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Dirichlet-to-Neumann or Poincaré-Steklov operator on fractals described by d -sets

KEVIN ARFI*, ANNA ROZANOVA-PIERRAT†

May 23, 2017

Abstract

In the framework of the Laplacian transport, describing by a Robin boundary-valued problem in an exterior domain in \mathbb{R}^n , we generalize the definition of the Poincaré-Steklov operator to d -set boundaries, $n-1 \leq d < n$, and give its spectral properties to compared to a the spectrum of the interior domain and also to a truncated domain, considered as an approximation of the exterior case. The well-posedness of the Robin boundary-valued problems for the truncated and exterior domains are obtained in the general framework of n -sets.

1 Introduction

Laplacian transports to and across irregular and fractal interfaces are ubiquitous in nature and industry: properties of rough electrodes in electrochemistry, heterogeneous catalysis, steady-state transfer across biological membranes (see [19, 20, 14, 18] and references therein). To model it there is a usual interest to consider truncated domains as an approximation of the exterior unbounded domain case.

Thus, in this paper, we consider Ω a domain of \mathbb{R}^n which can be

1. unbounded, and then we suppose that $\Omega = \mathbb{R}^n \setminus \overline{\Omega}_0$ for a bounded domain Ω_0 in \mathbb{R}^n (*i.e.* Ω is the exterior of a bounded domain);
2. bounded, and then we suppose that $\Omega = (\mathbb{R}^n \setminus \overline{\Omega}_0) \cap \Omega_1$ for bounded domains $\Omega_0 \subset \Omega_1$ in \mathbb{R}^n (*i.e.* Ω is the exterior domain of Ω_0 bounded by the boundary of Ω_1).

In what following for the boundaries of $\partial\Omega_0$ and $\partial\Omega_1$ (which are supposed to be disjoint $\partial\Omega_0 \cap \partial\Omega_1 = \emptyset$) are denoted by Γ and S respectively. Let us notice that $\Gamma \cup S = \partial\Omega$ (for the unbounded case $S = \emptyset$), see Fig. 1. To distinct the exterior Ω and the truncated domain, we will also denote it by Ω_S . The phenomenon of Laplacian transports to Γ

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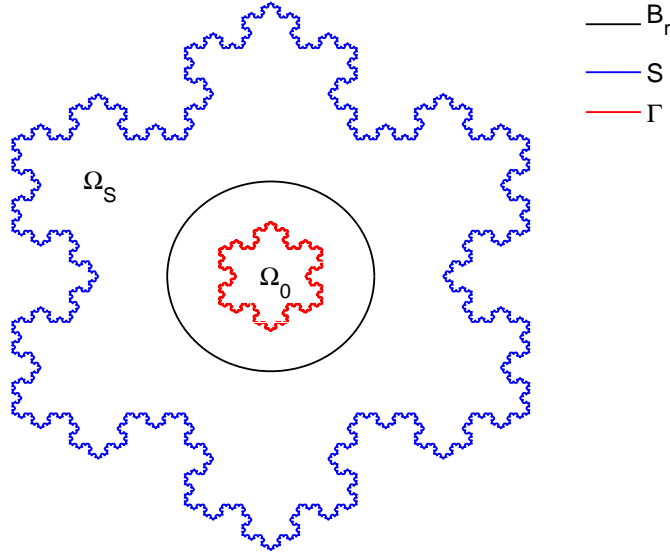


Figure 1: Example of the considered domains: Ω_0 (the Von Koch snowflake) is the bounded domain, bounded by a compact boundary Γ , which is a d -set (see Definition 3) with $d = \log 4 / \log 3 > n - 1 = 1$. The truncated domain Ω_S is between the boundary Γ and the boundary S (presented by the same Von Koch fractal as Γ). The boundaries Γ and S have no an intersection and here are separated by the boundary of a ball B_r of a radius $r > 0$. The domain, bounded by S , is called $\Omega_1 = \overline{\Omega_0} \cup \Omega_S$, and the exterior domain is $\Omega = \mathbb{R}^n \setminus \overline{\Omega_0}$.

can be described by the following boundary-valued problem:

$$\begin{aligned}
 -\Delta u &= 0, & x \in \Omega, \\
 \lambda u + \partial_\nu u &= \psi & \text{on } \Gamma, \\
 u &= 0 & \text{on } S,
 \end{aligned} \tag{1}$$

where $\partial_\nu u$ denotes the normal derivative of u , in some appropriate sense, $\lambda \in [0, \infty)$ is the resistivity of the boundary and $\psi \in L_2(\Gamma)$. For $S = \emptyset$ we impose Dirichlet boundary conditions at infinity. The case of a bounded Ω corresponds to the approximation of the exterior problem by a problem on a truncated domain in the sense of Theorem 8.

