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ALMOST PERIODIC MOTION OF A STRING VIBRATING AGAINST A STRAIGHT FIXED OBSTACLE

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1. INTRODUCTION

IN A PLANE with normalized coordinate system $0xu$, we consider the small oscillations of a vibrating string with fixed ends $(\pm 1/2, 0)$. The free oscillations of this string are perturbed by the presence of a fixed obstacle $\{u = -h\}$, against which it rebounds following a law which insures conservation of the energy ($0 \leq h < 1$).

At time $t = 0$, the string is at rest in a position $u(x, 0) = u_0(x)$ such that

$$u_0 \in H_0^1(-1/2, 1/2] \quad (1)$$

$$0 \leq u_0 \leq 1 \quad \text{on } [-1/2, 1/2]. \quad (2)$$

There exists a, b in $] -1/2, 1/2[$, $a \leq b$ such that

$$\left. \begin{array}{l} u_0 \text{ is non-decreasing and } u_0(x) < 1 \text{ on } [-1/2, a[\\ u_0(x) = 1 \quad \text{for } x \in [a, b] \\ u_0 \text{ is non-increasing and } u_0(x) < 1 \quad \text{on }]b, 1/2]. \end{array} \right\} \quad (3)$$

If there was no obstacle, the motion of the string would be described by the wave equation:

$$\square u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad (4)$$

a classical approximation following d'Alembert [1] for the small (2-periodic) oscillations around $u = 0$.

In case there is an obstacle, Amerio & Prouse [2, linear obstacle], then Schatzman [12, concave obstacle] studied a nonlinear version of (4), assuming that there is no longitudinal perturbation and no energy loss during the shocks. Following [8] and setting $\Omega =] -1/2, 1/2[$, one looks for solutions in the functional class

$$u \in C(R^+, H_0^1(\Omega)) \cap W^{1,\infty}(R^+, L^2(\Omega)) \quad (5)$$

satisfying the system

$$u \geq -h \quad \text{in } R^+ \times \Omega \quad (6)$$

$$\text{Supp}(\square u) \subset \{(x, t), u(x, t) > -h\} \quad (7)$$

$$\frac{\partial}{\partial x} \left\{ -2 \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial x} \right\} + \frac{\partial}{\partial t} \left\{ \left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right\} = 0. \quad (8)$$

Condition (8), which is a strengthened version of energy conservation, must hold in the sense of $\mathcal{D}'([0, +\infty[\times \Omega)$. It implies that the energy integral

$$\int_{\Omega} \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial^+ u}{\partial t} \right)^2 dx = \int_{\Omega} \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial^- u}{\partial t} \right)^2 dx$$

remains unchanged through the motion.

Notice that in case a shock takes place simultaneously along a set of non-zero measure in Ω , we cannot have

$$\frac{\partial u}{\partial t} \in C(R^-, L^2(\Omega)).$$

Schatzman studied the existence and uniqueness of a solution of (S) such that

$$u(x, 0) = u_0(x), \quad \frac{\partial^+ u}{\partial t}(x, 0) = v_0(x). \quad (9)$$

Actually, the existence and uniqueness of a solution for (S) + (9) is obtained in the functional class (5). The solution constructed satisfies the additional property:

$$\square u \geq 0 \quad \text{in } \mathcal{D}'([0, +\infty[\times \Omega) \quad (10)$$

which is used to get uniqueness, and is interpreted in [9] as equivalent to subsonic propagation of interactions.

In our case, since $\partial u / \partial t(x, 0) = 0$, we look for solutions (even as a function of t) defined on $R \times \bar{\Omega}$. Our purpose is to give a precise meaning to the physical idea of ‘vibrations’ against the obstacle.

We prove that the solution is strongly almost-periodic as a function from R to $H_0^1(\Omega)$, generally not exactly periodic in t . We also sketch out some simple results concerning the non-harmonic Fourier series of $u(x, t)$ with respect to t . Further computations in this direction are planned for the future.

2. STATEMENT OF THE RESULTS

Let us denote by (Σ) the system (5) \rightarrow (10) with $v_0 = 0$ and R^+ replaced by R everywhere. We recall $0 \leq h < 1$.

THEOREM 1.1. If u_0 satisfies (1)–(3), the solution of (Σ) is such that $u(t) = u(\cdot, t)$ is strongly almost-periodic as a function from R to $H_0^1(\Omega)$. Moreover, the map $u_0 \rightarrow u(t)$ is Lipschitz-continuous from $H_0^1(\Omega)$ to $L^\infty(R, H_0^1(\Omega))$.

Remark. In contrast with the case of equation (4), $u(t)$ is generally *not* time-periodic. More precisely, we have:

THEOREM 1.2. (a) $h = p/q$ with p, q integers, the motion is periodic with $p + q$ as a period if $p + q$ is even, and $2(p + q)$ as a period if $p + q$ is odd. In the special case when $u_0(-x) = u_0(x)$, we have always the period $p + q$.

In some cases the smallest period is smaller.

(b) If $h \notin \mathcal{Q}$, the motion is never periodic, except in the single case $u_0(x) = 1 - 2|x|$, where the motion has the period $1 + h$.

Remark. The aim of this paper is to give complete proofs of the results announced in [7], together with some more information.

Since u is almost-periodic with respect to t , for every $\lambda \in R$, the limit

$$\lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t u(x, \theta) d\theta = \hat{u}(x, \lambda)$$

exists. The set $\mathcal{E}(u) = \{\lambda \in R, \hat{u}(\cdot, \lambda) \neq 0\}$ is called the set of exponents of u , and in general (cf. [4]) is denumerable. Here, we have a more precise result.

THEOREM 1.3. The exponents of u lie in the additive subgroup of R generated by π and $2\pi/1 + h = \omega$.

More precisely:

(a) For general u_0 satisfying (1)–(3), we have

$$\mathcal{E}(u) \subset Z\pi + Z\omega. \quad (11)$$

(b) If u_0 is even, the motion of $u(0, t)$ can be developed as a generalized Fourier series:

$$u(0, t) = \lim_{N \rightarrow \infty} \left[\frac{1-h}{2} + \sum_{p=1}^N \sum_{k=-\infty}^{k=+\infty} \lambda_k^p \cos\{(2p+1)\omega + 2k\pi\}t \right] \quad (12)$$

where the infinite sums on the right-hand side converge uniformly and the convergence with respect to N is uniform for $t \in R$.

(c) In the special case $u_0(x) = 1 - 2|x + \lambda \sin 2\pi x|$, $|\lambda| \leq \frac{1}{2\pi}$

$$\lambda_k^p = -\frac{8(1+h)}{(2p+1)^2\pi^2} J_k[(2p+1)\lambda\omega] \quad \text{for } k > 0$$

$$\lambda_{-k}^p = (-1)^k \lambda_k^p \quad \text{for } k > 0$$

and

$$\lambda_0^p = -\frac{4(1+h)}{(2p+1)^2\pi^2} J_0[(2p+1)\lambda\omega]^\dagger.$$

3. SOME EXPLICIT FORMULAS

a. Case $u_0(x) = 1 - 2|x|$

The solution of (Σ) is easy to describe. For $0 \leq t \leq t^* = \frac{1+h}{2}$, the motion is given by

$$\bar{u}(x, t) = \begin{cases} 1 - 2 \sup\{|x|, t\} & \text{if } 0 \leq t \leq 1/2 \\ 1 - 2 \inf\{1 - |x|, t\} & \text{if } 1/2 \leq t \leq \frac{1+h}{2}. \end{cases} \quad (15)$$

† Here J_k is Bessel's function of order k .

At time $t^* = 1 + h/2$, the string hits the obstacle $\{u = -h\}$ along the line segment $\{u = -h, |x| \leq (1 - h)/2\}$, with a uniform velocity $v(t^*) = v^- = -2$.

Then the velocity is reversed into $v^+ = -v^- = +2$, and the motion proceeds backwards:

$$\tilde{u}(x, t) = \bar{u}(x, 1 + h - t), \forall t \in \left[\frac{1+h}{2}, 1+h \right]. \quad (16)$$

Thus $\partial \tilde{u} / \partial t(x, t)$ is discontinuous at $t = t^*$, while $\bar{u}(x, t)$ remains continuous with respect to $t \in R$. The motion \bar{u} is periodic in t with period $1 + h$.

Conditions (5) and (6) are easy consequences of (15), (16) as soon as (8) is checked (to insure strong continuity in $H_0^1(\Omega)$). It is obvious that $\bar{u} \in W^{1,\infty}(R \times \Omega)$ and moreover, almost everywhere on $R \times \Omega$, we have

$$\frac{\partial \bar{u}}{\partial t} \cdot \frac{\partial \bar{u}}{\partial x} = 0, \left(\frac{\partial \bar{u}}{\partial t} \right)^2 + \left(\frac{\partial \bar{u}}{\partial x} \right)^2 = 4. \quad (18)$$

Then (8) follows as an immediate consequence.

Finally, a lengthy but standard computation gives the expression of $\square \bar{u}$: for every $\varphi \in \mathcal{D}(R \times \Omega)$, we have

$$\langle \square \bar{u}, \varphi \rangle = 4 \sum_{m \in \mathbb{Z}} \int_{\{|x| \leq \frac{1-h}{2}\}} \varphi \left[x, (2m+1) \frac{1+h}{2} \right] dx. \quad (17)$$

This formula implies (7) and (10).

b. The 'regular' case

In addition to (1)–(3), we assume now

$$\left. \begin{aligned} u_0 &\in C^2([-1/2, 1/2]), \text{ and } u_0''(-1/2) = u_0''(1/2) = 0 \\ u_0'(a) = u_0'(b) = u_0''(a) = u_0''(b) &= 0. \end{aligned} \right\} \quad (19)$$

In order to compute $u(x, t)$, we generalize an idea used in [11] by Reder. We define a function $F: R \rightarrow R$ in several steps.

Step 1. If $x \in [-1/2, 1/2]$, we set

$$F(x) = \begin{cases} \frac{u_0(x) - 1}{2} & \text{if } -1/2 \leq x \leq a \\ 0 & \text{if } a \leq x \leq b \\ \frac{1 - u_0(x)}{2} & \text{if } b \leq x \leq 1/2. \end{cases} \quad (20)$$

It is clear from (19) that $F \in C^2([-1/2, 1/2])$.

Step 2. If $t \in [1/2, 3/2]$, we define

$$F(t) = 1 - F(1 - t). \quad (21)$$

It is clear that $F \in C^2([1/2, 3/2])$. Since $F''(1/2) = 0$, we have

$$F \in C^2([-1/2, 3/2]).$$

Step 3. We extend F on R by the condition

$$\forall t \in R, F(t + 2) = F(t) + 2. \quad (22)$$

We introduce $f(t) = F(t) - t$: then $f(t)$ is 2-periodic on R .

We claim that f and F are in $C^2(R)$, because $F'(-1/2) = F'(3/2)$ and $F''(-1/2) = F''(3/2)$.

The first condition is a consequence of (21), the second one follows from $F''(-1/2) = 0$.

Some other properties of F are summarized below.

LEMMA 3.1. F is nondecreasing: $R \rightarrow R$. In addition (21) holds for $t \in R$, and we have also $F(t) = -1 - F(-1 - t)$, $\forall t \in R$.

For every $x \in \bar{\Omega}$ and $t \in R$, we have

$$-1 \leq F(x + t) + F(x - t) \leq 1.$$

Proof. Formula (21) means that for every $\theta \in [-1/2, 3/2]$, we have

$$F(\theta) + F(1 - \theta) = 1.$$

Given $t \in R$, we choose $m \in Z$ and $\theta \in [-1/2, 3/2]$ such that $t = 2m + \theta$. Then:

$$F(t) + F(1 - t) = F(\theta) + F(1 - \theta) = 1.$$

Also, for $t \in R$, we have:

$$F(t) + F(-1 - t) = F(t) + F(1 - t) - 2 = -1.$$

Let $x \leq 1/2$: by (21), we have

$$F(x + t) + F(x - t) = F(x + t) - F(-x + t + 1) + 1.$$

But $x \leq 1/2 \Rightarrow -x + t + 1 \geq x + t$.

Thus we obtain: $F(x + t) + F(x - t) \leq 1$.

Also, if $x \geq -1/2$, we write

$$F(x + t) + F(x - t) = F(x - t) - F(-1 - x - t) - 1 \geq -1,$$

since $x - t \geq -1 - x - t$. ■

Since $-1/2 \leq F(x + t) + F(x - t)/2 \leq 1/2$, we introduce:

$$u(x, t) = \tilde{u} \left\{ \frac{F(x + t) + F(x - t)}{2}, \frac{F(x + t) - F(x - t)}{2} \right\} \quad (23)$$

for $(x, t) \in \bar{\Omega} \times R$.

LEMMA 3.2. u is the solution of (Σ) with initial conditions $(u_0, 0)$, and we have

$$\sup_{t \in R} |u(\cdot, t) - u'(\cdot, t)|_{H^1(\Omega)} \leq C |u_0 - u'_0|_{H^1(\Omega)} \quad (24)$$

for u_0, u'_0 'regular' satisfying (1-3).

Proof. Let us introduce for convenience

$$X = \frac{F(x+t) + F(x-t)}{2}, \quad T = \frac{F(x+t) - F(x-t)}{2}.$$

Then, in the sense of $\mathcal{D}'(\Omega \times R)$, we have

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial \bar{u}}{\partial X} \cdot \frac{F'(x+t) + F'(x-t)}{2} + \frac{\partial \bar{u}}{\partial T} \cdot \frac{F'(x+t) - F'(x-t)}{2} \\ \frac{\partial u}{\partial t} &= \frac{\partial \bar{u}}{\partial X} \cdot \frac{F'(x+t) - F'(x-t)}{2} + \frac{\partial \bar{u}}{\partial T} \cdot \frac{F'(x+t) + F'(x-t)}{2} \end{aligned} \right\} \quad (25)$$

Using (18), we get immediately that $\partial u/\partial t$ and $\partial u/\partial x$ are in $L^\infty(\Omega \times R)$ and moreover:

$$\left. \begin{aligned} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 &= 2[F'^2(x+t) + F'^2(x-t)] \\ \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial t} &= F'^2(x+t) - F'^2(x-t) \end{aligned} \right\} \quad (26)$$

The properties (5) and (6) are immediately checked. On the other hand, in the sense of $\mathcal{D}'(\Omega \times R)$, we obviously have

$$\begin{aligned} \frac{\partial}{\partial t} \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 \right\} &= 4[F'(x+t)F''(x+t) - F'(x-t)F''(x-t)] \\ \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial t} \right) &= 2[F'(x+t)F''(x+t) - F'(x-t)F''(x-t)] \end{aligned}$$

which implies (8).

Differentiating (25) in the distribution sense, we get after reduction:

$$\square u(x, t) = F'(x+t)F'(x-t) \square \bar{u}[X(x, t), T(x, t)], \quad (27)$$

this formula making sense because $\square \bar{u}$ is a measure and F' is continuous. From (27), we deduce that (7) and (10) are satisfied.

As a consequence of lemma 3.1, we have for every $t \in R$: $X(1/2, t) = 1/2$ and $X(-1/2, t) = -1/2$.

Thus $u(1/2, t) = 0$ and $u(-1/2, t) = 0$, and we have $u(t) \in H_0^1(\Omega)$ for all t .

To verify the initial condition, we remark that $u(x, 0) = \bar{u}(F(x), 0) = \bar{u}_0(F(x)) = 1 - 2|F(x)|$ and (20) implies $1 - 2|F(x)| = u_0(x)$ in $\bar{\Omega}$.

It is easy to check that

$$\frac{u(x, t) - u_0(x)}{t} \rightarrow 0 \text{ in } C(\bar{\Omega}) \text{ as } t \rightarrow 0, \text{ which implies } \frac{\partial u}{\partial t}(x, 0) = 0.$$

Finally, we can see that the mapping $u_0 \rightarrow f$ is Lipschitzian from $H_0^1(\Omega)$ to $H^1([0, 2])$. Since $\bar{u} \in W^{1,\infty}(\Omega \times R)$, the mapping:

$$f \rightarrow u(\cdot, t) \text{ is Lipschitz-continuous from } H^1([0, 2]) \text{ to } L^\infty(R, H^1(\Omega)).$$

Combining these two remarks, we get (24), which will appear a convenient tool to treat the more general case where u_0 is only in $H_0^1(\Omega)$. ■

c. *The general case*

If u_0 satisfies conditions (1–3), we can still define $F(t)$ as in the paragraph above. Then $F(t)$ is non-decreasing, in $H^1(R)$ and $f(t) = F(t) - t$ is 2-periodic.

We compute $u(x, t)$ by means of formula (23). To check that $u(x, t)$ is the solution of (Σ) starting from $(u_0, 0)$, we can choose a sequence of initials u_0^n satisfying (19) with $u_0^n \rightarrow u_0$ in $H_0^1(\Omega)$. According to (24), $u^n(x, t)$ will converge to $u(x, t)$ in $L^\infty(R, H_0^1(\Omega))$. A similar calculation shows that $\partial u^n / \partial t$ converges to $\partial u / \partial t$ in $L^\infty(R, L^2(\Omega))$. Thus all conditions (5) \rightarrow (10) are checked by density.

4. PROOF OF THEOREM 1.1

Because of (24), it is enough to check the almost-periodicity when u_0 is ‘regular’, so that $F \in C^2(R)$.

a. As a first step, we prove that $t \rightarrow u(\cdot, t)$ is almost-periodic in $C(\bar{\Omega})$, by using directly Bochner’s criterion.

Let (t_n) be any sequence of real numbers. We write:

$$\begin{aligned} t_n &= 2m_n + \rho_n & \rho_n &\in [0, 2] \\ 2m_n &= k_n(1 + h) + \sigma_n & \sigma_n &\in [0, 1 + h] \end{aligned}$$

where m_n and k_n are in \mathbb{Z} . Thus

$$u(x, t + t_n) = \bar{u} \left\{ \frac{F(x + t + \rho_n) + F(x - t - \rho_n)}{2}, \frac{F(x + t + \rho_n) - F(x - t - \rho_n)}{2} + \sigma_n \right\}$$

We may assume $\lim_{n \rightarrow \infty} \rho_n = \rho$ and $\lim_{n \rightarrow \infty} \sigma_n = \sigma$, by extracting a subsequence of (t_n) . Then, because of the uniform continuity of F and \bar{u} the sequence $u(x, t + t_{n_k})$ converges uniformly on $\bar{\Omega} \times R$ to

$$u^*(x, t) = \bar{u} \left\{ \frac{F(x + t + \rho) + F(x - t - \rho)}{2}, \frac{F(x + t + \rho) - F(x - t - \rho)}{2} + \sigma \right\}. \quad (28)$$

b. *Precompactness of the range of u in $H_0^1(\Omega)$*

Thanks to conservation of energy, $u(t)$ is bounded in $H_0^1(\Omega)$, thus weakly almost-periodic in $H_0^1(\Omega)$. According to [10], Theorem 2.11, the strong almost-periodicity is equivalent to precompactness of the range in $H_0^1(\Omega)$. Even in the ‘regular’ case, it is not so easy to check because we cannot approach u in $L^\infty(R, H_0^1(\Omega))$ by regular functions since \bar{u} has discontinuous derivatives. We need a technical lemma.

LEMMA 4.1. With the notations of (a), there exists a denumerable set $S \subset [-1/2, 1/2]$ such that

$$x \notin S \Rightarrow \lim_{k \rightarrow \infty} \frac{\partial u}{\partial x}(x, t_{n_k}) = \frac{\partial u^*}{\partial x}(x, 0). \quad (29)$$

Proof. Let us use the notations

$$X(x, t) = \frac{F(x + t) + F(x - t)}{2}, \quad T(x, t) = \frac{F(x + t) - F(x - t)}{2}$$

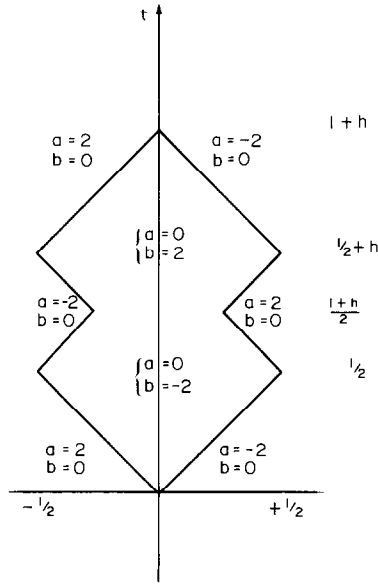


Fig. 1.

$$a(x, t) = \frac{\partial \bar{u}}{\partial x}(x, t) \quad (\text{for all } t \text{ and almost every } x)$$

$$b(x, t) = \frac{\partial \bar{u}}{\partial t}(x, t) \quad (\text{for all } x \text{ and almost every } t).$$

The functions a, b take only the values $0, -2$ and $+2$, their discontinuities lie on some curves of $\bar{\Omega} \times R$.

The periodicity of \bar{u}/t allows us to draw a picture in $\bar{\Omega} \times [0, 1+h]$ (cf. Fig. 1). For $x \in [-1/2, 1/2[$ and $0 < \delta < 1/2 - x$, we define

$$\alpha(x, t, \delta) = 1/\delta \{ \bar{u}[X(x+\delta, t) - T(x+\delta, t)] - \bar{u}[X(x, t), T(x+\delta, t)] \}$$

$$\beta(x, t, \delta) = 1/\delta \{ \bar{u}[X(x, t), T(x+\delta, t)] - \bar{u}[X(x, t), T(x, t)] \}$$

Also, for t fixed, we introduce two open sets

$$\omega = \left\{ x \in]-1/2, +1/2[, \frac{\partial X}{\partial x}(x, t) \neq 0 \right\}$$

$$\omega' = \left\{ x \in]-1/2, +1/2[, \frac{\partial T}{\partial x}(x, t) \neq 0 \right\}.$$

—First we notice that if $x \notin \omega$, then

$$X(x+\delta, t) - X(x, t) = o(\delta) \quad \text{when } \delta \rightarrow 0$$

Since \bar{u} is Lipschitzian in both variables, we conclude

$$\lim_{\delta \rightarrow 0} \alpha(x, t, \delta) = 0 \quad \text{in this case.}$$

—Let ω_n be the sequence of connected components of ω . For every n , we have either $\partial X/\partial x(x, t) > 0$ in ω_n , or $\partial X/\partial x(x, t) < 0$ in ω_n . Thus, in each ω_n , there are at most 2 values $(x_{n,k})_{k \in \{1, 2\}}$ such that (a) is discontinuous at $[X(x_{n,k}, t), T(x_{n,k}, t)]$. As a consequence, if $x \in \omega \setminus \bigcup_{n,k} \{x_{n,k}\}$ we find

$$\lim_{\delta \rightarrow 0} \alpha(x, t, \delta) = a[X, T] \frac{\partial X}{\partial x}.$$

In the same way, we obtain that except for denumerable values of x , we have

$$\lim_{\delta \rightarrow 0} \beta(x, t, \delta) = b[X, T] \frac{\partial T}{\partial x}.$$

As a consequence, there exists S_t denumerable such that, for every $x \in S_t$:

$$\frac{\partial}{\partial x} [u(x, t)] = a[X, T] \frac{\partial X}{\partial x} + b[X, T] \frac{\partial T}{\partial x}. \quad (30)$$

Let $S_{t_{n_k}} = S_k$. Then, for $x \in \bigcup_{k \in N} S_k$, we have

$$\forall k \in N, \frac{\partial}{\partial x} [u(x, t_k)] = a[X_k(x), T_k(x)] \frac{\partial X_k}{\partial x} + b[X_k(x), T_k(x)] \frac{\partial T_k}{\partial x}$$

where $X_k(x) = X(x, t_k)$ and $T_k(x) = T(x, t_k)$. Finally, let

$$\begin{cases} X_{\rho}(x) = X(x, \rho) \\ T_{\rho, \sigma}(x) = T(x, \rho) + \sigma. \end{cases}$$

By a very similar argument as above, we find S_∞ denumerable in $]-1/2, 1/2[$ such that

$$\begin{aligned} x \in \bigcup_{k \in N} S_k, x \in S_\infty &\Rightarrow \lim_{k \rightarrow \infty} \frac{\partial}{\partial x} [u(x, t_k)] \\ &= a[X_\rho, T_{\rho, \sigma}] \frac{\partial X_\rho}{\partial x} + b[X_\rho, T_{\rho, \sigma}] \frac{\partial T_{\rho, \sigma}}{\partial x}. \end{aligned}$$

But, except for $x \in S^*$ denumerable, this is precisely equal to $\partial/\partial x [u^*(x, 0)]$. Finally, (29) holds with $S = S^* \cup S_\infty \cup \bigcup_{k \in N} S_k$. ■

PROPOSITION 4.1. The range of $u(t)$ is precompact in $H_0^1(\Omega)$ [‘regular’ case].

Proof. Let (t_n) be an arbitrary sequence of reals. There exists a subsequence (t_{n_k}) , ρ and σ such that

$$u(x, t_{n_k}) \rightarrow u^*(x, 0) \quad \text{in } C(\bar{\Omega}).$$

By lemma: $\partial u/\partial x(x, t_{n_k}) \rightarrow \partial u^*/\partial x(x, 0)$ a.e. in Ω .

Since $\partial u/\partial x(x, t_{n_k})$ remains bounded in $L^\infty(\Omega)$ by (30), Lebesgue’s theorem implies

$$\int_{\Omega} \left| \frac{\partial u}{\partial x}(x, t_{n_k}) - \frac{\partial u^*}{\partial x}(x, 0) \right|^2 dx \xrightarrow{k \rightarrow \infty} 0.$$

Hence, $u(x, t_{n_k}) \rightarrow u^*(x, 0)$ in $H_0^1(\Omega)$. ■

End of the proof of Theorem 1.1. In the regular case, theorem 1.1 is now an obvious consequence of (a), proposition 4.1 and [10, theorem 2.11, p. 48], applied with $Y = C = C(\bar{\Omega})$ and $Z = H_0^1(\Omega)$.

In the general case, we can use (24) which shows that $u(x, t)$ is the limit in $L^\infty(R, H_0^1(\Omega))$ of solutions associated to ‘regular’ initials u_0^n . This finishes the proof of theorem 1.1, since the statement on $u_0 \rightarrow u(t)$ is actually a consequence of (24). ■

5. PROOF OF THEOREM 1.2

(a) Since $F(t) - t$ is 2-periodic and $\bar{u}(x, t)$ is $1 + h$ -periodic in t , the first assertion is an obvious consequence of formula (23).

If $u_0(-x) = u_0(x)$, then $b = -a$ and $F(t)$ is odd. Then $F(t) - t$ is 1-periodic, and the second assertion follows.

Actually, $F(t) = t + \alpha \sin 2n\pi t$ is non-decreasing as soon as $2n\pi|\alpha| < 1$, and corresponds to the initial datum $u_0(x) = 1 - 2|x + \alpha \sin 2n\pi x|$. In this case, $F(t) - t$ is $1/n$ -periodic.

If for instance $h = 1/n$, then $(n + 1)/n = 1 + h$ is a period for $u(\cdot, t)$, which is strictly (for $n \geq 2$) smaller than $n + 1$. For $n = 2p$, we get a period smaller than $n + 1 = p + q$.

(b) If $u(\cdot, t)$ is periodic with a period τ , then the functions

$$g_{\varepsilon, \alpha, \beta}(t) = \int_t^{t+\varepsilon} \int_\alpha^\beta \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial \theta} \right)^2 + 2 \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial \theta} \right] dx d\theta$$

are also τ -periodic, independently of $\varepsilon, \alpha, \beta$ for $-\frac{1}{2} \leq \alpha < \beta \leq \frac{1}{2}$. As a consequence of (26),

$$g_{\varepsilon, \alpha, \beta}(t) = 4 \int_t^{t+\varepsilon} \int_\alpha^\beta F'^2(x + \theta) dx d\theta.$$

Now it is obvious that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{4\varepsilon} g_{\varepsilon, \alpha, \beta}(t) = \int_\alpha^\beta F'^2(x + t) dx = \int_{t-\alpha}^{t+\beta} F'^2(\sigma) d\sigma.$$

Thus for every $l \in [0, 1]$, the function

$$\varphi(l, t) = \int_t^{t+l} F'^2(\sigma) d\sigma \text{ is } \tau\text{-periodic.}$$

Now we distinguish two possible cases:

1st case: $\tau \notin \mathcal{Q}$.

Since F' is 2-periodic, we must have $\varphi(l, t) = \varphi(l)$ for $l \in [0, 1]$. Since φ is a continuous function of l , it is immediate to deduce

$$\int_t^{t+l} F'^2(\sigma) d\sigma = k^2 l, \quad k \geq 0.$$

Then at every Lebesgue’s point of F'^2 , we must have $F'^2(t) = k^2$. Since $F' \geq 0$ a.e., this allows us to conclude that $F' \equiv k$ a.e. As a consequence of (22), we must have $k = 1$. Also by (21), $F(1/2) = 1/2$.

Thus $F(t) = t$, and $u_0(x) = 1 - 2|F(x)|$ implies $u_0(x) = 1 - 2|x|$.

2nd case: $\tau = p/q \in Q \setminus \{0\}$

Then $2q\tau = 2p$ is also a period for $u(\cdot, t)$.

On the other hand, by (23), we have:

$$u(x, t + 2p) = \bar{u} \left\{ \frac{F(x+t) + F(x-t)}{2}, \frac{F(x+t) - F(x-t)}{2} + 2p \right\}.$$

Choosing $t = 0$ and $x = a$ for instance, we get $u(a, 2p) = \bar{u}(0, 2p) = \bar{u}(0, 0) = 1$. The only possibility is $2p = m(1+h)$, $m \in Z \setminus \{0\}$.

Thus $h = 2p/m - 1 \in Q$.

6. THE EXPONENTS OF $u(t)$. (PROOF OF THEOREM 1.3)

(a) Let $\omega = 2\pi/1 + h$, we can write \bar{u} as a Fourier series in t :

$$\bar{u}(x, t) = \sum_{p=0}^{\infty} a_p(x) \cos(p\omega t)$$

Since $\partial \bar{u} / \partial t \in L^\infty(\bar{\Omega} \times R)$, this series is uniformly convergent on $\bar{\Omega} \times R$. The coefficients a_p are C^1 functions on $\bar{\Omega} = [-1/2, +1/2]$ and can be computed easily. Thus we have:

$$u(x, t) = \lim_{N \rightarrow +\infty} \sum_{p=0}^N a_p[X(x, t)] \cos[p\omega T(x, t)]$$

uniformly on $\bar{\Omega} \times R$ and we just need to examine the exponents of each term

$$a_p[X(x, t)] \cos[p\omega T(x, t)].$$

By a density argument, it is sufficient to do this in the 'regular' case. Then $F(t) - t = f(t)$ is a C^1 function, periodic with period 2, and we get a first development:

$$a_p[X(x, t)] = \sum_{k=0}^{\infty} \alpha_k^p(x) \cos k\pi t, \text{ with (for instance)}$$

$$\sum_{k=0}^{\infty} |\alpha_k^p|_\infty < +\infty.$$

As a second step, we may write:

$$\cos[p\omega T(x, t)] = \cos p\omega [t + \frac{1}{2}(f(x+t) - f(x-t))] = \cos p\omega t \beta(x, t) - \sin p\omega t \gamma(x, t).$$

The functions β and γ are in $C^1(\bar{\Omega} \times R)$ and thus:

$$\beta(x, t) = \sum [\beta_k^p(x) \cos k\pi t + \delta_k^p(x) \sin k\pi t]$$

$$\gamma(x, t) = \sum [\gamma_k^p(x) \cos k\pi t + \varepsilon_k^p(x) \sin k\pi t]$$

with

$$\sum_{k=0}^{\infty} (|\beta_k^p|_\infty + |\gamma_k^p|_\infty + |\delta_k^p|_\infty + |\varepsilon_k^p|_\infty) < +\infty.$$

Thus the Cauchy product of the two series defining $a_p[X(x, t)]$ and $\cos[p\omega T(x, t)]$ is absolutely

convergent in $C(\bar{\Omega})$, and after reduction, we are left with a development of the type:

$$a_p[X] \cos[p\omega t] = \sum_{k \in \mathbb{Z}} [\lambda_k(x) \cos(p\omega t + k\pi t) + \mu_k(x) \sin(p\omega t + k\pi t)]$$

which is strongly almost-periodic in $C(\bar{\Omega})$ with exponents contained in $\{p\omega\} + \mathbb{Z}\pi$ in the $L^2(\Omega)$ sense. Thus the exponents of $u(\cdot, t)$ in the sense of $L^2(\Omega)$ are in $Z\omega + \mathbb{Z}\pi$, and of course it is the same set of exponents than in $H_0^1(\Omega)$.

(b) In case u_0 is even and we take $x = 0$, some simplifications occur in the previous computations. We have

$$\begin{cases} a_0(0) = \frac{1-h}{2} \\ a_p(0) = -4(1+h) \frac{\sin^2(p/2)\pi}{p^2\pi^2}. \end{cases}$$

We also obtain

$$\cos(p\omega f(t)) = \sum_{k \geq 0} \beta_k^p \cos(2k\pi t) \quad (30)$$

$$\sin(p\omega f(t)) = \sum_{k \geq 0} \alpha_k^p \sin(2k\pi t) \quad (31)$$

with

$$\beta_k^p = 2 \int_0^1 \cos(p\omega f(t)) \cos(2k\pi t) dt, \quad \beta_0^p = \int_0^1 \cos(p\omega f(t)) dt \quad (32)$$

$$\alpha_k^p = 2 \int_0^1 \sin(p\omega f(t)) \sin(2k\pi t) dt, \quad (33)$$

and we deduce easily

$$u(0, t) = \lim_{N \rightarrow \infty} \left[\frac{1-h}{2} - 4(1+h) \Phi_N(t) \right], \text{ where}$$

$$\Phi_N(t) = \sum_{p=1}^N \frac{1}{(2p+1)^2\pi^2} \sum_{k \in \mathbb{Z}} A_k^p \cos\{(2p+1)\omega + 2k\pi\}t\}$$

and

$$A_k^p = \begin{cases} \frac{1}{2}(\beta_k^{2p+1} + \alpha_k^{2p+1}) & \text{if } k > 0 \\ \beta_0^{2p+1} & \text{if } k = 0 \\ \frac{1}{2}(\beta_{|k|}^{2p+1} - \alpha_{|k|}^{2p+1}) & \text{if } k < 0. \end{cases}$$

(c) Case $u_0(x) = 1 - 2|x + \lambda \sin 2\pi x|$

Then $f(t) = \lambda \sin 2\pi t$, and the condition $F' \geq 0$ is equivalent to $|\lambda| \leq 1/2\pi$.

Setting $c_p = \frac{1}{2}(2p + 1)\lambda\omega$, we obtain

$$A_k^p = 2J_k[(2p + 1)\lambda\omega] = 2 \sum_{r \geq 0} \frac{(-1)^r (c_p)^{2r+k}}{r!(r+k)!} \quad \text{if } k > 0$$

$$A_0^p = J_0[(2p + 1)\lambda\omega] = \sum_{r \geq 0} \frac{(-1)^r (c_p)^{2r}}{(r!)^2} \quad \text{if } k = 0$$

$$A_k^p = 2(-1)^k J_{|k|}[(2p + 1)\lambda\omega] = 2(-1)^k \sum_{r \geq 0} \frac{(-1)^r (c_p)^{2r+|k|}}{r!(r+|k|)!} \quad \text{if } k < 0.$$

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