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Game semantics as a singular functor, and definability as geometric realisation

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Abstract. Game semantics is a class of models of programming languages in which types are interpreted as games and programs as strategies. Though originally designed for sequential languages, its scope has recently been extended to concurrent ones. A salient feature of game semantics is the notion of innocence, which requires strategies to be determined by their values on a certain class of plays, called views.

In previous work, we have obtained a representation theorem for Tsukada and Ong’s categories of views and plays, in particular by constructing an embedding \mathbf{V} of views into a coslice of a certain presheaf category. We here exploit this result to exhibit an efficient categorical account of two crucial constructions of game semantics. First, we recover the interpretation of normal forms into innocent strategies as the singular functor associated to \mathbf{V} . Second, the corresponding geometric realisation functor yields the standard definability result saying that any innocent strategy is (isomorphic to) the interpretation of a normal form.

1 Introduction

Innocent game semantics [16, 12] has led to fully abstract models for a variety of functional languages, where programs are interpreted as strategies in a game. Concurrent extensions of game semantics have recently been designed, based on event structures [3] or on sheaves [9, 10, 6, 19].

An important result in games semantics asserts that the interpretation of normal forms induces a bijection with (some variant of) finite innocent strategies. The goal of this paper is to reconstruct this phenomenon categorically, in the setting of Tsukada and Ong’s games model [19], relying on the well-known Yoneda structure on categories [17, 21]. Admittedly, this is only a partial reconstruction, as composition of strategies through parallel composition plus hiding is missing from the picture. Still, we find the simplicity and algebraic topological nature of our construction worth communicating.

Our starting point is the representation theorem proved in [5], which (simplifying a bit for expository purposes) exhibits an embedding $\mathbf{P}_{A \vdash B}: \mathbb{P}_{A \vdash B} \hookrightarrow (A \vdash B)/\widehat{\mathcal{L}}$ of Tsukada and Ong’s category of plays $\mathbb{P}_{A \vdash B}$ over any arenas A and B into a coslice $(A \vdash B)/\widehat{\mathcal{L}}$ of a certain presheaf category $\widehat{\mathcal{L}}$. We here construct a further embedding $\mathbf{T}_{\sigma \vdash \tau}: \mathbf{NF}_{\sigma \vdash \tau} \hookrightarrow (A \vdash B)/\widehat{\mathcal{L}}$ of normal forms of type $\sigma \vdash \tau$ into the same coslice (assuming that the types σ and τ yield arenas A and B).

From there, using the standard categorical yoga of singular functors and geometric realisation, we are able to reconstruct (i) the translation of normal forms into innocent strategies and (ii) the ‘definability’ map, which associates to each (finitely presentable) innocent strategy X a normal form R such that $\llbracket R \rrbracket \cong X$. The former (i) is given by the composite

$$\text{NF}_{\sigma \vdash \tau} \xrightarrow{\mathbf{T}_{\sigma \vdash \tau}} ((A \vdash B)/\widehat{\mathbb{L}}) \xrightarrow{(\mathbf{V}^{A \vdash B})^*} \widehat{\mathbb{V}}_{A \vdash B}, \quad (1)$$

where $(\mathbf{V}^{A \vdash B})^*$ denotes the singular functor associated to the composite embedding

$$\mathbf{V}^{A \vdash B}: \mathbb{V}_{A \vdash B} \hookrightarrow \mathbb{P}_{A \vdash B} \hookrightarrow (A \vdash B)/\widehat{\mathbb{L}}.$$

The latter (ii) is the corresponding geometric realisation $\mathbf{V}_!^{A \vdash B}$. We prove that the composite (1) coincides with Tsukada and Ong’s interpretation of normal forms, and that $\mathbf{V}_!^{A \vdash B}(X)$ is indeed a normal form antecedent of X , up to isomorphism. Our proof of this last fact relies on a notion of *finite complex relative to $A \vdash B$* , which is defined following standard algebraic topological techniques [11]: we prove such complexes to coincide both (1) with the \mathbf{T} -image of normal forms and (2) with the $\mathbf{V}_!$ -image of finite innocent strategies.

Related work

Tsukada and Ong’s games draw inspiration from Mellès’s reformulation of innocence [14] and our sheaf-based notion of innocent and concurrent strategies [10]. There are other alternative representations of plays, e.g., Boudes’s *thick subtrees* [2], specific intersection types [7], or resource terms [20]. Ours is close in spirit. A main difference lies in our emphasis on morphisms between plays, which is of course key in stating the adjunction between geometric realisation and singular functor. A further difference is that our correspondence is less tight: the embedding lands into a much larger category. In a sense, this looseness is precisely what allows us to also embed normal forms and innocent strategies (using co-completeness).

Plan

We start in Section 2 by quickly reviewing some basic notation and a few categorical facts. We continue in Section 3 by recalling the representation theorem from [5] and explaining how plays map to the relevant coslice category. We continue in Section 4 by defining our functor from normal forms to the same coslice category. In Section 5, we construct the relevant singular functor and prove (Theorem 2) that it yields the correct interpretation of normal forms into innocent strategies. In Section 6, we construct the corresponding geometric realisation functor and show (Theorem 3) that it yields the expected antecedents for all finitely presentable innocent strategies. Finally, we conclude in Section 7.

2 Preliminaries

We often confuse natural numbers n with the corresponding finite sets $\{1, \dots, n\}$. For any category \mathcal{C} and object $c \in \mathcal{C}$, the *coslice* category c/\mathcal{C} has morphisms from c as objects, and as morphisms $(c \xrightarrow{f} d) \rightarrow (c \xrightarrow{g} e)$ all morphisms $h: d \rightarrow e$ in \mathcal{C} such that $h \circ f = g$. Furthermore, $\widehat{\mathcal{C}}$ denotes the category of *presheaves* on \mathcal{C} , i.e., contravariant functors to sets. For all such presheaves X , $f: c \rightarrow c'$, and $x \in X(c')$, we denote $X(f)(x)$ by $x \cdot f$ (resp. $f \cdot x$ for covariant functors). The Yoneda embedding is denoted by $y: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$. We will make one concrete use of the notion of a *finitely presentable* object in a category [1].

Given any functor $F: \mathcal{C} \rightarrow \mathcal{D}$, we denote precomposition by F^{op} by $\Delta_F: \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C}}$, which maps any $Y \in \widehat{\mathcal{D}}$ to the presheaf mapping any $c \in \mathcal{C}$ to $X(F(c))$. This should not be confused with $F^*: \mathcal{D} \rightarrow \widehat{\mathcal{C}}$, which we define to map any $d \in \mathcal{D}$ to the presheaf $c \mapsto \mathcal{D}(F(c), d)$. When \mathcal{D} is cocomplete, this F^* has a well-known left adjoint $F_!$, given by mapping any $X \in \widehat{\mathcal{C}}$ to $\int^c X(c) \cdot F(c)$. The latter is a *coend*, in which \cdot denotes iterated coproduct, i.e., $\underbrace{F(c) + \dots + F(c)}_{X(c) \text{ times}}$. E.g., when \mathcal{D}

is the category of sets and functions, the coend is the quotient of $\sum_c X(c) \cdot F(c)$ by the equivalence relation generated for all $f: c \rightarrow c'$, $x' \in X(c')$ and $u \in F(c)$ by $(x', f \cdot u) \sim (x' \cdot f, u)$. A particular, well-known case is *Yoneda reduction*: for all presheaves $X \in \widehat{\mathcal{C}}$, the canonical morphism $\int^c X(c) \cdot y_c \rightarrow X$ is an isomorphism.