When $\partial\Omega$ is regular (C^∞ or at least Lipschitz), what are the trace of $u \in H^1(\Omega)$ and the normal derivative $\partial_\nu u$ on Γ are well-known [30, 32]. The same thing with the properties of the Poincaré-Steklov or the Dirichlet-to-Neumann operator defined at manifolds with C^∞ boundaries [16, 40]. In the aim to generalize the Poincaré-Steklov operator to d -sets $n - 1 \leq d < n$ (the case $n - 1 < d < n$ contains the self-similar fractals), we firstly study the most general context (see Section 3), when the problem (1) is well-defined and its bounded variant (physically corresponding to a source at finite distance) can be viewed as an approximation of the unbounded case (corresponding to a source at infinity). The main extension and trace theorems recently obtained in the

framework of d -sets theory are presented and discussed in Section 2. After a short survey in Section 4 of known results on the spectral properties of the Poincaré-Steklov operator for a bounded domain, we introduce the Poincaré-Steklov operator A on d -sets for a bounded domain, its exterior domain and a truncated domain and relate their spectral properties (see Section 5). The two dimensional case differs from \mathbb{R}^n with $n \geq 3$ by the functional reason (see Subsection 3.2) and gives different properties of the spectrum of A (see Theorem 9), defined as an elliptic self-adjoint positive operator on $L_2(\Gamma)$, and in particular, different domains of its definition (see also Proposition 6 in Section 6). Specially, for the case of a d -set Γ (see Theorems 11 and 12), we justify the method, developed in [19], true for smooth boundaries, to find the total flux Φ across the interface Γ using the spectral decomposition of 1_Γ (belonging to the domain of A by Proposition 6) on the basis of eigenfunctions of the Dirichlet-to-Neumann operator $(V_k)_{k \in \mathbb{N}}$ in $L_2(\Gamma)$ and its eigenvalues $(\mu_k)_{k \in \mathbb{N}}$:

$$\Phi \propto \sum_k \frac{\mu_k (\mathbb{1}_\Gamma, V_k)_{L_2(\Gamma)}^2}{1 + \frac{\mu_k}{\lambda}}. \quad (2)$$

2 Elements of functional analysis and traces theorems on d -sets

Before to proceed to the generalization results, let us define the main notions and explain the functional context of d -sets. For instance, for the well-posedness result of problem (1) on “the most general” domains Ω in \mathbb{R}^n , we need to be able to say that for this Ω the extension operator $E : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ is continuous and the trace operator (to be defined, see Definition 6) $\text{Tr} : H^1(\Omega) \rightarrow \text{Im}(\text{Tr}(H^1(\Omega))) \subset L_2(\partial\Omega)$ is continuous and surjective.

Therefore, let us introduce the existing results about traces and extension domains in the framework of Sobolev spaces.

Definition 1 (*W_p^k -extension domains*) *A domain $\Omega \subset \mathbb{R}^n$ is called a W_p^k -extension domain ($k \in \mathbb{N}^*$) if there exists a bounded linear extension operator $E : W_p^k(\Omega) \rightarrow W_p^k(\mathbb{R}^n)$. This means that for all $u \in W_p^k(\Omega)$ there exists a $v = Eu \in W_p^k(\mathbb{R}^n)$ with $v|_\Omega = u$ and it holds*

$$\|v\|_{W_p^k(\mathbb{R}^n)} \leq C \|u\|_{W_p^k(\Omega)} \quad \text{with a constant } C > 0.$$

The classical results of Calderon-Stein [10, 39] say that every Lipschitz domain Ω is an extension domain for $W_p^k(\Omega)$ with $1 \leq p \leq \infty$, $k \in \mathbb{N}^*$.

This result was generalized by Jones [24] in the framework of (ε, δ) -domains:

Definition 2 (*(ε, δ) -domain [24, 26, 42]*) *An open connected subset Ω of \mathbb{R}^n is an (ε, δ) -domain, $\varepsilon > 0$, $0 < \delta \leq \infty$, if whenever $x, y \in \Omega$ and $|x - y| < \delta$, there is a rectifiable arc $\gamma \subset \Omega$ with length $\ell(\gamma)$ joining x to y and satisfying*

1. $\ell(\gamma) \leq \frac{|x-y|}{\varepsilon}$ and
2. $d(z, \partial\Omega) \geq \varepsilon|x-z| \frac{|y-z|}{|x-y|}$ for $z \in \gamma$.

This kind of domains are also called locally uniform domains [22]. For bounded domains, locally uniform domains, or (ε, δ) -domains, are equivalent (see [22] point 3.4) to the uniform domains, firstly defined by Martio and Sarvas in [34], for which there are no more restriction $|x - y| < \delta$ for two points of Ω .

