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Equity analysis of public investments : pure and mixed game-theoretic solutions

Heinz Schleicher

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EQUITY ANALYSIS OF PUBLIC INVESTMENTS :
PURE AND MIXED GAME-THEORETIC
SOLUTIONS

Heinz SCHLEICHER
April 1979

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EQUITY ANALYSIS OF PUBLIC INVESTMENTS :
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1. Introduction
2. Solutions proposed by various authors
 - 2.1. Separable Costs Remaining Benefits Method (SCRB)
 - 2.2. Minimum Costs Remaining Savings Method (MCRS)
 - 2.3. The Shapley Value
3. The Weighted Value
4. Mixed Solutions
 - 4.1. The Core exists
 - 4.1.1. A-fair and acceptable solution (type A)
 - 4.1.2. Acceptable and almost a-fair solution (type B)
 - 4.1.3. A-fair and forced acceptable solution (type C)
 - 4.2. The Core does not exist
 - 4.2.1. One-stage solutions
 - 4.2.1.1. Separable Costs Remaining Benefits Method
 - 4.2.1.2. The Weighted Value
 - 4.2.1.3. A-fair and acceptable solution
 - 4.2.2. Two-stage solutions
 - 4.2.2.1. Minimum Costs Remaining Savings Method
 - 4.2.2.2. Conditional a-fair and acceptable solution
 - 4.2.2.3. Conditional acceptable and almost a-fair solution
 - 4.2.2.4. Conditional a-fair and forced acceptable solution
5. Conclusion
6. Appendix
7. References

EQUITY ANALYSIS OF PUBLIC INVESTMENTS :
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1. INTRODUCTION

Contemporary cost-benefit and cost-effectiveness analysis of public investments often deal with complex situations. There may be not one purpose or objective but many as well as more than one decision-maker. In such cases there exists an efficiency and an equity problem to be solved. The efficiency problem deals with the optimal capacity of the public investment, given all purposes. The equity problem deals with the question of how to allocate costs among different purposes or objectives (cost allocation problem) and/or among the various decision-makers (cost sharing problem).

In section 2. a brief review will be given of some important traditional and contemporary procedures which deal with cost allocation or cost sharing problems in a game-theoretic frame-work. These methods do not always seem satisfactory. Thus, in section 3. a new solution concept is proposed : the weighted value of an n-person cooperative game. But the weighted value may be in contradiction to the core of the game. Then mixed solutions are suggested which reconcile both pure solution concepts (section 4). In section 4.1. it is supposed that the core exists. Then either the weighted value belongs to the core or it does not. In the second case a tax-subsidy scheme may be devised

such that the weighted value is modified to be in the core (section 4.1.2.), or the core is modified such that the weighted value is an element of this modified core (section 4.1.3.). If the core does not exist the weighted value is the solution (section 4.2.1.). However, one may calculate the fictitious strong ϵ -core and thus calculate conditional mixed solutions (section 4.2.2.).

2. SOLUTIONS PROPOSED BY VARIOUS AUTHORS

2.1. Separable Costs Remaining Benefits Method (SCRB)

Let N be the set of decision-makers or objectives and SC_N a coalition of decision-makers or objectives. Suppose $C(S)$, SC_N , is the cost function (or characteristic function of a n -person cost game). Let $B(S)$ be the benefits to a coalition of decision-makers. The vector $X = [x(1), \dots, x(n)]$ is the final cost allocation to the n decision-makers. Then there are 8 criteria ⁽¹⁾ which determine the final cost allocation, given the SCR. These criteria are :

- (1) $B(N) \geq C(N)$
- (2) $B(N) - C(N) \geq B(S) - C(S) \quad \forall S \subseteq N$
- (3) $B(i) \geq SC(i) \quad \forall i \in N$

where $SC(i) = C(N) - C(N - \{i\}) \quad \forall i \in N$

$SC(i)$ is the separable cost of decision-maker $i \in N$.

These three criteria determine economic efficiency or the optimal capacity of the project.

(1) Heaney (1978). These 8 criteria are in fact based on Ransmeier's criteria (1942) and on the various techniques proposed by the Inter-Agency Committees (Federal Inter-Agency Committee, 1950; Inter-Agency Committee, 1958; Water Resources Council, 1962).

The equity or cost allocation criteria are the following.

$$(4) \quad x(i) \leq \text{Min} [B(i), C(i)] \quad \forall i \in N$$

$$(5) \quad \sum_{i \in S} x(i) \leq \text{Min}[B(S), C(S)] \quad \forall S \subset N$$

$$(6) \quad x(i) \geq SC(i) \quad \forall i \in N$$

Let NSC be the non separable costs defined as

$$NSC = C(N) - \sum_{i \in N} SC(i)$$

and let $\beta(i)$, $i \in N$, be an apportioning factor defined as

$$\beta(i) = \frac{[\text{Min}(B(i), C(i)) - SC(i)]}{\sum_{i \in N} [\text{Min}(B(i), C(i)) - SC(i)]} \quad \forall i \in N$$

and $\sum_{i \in N} \beta(i) = 1.$

Then

$$(7) \quad x(i) = SC(i) + \beta(i) (NSC)$$

and

$$(8) \quad \sum_{i \in N} x(i) = C(N)$$

Conditions (4), (5) and (8) define the core of this cost game.

2.1.1. Definition : *The core of a cost-game is defined as the set of imputations such that*

$$\sum_{i \in S} x(i) \leq C(S) \quad \forall S \subset N$$

and $\sum_{i \in N} x(i) = C(N)$

As Heaney has pointed out, condition (3) is reasonable for convex cost games.

2.1.2. Definition : A cost game is convex if

$$C(SU\{i\}) - C(S) \geq C(TU\{i\}) - C(T) \\ \forall i \in N \text{ and } \forall S \subseteq T \subseteq N - \{i\}$$

In this case marginal cost, or separable cost, is lowest. However, if the game is not convex, $SC(i)$ is generally higher than the lowest marginal cost which may induce inefficiencies if one still applies criterion (3). It is one of the reasons why Heaney ⁽²⁾ suggests his "Minimum Cost Remaining Savings Method".

2.2. Minimum Cost Remaining Savings Method (MCRS)

There is another reason why SCRB may not be a reasonable method for allocating costs among decision-makers or objectives. Condition (4) states that

$$(4) \quad x(i) \leq \min[B(i), C(i)] \quad \forall i \in N$$

Now $C(i)$ may be unfeasible with respect to the core. However, this is true only for non-convex games ⁽³⁾. But in general, (4) has to be changed into (4')

$$(4') \quad x(i) \leq \min[B(i), x(i)_{\max}] \quad \forall i \in N$$

(2) Heaney (1978)

(3) Schleicher (1978 I)

This, in addition to the redefinition of $SC(i)$ in (3), to (3')

$$(3') \quad B(i) \geq x(i)_{\min} \quad \forall i \in N$$

changes $\beta(i)$ into

$$\beta'(i) = \frac{[\text{Min}(B(i), x(i)_{\max})] - x(i)_{\min}}{\sum_{i \in N} [\text{Min}(B(i), x(i)_{\max}) - x(i)_{\min}]} \quad \forall i \in N$$

RC is now defined as

$$RC = C(N) - \sum_{i \in N} x(i)_{\min}$$

and (7) becomes (7')

$$(7') \quad x(i) = x(i)_{\min} + \beta'(i) (RC)$$

$x(i)_{\min}$ and $x(i)_{\max}$ are determined by linear program(1) and (2), respectively (4).

$$\begin{array}{lll} \text{Program (1)} & \text{Min } x(i) & \forall i \in N \\ & \text{s.t. } x(i) \leq \text{Min} [B(i), C(i)] & \forall i \in N \\ & \sum_{i \in S} x(i) \leq \text{Min} [B(S), C(S)] & \forall S \subset N \\ & \sum_{i \in N} x(i) \geq C(N) & \\ & x(i) \geq 0 & \forall i \in N \end{array}$$

(4) Heaney (1978). These programs are based on Bondareva (1963), Charnes and Kortanek (1966), Shapley (1967), Scarf (1967).

Shapley has shown that this value exists and is unique (7).

Loehman, Pingry and Whinston (8) have used the Shapley value as a cost allocation device. These authors applied it to river pollution caused by three cities.

However, there are two major drawbacks associated with the Shapley value. The first : it may not be in the core. As the core concept expresses individual and group rationality in its purest sense which amounts to free-market bargaining, rational players would refuse a cost allocation which does not belong to the core. Thus one will have to find a method which makes both of them compatible if one thinks of the Shapley value as a solution to the equity problem. This method should imply a minimum necessary intervention in the power structure of a given situation. The instrument will be a tax-subsidy system (section 4.1.).

The second major disadvantage, at least for a large class of public investments, is the partially equitarian nature of the Shapley value. To see this consider the following interpretation of the Shapley value (9) (weighted value). One calculates the net value of each coalition and apportions it equally to all members of the coalition. The net value of a coalition S , where $N = \{1, \dots, 3\}$, is defined as $v(S) - v(S - \{i\}) - v(\{i\})$.

(7) Shapley (1953)

(8) Loehman, Pingry, Whinston (1973)

(9) Oral communication with M.Maschler (1976)

3. THE WEIGHTED VALUE (W.V.(Γ))

Let

$$C_S^i = [C(\{i\}) - C(\{\emptyset\})] \quad \text{for } S = \{i\} \\ i = 1, \dots, 3$$

$$C_S^i = \frac{1}{|S|} [C(S) - C(\{i\}) - C(\{j\})] \\ \text{for } |S| = 2 \\ i \in S \\ j \in S \\ i \neq j$$

$$C_N^i = \frac{1}{|N|} [C(N) - \sum_{i=1}^n \sum_{\substack{S \subseteq N - \{i\} \\ S \ni i}} C_S^i]$$

and $C_S^i \geq 0$ for $i = 1, \dots, 3$
 $S \subseteq N$

$x(i)$ is now defined as

$$x(i) = \sum_{\substack{S \subseteq N \\ S \ni i}} C_S^i \quad \text{and} \quad \sum_{i \in N} x(i) = C(N)$$

It turns out that the classical Shapley value and this weighted value are identical for three-person (cost) games ⁽¹⁰⁾. However, this is no more true for $n > 3$. For $n > 3$ the two values generally differ.

To compute the weighted value for $n > 3$ one simply adds the following condition.

$$C_S^i = \frac{1}{|S|} [C(S) - \underset{i \in S}{\text{Min}} \{C(S) - \{i\}\} - \underset{i \in S | C(S) - \{i\} = \text{Min}}{\text{Max}} \{C(\{i\})\}]$$

$$\text{for } 3 \leq |S| \leq |N| - 1 \\ i \in S$$

$$\text{and } C_S^i \geq 0$$

The last term $\text{Max} \{C(\{i\})\}$ takes care of the fact that there may be more than one coalition $S - \{i\}$ with the same value. In this case one has to choose a proper $\{i\}$. It is reasonable to select $\text{Max} \{C(\{i\})\}$ because otherwise one would distribute a net gain among the members of the coalition which is too small. C_S^i should be the result of the best opportunity coalition S has.

The salient feature of this weighted value is that it clearly shows : (1) that the Shapley value is based on the power of each coalition and (at least for the three-person case): (2) that it is based on an equalitarian distribution of the net gains of a coalition. This second feature may be justified if the players are homogeneous and of equal size or if the net gains may not be apportioned according to size, productivity or any other measure.

(10) Schleicher (1978 II, annexe 2)

However, if there exists such a common measure then the equitarian rule should be disposed of in favor of an apportioning rule which takes into account the individual contribution of each member to the coalitional net gain. Fortunately, for many public investments such a measure exists for instance for waste water treatment facilities : the amount of m³ of a standardized polluted water. If several cities join to build a common sewage treatment plant then the economies of scale may be apportioned according to the size (in terms of the amount of m³ of sewage of a certain degree of pollution) of the cities.

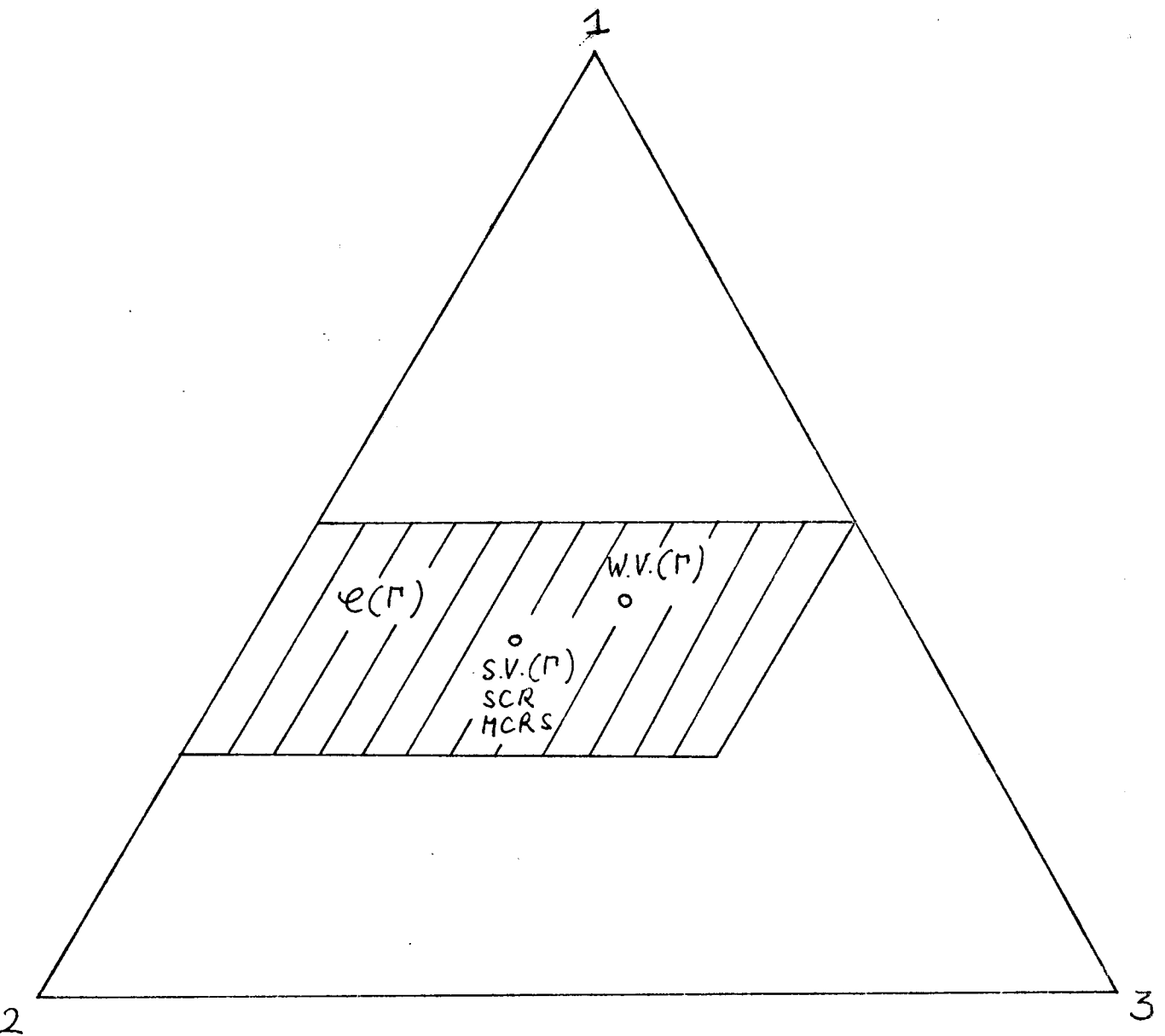
For this class of public investments the weighted value is a perfect solution, in the sense that the unique power, which amounts to the productivity or to the economies of cost of a group of cities, determines the final imputation of costs. The value as well as the net gain of a coalition are based on productivity alone. The apportionment of the net gain within a coalition to its members will be according to their productivity. To calculate the productivity one makes a simple linearity assumption. The weighted value thus is a method of cost sharing which should be easily applicable in liberal economies.

Consider the following simple example of a three city cost game. Three cities may join together to build a common sewage treatment plant. The total cost (value) of each coalition may be divided into the costs of pipes between cities and the cost of constructing the plant. This dichotomy may prove useful in allocating the costs to the cities. The game Γ is such divided in two subgames, Γ^1 , the game of pipes and, Γ^2 , the game of the treatment plant. The cost figures are given by table 3.1.

Table 3.1. : The three city cost game (example 1)

Coalition	Number of inhabitants	Total cost (4)+(5)	Cost of pipes	Cost of plant	Coalitional net gains (total costs)	Coalitional net gains (cost of plant)
(1)	(2)	= (3)	(4)	(5)	(6)	(7)
{1}	100	8	-	8	-	-
{2}	200	12	-	12	-	-
{3}	100	8	-	8	-	-
{12}	300	16	6	10	4	10
{13}	200	16	4	12	0	4
{23}	300	12	2	10	8	10
{123}	400	16	8	8	12	20

Figure 3.1. : Solutions (Example 1)



$$C(\Gamma) = (4 \leq x(1) \leq 8, \quad 0 \leq x(2) \leq 12, \quad 0 \leq x(3) \leq 8)$$

$$SCR_B = (6, 6, 4)$$

$$MCRS = (6, 6, 4)$$

$$s.v.(\Gamma) = (6, 6, 4)$$

$$w.v.(\Gamma) = (6,66 ; 4 ; 5,33)$$

The total cost function, $C(\Gamma)$, is sub-additive.
The game is convex for :

$$\begin{aligned} C(\{12\}) - C(\{2\}) &\geq C(\{123\}) - C(\{23\}) && \text{for } i = 1 \\ C(\{13\}) - C(\{3\}) &\geq C(\{123\}) - C(\{23\}) && \text{for } i = 1 \\ C(\{12\}) - C(\{1\}) &\geq C(\{123\}) - C(\{13\}) && \text{for } i = 2 \\ C(\{23\}) - C(\{3\}) &\geq C(\{123\}) - C(\{13\}) && \text{for } i = 2 \\ C(\{13\}) - C(\{1\}) &\geq C(\{123\}) - C(\{12\}) && \text{for } i = 3 \\ C(\{23\}) - C(\{2\}) &\geq C(\{123\}) - C(\{12\}) && \text{for } i = 3 \end{aligned}$$

Thus the core exists⁽¹¹⁾. It is (see figure 3.1.)

$$C(\Gamma) = (4 \leq x(1) \leq 8, 0 \leq x(2) \leq 12, 0 \leq x(3) \leq 8)$$

The Shapley value may be calculated as follows
(table 3.2.)

Table 3.2. : Shapley value of the
three city cost game (example 1)

Coalition	City		
	1	2	3
{1} {2} {3}	8	12	8
{12}	- 2	- 2	-
{13}	0	-	0
{23}	-	- 4	- 4
{123}	0	0	0
$\sum_{S \subseteq N} C_S^i = x(i)$	6	6	4

(11) Shapley (1971)

The Weighted value may be calculated in an analogous manner. However, one may decompose Γ into Γ^1 and Γ^2 , and calculate two weighted values, $W.V.(\Gamma^1)$ and $W.V.(\Gamma^2)$. This is done in tables 3.3. and 3.4.

Table 3.3. : Weighted value of Γ^1
(Example 1)

Coalitions	City		
	1	2	3
{1} {2} {3}	0	0	0
{12}	2	4	-
{13}	2	-	2
{23}	-	1,33	0,66
{123}	- 1	- 2	- 1
$\sum_{S \subseteq N} C_S^i = x(i)$	3	3,33	1,66

Table 3.4. : Weighted value of Γ^2
(Example 1)

Coalitions	City		
	1	2	3
{1} {2} {3}	8	12	8
{12}	- 3,33	- 6,66	-
{13}	- 2	-	- 2
{23}	-	6,66	- 3,33
{123}	+ 1	+ 2	+ 1
$\sum_{S \subseteq N} C_S^i = x(i)$	3,66	0,66	3,66

In an analogous manner one might compute $S.V.(\Gamma^1)$ and $S.V.(\Gamma^2)$. The results are presented in table 3.5.

Table 3.5. : Comparison of $S.V.(\Gamma)$ and $W.V.(\Gamma)$ (Example 1)

Solution	City		
	1	2	3
$S.V.(\Gamma)$	6	6	4
$S.V.(\Gamma^1)$	3,66	2,66	1,66
$S.V.(\Gamma^2)$	2,33	3,33	2,33
$W.V.(\Gamma)$	6,66	4	5,33
$W.V.(\Gamma^1)$	3	3,33	1,66
$W.V.(\Gamma^2)$	3,66	0,66	3,66

These results show that both, $S.V.(\Gamma^2)$ and $W.V.(\Gamma^2)$ are symmetric with respect to the cities (players). However, while $S.V.(\Gamma^2)$ does take into account the economies of cost caused by (big) city 2 on a smaller scale, $W.V.(\Gamma^2)$ does it in proportion to the size of city 2. Analogously city 2 supports a higher share of the cost of the pipes in $W.V.(\Gamma^1)$ than in $S.V.(\Gamma^1)$. This because of the hypothesis that if a relatively larger share of the economies of cost of the plant is allocated to city 2, it should bear an equally proportionally larger share of the cost of the pipes. On the whole $W.V.(\Gamma)$ favors the bigger city 2 and disadvantages cities 1 and 3, while $S.V.(\Gamma)$ does not. This shows clearly that $W.V.(\Gamma)$ is based on the productivity of a player, or what is equivalent in this example, on the economies of cost due to that player.

As $S.V.(\Gamma)$ and $W.V.(\Gamma)$ belong to the core in this example 1 the equity problem may be regarded as resolved. However, this is not generally true. To take care of such situations the concept of a mixed solution will be proposed.

4. MIXED SOLUTIONS

4.1. The core exists

Suppose the core exists and does not contain the weighted value. As the core implies individual and group rationality in a n-person cooperative situation and the weighted value involves equity considerations, if equity or fairness is based on allocational aspects of individual or group productivity, one wants both solution concepts to be at least partially realized. However, this may only be possible by outside intervention (by the central or regional government). From a liberal point of view this intervention should be as imperceptible as possible. We shall suggest tax-subsidy systems which will be more and more compelling for the local governments. This will lead to a hierarchy of three mixed solutions either more in favor of the core or of the weighted value.

4.1.1. A-fair and acceptable solution (type A)

Suppose $W.V.(\Gamma) \notin C(\Gamma)$. Then it may be that $x(i) > x(i)_{\max}$, for some $i \in N$. In this case a rational player i would never accept $W.V.(\Gamma)$. To make him still comply with $W.V.(\Gamma)$, one would have to subsidize him by an amount $x(i)_{\max} - x(i) = \text{Sub}(i)$. The total amount of subsidies is then

$$\sum_i \text{Sub}(i) = \sum_i (x(i)_{\max} - x(i))$$

$$\forall i \mid x(i) > x(i)_{\max} \text{ and } i \in N$$

and

$$\sum_i \text{Sub}(i) < 0$$

The system of subsidies is open. It is the central or the regional government which will have to grant these payments.

The hierarchy of pure solutions within this type A of a mixed solution is such that the weighted value is preferred to the core.

type A : $W.V.(\Gamma) \succ C(\Gamma)$

While $W.V.(\Gamma)$ is fully respected, one takes only care of the upper bounds of $C(\Gamma)$. Each city for which $x(i) < x(i)_{\min}$ obtains a windfall gain which is certainly not justified by the core.

The mixed solution of type A is called a-fair in the first place because it guarantees the weighted value which is based on allocational equity. It is called acceptable in the second place because the core is realized at least for those local governments which would be hurt otherwise.

This solution of type A implies a least degree of outside intervention to make the weighted value (to tally) and the core (at least partially) compatible .

4.1.2. Acceptable and almost a-fair solution (type B)

In a further step toward more stringent interventionism one may, in addition to the subsidies, tax all local governments $j \in N | x(j) < x(j)_{\min}$. This would assure the validity of the core as far as its lower and upper bounds are concerned. The individual tax is

$$\text{Tax } (j) = (x(j)_{\min} - x(j))$$

and the total amount of taxes is

$$\sum_j \text{Tax}(j) = \sum_j (x(j)_{\min} - x(j))$$

$$\forall j | x(j)_{\min} > x(j) \text{ and } j \in N$$

and

$$\sum_j \text{Tax}(j) > 0$$

Such a combined tax-subsidy system may be balanced, in deficit or in surplus. In the first and third case the system is closed. It is clear that solutions of type A and B require that taxes and subsidies are effectively paid. Otherwise the system would not work.

The hierarchy of pure solutions within the solution of type B is such that the core is preferred to the weighted value.

Type B : $C(\Gamma) \leftarrow W.V.(\Gamma)$

Now the upper and lower bounds of the core are assured. Indeed, the weighted value is partially contained in the core by the tax-subsidy system. For all $j | x(j)_{\min} > x(j)$ and $j \in N$ the weighted value is violated. This is the reason why the type B solution is acceptable in the first place and almost a-fair in the second place.

Consider the following example 2 of a three city cost game :

$$\begin{array}{lll} C(\{1\}) = 14 & C(\{2\}) = 15 & C(\{3\}) = 14 \\ C(\{12\}) = 18 & C(\{13\}) = 16 & C(\{23\}) = 16 \\ & C(\{123\}) = 20 & \end{array}$$

This game has a core $C(\Gamma)$. It is

$$C(\Gamma) = (4 \leq x(1) \leq 14, 4 \leq x(2) \leq 14, 2 \leq x(3) \leq 12)$$

Suppose $W.V.(\Gamma) = (3, 3, 14)$. Then $W.V.(\Gamma) \notin C(\Gamma)$ and the solution of type B requires a tax on city 1 and 2 of 1 unit each and a subsidy of - 2 units to city 3.

4.1.3. A-fair and forced acceptable solution (type C)

Instead of modifying $W.V.(\Gamma)$ by a tax-subsidy scheme such that $W.V.(\Gamma) \in C(\Gamma)$ one may modify the core by a tax-subsidy system such that the modified core

$C'(\Gamma) \supset W.V.(\Gamma)$. Indeed, one may either change $W.V.(\Gamma)$ such that it is in the core or one may modify the core, in a much stronger way, such that it contains the weighted value. There exists a duality between a solution of type B and a solution of type C. This duality may be seen (see figure 4.1.3.1.) by example 2 (section 4.1.2.). The cost function is

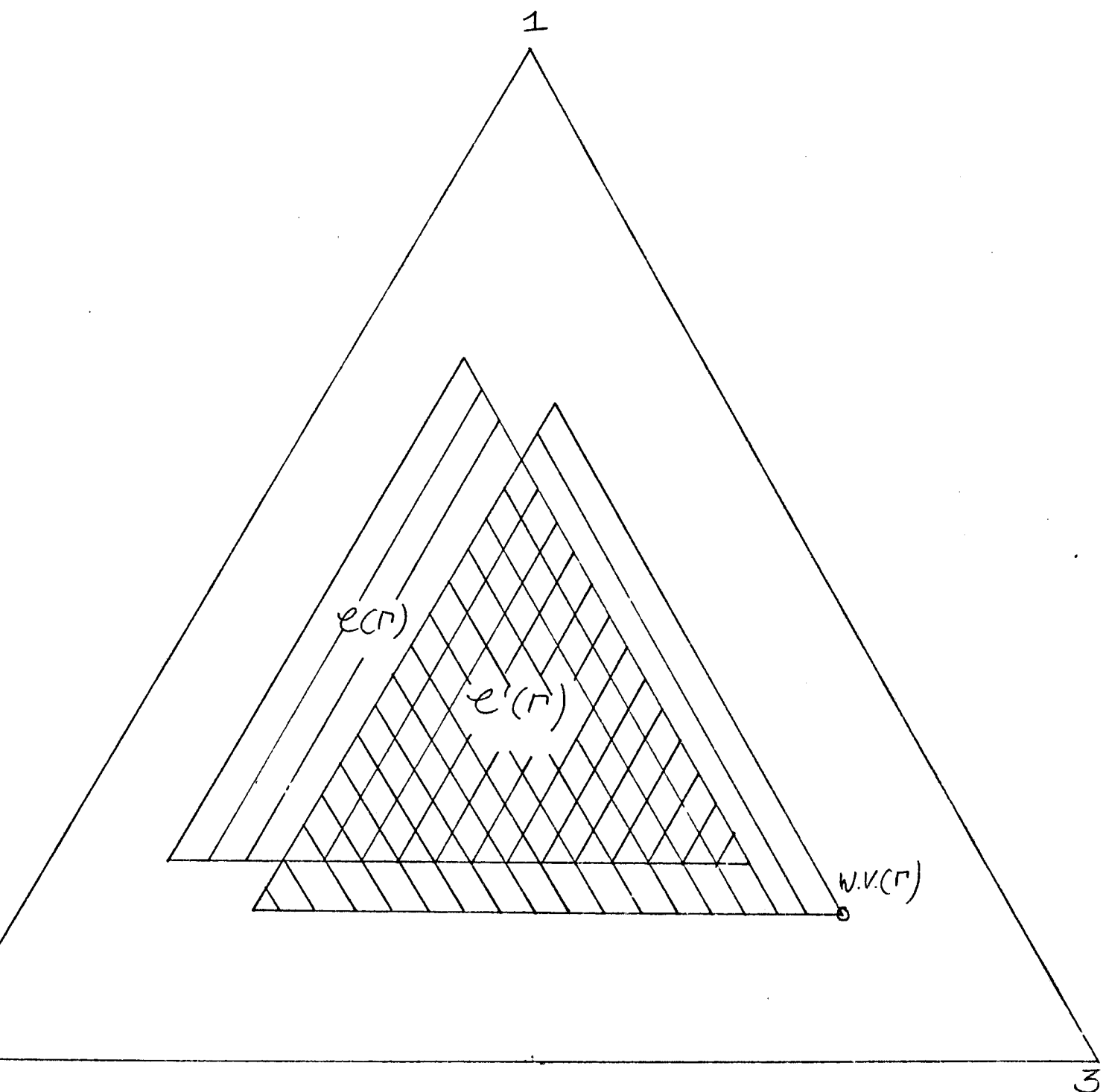
$$\begin{array}{lll} C(\{1\}) = 14 & C(\{2\}) = 15 & C(\{3\}) = 14 \\ C(\{12\}) = 18 & C(\{13\}) = 16 & C(\{23\}) = 16 \\ & C(\{123\}) = 20 & \end{array}$$

The core is

$$C(\Gamma) = (4 \leq x(1) \leq 14, 4 \leq x(2) \leq 14, 2 \leq x(3) \leq 12).$$

Suppose $W.V.(\Gamma) = (3, 3, 14)$. Then $C'(\Gamma) \mid C'(\Gamma) \ni W.V.(\Gamma)$ is computed by using the following tax-subsidy system (for the construction of these equations see appendix) :

Figure 4.1.3.1.: Quality of type B and C solutions
(Example 2)



$$\text{///} = e(r)$$

$$\text{///} = e'(r)$$

$$\text{Tax (ij)} = (x(i) - x(i)_{\min}) + (x(j) - x(j)_{\min}) = -2$$

$$\text{Tax (ik)} = (x(i) - x(i)_{\min}) + (x(k) - x(k)_{\max}) = 1$$

$$\text{Tax (jk)} = (x(j) - x(j)_{\min}) + (x(k) - x(k)_{\max}) = 1$$

Subsidies are defined as negative taxes.

The modified cost function $C'(\Gamma)$, is

$$C'(\{1\}) = 14 \quad C'(\{2\}) = 15 \quad C'(\{3\}) = 14$$

$$C'(\{12\}) = 16 \quad C'(\{13\}) = 17 \quad C'(\{23\}) = 17$$

$$C'(\{123\}) = 20$$

$$\text{and } C'(\Gamma) = (3 \leq x(1) \leq 13, 3 \leq x(2) \leq 13, 4 \leq x(3) \leq 14)$$

Thus $C'(\Gamma) \ni \text{W.V.}(\Gamma)$.

This particular example shows several features which are not generally true for three-person cost games. First, the tax-subsidy system is closed. In the general case it may be open (deficit) or closed (o-balance or surplus). Second, the duality is such that the tax (subsidy) in type B solution (type C solution) is exactly equal to the subsidy (tax) in type C solution (type B solution). This strict duality is only fulfilled in very special cases (see appendix).

Type C solution implies the following hierarchy of pure solutions :

$$\text{Type C} : \text{W.V.}(\Gamma) \leftarrow C(\Gamma).$$

This preference relation is based on a different environment than in the case of the solution of type A.

Here, the situation of the power of groups is changed by a tax-subsidy system, if necessary. In fact this tax-subsidy system is a threat which will be realized only if there is no compliance by the local governments. Thus, if they act rationally, the threat needs never to be realized. In solution of type A no tax is levied, some of the local governments are just persuaded (by subsidies) to accept $W.V.(\Gamma)$.

Again solution of type C is called a-fair because it is based on allocational fairness. It is forced acceptable because the core $C(\Gamma)$ is changed, by direct intervention into $C'(\Gamma)$ and thus becomes necessarily acceptable.

4.2. The core does not exist

If the core does not exist two approaches may be chosen to find a solution to the equity problem : one accepts the fact of non-existence of the core and one computes the SCRB solution, the weighted value, $W.C.(\Gamma)$, or the solution of type A (one-stage solutions). All three of them do not depend on the existence of a core. The other approach (two-stage solutions) would be to compute a (strong) ϵ -core, which is entirely fictitious, and superimposes on it either the MCRS-solution, or solutions of type A, B or C. In the subsequent sections all these possibilities will be discussed separately.

4.2.1. One-stage solutions

4.2.1.1. Separable Costs Remaining Benefits Solution

This solution is in no way different from the one discussed in section 2.1. By definition it does not depend on the existence of a core.

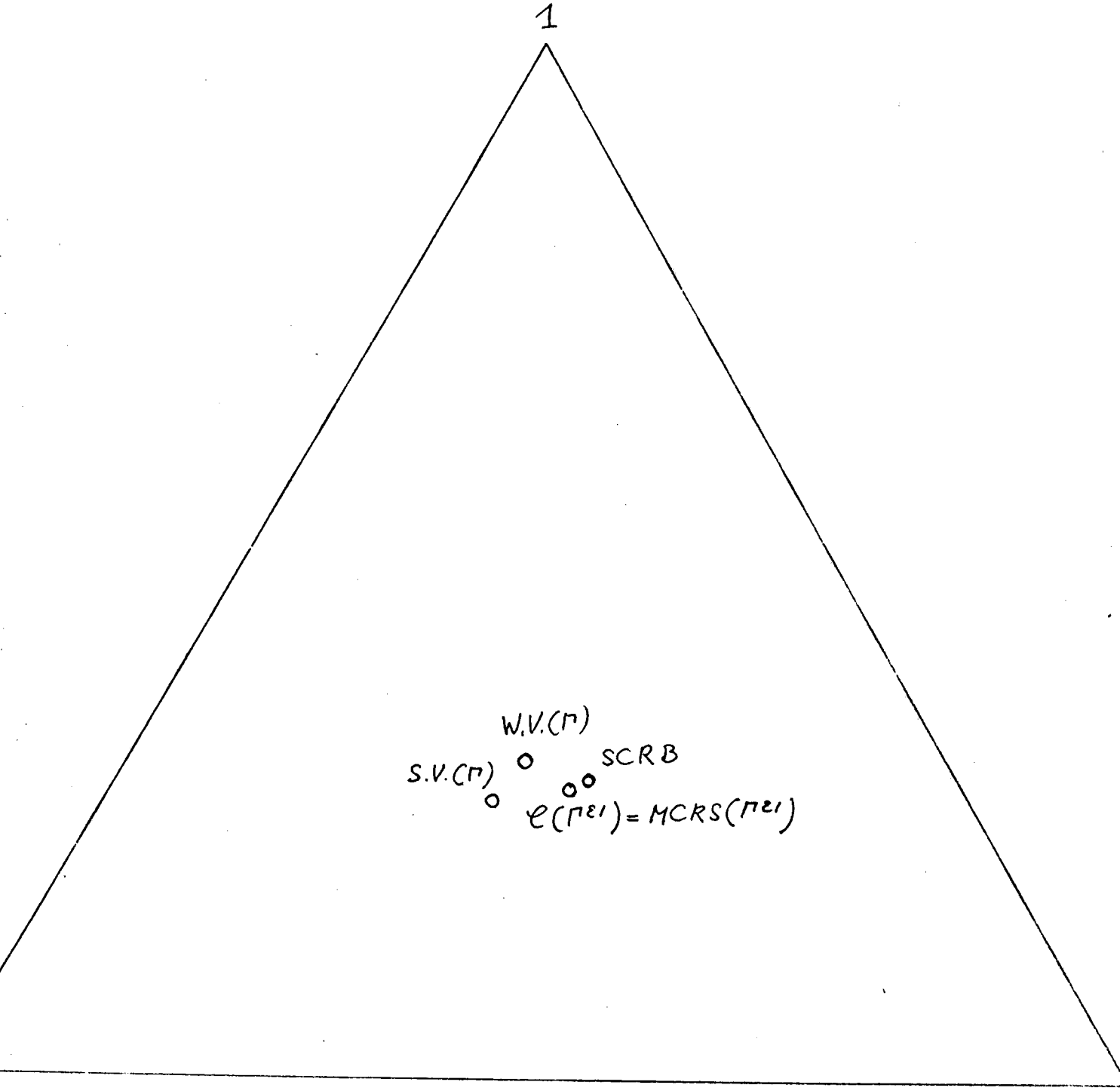
Consider the following example 3 of a three city cost game (table 4.2.1.1.1.)

Table 4.2.1.1.1.: The three city cost game
(example 3)

Coalition	Number of inhabitants	Total cost (4)+(5) = (3)	Cost of pipes (4)	Cost of plant (5)	Coalitional net gains (total costs) (6)	Coalitional net gains (cost of plant) (7)
(1)	(2)	= (3)	(4)	(5)	(6)	(7)
{1}	100	8	-	8	-	-
{2}	200	12	-	12	-	-
{3}	100	8	-	8	-	-
{12}	300	12	2	10	8	10
{13}	200	13	1	12	3	4
{23}	300	14	4	10	6	10
{123}	400	20	12	8	8	20

This game is not convex. The core does not exist, for $C(\{12\}) + C(\{13\}) + C(\{23\}) < 2C(\{123\})$. The SCRB solution (see figure 4.2.1.2.1.) of this game is the vector (5,71 ; 6,29 ; 8).

Figure 4.2.1.2.1. : Solutions (Example 3)



$$C(\Gamma^E) = (5,65 ; 6,68 ; 7,67)$$

$$S.V.(\Gamma) = (5,5 ; 8 ; 6,5)$$

$$W.V.(\Gamma) = (6,08 ; 7,16 ; 6,74)$$

$$SCR B = (5,71 ; 6,29 ; 8)$$

$$MCRS(\Gamma^E) = (5,65 ; 6,68 ; 7,67)$$

4.2.1.2. The Weighted Value

The weighted value is independent of the core. Given the three city cost game (example 3 section 4.2.1.1.) it may be easily computed. If one decomposes Γ into Γ^1 , the game of pipes, and Γ^2 , the game of the plant, one obtains $W.V.(\Gamma^1) = (2,42; 6,5; 3,08)$ and $W.V.(\Gamma^2) = (3,67; 0,66; 3,67)$. Thus $W.V.(\Gamma) = W.V.(\Gamma^1) + W.V.(\Gamma^2) = (6,09; 7,16; 6,75)$.

4.2.1.3. A-fair and acceptable solution (type A)

Even if the core does not exist one or several (in this example a coalition of 2) local governments may refuse the weighted value as a cost-sharing method. To see this clearly one has to compare the cost function, $C^{W.V.}(\Gamma)$, (characteristic function) of the inessential game $\Gamma^{W.V.}$, with the cost function, $C(\Gamma)$, of the game Γ (example 3, section 4.2.1.1.)

$$\begin{aligned} C^{W.V.}(\{1\}) &= 6,09 & C^{W.V.}(\{2\}) &= 7,16 & C^{W.V.}(\{3\}) &= 6,75 \\ C^{W.V.}(\{12\}) &= 13,25 & C^{W.V.}(\{13\}) &= 12,84 & C^{W.V.}(\{23\}) &= 14 \\ & & C^{W.V.}(\{123\}) &= 20 & & \end{aligned}$$

and

$$\begin{aligned} C(\{1\}) &= 8 & C(\{2\}) &= 12 & C(\{3\}) &= 8 \\ C(\{12\}) &= 12 & C(\{13\}) &= 13 & C(\{23\}) &= 14 \\ & & C(\{123\}) &= 20 & & \end{aligned}$$

One sees clearly that coalition $\{12\}$ will never accept $W.V.(\Gamma)$ as a cost-sharing method as $13,25 > 12$. Thus one will have to subsidize $\{12\}$ by an amount of

$$\text{Sub } (12) = C(\{12\}) - C^{W.V.}(\{12\}) = -1,25$$

in order to make $W.V.(\Gamma)$ generally acceptable.

A new problem occurs. How to divide the subsidy among the cities 1 and 2 ? Three procedures are possible. The last one corresponds only to the rationale of the cost sharing method proposed in this paper : the allocation of cost according to productivity.

The first procedure exists of dividing the subsidy equally between city one and two, the second of a division according to the costs $C^{W.V.}(\{i\})$, for $i=1,2$. This implies that a city with relatively higher costs gets a relatively highershare of the subsidy. However, this is contrary to the productivity-approach to cost sharing. Consequently, we propose the contrary : the city with a relatively lower share of the total cost gets a relatively higher share of the subsidy.

The subsidies (example 3) may now be computed.

$$\text{Sub}(1) = \frac{x(2)}{\sum_{i=1} x(i)} [C(\{12\}) - C^{W.V.}(\{12\})] = \frac{7,16}{13,25}(-1,25) = -0,68$$

$$\text{Sub}(2) = \frac{x(1)}{\sum_{i=1} x(i)} [C(\{12\}) - C^{W.V.}(\{12\})] = \frac{6,09}{13,25}(-1,25) = -0,57$$

City 1 with a lower cost share obtains a higher subsidy than city 2 and vice versa.

This rule may be generalized for the case $n \geq 3$.
 Let SCN and $S \geq 3$, and $C(S) < C^{W.V.}(S)$, then

$$\text{Sub}(i) = \frac{\sum_{i \in S} x(i) - x(i)}{\sum_{i \in S} (\sum_{i \in S} x(i) - x(i))} [C(S) - C^{W.V.}(S)]$$

$\forall i \in S$

If there exists more than one $S | C(S) < C^{W.V.}(S)$, then

$$\text{Sub}(i) = \text{Max}_{SCN | C(S) < C^{W.V.}(S)} \{ \text{Sub}(i)(S) \}$$

4.2.2. Two-stage Solutions

If the core, a set of solutions which satisfies all individual members and coalitions, does not exist one may create a fictitious one by a (fictitious) tax system. The non-existence of the core reveals the fact that intermediate coalitions are too powerful with respect to their bargaining power. In our example, their joint costs are too low to be undercut by a cost allocation of the grand coalition. To make them less powerful one just may (threaten to) tax them if they cooperate independently. However, by this method one abandons the liberal idea of the core.

The problem is now : How to tax the intermediate coalitions ? Among many possibilities we propose two, the second of which will only be applied in the subsequent sections.

The notion of a strong ϵ -core requires that each proper coalition $S \subset N$, without $S = \{i\}$, $i=1, \dots, n$, is taxed¹² by an amount $\epsilon C(S)$, where $C(S)$ is the cost function of the original game Γ and $\epsilon > 0$. However, we propose to tax each intermediate coalition by an amount of $\epsilon' C^{W.V.}(S)$, where $C^{W.V.}(S)$ is the cost function of the inessential game $\Gamma^{W.V.}$ (see section 4.2.1.3.). $\Gamma^{W.V.}$ is a measure of the "true" costs, that is, it implies the economies of costs due to each participant.