3 Representing plays

In this section, we recall Tsukada and Ong's games and our representation theorem.

3.1 Brief review of Tsukada-Ong games

Tsukada and Ong's games are based on the standard notion of arena, of which we adopt the following innocuous simplification:

Definition 1. *An arena is a simple, countable forest, i.e., a directed, simple graph in which all vertices are uniquely reachable from a unique root (= vertex without a parent).*

Remark 1. For size reasons, let us interpret 'countable' in the restricted sense that vertices are all natural numbers. This entails that arenas form a *set*.

Example 1. Anticipating a little, the considered simply-typed λ -calculus will have one basic type o , and $\mathbb{B} := (o \rightarrow o \rightarrow o)$ will be interpreted as the arena below left

$$m_2 \longleftarrow \overbrace{m_1 \longleftarrow m_0} \qquad m_2 \longleftarrow \overbrace{m_1 \longleftarrow m_0 \longleftarrow m \longleftarrow m'}$$

An example type of higher rank is $(\mathbb{B} \rightarrow o) \rightarrow o$, which yields the arena above right.

Notation 1 In any arena, vertices will be called moves, and roots will be deemed initial. We denote by \sqrt{A} the set of roots of A . If A is an arena and m is a move in A , then $A_{/m}$ is the forest strictly below m , and $A \cdot m$ denotes $A_{/m}$ when $m \in \sqrt{A}$. Any forest A is a coproduct of trees, so that $A \cong \sum_{m \in \sqrt{A}} T_m$ where each T_m is a tree. For any arena A and $m \notin A$, we denote by $m.A$ the unique tree T such that $\sqrt{T} = \{m\}$ and $T \cdot m = A$. Thus, any forest may be written as $A = \sum_{m \in \sqrt{A}} m.(A \cdot m)$. The ownership of any vertex $m \in A$ is O (for Opponent) if the length of the unique path from a root to m is even, and P (for Proponent) otherwise. So, e.g., all roots have ownership O .

Example 2. The arenas of Example 1 may be denoted by $m_0.(m_1+m_2)$ and $m'.m.m_0.(m_1+m_2)$, respectively.

More generally, we will consider a simply-typed λ -calculus with one base type o , whose types are inductively interpreted as arenas:

$$\llbracket o \rrbracket = m_o.\emptyset \quad \llbracket \sigma \rightarrow \tau \rrbracket = \sum_{q \in \sqrt{\llbracket \tau \rrbracket}} q.(\llbracket \sigma \rrbracket + \llbracket \tau \rrbracket \cdot q).$$

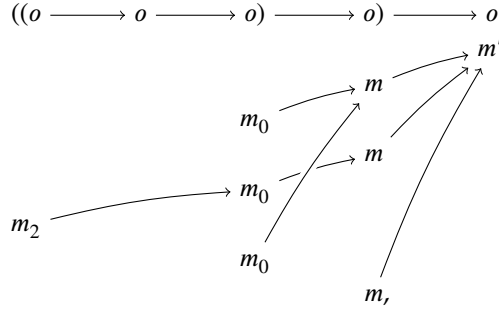
Notation 2 In the sequel, we often leave $\llbracket - \rrbracket$ implicit, i.e., τ sometimes implicitly denotes $\llbracket \tau \rrbracket$. Let us moreover observe that any type has the shape $\tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow o$, and that this is interpreted (up to isomorphism of forests) as $m_o.(\sum_i \llbracket \tau_i \rrbracket)$. Thus, $\tau \cdot m_o = \sum_i \llbracket \tau_i \rrbracket$. Such sums of interpretations of types will occur frequently in the sequel, and we adopt the convention of denoting by m_i the root of the i th term of the sum.

The next step is to define categories of plays. Let us fix arenas A and B , and let $A \multimap B$ denote the empty arena \emptyset if $B = \emptyset$, and otherwise the simple graph obtained by adding to $A + B$ an edge $b \rightarrow a$ for all $b \in \sqrt{B}$ and $a \in \sqrt{A}$. The notion of ownership straightforwardly extends to $A \multimap B$: it is left unchanged in B but reversed in A .

For any A and B , the category $\mathbb{P}_{A \multimap B}$ of plays has as objects all alternating, P-visible justified sequences over $A \multimap B$, as is standard in game semantics. Briefly, one first defines *justified sequences* as sequences of moves in $A \multimap B$ with valid pointers. Then, one defines the *view* of any move in any justified sequence. *Preplays* are then defined as alternating, P-visible justified sequences, where P-visibility means that any P-move points into its view. Finally, *plays* are preplays of even length.

A peculiarity is that $\mathbb{P}_{A \multimap B}$, following Melliès [14], has more morphisms than prescribed by the prefix ordering. It in particular incorporates permutations of independent moves. Specifically, morphisms $f: P \rightarrow Q$ of preplays are defined as injective maps between moves preserving *OP blocks*: if $2i$ is the index of a Proponent move in P , then $f(2i) = f(2i-1) + 1$. In words, if $P = (m_1 \dots m_n)$ and $Q = (m'_1 \dots m'_n)$, the ‘image’ of m_{2i-1} (an O-move) is $m'_{f(2i-1)}$, and the condition says that m_{2i} , the successor of m_{2i-1} , is mapped to the successor of $m'_{f(2i-1)}$, i.e., $m'_{f(2i-1)+1}$.

Example 3. An example play on the arena corresponding to $(\mathbb{B} \rightarrow o) \rightarrow o$ is



where time flows downwards (so the play really is $m' m m_0 m m_0 m_2 m_0 m$) and arrows denote justification pointers.

Proposition 1. *Preplays and morphisms between them form a category $\mathbb{P}\mathbb{P}_{A \vdash B}$, with composition given by composition of underlying maps.*

Definition 2. *The category $\mathbb{P}_{A \vdash B}$ is the full subcategory of $\mathbb{P}\mathbb{P}_{A \vdash B}$ spanning plays. A view on (A, B) is a non-empty play in which all Opponent moves point to their predecessors. Let $\mathbf{I}_{A \vdash B}: \mathbb{V}_{A \vdash B} \rightarrow \mathbb{P}_{A \vdash B}$ denote the corresponding full subcategory embedding.*

3.2 Plays as proof trees

In order to explain the design of the category \mathbb{L} over which our representation theorem will take presheaves, let us take a detour through a sequent calculus description of Tsukada-Ong plays. Mellies [15] has revealed a link between plays (and strategies) in game semantics and a simple intuitionistic sequent calculus. We consider a simple non-linear variant, in which formulas are game semantical arenas. In our sequent calculus, an arena $A = \sum_i m_i.A_i$ is understood as a logical formula much like $\bigwedge_i \neg A_i$. The rules are then a mere adaptation of Mellies's.