Thanks to Jones [24], it is known that any (ε, δ) -domain in \mathbb{R}^n is a W_p^k -extension domain for all $1 \leq p \leq \infty$ and $k \in \mathbb{N}^*$. Moreover, for a bounded finitely connected domain $\Omega \subset \mathbb{R}^2$, Jones [24] proved that Ω is a W_p^k -extension domain ($1 \leq p \leq \infty$ and $k \in \mathbb{N}^*$) iff Ω is a (ε, ∞) -domain for some $\varepsilon > 0$ iff the boundary $\partial\Omega$ consists of finite number of points and quasi-circles. However, it is no more true for $n \geq 3$, i.e. there are W_p^1 -extension domains which are not locally uniform [24] (in addition, a (ε, δ) -domain in \mathbb{R}^n with $n \geq 3$ is not necessary a quasi-sphere).

To discuss general properties of locally uniform domains, let us introduce Ahlfors d -regular sets or d -sets:

Definition 3 (Ahlfors d -regular set or d -set [26, 42, 27]) Let Γ be a closed subset of \mathbb{R}^n and $0 < d \leq n$. A positive Borel measure m_d with support Γ is called a d -measure of Γ if, for some positive constants $c_1, c_2 > 0$,

$$c_1 r^d \leq m_d(\Gamma \cap B_r(x)) \leq c_2 r^d, \quad \text{for } \forall x \in \Gamma, 0 < r \leq 1,$$

where $B_r(x) \subset \mathbb{R}^n$ denotes the Euclidean ball centered at x and of radius r . The set Γ is a d -set if there exists a d -measure on Γ .

Thanks to Ref. [[33], p.30], any two d -measures on Γ are equivalent. Henceforth, Γ is endowed with the d -dimensional Hausdorff measure and $L_p(\Gamma)$ is defined with respect to this measure as well.

From [42], it is known that

- All (ε, δ) domains in \mathbb{R}^n are n -sets (d -set with $d = n$):

$$\exists c > 0 \quad \forall x \in \overline{\Omega}, \forall r \in]0, \delta[\cap]0, 1[\quad \mu(B_r(x) \cap \Omega) \geq C \mu(B_r(x)) = cr^n,$$

where $\mu(A)$ denotes the Lebesgue measure of a set A . This property is also called the measure density condition [21]. Let us notice that a n -set Ω cannot be “thin” close to its boundary $\partial\Omega$.

- If Ω is an (ε, δ) -domain and $\partial\Omega$ is a d -set ($d < n$) then $\overline{\Omega} = \Omega \cup \partial\Omega$ is an n -set.

In particular, a Lipschitz domain Ω of \mathbb{R}^n is an (ε, δ) -domain and also an n -set [42]. But not every n -set is an (ε, δ) -domain: adding an in-going cusp to an (ε, δ) -domain we obtain an n -set which is not an (ε, δ) -domain anymore. Self-similar fractals (e.g., von Koch’s snowflake domain) are examples of (ε, ∞) -domains with the d -set boundary [12, 42], $d > n - 1$. From [26] p.39, it is also known that all closed d -sets with $d > n - 1$ preserve Markov’s local inequality:

Definition 4 (Markov’s local inequality) A closed subset V in \mathbb{R}^n preserves Markov’s local inequality if for every fixed $k \in \mathbb{N}^*$, there exists a constant $c = c(V, n, k) > 0$, such that

$$\max_{V \cap B_r(x)} |\nabla P| \leq \frac{c}{r} \max_{V \cap B_r(x)} |P|$$

for all polynomials $P \in \mathcal{P}_k$ and all closed balls $B_r(x)$, $x \in V$ and $0 < r \leq 1$.

For instance, self-similar sets that are not a subset of any $(n - 1)$ -dimensional subspace of \mathbb{R}^n , the closure of a domain Ω with Lipschitz boundary and also \mathbb{R}^n itself preserve Markov's inequality (see Refs. [42, 28]). The geometrical characterization of sets preserving Markov's local inequality was initially given in [25] (see Theorem 1.3) and can be simply interpreted as sets which are not too flat anywhere. It can be illustrated by the following theorem of Wingren [43]

Theorem 1 *A closed subset V in \mathbb{R}^n preserves Markov's local inequality if and only if there exists a constant $c > 0$ such that for every ball $B_r(x)$ centered in $x \in V$ and with the radius $0 < r \leq 1$, there are $n + 1$ affinely independent points $y_i \in V \cap B_r(x)$, $i = 1, \dots, n + 1$, such that the n -dimensional ball inscribed in the convex hull of y_1, y_2, \dots, y_{n+1} , has radius not less than cr .*

Smooth manifolds in \mathbb{R}^n of dimension less than n are examples of "flat" sets not preserving Markov's local inequality.

