on specifications which are peculiar to their respective fields of study. We show here that they are particular cases of the economic calculus of optimizing a fuzzy objective function under a fuzzy constraint. The problem boils down to finding, within the set of possible decisions that the agent can take a priori, that decision which best satisfies the fuzzy objective function as well as the fuzzy constraint.

2.2.1.1. From now onwards, we put  $\underline{M} = [0,1]$ .

Let  $\underline{H}$  be a countable set of possible actions denoted by  $k^h$ , with  $k = 1, \dots, z$ , and which are realizable within a set of localities designated by  $p$ , with  $p = 1, \dots, s$ . We have  $\underline{H} = \{k^h\}$ .

A fuzzy objective is a fuzzy subset of  $\underline{H}$  denoted by  $H$ , such that:

$$H = \left\{ k^h, \mu_H ; \forall k^h \in \underline{H} : \mu_H(k^h) \in [0,1] \right\}.$$

A fuzzy constraint is a fuzzy subset,  $C$ , of  $\underline{H}$ , such that:

$$C = \left\{ k^h, \mu_C ; \forall k^h \in \underline{H} : \mu_C(k^h) \in [0,1] \right\}.$$

A fuzzy decision is a fuzzy subset,  $F$ , of  $\underline{H}$ , such that:

$$F = \left\{ k^h, \mu_F ; \forall k^h \in \underline{H} : \mu_F(k^h) \in [0,1] \right\}.$$

By definition, an optimal fuzzy decision is the intersection (when this is not empty) of a fuzzy objective (fuzzy subset of possible actions totally preordered with the aid of a choice criterion) and a fuzzy constraint (fuzzy subset of more or less efficient actions).

Thus:  $F = H \cap C$ , and the problem reduces to finding:

$$\sup_{k^h \in \underline{H}} \mu_F(k^h) = \sup_{k^h \in \underline{H}} [\mu_H(k^h) \wedge \mu_C(k^h)]$$

2.2.1.2. It can be shown that [23][35] :

$$\sup_{\substack{p_h \in \underline{H} \\ k}} \mu_F(p_h) = \sup_{\alpha \in [0,1]} [\alpha \wedge \sup_{\substack{p_h \in \underline{C}_\alpha \\ k}} \mu_H(p_h)]$$

where

$$\underline{C}_\alpha = \left\{ \substack{p_h \\ k} ; \substack{p_h \in \underline{H} \\ k} , \mu_C(\substack{p_h \\ k}) \geq \alpha \right\} \text{ with } \alpha \in [0,1].$$

$\underline{C}_\alpha$  is called the set of  $\alpha$ -cuts of the fuzzy subset C.

By this theorem, the problem of finding the fuzzy optimal decision is reduced to that of finding the extremum of a scalar function.

Let  $\varphi$  be a function, such that:

$$\begin{array}{ccc} \varphi : [0,1] & \longmapsto & [0,1] \\ \alpha & \longmapsto & \varphi(\alpha) = \sup_{\substack{p_h \in \underline{C}_\alpha \\ k}} \mu_H(p_h) \end{array}$$

and let  $\Psi$  be a function, such that:

$$\begin{array}{ccc} \Psi : [0,1] & \longmapsto & [0,1] \\ \alpha & \longmapsto & \Psi(\alpha) = \alpha \wedge \Psi(\alpha) . \end{array}$$

Then:

$$\sup_{\substack{p_h \in \underline{H} \\ k}} \mu_F(p_h) = \sup_{\alpha \in [0,1]} \Psi(\alpha)$$

After establishing that the function  $\varphi$  has the following properties:

$$(1) \quad \varphi(0) = \sup_{\substack{p_h \in \underline{H} \\ k}} \mu_H(p_h)$$

$$(2) \quad \alpha \leq \beta \implies \varphi(\alpha) \geq \varphi(\beta) ,$$

it can be shown [23][35] that, if  $\varphi$  is continuous and decreasing over  $[0,1]$ , then:

(i)  $\varphi$  has a fixed point, that is:

$$\exists \bar{\alpha} \in [0,1] / \varphi(\bar{\alpha}) = \bar{\alpha} \text{ and } \bar{\alpha} = \sup_{\substack{p_h \in \underline{C}_{\bar{\alpha}} \\ k}} \mu_H(p_h)$$

$$(ii) \quad \sup_{\substack{p_h \in \underline{H} \\ k}} \mu_F(p_h) = \bar{\alpha} .$$

Using this theorem, the solution to the problem of finding  $\sup_{\substack{p_h \in \underline{H} \\ k}} \mu_F(\frac{p_h}{k})$  amounts to finding  $\bar{\alpha} = \sup_{\substack{p_h \in \underline{C} \\ k}} \mu_H(\frac{p_h}{k})$ .

2.2.1.3. We know that:

$$\begin{aligned} \sup_{\substack{p_h \in \underline{H} \\ k}} \mu_F(\frac{p_h}{k}) &= \sup_{\substack{p_h \in \underline{H} \\ k}} [ \mu_H(\frac{p_h}{k}) \wedge \mu_C(\frac{p_h}{k}) ] \\ &= \sup_{\substack{p_h \in \text{supp}\underline{C} \\ k}} [ \mu_H(\frac{p_h}{k}) \wedge \mu_C(\frac{p_h}{k}) ] \end{aligned}$$

where  $\text{supp}\underline{C}$  stands for the support of  $C$ , that is, the subset, such that:

$$\text{supp}\underline{C} = \left\{ \frac{p_h}{k} ; \forall \frac{p_h}{k} \in \underline{H}, \mu_C(\frac{p_h}{k}) > 0 \right\}.$$

By virtue of propositions (i) and (ii):

$$\sup_{\substack{p_h \in \text{supp}\underline{C} \\ k}} [ \mu_H(\frac{p_h}{k}) \wedge \mu_C(\frac{p_h}{k}) ] = \sup_{\substack{p_h \in \underline{C} \\ k}} \mu_H(\frac{p_h}{k}),$$

and,  $\frac{\underline{C}}{\bar{\alpha}} \subset \text{supp}\underline{C}$ , since  $\bar{\alpha} \neq 0$ .

In order to calculate a supremum, we have to find an appropriate set  $\underline{A}$ , such that:

$$\frac{\underline{C}}{\bar{\alpha}} \subset \underline{A} \subset \text{supp}\underline{C}.$$

To this end, we use the following result [23][35]:

$$\sup_{\substack{p_h \in \underline{A} \\ k}} \mu_H(\frac{p_h}{k}) = \sup_{\substack{p_h \in \underline{H} \\ k}} \mu_F(\frac{p_h}{k})$$

with  $\underline{A} = \left\{ \frac{p_h}{k} ; \frac{p_h}{k} \in \underline{H}, \mu_C(\frac{p_h}{k}) \geq \mu_H(\frac{p_h}{k}) \right\}$ .

2.2.2. Finally, it remains to study the continuity conditions for the function  $\varphi(\alpha) = \sup_{\substack{p_h \in \underline{C}_\alpha \\ k}} \mu_H(\frac{p_h}{k})$ .

To this end, we impose the condition of strict convexity on the fuzzy subset  $C$ . A fuzzy subset,  $C$ , is strictly convex if and only if the membership function is strictly quasi-concave, that is, if and only if:

$$\forall p_h \in \underline{K}, \forall q_h \in \underline{K}, p_h \neq q_h,$$

$$[\mu_C(q_h) > \mu_C(p_h)] \implies \forall \lambda \in ]0,1[ ,$$

$$\mu_C[\lambda p_h + (1 - \lambda) q_h] > \mu_C(p_h) .$$

A theorem due to ASAI and TANAKA [23] states, under the above hypothesis, a sufficient condition for the continuity of  $\varphi(\alpha)$  :

if the fuzzy subset, C, is strictly convex, then

$$\varphi(\alpha) = \sup_{\substack{p_h \in C_\alpha}} \mu_H(p_h) \quad \text{is continuous .}$$

2.2.3. Thus, at equilibrium (which is also an optimum), the fuzzy decision optimizes the imprecise objective-function under the fuzzy limitation constraint placed on the possible actions. In the economic space, the locality, where the optimal action, compatible with the restriction imposed on the set of possible actions, is realized, is simultaneously determined.

### 3 - CONCLUSION

3.1. We have verified that the fuzziness which characterizes the real world should no more constitute a setback to scientific analysis. Fuzzy economic spaces can be the subject of a theory presenting the same rigour as the traditional theory of precise economic spaces.

3.2. The general character of the theory of fuzzy economic spaces is established, for the mathematical model which underlies it contains, as particular cases, the theory of sure sets and that of random sets. It is thus unnecessary to match fuzzy spaces against precise spaces. The latter are limiting cases of the former. The theory of fuzzy economic spaces embodies that of ordinary spaces.

3.3. However, interesting though it may be, this aptitude for generality is not the principal quality of the theory of fuzzy economic spaces. It should be emphasized that the latter contains new and considerable results: it renews the theory of value, it modifies the theory of spatial behaviours and it deepens the description of economic universes and of their configurations.

In addition, on all these points, the analysis gains in realism without loss of rigour.

3.4. Certainly, numerous and delicate researches should still be pursued in order to cover a domain as vast as this: the elaboration of fuzzy spatial theories of equilibrium, of the optimum and of disequilibrium and the introduction of time and some dynamics seem, notably, to be tasks that should have priority. But, we can anticipate that the analysis of fuzzy economic spaces, which is based on axiomatic foundations and which has at its disposal the efficient and powerful tools that it needs, will enjoy the theoretical developments that it solicits and bring its assistance to the solution of decision problems.

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