An arena sequent is a list of arenas, possibly with a distinguished arena, denoted by $A_1, \dots, A_n \vdash$, resp. $A_1, \dots, A_n \vdash A$, and our sequent calculus has the following inference rules:

$$\frac{[\Lambda_{(\Gamma \vdash A), m}]}{\Gamma, A \cdot m \vdash} \quad \frac{[\@_{(A_1, \dots, A_n \vdash), k, m}]}{A_1, \dots, A_n \vdash A_k \cdot m} \quad \frac{[\text{SUM}_S]}{S \quad \dots \quad S} \quad \left(\begin{array}{l} k \in n, \\ m \in \sqrt{A_k} \end{array} \right)$$

Remark 2. The inference rules are designed to model plays rather than simply-typed terms. Indeed, the $\Lambda_{(\Gamma \vdash A), m}$ rule only handles one of the possibly many roots of A . We will however see below that the SUM rule allows to also interpret normal forms.

Notation 3 *All rules are annotated with their conclusion, which will be omitted when clear from context.*

$$\frac{\frac{\mathbf{D}(p')}{A, B \cdot m_0 \vdash C_{m_0} \cdot m_1}}{A, B \cdot m_0 \vdash} \text{@}_{k, m_1} \quad \frac{\frac{\mathbf{D}_{\Gamma, B \cdot m_0 \vdash C \cdot m_1}(p')}{\Gamma, B \cdot m_0 \vdash C \cdot m_1}}{\Gamma, B \cdot m_0 \vdash} \text{@}_{k, m_1}$$

$$\frac{A, B \cdot m_0 \vdash}{A \vdash B} \Lambda_{m_0} \quad \frac{\Gamma, B \cdot m_0 \vdash}{\Gamma \vdash B} \Lambda_{m_0}$$

where $k = 1$ if $m_1 \in \sqrt{A}$, $k = 2$ otherwise, and \mathbf{D} denotes the map from plays to derivations that we are trying to define. However, the recursive call does not quite typecheck, because $\mathbf{D}(p')$ has conclusion $C_{m_0} \vdash C_{m_0} \cdot m_1$ instead of $A, B \cdot m_0 \vdash C_{m_0} \cdot m_1$. So we need to generalise a bit:

Notation 4 *Let*

$$!(A_1, \dots, A_n \vdash) = \left(\sum_{i \in n} A_i \vdash \right) \quad \text{and} \quad !(A_1, \dots, A_n \vdash B) = \left(\sum_{i \in n} A_i \vdash B \right).$$

We define, for any $S = (\Gamma \vdash B) = (A_1, \dots, A_n \vdash B)$, a map \mathbf{D}_S mapping plays over $!S$ to derivations over S . This time we may correctly define $\mathbf{D}_S(m_0 m_1 p')$ as above right, where $(k, C) = (i, A_i)$ if $m_1 \in \sqrt{A_i}$ and $(k, C) = (n + 1, B \cdot m_0)$ if $m_1 \in \sqrt{B \cdot m_0}$ (leaving implicit the isomorphism $\mathbb{T}_{!S} \cong \sum_{m_0 m_1} \mathbb{P}_{!(\Gamma, B \cdot m_0 \vdash C \cdot m_1)}$).

$$\text{Finally, we translate } p = \sum_j t_j \in \mathbb{P}_S \text{ using the SUM rule: } \frac{\dots \frac{\mathbf{D}_S(t_j)}{S} \dots}{S}.$$

In particular, the empty play is interpreted as the nullary sum.

3.3 Proof trees as presheaves

We have thus interpreted plays into sequent calculus derivations. The latter, being tree-like structures, are easily viewed as presheaves over a certain category:

Definition 3. *Let \mathbb{L} denote the free category on the graph with*

- a vertex for each arena sequent, plus one for each rule $\Lambda_{(\Gamma \vdash A), m}$ and $\text{@}_{\Gamma, k, m}$
- for each rule ρ with conclusion S and premise S' , edges $S \xrightarrow{t} \rho \xleftarrow{s} S'$.

Remark 3. In [5], a slightly more complicated variant of \mathbb{L} is presented, which allows to represent not only plays, but also *interaction sequences*, the basic ingredient for parallel composition. The present definition is simpler, and sufficient for our purposes.

Let us now inductively interpret derivations π of conclusion S into morphisms $S \rightarrow \llbracket \pi \rrbracket$ in $\widehat{\mathbb{L}}$, which will yield the desired interpretation of plays into $S/\widehat{\mathbb{L}}$ by composition. We again start with an example.

Example 6. The derivation of Example 5 will be interpreted as the presheaf U with:

- for each arena sequent S , an element of type S for each occurrence of S in the derivation (except that the premises and conclusion of each SUM rule are equated),
- for each non-SUM rule ρ , an element of type ρ for each occurrence of ρ ,
- for each $r \in U(\rho)$, $U(t)(r)$ is the occurrence of the conclusion of ρ corresponding to r , and $U(s)(r)$ is the occurrence of the premise of ρ corresponding to r .

A detailed definition of U is in the first two parts of Figure 1. Elements are in the first part; and the action of morphisms is defined in the second part. Reading this in full detail is of course not mandatory, but having a look may be useful for setting things straight. We introduce elements following the proof bottom-up and breadth-first, and proceed similarly for equations. An efficient way of representing all this data is to depict the *category of elements* of U , which has as objects all pairs (c, x) with $x \in U(c)$, and as morphisms $(c, x) \rightarrow (c', x')$ all morphisms $f: c \rightarrow c'$ such that $x' \cdot f = x$. This category is displayed on the left in the last part of Figure 1 (displaying just x instead of (c, x) for readability). The element pointed by $(A \vdash B)$ is the bottom one, $s_{\vdash m'.m.B}$.