The interest to work with d -sets boundaries preserving Markov's inequality (thus $0 < d < n$), related in [9] with Sobolev-Gagliardo-Nirenberg inequality, is to ensure the regular extensions $W_p^k(\Omega) \rightarrow W_p^k(\mathbb{R}^n)$ with $k \geq 2$ (actually the condition applies the continuity of the extension $C^\infty(\Omega) \rightarrow C^\infty(\mathbb{R}^n)$). For the extensions of minimal regularity $k = 1$ (see in addition the Definition of Besov space Def.3.2 in [23] with the help of the normalized local best approximation in the class of polynomials P_{k-1} of the degree equal to $k - 1$) Markov's inequality is trivially satisfied.

Recently, Hajłasz, Koskela and Tuominen [21] have proved that every W_p^k -extension domain in \mathbb{R}^n for $1 \leq p < \infty$ and $k \geq 1$, $k \in \mathbb{N}$ is a n -set. In addition, they proved that any n -set, for which $W_p^k(\Omega) = C_p^k(\Omega)$ (with norms' equivalence), is W_p^k -extension domain for $1 < p < \infty$ (see [21] also for the results for $p = 1$ and $p = \infty$). By $C_p^k(\Omega)$ is denoted the space of the fractional sharp maximal functions:

Definition 5 *For a set $\Omega \subset \mathbb{R}^n$ of positive Lebesgue measure,*

$$C_p^k(\Omega) = \{f \in L_p(\Omega) | f_{k,\Omega}^\sharp(x) = \sup_{r>0} r^{-k} \inf_{P \in \mathcal{P}^{k-1}} \frac{1}{\mu(B_r(x))} \int_{B_r(x) \cap \Omega} |f - P| dy \in L^p(\Omega)\}$$

with the norm $\|f\|_{C_p^k(\Omega)} = \|f\|_{L_p(\Omega)} + \|f_{k,\Omega}^\sharp\|_{L_p(\Omega)}$.

The question about W_p^k -extension domains is equivalent to the question of the continuity of the trace operator $\text{Tr} : W_p^k(\mathbb{R}^n) \rightarrow W_p^k(\Omega)$.

Let us generalize of the notion of the trace:

Definition 6 *For an arbitrary open set Ω of \mathbb{R}^n , the trace operator Tr is defined [26, 8, 29] for $u \in L_1^{loc}(\Omega)$ by*

$$\text{Tr}u(x) = \lim_{r \rightarrow 0} \frac{1}{m(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u(y) dy,$$

where m denotes the Lebesgue measure. The trace operator Tr is considered for all $x \in \overline{\Omega}$ for which the limit exists.

We summarize useful in the following results (see [26, 42, 27, 21]) for the trace and the extension operators (see [38] for more general results for the case $p > n$):

Theorem 2 *Let $1 < p < \infty$, $k \in \mathbb{N}^*$ be fixed. Let Ω be a domain in \mathbb{R}^n such that*

- *if $n = 2$, Ω is a locally uniform domain (the boundary $\partial\Omega$ consists of finite number of points and quasi-circles),*
- *if $n \geq 3$, Ω is a n -set, such that $W_p^k(\Omega) = C_p^k(\Omega)$ as sets with equivalent norms (for example, a (ε, δ) -domain), with a closed d -set boundary $\partial\Omega$, $0 < d < n$, preserving local Markov's inequality.*

Then, for $\beta = k - (n - d)/p > 0$, the following trace operators (see Definition 6)

1. $\text{Tr} : W_p^k(\mathbb{R}^n) \rightarrow B_\beta^{p,p}(\partial\Omega) \subset L_p(\partial\Omega)$,
2. $\text{Tr}_\Omega : W_p^k(\mathbb{R}^n) \rightarrow W_p^k(\Omega)$,
3. $\text{Tr}_{\partial\Omega} : W_p^k(\Omega) \rightarrow B_\beta^{p,p}(\partial\Omega)$

are linear continuous and surjective with linear bounded right inverse, i.e. extension, operators $E : B_\beta^{p,p}(\partial\Omega) \rightarrow W_p^k(\mathbb{R}^n)$, $E_\Omega : W_p^k(\Omega) \rightarrow W_p^k(\mathbb{R}^n)$ and $E_{\partial\Omega} : B_\beta^{p,p}(\partial\Omega) \rightarrow W_p^k(\Omega)$.

Note that for $d = n - 1$, one has $\beta = \frac{1}{2}$ and $B_{\frac{1}{2}}^{2,2}(\partial\Omega) = H^{\frac{1}{2}}(\partial\Omega)$ as usual in the case of the classical results [30, 32] for Lipschitz boundaries $\partial\Omega$. In addition, for $u, v \in H^1(\Omega)$ with $\Delta u \in L_2(\Omega)$, the Green formula still holds in the framework of dual Besov spaces on a closed d -set boundary of Ω (see also [29, 11] for the von Koch case in \mathbb{R}^2):

$$\int_\Omega v \Delta u dx = \left\langle \frac{\partial u}{\partial \nu}, \text{Tr} v \right\rangle_{(B_\beta^{2,2}(\partial\Omega))', B_\beta^{2,2}(\partial\Omega)} - \int_\Omega \nabla v \nabla u dx, \quad (3)$$

where the dual Besov space $(B_\beta^{2,2}(\partial\Omega))' = B_{-\beta}^{2,2}(\partial\Omega)$ is introduced in [27].

Proposition 1 *Let Ω be a domain of \mathbb{R}^n either satisfying the conditions of Theorem 2 or be truncated domain of a domain Ω_0 by a domain Ω_1 such that Ω_i satisfies the conditions of Theorem 2 for $i = 0, 1$ ($\Omega_0 \subset \Omega_1$, $\Omega = \Omega_1 \cap (\mathbb{R}^n \setminus \Omega_0)$). Then $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$: $H^1(\Omega) \subset\subset L_2(\Omega)$.*

Proof. By assumptions of the proposition, thanks to Theorem 2, if $\Omega = \mathbb{R}^n \setminus \Omega_0$, then there exists linear bounded operator $E_\Omega : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ (let us remark that, by Jones [24], if Ω_0 is a (ε, δ) -domain then $\mathbb{R}^n \setminus \Omega_0$ is also a (ε, δ) -domain which applies Theorem 2). By the same way, there exist linear bounded operators $E_{\Omega_i} : H^1(\Omega_i) \rightarrow H^1(\mathbb{R}^n)$ and $E_{\mathbb{R}^n \setminus \Omega_i} : H^1(\Omega_i) \rightarrow H^1(\mathbb{R}^n)$. Let us prove that for the truncated domain Ω_S the extension operator $E_{\Omega_S \rightarrow \Omega} : H^1(\Omega_S) \rightarrow H^1(\Omega)$ is a linear bounded operator. It follows from the fact that it is possible to extend Ω_1 to \mathbb{R}^n and that the properties of the extension are local, i.e. depend on the properties of the boundary $S = \partial\Omega_1$, which has no intersection with $\Gamma = \partial\Omega_0$. For instance, if $S \in C^1$, then we can use the standard "reflection method" (as for instance in [1] Proposition 4.4.2). More precisely, we have to

use a finite open covering $(\omega_i)_i$ of S such that for all i $\omega_i \cap \overline{\Omega_0} = \emptyset$. The compactness of S and the fact that $S \cap \Gamma = \emptyset$ ensure that such a covering exists. In the case of a d -set boundary we use the Whitney extension method.

Hence, using Theorem 2, there exists a linear bounded operator $A : H^1(\Omega) \rightarrow H^1(\Omega_1)$ as a composition of the extension operator $E_\Omega : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ and the trace operator $Tr_{\Omega_1} : H^1(\mathbb{R}^n) \rightarrow H^1(\Omega_1)$ ($A = E_\Omega \circ Tr_{\Omega_1}$). Let define a parallelepiped Π in the way as $\Omega \subsetneq \Pi$, $\Pi = \{x = (x_1, \dots, x_n) \mid 0 < x_i < d_i\}$. Let

$$\Pi = \sqcup_{i=1}^{N^n} \Pi_i, \text{ where } \Pi_i = \otimes_{k=1}^n [a_i, a_i + \frac{d_k}{N}].$$

Consequently, the operator $B = E_{\Omega_S \rightarrow \Omega} \circ A \circ E_{\Omega_1 \rightarrow \Pi} : H^1(\Omega_S) \rightarrow H^1(\Pi)$ is a linear bounded operator as a composition of linear bounded operators. Thus, we can apply the classical proof of the compact embedding of $H^1(\Omega_S)$ to $L_2(\Omega_S)$, which we give for the convenience [35]. Indeed, let (u_m) is a bounded sequence in $H^1(\Omega_S)$. Thanks to the boundness of B , we can extend u_m from Ω_S to a parallelepiped Π , containing Ω_S , such that the extensions \tilde{u}_m

$$\tilde{u}_m \in H^1(\Pi), \quad \tilde{u}_m|_\Omega = u_m, \quad \|u_m\|_{H^1(\Omega_S)} \leq \|\tilde{u}_m\|_{H^1(\Pi)}$$

and in addition there exists a constant $C(\Omega, \Pi)$ independing on u_m , such that

$$\|\tilde{u}_m\|_{H^1(\Pi)} \leq C(\Omega, \Pi) \|u_m\|_{H^1(\Omega_S)}.$$

Thus, the sequence (\tilde{u}_m) is also a bounded sequence in $H^1(\Pi)$. Since the embedding $H^1(\Pi)$ to $L_2(\Pi)$ is continuous, the sequence (\tilde{u}_m) is also bounded in $L_2(\Pi)$.

Thanks to [35] p. 283, in Π there holds the following inequality for all $u \in H^1(\Pi)$:

$$\int_{\Pi} u^2 dx \leq \sum_{i=1}^{N^n} \frac{1}{|\Pi_i|} \left(\int_{\Pi_i} u dx \right)^2 + \frac{n}{2} \int_{\Pi} \sum_{k=1}^n \left(\frac{d_k}{N} \right)^2 \left(\frac{\partial u}{\partial x_k} \right)^2 dx. \quad (4)$$

On the other hand, $L_2(\Pi)$ is a Hilbert space, thus weak* topology on it is equal to the weak topology. Moreover, as L_2 is separable, all closed bounded sets in $L_2(\Pi)$ are weakly sequentially compact (or compact in the weak topology since here the weak topology is metrizable). To simplify the notations, we will simply write u_m for $\tilde{u}_m \in L_2(\Pi)$. Consequently, the sequence (u_m) is weakly sequentially compact in $L_2(\Pi)$ and we have

$$\exists (u_{m_k}) \subset (u_m) : \exists u \in L_2(\Pi) \quad u_{m_k} \rightharpoonup u.$$

Here u is an element of L_2 , not necessarily in H^1 .

As $(L_2(\Pi))^* = L_2(\Pi)$, by the Riesz representation theorem,

$$u_{m_k} \rightharpoonup u \in L_2(\Pi) \quad \Leftrightarrow \quad \forall v \in L_2(\Pi) \quad \int_{\Pi} (u_{m_k} - u) v dx \rightarrow 0.$$

Since (u_{m_k}) is a Cauchy sequence in weak topology on L_2 , if we take $v = \mathbb{1}_{\Pi}$, then

$$\int_{\Pi} (u_{m_k} - u_{m_j}) dx \rightarrow 0 \quad \text{for } k, j \rightarrow +\infty.$$

Thus, using (4), for two members of the sub-sequence u_{m_k} with sufficiently large ranks p and q , we have

$$\begin{aligned} \|u_p - u_q\|_{L_2(\Omega_S)}^2 &\leq \|u_p - u_q\|_{L_2(\Pi)}^2 \\ &\leq \sum_{i=1}^{N^n} \frac{1}{|\Pi_i|} \left(\int_{\Pi_i} (u_p - u_q) dx \right)^2 + \frac{n}{2N^2} \sum_{k=1}^n d_k^2 \left\| \frac{\partial u_p}{\partial x_k} - \frac{\partial u_q}{\partial x_k} \right\|_{L_2(\Pi)}^2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Here we have chosen N such that

$$\frac{n}{2N^2} \sum_{k=1}^n d_k^2 \left\| \frac{\partial u_p}{\partial x_k} - \frac{\partial u_q}{\partial x_k} \right\|_{L_2(\Pi)}^2 < \frac{\varepsilon}{2}.$$

Consequently, (u_{m_k}) is a Cauchy sequence in $L_2(\Omega_S)$, and thus converges strongly in $L_2(\Omega_S)$.

□

Theorem 3 (Compactness of the trace) *Let Ω_0 and Ω_1 be bounded domains of \mathbb{R}^n , satisfying conditions of Theorem 2, with d -sets boundaries Γ and S respectively ($n-1 \leq d_\Gamma, d_S < n$) such that $\Omega_0 \subset \Omega_1$ and $\Gamma \cap S = \emptyset$. Let Ω be the exterior or the truncated (by Ω_1) domain to Ω_0 . Then there exist linear trace operators*

$$\text{Tr}_\Gamma : H^1(\Omega) \rightarrow L_2(\Gamma) \quad \text{and} \quad \text{Tr}_S : H^1(\Omega) \rightarrow L_2(S)$$

which are compact. Moreover, $\text{Im}(\text{Tr}_\Gamma(H^1(\Omega))) = B_{\beta_\Gamma}^{2,2}(\Gamma)$ for $\beta_\Gamma = 1 - \frac{n-d_\Gamma}{2} > 0$ and $\text{Im}(\text{Tr}_S(H^1(\Omega))) = B_{\beta_S}^{2,2}(S)$ for $\beta_S = 1 - \frac{n-d_S}{2} > 0$.

Proof. To prove the compactness of the trace operator, we can apply the proof of Arendt [5] **Proposition 8.1**. We consider $\text{Tr} \equiv \text{Tr}_\Gamma$. For Tr_S the proof is identically the same.

Let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence in $(H^1(\Omega), \|\cdot\|_{H^1(\Omega)})$. Passing to a sub-sequence if necessary, there exists $u \in H^1(\Omega)$ such that $u_n \xrightarrow{H^1(\Omega)} u$ and $u_n \xrightarrow{L_2(\Omega)} u$ since the compact embedding $H^1(\Omega) \subset\subset L_2(\Omega)$. Without loss of generality, we can assume that $u = 0$. Let us now show that $\text{Tr}u_n \xrightarrow{L_2(\Gamma)} 0$.

Let $\varepsilon > 0$ and $M > 0$ such that for all $n \in \mathbb{N}$ $\|\nabla u_n\|_{L_2(\Omega)}^2 \leq M$. Thanks to Theorem 2, since $\text{Im} \text{Tr} \subset L_2(\Gamma)$, we know the continuity of the trace, which can be read: there exists $C > 0$ such that for all $u \in H^1(\Omega)$ $\|\text{Tr}u\|_{L_2(\Gamma)} \leq C\|u\|_{H^1(\Omega)}$, or equivalently [5], for all fixed $\beta > 0$ there exists $c(\beta) > 0$ such that

$$\forall u \in H^1(\Omega) \quad \beta \|\text{Tr}u\|_{L_2(\Gamma)}^2 \leq \|\nabla u\|_{L_2(\Omega)}^2 + c(\beta) \|u\|_{L_2(\Omega)}^2.$$

Therefore, with $\beta = \frac{M}{\varepsilon}$ we have for all $n \in \mathbb{N}$ $\|\text{Tr}u_n\|_{L_2(\Gamma)}^2 \leq \varepsilon + c\|u_n\|_{L_2(\Omega)}^2$ for some $c > 0$, and consequently $\limsup \|\text{Tr}u_n\|_{L_2(\Gamma)}^2 \leq \varepsilon$. Therefore $\text{Tr}u_n \xrightarrow{L_2(\Gamma)} 0$ which completes the proof. □

3 Well-posedness of Robin boundary problem for the Laplace equation

3.1 Well-posedness on a truncated domain

Let us start by a well-posedness of problem (1) for a truncated domain Ω introduced in Section 1. Therefore, Ω is a bounded domain with a d_Γ -set boundary Γ , $n-1 \leq d_\Gamma < n$ ($n \geq 2$), with the Robin boundary condition for $\lambda > 0$ and $\psi \in L_2(\Gamma)$, and a d_S -set boundary S , $n-1 \leq d_S < n$, with the homogeneous Dirichlet boundary condition.

Let us denote $\tilde{H}^1(\Omega) := \{u \in H^1(\Omega) : \text{Tr}_S u = 0\}$. Note that, thanks to Theorem 3, the continuity of the map Tr_S ensures that $\tilde{H}^1(\Omega)$ is a Hilbert space. Therefore, thanks to Proposition 1, as $H^1(\Omega) \subset\subset L_2(\Omega)$, following for instance the proof of Evans [13] (see section 5.8.1 Theorem 1) we obtain

Proposition 2 (Poincaré inequality) *For all $v \in \tilde{H}^1(\Omega)$ there exists $C > 0$, depending only on Ω and n , such that*

$$\|v\|_{L_2(\Omega)} \leq C \|\nabla v\|_{L_2(\Omega)}.$$

Therefore the semi-norm $\|\cdot\|_{\tilde{H}^1(\Omega)}$, defined by $\|v\|_{\tilde{H}^1(\Omega)} = \|\nabla v\|_{L_2(\Omega)}$, is a norm, which is equivalent to $\|\cdot\|_{H^1(\Omega)}$ on $\tilde{H}^1(\Omega)$.

Remark 1 *Let us denote $\langle v \rangle = \frac{1}{\text{Vol}(\Omega)} \int_\Omega v dx$. Since Ω is the W_p^1 -extension domain and, as Ω is bounded, the Reillich-Kondrachov type theorem [13] (see section 5.7) ensures $W_p^1(\Omega) \subset\subset L_p(\Omega)$ for all $1 \leq p \leq \infty$, then the Poincaré inequality can be generalized with the same proof to $W_p^1(\Omega)$ for all $1 \leq p \leq \infty$:*

$$\forall v \in W_p^1(\Omega) \exists C = C(\Omega, p, n) > 0 : \quad \|v - \langle v \rangle\|_{L^p(\Omega)} \leq C \|\nabla v\|_{L^p(\Omega)}.$$

Consequently, using these properties of $\tilde{H}^1(\Omega)$, we have the well-posedness of problem (1):

Theorem 4 *Let $\Omega \subset \mathbb{R}^n$ be a truncated domain, introduced in Theorem 3, with $n \geq 2$. For all $\psi \in L_2(\Gamma)$ and $\lambda \geq 0$, there exists a unique weak solution $u \in \tilde{H}^1(\Omega)$ of problem (1) such that*

$$\forall v \in \tilde{H}^1(\Omega) \quad \int_\Omega \nabla u \nabla v dx + \lambda \int_\Gamma \text{Tr}_\Gamma u \text{Tr}_\Gamma v dm_{d_\Gamma} = \int_\Gamma \psi \text{Tr}_\Gamma v dm_{d_\Gamma}. \quad (5)$$

Therefore, for all $\lambda \in [0, \infty[$ and $\psi \in L_2(\Gamma)$ the operator $B_\lambda(S) : \psi \in L_2(\Gamma) \mapsto u \in \tilde{H}^1(\Omega)$ with u , the solution of the variational problem (5), then

1. $B_\lambda(S)$ is a linear compact operator;
2. $B_\lambda(S)$ is positive: if $\psi \geq 0$ from $L_2(\Gamma)$, then for all $\lambda \in [0, \infty[$ $B_\lambda \psi = u \geq 0$;
3. $B_\lambda(S)$ is monotone: if $0 \leq \lambda_1 < \lambda_2$, then for all $\psi \geq 0$ from $L_2(\Gamma)$ it holds $B_{\lambda_2}(S)\psi = u_{\lambda_2} \leq u_{\lambda_1} = B_{\lambda_1}(S)\psi$;

4. If $\lambda \in [0, \infty[$ then $0 \leq B_\lambda(S)\mathbf{1}_\Gamma \leq \frac{1}{\lambda}\mathbf{1}_\Omega$;

Proof. It's a straightforward application of the Lax-Milgram theorem. The continuity of the two forms is ensured by the continuity of the trace operator Tr_Γ (see Theorem 3). The coercivity of the symmetric bilinear form is ensured by the Poincaré inequality (see Proposition 2). To prove the properties of the operator $B_\lambda(S)$ it is sufficient to replace $W^D(\Omega)$ by $\tilde{H}^1(\Omega)$ in the proof of Theorem 7. \square

3.2 Functional spaces for the exterior problem

To be able to prove the well-posedness of problem (1) on an exterior domain with Dirichlet boundary conditions at infinity, we extend the notion of $(\tilde{H}^1, \|\cdot\|_{\tilde{H}^1})$ to the unbounded domains. If Ω is an exterior domain of a bounded domain Ω_0 , the usual Poincaré inequality does not hold anymore and, hence, we don't have Proposition 2. For this purpose, we use [31, 6] and define for $\Omega = \mathbb{R}^n \setminus \overline{\Omega_0}$, satisfying the conditions of Theorem 2,

$$W(\Omega) := \{u \in H_{loc}^1(\Omega), \int_{\Omega} |\nabla u|^2 dx < \infty\}.$$

Remark 2 Let us fix a $r_0 > 0$ in the way that there exists $x \in \mathbb{R}^n$ such that $\overline{\Omega_0} \subset B_{r_0}(x) = \{y \in \mathbb{R}^n \mid |x - y| < r_0\}$ and for all $r \geq r_0$ define $\Omega_r = \Omega \cap B_r(x)$. Thanks to Remark 1, locally we always have the Poincaré inequality :

$$\forall u \in W(\Omega) \quad \|u - \langle u \rangle\|_{L_{loc}^2(\Omega)} \leq C_{loc} \|\nabla u\|_{L_{loc}^2(\Omega)} \leq C_{loc} \|\nabla u\|_{L_2(\Omega)} < \infty,$$

which implies that for all $r \geq r_0$ the trace $u|_{\Omega_r} \in H^1(\Omega_r)$ (see Theorem 3). Therefore, as in [6], it is still possible to consider (but we don't need it)

$$W(\Omega) = \{u : \Omega \rightarrow \mathbb{C} \mid u \text{ is measurable, } \forall r > r_0 \ u|_{\Omega_r} \in H^1(\Omega_r) \text{ and } \int_{\Omega} |\nabla u|^2 < \infty\}.$$

Thanks to G. Lu and B. Ou (see [31] **Theorem 1.1** with $p = 2$), we have

Theorem 5 Let $u \in W(\mathbb{R}^n)$ with $n \geq 3$. Then there exists the following limit:

$$(u)_\infty = \lim_{R \rightarrow \infty} \frac{1}{|B_R|} \int_{B_R} u dx.$$

Moreover, there exists a constant $c > 0$, depending only on the dimension n , but not on u , such that:

$$\|u - (u)_\infty\|_{L_{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq c \|\nabla u\|_{L_2(\mathbb{R}^n)}.$$

In [31] Section 5, G. Lu and B. Ou extend this result to exterior domains with a Lipschitz boundary. Their proof is based on the existence of a continuous extension operator. Therefore, thanks to Theorem 2, we generalize Theorem 5.2 and Theorem 5.3 of G. