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On the consistency of the commodity-based technology in make-use model of production

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ABSTRACT. In the Stone make-use model, the industry-based technology is consistent because his solution can be explained in variational terms inside a circuit. However, the alternative model, the commodity-based technology, is not economically realistic: it never corresponds to a circuit, even if an exact solution can be found when the number of commodities is equal to the number of industries. This model hesitates between a supply-driven and a demand-driven model but when it is converted into a true supply-driven one, it retrieves a consistency.

I. Introduction

The input-output model has a variant, developed by Stone (1961), where two rectangular homogenous matrices are considered:

- The Use matrix, denoted **U**, with industries as columns and commodities as rows and with final demand as supplementary column and added value as supplementary row; it indicates how much of each commodity each industry buys to produce.

$$\begin{matrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} & \begin{matrix} e_1 & q_1 \\ e_2 & q_2 \\ e_3 & q_3 \end{matrix} \\ \begin{matrix} w_1 & w_2 \\ x_1 & x_2 \end{matrix} & \end{matrix}$$

- The Make matrix, denoted **V**, with industries as rows and commodities as columns, indicating how many of each commodity an industry produces.

$$\begin{matrix} \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{bmatrix} & \begin{matrix} x_1 \\ x_2 \end{matrix} \\ \begin{matrix} q_1 & q_2 & q_3 \end{matrix} & \end{matrix}$$

One has four accounting identities: $x_i = \sum_{j=1}^{m} v_{ij}$ for all i, $x_j = \sum_{i=1}^{m} u_{ij} + w_j$ for all j and $q_i = \sum_{j=1}^{n} u_{ij} + e_i$ for all i, $q_j = \sum_{i=1}^{n} v_{ij}$ for all j, where w_j is the value added of industry j and e_i is the amount of commodity i sold to final demand; that is:

(1) $$\mathbf{x} = \mathbf{V}\,\mathbf{s}$$

(2) $$\mathbf{x} = \mathbf{U}'\,\mathbf{s} + \mathbf{w}$$

(3) $$\mathbf{q} = \mathbf{U}\,\mathbf{s} + \mathbf{e}$$

(4) $$\mathbf{q} = \mathbf{V}'\,\mathbf{s}$$

These identities require homogeneity by row and columns, so all terms are expressed in value. Are defined technical coefficients: $a^u_{ij} = \frac{u_{ij}}{x_j}$, or $\mathbf{A}^u = \mathbf{U}\,\hat{\mathbf{x}}^{-1}$. In this model, one can pose two alternative hypotheses about matrix **V**:

- The total output q_j of a commodity j is supplied by any industry i in fixed proportions, i.e., the *commodity-output proportion* $d_{ij} = \frac{v_{ij}}{q_j}$ is fixed (case referred as *technology based on industries*), that is:

(5) $$\mathbf{D} = \mathbf{V}\,\hat{\mathbf{q}}^{-1}$$

- The total output x_i of any industry i is composed of commodities in fixed proportions, i.e., the *industry output proportion* $c_{ij} = \frac{v_{ij}}{x_i}$ is fixed (case referred as *technology based on commodities*), that is:

(6) $$\mathbf{C} = \hat{\mathbf{x}}^{-1}\,\mathbf{V}$$

In both cases, the model has a solution, if the number of commodities is equal to the number of industries (Miller and Blair, 1985, pp. 174). In this short note, I will show that the economic interpretation of the commodity based technology is problematic, unless the model is assumed to be supply-driven.

II. The circuit in the traditional Leontief model

The central equation of the traditional input-output economics (Leontief, 1936) is:

(7) $$\mathbf{x} = \mathbf{A}\,\mathbf{x} + \mathbf{f}$$

where $a_{ij} = \frac{z_{ij}}{x_j}$ is the technical coefficient, x_j is the output of sector j, f_i is the final demand of commodity i, z_{ij} indicates how much of commodity i is bought by sector j, that is the flow from i to j, and v_j is the added value of sector j, with the unique matrix:

$$\begin{matrix} \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} & \begin{matrix} f_1 & x_1 \\ f_2 & x_2 \\ f_3 & x_3 \end{matrix} \\ \begin{matrix} v_1 & v_2 & v_3 \\ x_1 & x_2 & x_3 \end{matrix} & \end{matrix}$$

The model can be solved by simply obtaining from equation (7):

(8) $$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{f}$$

but it must also support a reasoning based on impulsions of demand, that is a differentiation of equation (7): $\Delta f_j^{(0)} \rightarrow \Delta x_j^{(0)} = \Delta f_j^{(0)} \rightarrow \Delta x_i^{(1)} = \Delta z_{ij}^{(1)} = a_{ij}\,\Delta x_j^{(0)}$. So, the total increase of the output of sector i is: $\Delta x_i^{(1)} = \sum_{j=1}^{n} a_{ij}\,\Delta x_j^{(0)}$. This continues at steps 2, ..., etc., and at step k: $\Delta x_i^{(k)} = \sum_{j=1}^{n} a_{ij}\,\Delta x_j^{(k-1)}$ that is $\Delta\mathbf{x}^{(k)} = \mathbf{A}\,\Delta\mathbf{x}^{(k-1)} = \mathbf{A}^k\,\Delta\mathbf{x}^{(0)} = \mathbf{A}^k\,\Delta\mathbf{f}$, so the total increase of output is given by: $\Delta\mathbf{x} = \sum_{k=1}^{n} \Delta\mathbf{x}^{(k)} = \left(\sum_{k=1}^{n} \mathbf{A}^k\right)\Delta\mathbf{f} \underset{k\rightarrow\infty}{\rightarrow} (\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{f}$ and equation (8) is retrieved. This is well known but it must be reminded: if the second type of solution is impossible, the first cannot be interpreted and it is only a vain exercise without economical meaning. Now, this second reasoning describes a circular process: a production of a sector generates a demand of some intermediate commodities described by the technical coefficients, what generates at its turn a production of the concerned sectors (remembering that the bijective correspondence sector-products is assumed).

III. The circuit with the technological hypotheses of the make-use

For the make-use model, all is in the plausibility of the circular process as described by both alternative hypotheses. Either the process is plausible and the solution of the model has an economic meaning, either it is not.

With the industry-based hypotheses, commodities are produced by industries following the commodity-output-coefficients d_{ij} found from the Make matrix, and industries demand products by the mean of technical coefficients a^u_{ij} found from the Use matrix, that is they are found by the product of two matrices, the matrix of technical coefficients, $\mathbf{A}^u$, and the Markovian matrix commodity-output-proportion matrix $\mathbf{D}$, following the model $\mathbf{q} = \mathbf{A}^u\ \mathbf{D}\ \mathbf{q} + \mathbf{e}$ (Miller and Blair, 1985, p. 167) [1]:

(9) $$\mathbf{A}^u\ \mathbf{D} = (\mathbf{U}\ \hat{\mathbf{x}}^{-1})\ (\mathbf{V}\ \hat{\mathbf{q}}^{-1}) = \mathbf{U}\ \langle \mathbf{V}\ \mathbf{s}\rangle^{-1}\ \mathbf{V}\ \langle \mathbf{V}'\ \mathbf{s}\rangle^{-1}$$

A final demand $\Delta e_j^{(0)}$ of commodity j is an equal need for commodity j: $\Delta q_j^{(0)} = \Delta e_j^{(0)}$. This generates an increase in the production of industry i: $\Delta x_i^{(1)} = d_{ij}\ \Delta e_j^{(0)}$; so, to the total, industry i has to produce: $\Delta x_i^{(1)} = \sum_{j=1}^{m} d_{ij}\ \Delta e_j^{(1)}$. Then, the additional production of industry i generates the need of intermediate goods, that is for commodity l: $\Delta q_l^{(1)} = \Delta u_{il}^{(1)} = a^u_{il}\ \Delta x_i^{(1)}$. To the total, the need for the intermediate commodity l is: $\Delta q_l^{(1)} = \sum_{i=1}^{n} a^u_{il}\ \Delta x_i^{(1)}$. And the circuit is closed and starts again with this demand $\Delta q_l^{(1)}$ of commodity l. At step k, one has: $\Delta x_i^{(k)} = \sum_{j=1}^{m} d_{ij}\ \Delta e_j^{(k-1)}$, that is $\Delta \mathbf{x}^{(k)} = \mathbf{D}\ \Delta \mathbf{e}^{(k-1)}$, and $\Delta q_l^{(k)} = \sum_{i=1}^{n} a^u_{il}\ \Delta x_i^{(k)}$, that is

[1] This works even for rectangular matrices. However, most formulae of the model as exposed in Miller and Blair (1985) require that the number of commodity is equal to the number of industries: matrices Make and Use must be square, what is much restricting.

$\Delta \mathbf{q}^{(k)} = \mathbf{A}^u \, \Delta \mathbf{x}^{(k)}$. Finally, $\Delta \mathbf{q}^{(k)} = \mathbf{A}^u \, \mathbf{D} \, \Delta \mathbf{e}^{(k-1)}$ and equation (9) is retrieved. Obviously, the process could begin by a demand to an industry instead of a demand of a commodity.

With the commodity-based technology, industries demand commodities -- as above -- by the technical coefficients, but these commodities are assumed to be produced by industries following the industry output proportions, c_{ij}. If we want to translate it in terms of circuit, the process could begin by a final demand of commodity *j*: $\Delta e_j^{(0)} \rightarrow \Delta q_j^{(0)} = \Delta e_j^{(0)}$. Unlike **D**, the **C** matrix does **not** indicate what industry will produce this commodity; but if it is a particular industry *i*, arbitrary chosen, that produces this commodity *j*, that is $\Delta x_i^{(1)} = \frac{\Delta q_j^{(0)}}{c_{ij}}$, then *i* will have to increase its production of **all** the other commodities *l* that it produces usually: $\Delta q_l^{(1)} = c_{il} \, \Delta x_i^{(1)}$. All this is highly unrealistic! If the process begins by the demand $\Delta x_i^{(0)}$ to an industry *i*, then this generates a demand of intermediate commodities *j* by the mean of the technical coefficients a_{ji}^u: $\Delta q_j^{(1)} = a_{ji}^u \, \Delta x_i^{(0)}$; but after this, the above problem of allocation toward the industries occurs again.

Alternately, this could mean that an industry *i*, that produces commodities following coefficients c_{ij}, also demands intermediate commodities following the technical coefficients. So, a demand $\Delta x_i^{(0)}$ to an industry *i* implies that this industry increases its production of **all** commodities following $\Delta q_j^{(1)} = c_{ij} \, \Delta x_i^{(0)}$ and, simultaneously, it demands some intermediate commodities following the technical coefficients: $\Delta q_l^{(1)} = a_{li}^u \, \Delta x_i^{(0)}$. But this is not a circular process because after this demand of intermediate commodities, there is no continuation.

The difficulty met with both interpretations is linked to the fact that one has to inverse matrix **C** to deduce the commodity-by-commodity balance accounting identity with the commodity-based technology: $\mathbf{q} = \mathbf{A}^u \, \mathbf{C}^{-1} \, \mathbf{q} + \mathbf{e}$ (formula 5-18 of Miller and Blair (1985, pp. 169), what requires again all matrices to be square. In any case, this behavior corresponds to a push-process, similar to those assumed by Ghosh with its hypothesis of a supply-driven model in the traditional input-output economics. It appears to be contradictory with the existence of technical coefficients in the Use matrix, i.e., a demand-driven hypothesis. However, it is possible to reverse the perspective. Replace the technical coefficient matrix $\mathbf{A}^u$ by a matrix of allocation coefficients, $b_{ij}^u = \frac{u_{ij}}{q_i}$, that is:

(10) $$\mathbf{B}^u = \hat{\mathbf{q}}^{-1}\,\mathbf{U}$$

From (6), it follows $\mathbf{V} = \hat{\mathbf{x}}\,\mathbf{C}$, what reported in (4) gives:

(11) $$\mathbf{q} = \mathbf{C}'\,\mathbf{x}$$

From (10), it comes $\hat{\mathbf{q}}\,\mathbf{B}^u = \mathbf{U}$ and by reporting it in (2) it is obtained $\mathbf{x} = \mathbf{B}^{u\prime}\,\hat{\mathbf{q}}\,\mathbf{s} + \mathbf{w} = \mathbf{B}^{u\prime}\,\mathbf{q} + \mathbf{w}$; so, the equation of the model is obtained by reporting (11) in this last equation:

(12) $$\mathbf{x} = \mathbf{B}^{u\prime}\,\mathbf{C}'\,\mathbf{x} + \mathbf{w}$$

Now, the push-process is set in complete conformity with a supply-driven model and it is circular. A supply of a commodity j generates an output of all industries as indicated by $\mathbf{B}^u$, in the Ghosh way, then the industries sell commodities following the proportions indicated by the coefficients c_{ij}. The initial increase of the value added v_i of an industry i generates an equal increase in the output of this industry: $\Delta x_i^{(0)} = \Delta v_i^{(0)}$. By the matrix $\mathbf{C}$, this generates an increase in the supply of all commodities: $\Delta q_j^{(1)} = c_{ij}\,\Delta x_i^{(0)}$, that is, to the total the increase in the supply of commodity j is: $\Delta q_j^{(1)} = \sum_{i=1}^{n} c_{ij}\,\Delta x_i^{(0)}$. This supplementary supply of a commodity j induces an increase of the output of all industries l following $\mathbf{B}^u$: $\Delta q_j^{(1)} \rightarrow \Delta x_l^{(1)} = b_{jl}^u\,\Delta q_j^{(1)}$, so, to the total, the industry l increases its output of $\Delta x_l^{(1)} = \sum_{j=1}^{m} b_{jl}^u\,\Delta q_j^{(1)}$. At step k, , one has: $\Delta q_j^{(k)} = \sum_{i=1}^{n} c_{ij}\,\Delta x_i^{(k-1)}$ and $\Delta x_l^{(k)} = \sum_{j=1}^{m} b_{jl}^u\,\Delta q_j^{(k)}$, that is $\Delta\mathbf{q}^{(k)} = \mathbf{C}'\,\Delta\mathbf{x}^{(k-1)}$ and $\Delta\mathbf{x}^{(k)} = \mathbf{B}^{u\prime}\,\Delta\mathbf{q}^{(k)}$, so $\Delta\mathbf{x}^{(k)} = \mathbf{B}^{u\prime}\,\mathbf{C}'\,\Delta\mathbf{x}^{(k-1)}$. This is in conformity with the corresponding model (12): the supply-driven commodity-based-technology model is consistent [2].

IV. Conclusion

In the Stone make-use model, the industry-based technology is consistent because his solution can be explained in variational terms inside a circuit. However, the alternative model, the commodity-based technology, is not economically realistic: it never corresponds to a circuit, even if an exact solution can be found when the number of commodities is equal to the number of industries. This model hesitates between a supply-driven and a demand-driven model but when it is converted into a true supply-driven one, it retrieves a consistency.

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[2] All this is irrespective of the discussion about the artificial character of a supply-driven model: it is just to demonstrate that the commodity-based technology is inconsistent, hesitating between a supply-driven and a demand-driven-model.