Let us now define our interpretation in full generality, by induction on the given derivation. The Λ and $@$ rules are easy to interpret. Indeed, given any

proof π of S' , if ρ is one of Λ or $@$, we interpret $\frac{\pi}{S} \rho$ as the bottom row below left

$$\begin{array}{ccc}
 S' & \longrightarrow & \llbracket \pi \rrbracket \\
 \downarrow & & \downarrow \\
 S & \longrightarrow & \rho \bullet \llbracket \pi \rrbracket
 \end{array}
 \qquad
 \begin{array}{ccc}
 \sum_{i \in n} S & \longrightarrow & \sum_{i \in n} \llbracket \pi_i \rrbracket \\
 \downarrow & & \downarrow \\
 S & \longrightarrow & \bigvee_i \llbracket \pi_i \rrbracket,
 \end{array}$$

where the marked square is as pushout in $\widehat{\mathbb{L}}$. Similarly, given proofs π_1, \dots, π_n of

$$\frac{\frac{\llbracket \pi_1 \rrbracket}{S} \quad \dots \quad \frac{\llbracket \pi_n \rrbracket}{S}}{S} \text{SUM}$$

we interpret $\frac{\llbracket \pi_1 \rrbracket}{S} \quad \dots \quad \frac{\llbracket \pi_n \rrbracket}{S} \text{SUM}$ as the bottom row above right.

We now have the object part of our representation: the composite

$$\mathbf{P}_S: \mathbb{P}_{!S} \rightarrow \text{Derivations}_S \rightarrow S/\widehat{\mathbb{L}},$$

for all arena sequents S . For any play $p \in \mathbb{P}_{!S}$, there is a bijection between the moves of p and the elements of $\mathbf{P}_S(p)$ of type Λ and $@$. Given a morphism $f: p \rightarrow q$, this directly induces a candidate morphism $\mathbf{P}_S(f): \mathbf{P}_S(p) \rightarrow \mathbf{P}_S(q)$ on such elements. Clearly, there is at most one extension of this candidate to all elements that qualifies as a proper morphism. Indeed, for all elements $x \in \mathbf{P}_S(p)(S)$, there exists some Λ or $@$ element $y \in \mathbf{P}_S(p)(\rho)$ and map $\partial \in \{s, t\}$ such that $x = y \cdot \partial$; naturality thus imposes

$$\mathbf{P}_S(f)(x) = \mathbf{P}_S(f)(y \cdot \partial) = \mathbf{P}_S(f)(y) \cdot \partial.$$

$$\begin{aligned}
U(\vdash m'.m.\mathbb{B}) &= \{s_{\vdash m'.m.\mathbb{B}}\} & U(\Lambda_{m'}) &= \{x_{m'}\} & U(m.\mathbb{B} \vdash) &= \{s_{m.\mathbb{B} \vdash}\} \\
U(@_{(m.\mathbb{B} \vdash),1,m}) &= \{x_m\} & U(m.\mathbb{B} \vdash \mathbb{B}) &= \{s_{m.\mathbb{B} \vdash \mathbb{B}}\} & U(\Lambda_{(m.\mathbb{B} \vdash \mathbb{B}),m_0}) &= \{x_{m_0}^1, x_{m_0}^2\} \\
U(m.\mathbb{B}, m_1 + m_2 \vdash) &= \{s_{m.\mathbb{B}, m_1 + m_2 \vdash}^1, s_{m.\mathbb{B}, m_1 + m_2 \vdash}^2\} & U(@_{(m.\mathbb{B}, m_1 + m_2 \vdash), 2, m_1}) &= \{x_{m_1}\} \\
U(@_{(m.\mathbb{B}, m_1 + m_2 \vdash), 2, m_2}) &= \{x_{m_2}\} & U(m.\mathbb{B}, m_1 + m_2 \vdash \emptyset) &= \{s_{m.\mathbb{B}, m_1 + m_2 \vdash \emptyset}^1, s_{m.\mathbb{B}, m_1 + m_2 \vdash \emptyset}^2\}
\end{aligned}$$

$$\begin{aligned}
x_{m'} \cdot t &= s_{\vdash m'.m.\mathbb{B}} & x_{m'} \cdot s &= s_{m.\mathbb{B} \vdash} = x_m \cdot t & x_m \cdot s &= s_{m.\mathbb{B} \vdash \mathbb{B}} = x_{m_0}^1 \cdot t = x_{m_0}^2 \cdot t \\
x_{m_0}^1 \cdot s &= s_{m.\mathbb{B}, m_1 + m_2 \vdash}^1 = x_{m_1} \cdot t & x_{m_0}^2 \cdot s &= s_{m.\mathbb{B}, m_1 + m_2 \vdash}^2 = x_{m_2} \cdot t & x_{m_1} \cdot s &= s_{m.\mathbb{B}, m_1 + m_2 \vdash \emptyset}^1 \\
& & & & x_{m_2} \cdot s &= s_{m.\mathbb{B}, m_1 + m_2 \vdash \emptyset}^2
\end{aligned}$$

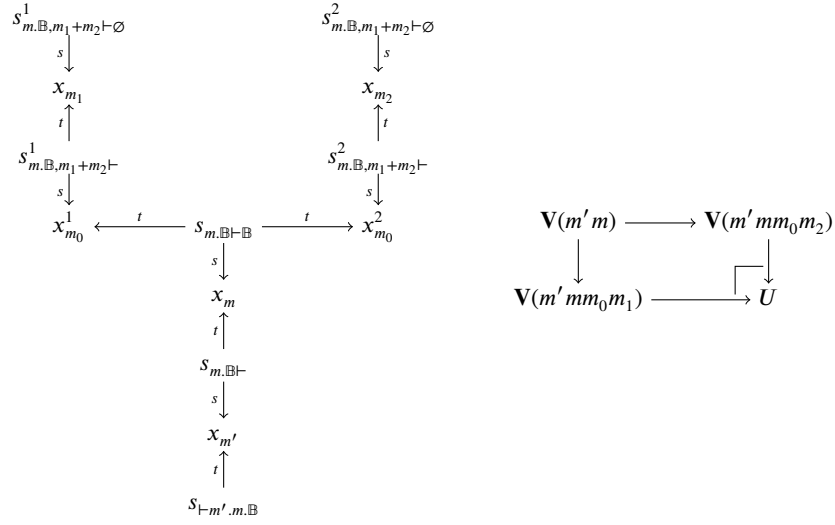


Fig. 1. Presheaf for the derivation of Example 5

It remains to verify that for any other y' and ∂' such that $x = y' \cdot \partial'$, we have $y \cdot \partial = y' \cdot \partial'$, which indeed holds. This ends the definition of \mathbf{P}_S , and we may at last state:

Theorem 1 ([5]). *For all arena sequents S , there is an embedding $\mathbf{P}_S: \mathbb{P}_{!S} \rightarrow S/\widehat{\mathbb{L}}$ of the category of Tsukada-Ong plays over $!S$ into a coslice of $\widehat{\mathbb{L}}$ under S . The composite embedding \mathbf{V}_S defined by $\mathbb{V}_{!S} \xrightarrow{\mathbf{I}_{!S}} \mathbb{P}_{!S} \xrightarrow{\mathbf{V}_S} S/\widehat{\mathbb{L}}$ is full.*

A more detailed presentation and a proof of the case $S = (A \vdash B)$ are available in [5].

Example 7. The presheaf U of Example 6 may alternatively be described as the (implicitly pointed) pushout on the right in the last part of Figure 1.

4 Representing normal forms

Our next step is to extend the representation theorem to normal forms in the simply-typed, non-deterministic λ -calculus studied by Tsukada and Ong. Briefly, it is simply-typed λ -calculus with one base type o , one constant \perp of type o , plus non-deterministic sum, denoted by $M + N$. Up to may testing equivalence, all terms are equivalent to some normal form. Tsukada and Ong define these as the class of typed terms covered by the following grammar:

$$R ::= \lambda x_1 \dots x_p. \perp \mid Q_1 + \dots + Q_{n+1} \quad Q ::= \lambda x_1 \dots x_p. y R_1 \dots R_k,$$

with $n, p, k \in \mathbb{N}$ and y is fully applied (so the grammar is implicitly typed). Let us start by observing that normal forms may be defined in natural deduction style:

$$\frac{\gamma, x_1 : \sigma_1, \dots, x_p : \sigma_p \vdash \langle R_1, \dots, R_k \rangle : \tau_1, \dots, \tau_k \quad (y'(y) = \tau_1 \rightarrow \dots \rightarrow \tau_k \rightarrow o)}{\gamma \vdash \lambda x_1 \dots x_p. y R_1 \dots R_k : \sigma_1 \rightarrow \dots \rightarrow \sigma_p \rightarrow o} \quad \frac{\dots \quad \gamma \vdash Q_i : \sigma \quad \dots \quad (i \in n) \quad \dots \quad \gamma \vdash R_j : \tau_j \quad \dots \quad (j \in k)}{\gamma \vdash \sum_{i \in n} Q_i : \sigma \quad \gamma \vdash \langle R_1, \dots, R_k \rangle : \tau_1, \dots, \tau_k},$$

where $\gamma' = (\gamma, x_1 : \sigma_1, \dots, x_p : \sigma_p)$.

Remark 4. This slightly differs from Tsukada and Ong's presentation, but clearly yields the same terms. Indeed, they distinguish a particular case of the second rule when $n = 0$, which they denote by $\lambda x_1 \dots x_p. \perp$. The only thing that matters here is that our nullary sum will be interpreted in the same way.

Our interpretation of normal forms is given by showing that, up to the interpretation of simple types as arenas, the natural deduction rules are derivable in the sequent calculus of Section 3.2. Let us start with an example.

Example 8. Recall the play (and resource term) of Example 5, $\lambda f.f[\lambda ab.a, \lambda a'b'.b']$ of type $(\mathbb{B} \rightarrow o) \rightarrow o$. Turning the multiset of arguments into a sum, we obtain the normal form $R := \lambda f.f(\lambda ab.a + \lambda a'b'.b')$, which is interpreted as the exact same derivation. This differs from what we would obtain by lifting the sum, i.e., from $R' := (\lambda f.f(\lambda ab.a)) + (\lambda f.f(\lambda a'b'.b'))$:

$$\frac{\frac{\frac{\frac{\frac{m.\mathbb{B}, m_1 + m_2 \vdash \emptyset}{\text{SUM}}}{m.\mathbb{B}, m_1 + m_2 \vdash \emptyset} @_{2,m_1}}{\frac{m.\mathbb{B} \vdash \mathbb{B}}{\Lambda_{m_0}}} @_{1,m}}{m.\mathbb{B} \vdash \mathbb{B}} @_{1,m}}{\vdash m'.m.\mathbb{B}} \Lambda_{m'}}{\vdash m'.m.\mathbb{B}} \text{SUM} \quad \frac{\frac{\frac{\frac{\frac{m.\mathbb{B}, m_1 + m_2 \vdash \emptyset}{\text{SUM}}}{m.\mathbb{B}, m_1 + m_2 \vdash \emptyset} @_{2,m_2}}{\frac{m.\mathbb{B} \vdash \mathbb{B}}{\Lambda_{m_0}}} @_{1,m}}{m.\mathbb{B} \vdash \mathbb{B}} @_{1,m}}{\vdash m'.m.\mathbb{B}} \Lambda_{m'}}{\vdash m'.m.\mathbb{B}} \text{SUM}.$$

What these derivations become when interpreted into presheaves should be clear.

Let us now consider the general case. The second rule, for $R = \sum_{i \in n} Q_i$, will be interpreted straightforwardly using the SUM rule, so that $\mathbf{T}(R)$ will roughly be the derivation below left:

$$\mathbf{T}(\sum_i Q_i) = \frac{\dots \frac{\mathbf{T}(Q_i)}{\gamma \vdash \sigma} \dots \quad (i \in n)}{\gamma \vdash \sigma} \text{SUM} \quad \frac{\frac{\mathbf{T}\langle R_1, \dots, R_k \rangle}{\gamma, \sum_{i \in p} \sigma_i \vdash \sum_{j \in k} \tau_j} @}{\gamma, \sum_{i \in p} \sigma_i \vdash} @_{m_o}}{\gamma \vdash m_o.(\sum_{i \in p} \sigma_i)} \Lambda_{m_o}.$$

Now, we would like to derive the rule for $Q = \lambda x_1 \dots x_p. y R_1 \dots R_k$ by something like the derivation above right (assuming y has type $\tau_1 \rightarrow \dots \rightarrow \tau_k \rightarrow o$). There is one little glitch, however: we expect to make a recursive call on $\gamma, x_1 : \sigma_1, \dots, x_p : \sigma_p \vdash \langle R_1, \dots, R_k \rangle : \tau_1, \dots, \tau_k$, but the context we get is $\gamma, \sum_{i \in p} \sigma_i \vdash \sum_{j \in k} \tau_j$: some arenas are grouped together, according to the λ -abstraction that introduced them. We thus need an additional parameter.

Definition 4. A grouping of a typing context $\gamma = (x_1 : \sigma_1, \dots, x_n : \sigma_n)$ consists of a natural number $K \in \mathbb{N}$ and a monotone map $g : n \rightarrow K$. Given such a g , the grouped context $g[\gamma]$ is the list of arenas of length K whose i th element is $\sum_{l \in g^{-1}(i)} \llbracket \sigma_l \rrbracket$ (where the sum is ordered according to γ). We generalise the notation $g[\gamma]$ to sequents by posing $g[\gamma \vdash \tau_1, \dots, \tau_k] = (g[\gamma] \vdash \sum_{j \in k} \tau_j)$. Similarly, for any sequent S , $!S$ denotes $!id_{|S|}[S]$, i.e., we apply $!$ to the ungrouped arena sequent corresponding to S .

We thus define for all sequents $S = (\gamma \vdash \delta)$ and groupings g a map $\mathbf{T}_{g,S} : \mathbf{NF}_S \rightarrow g[S]/\hat{\mathbb{L}}$. First, $\mathbf{T}_{g,S}\langle R_1, \dots, R_k \rangle$ should map any tuple $\langle R_1, \dots, R_k \rangle$ with $\gamma \vdash R_j : \tau_j$

for all $j \in k$ to some derivation of $g[\gamma] \vdash \sum_{j \in k} \llbracket \tau_j \rrbracket$. We straightforwardly let:

$$\mathbf{T}_{g,S} \langle R_1, \dots, R_k \rangle = \frac{\dots \frac{\frac{\mathbf{T}_{g,S}^+(Q_l^j)}{g[\gamma], \tau_j \cdot m_o \vdash}}{g[\gamma] \vdash \sum_{j \in k} \tau_j} \Lambda_{m_j} \dots}{g[\gamma] \vdash \sum_{j \in k} \tau_j} \text{SUM}, \quad (j \in k, l \in n_j)$$

where we decompose each R_j as $\sum_{l \in n_j} Q_l^j$, with $\gamma \vdash R_j : \tau_j$ and $\tau = (\alpha_1 \rightarrow \dots \rightarrow \alpha_{p_j} \rightarrow o)$, for all $j \in k$. We cannot directly use $\mathbf{T}(Q_l^j)$ in the recursive call, however, as this would have conclusion $g[\gamma] \vdash \tau_j$ instead of $g[\gamma] \vdash \sum_{j \in k} \tau_j$. We thus define an intermediate interpretation, \mathbf{T}^+ , mapping normal forms $\gamma \vdash Q : \sigma$ to derivations of $g[\gamma], \sigma \cdot m_o \vdash$, by:

$$\mathbf{T}_{g,S}^+(\lambda x_1 \dots x_p. y R_1 \dots R_k) = \frac{\frac{\mathbf{T}_{g+!, (\gamma, x_1 : \sigma_1, \dots, x_p : \sigma_p \vdash \tau_1, \dots, \tau_k)} \langle R_1, \dots, R_k \rangle}{g[\gamma], \sum_{i \in p} \sigma_i \vdash \sum_{j \in k} \tau_j}}{g[\gamma], \sum_{i \in p} \sigma_i \vdash} @_{k_y, m_{i_y}},$$

where $g+! : n + p \rightarrow K + 1$ is obtained by coproduct, k_y is unique such that $y \in g^{-1}(k_y)$, and i_y is the index of y therein. Concretely, letting $\gamma' := (\gamma, x_1 : \sigma_1, \dots, x_p : \sigma_p) = (z_1 : \alpha_1, \dots, z_N : \alpha_N)$ and $\Gamma := (g[\gamma], \sum_{i \in p} \sigma_i)$, the k_y th arena in Γ is $\sum_{l \in (g+!)^{-1}(k_y)} \alpha_l$, and $(\sum_{l \in (g+!)^{-1}(k_y)} \alpha_l) \cdot m_{i_y} = \alpha_y \cdot m_o$, which must have the form $\sum_{j \in k} \tau_j$.

We thus have defined maps

$$\mathbf{T}_{g,S} : \text{NF}_S \rightarrow g[S]/\widehat{\mathbb{L}},$$

for all sequents S and groupings $g : |S| \rightarrow K$.

Definition 5. For all sequents $S = (\gamma \vdash \sigma)$, let $\mathbf{T} = \mathbf{T}_{\text{id}_{|\gamma|}, S} \langle - \rangle : \text{NF}_S \rightarrow S/\widehat{\mathbb{L}}$.

The set NF_S is here viewed as a discrete category, and \mathbf{T} is evidently injective, hence trivially an embedding.

5 Game semantics as a singular functor

At this stage, for all sequents $S = (\gamma \vdash \delta)$, we know how to embed views and plays over $!g[S]$ and normal forms over S into the coslice category $g[S]/\widehat{\mathbb{L}}$. We may now show how this allows us to automatically translate normal forms into innocent strategies. Let us first recall what innocent strategies are. They standardly come in two flavours, so let us disambiguate:

Definition 6. The category of behaviours over (A, B) is the presheaf category $\widehat{\mathbb{V}}_{A \vdash B}$.

Similarly, the category of strategies over (A, B) is the presheaf category $\widehat{\mathbb{P}}_{A \vdash B}$. A strategy is innocent iff it is in the essential image of right Kan extension along \mathbf{I}^{op} .

Now, for any sequent $S = (\gamma \vdash \delta)$ and compatible grouping g , simply unfolding the standard Yoneda structure on categories yields the solid part of the diagram below (leaving some dependencies on S and g implicit for readability):

$$\begin{array}{ccc}
 \mathbb{V}_{!g[S]} & \xrightarrow{\mathbf{V}} & g[S]/\widehat{\mathbb{L}} \longleftarrow \mathbf{T} \quad \text{NF}_S \\
 \downarrow \mathbf{y} & \xRightarrow{\chi^{\mathbf{V}}} & \mathbf{V}^* \swarrow \\
 \widehat{\mathbb{V}}_{!g[S]} & \xleftarrow{\Delta_{\theta^g, S}} & \widehat{\mathbb{V}}_{!S} \longleftarrow \llbracket - \rrbracket
 \end{array} \quad (2)$$

where $\chi^{\mathbf{V}}$ is a left extension and absolute left lifting of \mathbf{y} [17]. Concretely,

$$\mathbf{V}^*(p)(v) = \mathbb{V}_{!g[S]}(\mathbf{V}(v), p),$$

and $(\chi^{\mathbf{V}})_{v'}: \mathbb{V}_{!g[S]}(v', v) \rightarrow (g[S]/\widehat{\mathbb{L}})(\mathbf{V}(v'), \mathbf{V}(v))$ is merely \mathbf{V} on morphisms. This automatically yields our candidate interpretation, namely the composite $\mathbf{V}^* \circ \mathbf{T}$. Furthermore, the arenas composing $!S$ and $!g[S]$ are isomorphic, and thus induce an isomorphism $\theta^{g, S}: \mathbb{V}_{!g[S]} \rightarrow \mathbb{V}_{!S}$ between the corresponding categories of views, hence (by precomposition) an isomorphism between the corresponding presheaf categories, which we denote by $\Delta_{\theta^{g, S}}: \widehat{\mathbb{V}}_{!S} \rightarrow \widehat{\mathbb{V}}_{!g[S]}$. Letting $\llbracket - \rrbracket$ denote Tsukada and Ong's interpretation, we obtain the dashed arrows in (2). We have:

Theorem 2. For any sequent $S = (\gamma \vdash \delta)$ with grouping g and normal form $\gamma \vdash \vec{R}: \delta$, we have $\mathbf{V}^*(\mathbf{T}(\vec{R})) \cong \Delta_{\theta^{g, S}} \llbracket \vec{R} \rrbracket$, where $\llbracket \vec{R} \rrbracket = \langle \llbracket R_1 \rrbracket, \dots, \llbracket R_k \rrbracket \rangle$.

Concretely, this means that $\Delta_{\theta^{g, S}}(\llbracket \vec{R} \rrbracket)(v) = \llbracket \vec{R} \rrbracket(\theta^{g, S}(v)) \cong (g[S]/\widehat{\mathbb{L}})(\mathbf{V}(v), \mathbf{T}(\vec{R}))$ for all v .

Example 9. Recall the play U of Figure 1 (and Examples 5 and 6). From the description of U as a pushout, we see that it has three views: $m'm$, $m'mm_0m_1$, and $m'mm_0m_2$. The intuitive difference between R and R' is that R feeds f with a non-deterministic argument, so calls to it from f may play either like $\lambda ab.a$ or $\lambda a'b'.b'$. This is not the case in R' , which non-deterministically chooses between two functions, the first of which calls its argument f on $\lambda ab.a$ whilst the second calls it on $\lambda a'b'.b'$.

This shows up already on $m'm$, since $\mathbf{V}^*(\mathbf{T}(R))(m'm) \cong 1$ and $\mathbf{V}^*(\mathbf{T}(R'))(m'm) \cong 2$. And it further leads R' to reject some plays accepted by R . E.g., denoting by $\prod_{\mathbf{I}}(S)$ the (innocent) strategy associated to any behaviour S , we in particular have

$$\begin{aligned}
 \prod_{\mathbf{I}}(\mathbf{V}^*(\mathbf{T}(R)))(U) &\cong \mathbf{V}^*(\mathbf{T}(R))(m'mm_0m_1) \times_{\mathbf{V}^*(\mathbf{T}(R))(m'm)} \mathbf{V}^*(\mathbf{T}(R))(m'mm_0m_2) \\
 &\cong 1 \times_1 1 \\
 &\cong 1,
 \end{aligned}$$

$$\begin{aligned}
& \text{but} \\
\prod_1(\mathbf{V}^*(\mathbf{T}(R')))(U) & \cong \mathbf{V}^*(\mathbf{T}(R'))(m' mm_0 m_1) \times_{\mathbf{V}^*(\mathbf{T}(R'))(m' m)} \mathbf{V}^*(\mathbf{T}(R'))(m' mm_0 m_2) \\
& \cong 1 \times_2 1 \\
& \cong 0,
\end{aligned}$$

where $1 \times_2 1$ denotes the pullback of $1 \rightrightarrows 2 \xleftarrow{2} 1$, which is indeed empty.

Let us return to the theorem.

Proof (Proof sketch). We proceed by induction. Let us start by recalling the following characterisation of $\llbracket \vec{R} \rrbracket$: let us assume $\gamma \vdash \langle R_1, \dots, R_k \rangle : \sigma_1, \dots, \sigma_{k'}$ with $R_j = \sum_{l \in n_j} Q^{j,l}$ for all $j \in k$, and $Q^{j,l} = \lambda x_1 : \sigma_1^j \dots x_{p^j} : \sigma_{p^j}^j . y^{j,l} R_1^{j,l} \dots R_{k^j,l}^{j,l}$ for all $j \in k$ and $l \in n_j$. Then, letting $\gamma_j = (x_1 : \sigma_1^j, \dots, x_{p^j} : \sigma_{p^j}^j)$, $(\gamma, \gamma_j)(y^{j,l}) = \tau_1^{j,l} \rightarrow \dots \rightarrow \tau_{k^j,l}^{j,l} \rightarrow o$, and $S^{j,l} = (\gamma, \gamma_j \vdash \tau_1^{j,l}, \dots, \tau_{k^j,l}^{j,l})$ we have

$$\llbracket \vec{R} \rrbracket = \sum_{j \in k} \sum_{l \in n_j} (m_j m_{y^{j,l}} \triangleright \langle \llbracket R_1^{j,l} \rrbracket, \dots, \llbracket R_{k^j,l}^{j,l} \rrbracket \rangle),$$

where $m_{y^{j,l}}$ denotes the move corresponding to y in $!(\gamma \vdash \sigma_1, \dots, \sigma)$ and $m_j m_{y^{j,l}} \triangleright S$ is defined in [18]. Roughly, $m_j m_{y^{j,l}} \triangleright S$ denotes the strategy playing $m_j m_{y^{j,l}}$ and then S . We thus have

$$\Delta_{\theta g, S} \llbracket \vec{R} \rrbracket = \sum_{j \in k} \sum_{l \in n_j} (m_j m_{i_{y^{j,l}}, k_{y^{j,l}}} \triangleright \langle \Delta_{\theta g^+, S^{j,l}} \llbracket R_1^{j,l} \rrbracket, \dots, \Delta_{\theta g^+, S^{j,l}} \llbracket R_{k^j,l}^{j,l} \rrbracket \rangle),$$

where $i_{y^{j,l}}$ denotes the unique index in K such that $g(y^{j,l}) = i_{y^{j,l}}$, and $k_{y^{j,l}}$ denotes the index of $y^{j,l}$ in $g^{-1}(i_{y^{j,l}})$. But by induction hypothesis and definition of \mathbf{V}^* and \mathbf{T} , this is isomorphic to $\mathbf{V}^*(\mathbf{T}(\vec{R}))$.

6 Definability as geometric realisation

Let us now consider definability. For $S = (\gamma \vdash \delta)$, given any $X \in \widehat{\mathbb{V}}_{!g[S]}$, we thus seek a normal form $\gamma \vdash \vec{R} : \delta$ such that $\llbracket \vec{R} \rrbracket \cong X$. Our candidate relies on:

Definition 7. *The geometric realisation functor $\mathbf{V}_! : \widehat{\mathbb{V}}_{!g[S]} \rightarrow (g[S]/\widehat{\mathbb{L}})$ is defined (up to unique commuting isomorphism) as the left Kan extension of \mathbf{V} along \mathbf{y} .*

As is well-known, this is well defined because $g[S]/\widehat{\mathbb{L}}$ is cocomplete, and we have $\mathbf{V}_!(X) \cong \int^v X(v) \cdot \mathbf{V}(v)$. Furthermore, $\mathbf{V}_!$ is left adjoint to \mathbf{V}^* . Our second main result is:

Theorem 3. *For any $X \in \widehat{\mathbb{V}}_{!g[S]}$, i.e., any finitely presentable presheaf on $\mathbb{V}_{!g[S]}$, $\mathbf{V}_!(X)$ is the \mathbf{T} -image of a normal form \vec{R} , and furthermore $\mathbf{V}^*(\mathbf{V}_!(X)) \cong X$, so that in particular $\mathbf{V}^*(\mathbf{T}(\vec{R})) \cong X$.*

Example 10. Recall the normal forms R and R' of Example 8. By the standard characterisation of left Kan extensions as colimits, $\mathbf{V}_!(R)$ and $\mathbf{V}_!(R')$ respectively correspond to the pushouts below:

$$\begin{array}{ccc}
\mathbf{V}_1(m'm) & \longrightarrow & \mathbf{V}_1(m'mm_0m_2) \\
\downarrow & & \downarrow \\
\mathbf{V}_1(m'mm_0m_1) & \longrightarrow & \mathbf{V}_1(R)
\end{array}
\quad
\begin{array}{ccc}
\mathbf{V}_1(\emptyset) & \longrightarrow & \mathbf{V}_1(m'mm_0m_2) \\
\downarrow & & \downarrow \\
\mathbf{V}_1(m'mm_0m_1) & \longrightarrow & \mathbf{V}_1(R')
\end{array}$$

which indeed are R and R' .

Returning to the theorem, the second point is relatively easy. Indeed, for all $v \in \mathbb{V}$,

$$\begin{aligned}
\mathbf{V}^*(\mathbf{V}_1(X))(v) &\cong (g[S]/\widehat{\mathbb{L}})(\mathbf{V}(v), \mathbf{V}_1(X)) \\
&\cong (g[S]/\widehat{\mathbb{L}})(\mathbf{V}(v), f^{v'} X(v') \cdot \mathbf{V}(v')) \\
&\cong f^{v'} X(v') \cdot (g[S]/\widehat{\mathbb{L}})(\mathbf{V}(v), \mathbf{V}(v')) \\
&\cong X(v).
\end{aligned}$$

The last isomorphism holds by Yoneda reduction and full faithfulness of \mathbf{V} ; and the penultimate one by:

Lemma 2. For any sequent S , $v \in \mathbb{V}_{g[S]}$, and $X \in \widehat{\mathbb{V}}_{g[S]}$, the canonical map

$$f^{v'} X(v') \cdot (g[S]/\widehat{\mathbb{L}})(\mathbf{V}(v), \mathbf{V}(v')) \rightarrow (g[S]/\widehat{\mathbb{L}})(\mathbf{V}(v), \mathbf{V}_1(X))$$

is an isomorphism.

Proof (Proof sketch). Let $\mathbf{V}_1(X) = (g[S] \xrightarrow{t} R)$ and consider the binary relation $<$ on the elements of R defined by $x < y$ iff $x = y \cdot t$ or $x \cdot s = y$. Let now \leq denote the reflexive transitive closure of $<$, which forms a partial order. For any arena sequent $S' = (\gamma' \vdash \sigma')$ and $x \in R(S')$, the elements below x determine a view v_x , an element $y_x \in X(v_x)$, and an inclusion $i_x: \mathbf{V}(v_x) \hookrightarrow (X(v_x) \cdot \mathbf{V}(v_x)) \hookrightarrow \mathbf{V}_1(X)$. Now, consider any map $f: \mathbf{V}(v) \rightarrow \mathbf{V}_1(X)$. Letting x denote the image of the top element of $\mathbf{V}(v)$, we obtain that $v = v_x$ and $i_x = f$, so f factors as $\mathbf{V}(v) \cong \mathbf{V}(v_x) \xrightarrow{i_x} \mathbf{V}_1(X)$. One then verifies that this provides a unique antecedent to f .

It remains to show that $\mathbf{V}_1(X)$ is indeed the image of a normal form. For this, the following notion will be helpful:

Definition 8. For any $X \in \widehat{\mathbb{L}}$, a finite complex (relative to X) is a finite composite $X \rightarrow Y$ of pushouts of maps of the form $S \rightarrow \mathbf{V}(m_0m_1)$, for all arena sequents S and $m_0m_1 \in \mathbb{V}_{S'}$:

- the identity id_X is a finite complex relative to X ;
- and for all finite complexes $f: X \rightarrow Y$ and $m_0m_1 \in \mathbb{V}_{S'}$, the bottom row of

$$\begin{array}{ccc}
S & \longrightarrow & \mathbf{V}(m_0m_1) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \longrightarrow Z
\end{array}$$

is again a finite complex.

The term ‘complex’ refers to a standard construction in cofibrantly generated factorisation systems [11]. Intuitively, for any arena sequent S , finite complexes relative to S are obtained from S by grafting finitely many OP blocks. The following should thus seem natural:

Lemma 3. *The essential image of $\mathbf{T}: \mathbf{NF}_S \rightarrow g[S]/\widehat{\mathbb{L}}$ has as objects all finite complexes relative to $g[S]$.*

Proof (Proof sketch). That any normal form is mapped by \mathbf{T} to a finite complex follows by induction from the very definition of \mathbf{T} . Conversely, we also proceed by induction on the length of the given complex. The only non-trivial point is that when we add one composite, we need to find the occurrence in the corresponding term where to graft the corresponding OP block.

A handy property of finite complexes is:

Lemma 4. *Finite relative complexes are stable under pushout.*

Proof. We proceed by induction on the length of one of the given complexes. For the induction step, consider any finite complex $f: X \rightarrow Y$ relative to X , pushout $g: Y \rightarrow Z$ of a generating map $S \rightarrow \mathbf{V}(m_0m_1)$, and any map $h: X \rightarrow X'$. We need to show that the bottom row of

$$\begin{array}{ccccc}
 & & S & \longrightarrow & \mathbf{V}(m_0m_1) \\
 & & \downarrow & & \downarrow \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
 h \downarrow & & \downarrow & & \downarrow \\
 X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z'
 \end{array}$$

is a finite complex relative to X' . But by induction hypothesis f' is, and by the pushout lemma, g' is a pushout of $S \rightarrow \mathbf{V}(m_0m_1)$, hence the result.

Because finite colimits reduce to pushouts and initial object, and because the forgetful functor $X/\widehat{\mathbb{L}} \rightarrow \widehat{\mathbb{L}}$ creates pushouts, we have:

Corollary 1. *Finite complexes relative to X , with relative complexes as morphisms between them, form a subcategory of $X/\widehat{\mathbb{L}}$ which is closed under finite colimits.*

We may finally prove that $\mathbf{V}_1(X)$ is the image of a normal form: \mathbf{V}_1 , being a left adjoint, preserves colimits, so $\mathbf{V}_1(X)$ is a finite colimit of objects of the form $\mathbf{V}(v)$, which clearly are finite complexes. So $\mathbf{V}_1(X)$ is a finite colimit of finite complexes, hence itself a finite complex by Corollary 1, hence isomorphic to the image of a normal form by Lemma 3.

7 Conclusion and perspectives

Based on the embedding of plays into a coslice of a presheaf category, we have recovered (1) the interpretation of normal forms as innocent strategies as a singular functor and (2) definability as the corresponding geometric realisation. We find intriguing that the whole coslice may in fact be assigned an interpretation into innocent strategies, thus wildly generalising the interpretation of normal forms. E.g., this flexibility might be useful to handle recursive definitions, interpreting them as cyclic presheaves. On such generalised terms M , geometric realisation will however not return something isomorphic to M in general, only some sort of unfolding [4].

As mentioned in the introduction, a fundamental aspect of game semantics which is not treated here is composition of strategies, in particular associativity, and stability of innocent strategies. In ongoing work, we use presheaf-based techniques, in particular the *exact* squares of Guitart [8], to give alternative, streamlined proofs of these results.

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