The computation of the strong ϵ' -core is simple. First one has to compute a minimum ϵ' such that a ϵ' -core exists (program 3) and then one applies this ϵ' to program (1) and (2) which become program (1') and (2').

$$\begin{array}{ll}
 \text{Program (3)} & \text{Min } \epsilon' \\
 & \text{s.t. } x(i) \leq C(i) \quad \forall i \in N \\
 & \sum_{i \in S} x(i) \leq C(S) + \epsilon' C^{W.V.}(S) \quad \forall S \subset N \\
 & \sum_{i \in N} x(i) = C(N) \\
 & x(i) \geq 0 \quad \forall i \in N
 \end{array}$$

Suppose ϵ'^* is the solution to program (3) then one obtains X_{\min} by program (1') and X_{\max} by program (2').

$$\begin{array}{ll}
 \text{Program (1')} & \text{Min } x(i) \\
 & \text{s.t. } x(i) \leq C(i) \quad \forall i \in N \\
 & \sum_{i \in S} x(i) \leq C(S) + \epsilon'^* C^{W.V.}(S) \quad \forall S \subset N \\
 & \sum_{i \in N} x(i) = C(N) \\
 & x(i) \geq 0 \quad \forall i \in N
 \end{array}$$

(12) This approach is chosen by Heaney (1978)

$$\begin{array}{ll}
\text{Program (2')} & \text{Max } x(i) \\
& \text{s.t. } x(i) \leq C(i) \quad \forall i \in N \\
& \quad \sum_{i \in S} x(i) \leq C(S) + \varepsilon' * C^{W.V.}(S) \quad \forall S \subset N \\
& \quad \sum_{i \in N} x(i) = C(N) \\
& \quad x(i) \geq 0 \quad \forall i \in N
\end{array}$$

In example 3 (section 4.2.1.1.), $\varepsilon'^* = 0,025$, and the strong ε' -core is a point $C(\Gamma^{\varepsilon'}) = (5,65; 6,68; 7,67)$.

The second stage of the following two-stage solutions consists now of superimposing the minimum cost remaining savings method or the solutions of type A, B or C on this ε' -core.

4.2.2.1. Minimum Cost Reamaining Savings Method

In example 3 (section 4.2.1.1.) the strong ε' -core is a point. Thus $x(i)_{\min} = x(i)_{\max}$, $\forall i \in N$. Consequently, $\beta(i) = 0$, $\forall i \in N$ and $x(i) = x(i)_{\min} = x(i)_{\max}$, $\forall i \in N$. The MCRS-solution is then $(5,65; 6,68; 7,67)$.

One sees clearly that this solution becomes acceptable only by force, that is by the threat of an appropriate tax system. Each intermediate coalition is worse off than in the original game Γ . One obtains $(5,65 + 6,68 = 12,33 > 12 = C(\{12\}); 5,65 + 7,67 = 13,32 > 13 = C(\{13\}); 6,68 + 7,67 = 14,35 > 14 = C(\{23\})$. However, this is the (coalitional) price for a societal gain which is implied by the sub-additivity of the cost-function of the original game Γ .

4.2.2.2. Conditional a-fair and acceptable Solution

One superimposes the solution of the type A on the strong ε' -core, $C(\Gamma^{\varepsilon'})$. The state or regional government threatens to tax the local governments if they do not comply with the strong ε' -core. The local governments have the choice to either accept or reject it. In the first case each local government for which $x(i)_{\max \varepsilon'} < x(i)$, receives a subsidy $\text{Sub}(i) = x(i)_{\max \varepsilon'} - x(i)$. Thus the upper bounds of $C(\Gamma^{\varepsilon'})$ are assured. In the second case the threat is put in effect with the result that $C(\Gamma^{\varepsilon'})$ is guaranteed. Then the subsidies are paid. Thus the local governments have no other choice than to accept $C(\Gamma^{\varepsilon'})$ as a basis for the solution of type A. It is also clear that some local government(s) will be worse off due to the non-existence of a (non-imposed) core.

In example 3 (section 4.2.1.1.) the vector of subsidies is $(-0,44; -0,48; 0)$, the strong ε' -core, $C(\Gamma^{\varepsilon'}) = (5,65; 6,68; 7,67)$ and the weighted value $W.V.(T) = (6,09; 7,16; 6,75)$. As $5,65 + 6,68 = 12,33 > 12 = C(\{12\})$, $5,65 + 6,75 = 12,40 < 13 = C(\{13\})$, $6,68 + 6,75 = 13,43 < 14 = C(\{23\})$ it is city 1 or 2, or both which are disadvantaged. This solution is called conditionally a-fair and acceptable as it is a solution of type A, given the fictitious strong ε' -core.

4.2.2.3. Conditional acceptable and almost a-fair solution

Every thing which has been said about the solution of type B (section 4.1.2.) may be applied to this section. The point or the set of reference is now the strong ε' -core, $C(\Gamma^{\varepsilon'})$. This is the reason why we call this solution conditionally acceptable and almost a-fair. While the tax-system which leads to the ε' -core is to be conceived as

a threat, the mixed tax-subsidy system to guarantee $C(\Gamma^{\varepsilon'}) \ni W.V.(\Gamma)$ is real. Thus the local governments have in fact no other choice than to accept the ε' -core.

In example 3 (section 4.2.1.1.) the vector of subsidies is $(-0,44; -0,48; 0)$ and the vector of taxes is $(0; 0; 0,92)$. The effective costs to the coalitions are :

$$5,65 + 6,68 = 12,33 > 12 = C(\{12\})$$

$$5,65 + 7,67 = 13,32 > 13 = C(\{13\})$$

$$6,68 + 7,67 = 14,35 > 14 = C(\{23\})$$

Hence in this particular example the local governments would (weakly) prefer the conditional solution of type A.

4.2.2.4. Conditional a-fair and forced acceptable solution

What has been said about solution of type C (section 4.1.3.) may be applied here. The strong ε' -core, $C(\Gamma^{\varepsilon'})$, is to be modified by a mixed tax-subsidy system to $C'(\Gamma^{\varepsilon'})$, such that $C'(\Gamma^{\varepsilon'}) \ni W.V.(\Gamma)$. The dual mixed tax-subsidy system in example 3 is the vector $(0,44; 0,48; -0,92)$. The effective costs to the coalitions are

$$6,09 + 7,16 = 13,25 > 12 = C(\{12\})$$

$$6,09 + 6,75 = 12,84 < 13 = C(\{13\})$$

$$7,16 + 6,75 = 14,1 > 14 = C(\{23\})$$

One sees immediately that local governments 1 and 2 prefer conditional solution of type A or B over type C, and local government 3 prefers the solutions of type A or C over B.

5. CONCLUSION

It has been shown in this article that there exist different types of solutions (see table 5.1.) to cost sharing of public investments with more than one decision-maker involved. If one uses a game-theoretic frame-work it would be too narrow to propose just the Shapley-value or the core as a pure solution concept. Generally one will have to suggest mixed solutions, which are based on the strategic situation as well as on some equity principle, if one wants to respond to a large set of possibilities suitable to local governments. This mixed approach shows the fruitfulness and the richness of game theory to deal with different institutional environments and thus the power of mathematical institutional economics.

The pure solution concepts used in this paper are the core which expresses individual and group rationality in "free-market" bargaining, and the weighted value as an equity principle which is based on productivity, or economies of cost. Thus, as regards cost games, both pure solutions are perfect, as the sole measure of cost allocation is individual or group productivity.

If $W.V.(\Gamma) \in C(\Gamma)$, then the problem of cost sharing is solved. If $W.V.(\Gamma) \notin C(\Gamma)$ and $C(\Gamma) \neq \emptyset$ then the solution of type A seems a priori reasonable. If $C(\Gamma) = \emptyset$ then solution of type A is still reasonable. These solutions have the advantage of being relatively easy to compute. However, all the other mixed and pure solution concepts proposed in this paper (except for the Shapley-value, SCRB and MCRS) are perfect in the above sense and equally reasonable in a given institutional frame-work. The political situation may be such that only somewhat more complex compromises, represented by the various mixed solution concepts, may be admissible.

Table 5.1. : Types of solutions of cost sharing games

Core	Relation between the core and the weighted value	Pure and mixed solutions
C(Γ) exists		Separable costs remaining benefits method (SCRB) Minimum cost remaining savings method (MCRS)
	S.V. (Γ) \in C(Γ) W.V. (Γ) \in C(Γ)	Shapley value Weighted value
	W.V. (Γ) \notin C(Γ)	A-fair and acceptable solution (type A) Acceptable and almost a-fair solution (type B) A-fair and forced acceptable solution (type C)
C(Γ) does not exist		Separable costs remaining benefits method (SCRB) Weighted value A-fair and acceptable solution (type A)
	W.V. (Γ) \in C($\Gamma^{\epsilon'}$)	Minimum cost remaining savings method (MCRS) Conditional a-fair and acceptable solution Conditional acceptable and almost a-fair solution
	W.V. () \in C'($\Gamma^{\epsilon'}$)	Conditional a-fair and forced acceptable solution

6. APPENDIX

Construction of a tax-subsidy system
(solution of type C)

Suppose the three-person cost game Γ is given by its cost-function $C(\Gamma)$. Suppose the core exists and is

$$C(\Gamma) = (x(1)_{\min} \leq x(1) \leq x(1)_{\max}, x(2)_{\min} \leq x(2) \leq x(2)_{\max}, x(3)_{\min} \leq x(3) \leq x(3)_{\max})$$

and the weighted value is

$$W.V.(\Gamma) = (x(1), x(2), x(3))$$

Define a sub-class of Γ such that

$$x(i) \leq x(i)_{\min} \quad \text{or} \quad x(i) \geq x(i)_{\max}$$

for $i = 1, 2, 3$, and thus $W.V.(\Gamma) \notin C(\Gamma)$. To calculate the solution of type C one must define a tax-subsidy system such that Γ is modified to Γ' . Suppose the core of Γ' is

$$C(\Gamma') = (x'(1)_{\min} \leq x'(1) \leq x'(1)_{\max}, x'(2)_{\min} \leq x'(2) \leq x'(2)_{\max}, x'(3)_{\min} \leq x'(3) \leq x'(3)_{\max})$$

$$\text{Let } C(\{i\}) \geq C(\{ijk\}) - (x(j)'_{\min} + x(k)'_{\min})$$

for $i, j, k = 1, 2, 3$

and $i \neq j \neq k$

Consider the subclass of Γ , such that

$$x(i) \leq x(i)_{\min}, \quad x(j) \leq x(j)_{\min}, \quad x(k) \geq x(k)_{\max}$$

for $i, j, k = 1, 2, 3$

and $i \neq j \neq k$

Then

Proposition : The following tax-subsidy system guarantees that $W.V.(\Gamma) \in C(\Gamma')$.

$$\text{Tax}(ij) = (x(i) - x(i)_{\min}) + (x(j) - x(j)_{\min})$$

$$\text{Tax}(ik) = (x(i) - x(i)_{\min}) + (x(k) - x(k)_{\max})$$

$$\text{Tax}(jk) = (x(j) - x(j)_{\min}) + (x(k) - x(k)_{\max})$$

Remark : $\text{Tax}(\dots) < 0$ is a subsidy, $\text{Tax}(\dots) > 0$ is a tax.

Proof : The cost function of Γ is $C(\Gamma)$

$$C(\{i\}) = a \quad C(\{j\}) = b \quad C(\{k\}) = c$$

$$C(\{ij\}) = x(i)_{\min} + x(j)_{\max} = x(i)_{\max} + x(j)_{\min}$$

$$C(\{ik\}) = x(i)_{\min} + x(k)_{\max} = x(i)_{\max} + x(k)_{\min}$$

$$C(\{jk\}) = x(j)_{\min} + x(k)_{\max} = x(j)_{\max} + x(k)_{\min}$$

$$C(\{ijk\}) = d$$

The cost-function of Γ' , $C'(\Gamma')$ is

$$C'(\{i\}) = a \quad C'(\{j\}) = b \quad C'(\{k\}) = c$$

$$C'(\{ij\}) = (x(i)_{\min} + x(j)_{\max}) + (x(i) - x(i)_{\min})$$

$$+ (x(j) - x(j)_{\min})$$

$$= x(i) + x(j) + (x(j)_{\max} - x(j)_{\min})$$

$$C'(\{ik\}) = (x(i)_{\min} + x(k)_{\max}) + (x(i) - x(i)_{\min})$$

$$+ x(k) - x(k)_{\max} = x(i) + x(k)$$

$$C'(\{jk\}) = (x(j)_{\min} + x(k)_{\max}) + (x(j) - x(j)_{\min})$$

$$+ (x(k) - x(k)_{\max}) = x(j) + x(k)$$

$$C'(\{ijk\}) = d$$

Then because of the existence ⁽¹³⁾ of the core

$$x(i) \geq x'(i)_{\min} = d - (x(j) + x(k))$$

$$x(j) \geq x'(j)_{\min} = d - (x(i) + x(k))$$

$$x(k) \geq x'(k)_{\min} = d - (x(i) + x(j) + (x(j)_{\max} - x(j)_{\min}))$$

and because of the previous assumption

$$x(i) \leq x'(i)_{\max} = d - \{[d - (x(i) + x(k))] + [d - (x(i) + x(j) + x(j)_{\max} - x(j)_{\min})]\}$$

$$= d - x(j) - x(k) + x(j)_{\max} - x(j)_{\min}$$

$$= x(i) + x(j)_{\max} - x(j)_{\min}$$

(13) See Schleicher (1978 II, annexe 1)

$$\begin{aligned}
x(j) &\leq x'(j)_{\max} = d - \{[d - (x(j) + x(k))]\} \\
&\quad + [d - (x(i) + x(j) + x(j)_{\max} - x(j)_{\min})] \} \\
&= d - x(i) - x(k) + x(j)_{\max} - x(j)_{\min} \\
&= x(j) + x(j)_{\max} - x(j)_{\min}
\end{aligned}$$

$$\begin{aligned}
x(k) &\leq x'(k)_{\max} = d - \{[d - (x(j) + x(k))]\} \\
&\quad + [d - (x(i) + x(k))]\} \\
&= d - x(i) - x(j) = x(k) \qquad \text{Q.E.D.}
\end{aligned}$$

The proposition is also true for the following subclass of Γ :

$$x(i) > x(i)_{\min} = x(i)_{\max}$$

$$x(j) > x(j)_{\min} = x(j)_{\max}$$

$$x(k) \leq x(k)_{\min} = x(k)_{\max}$$

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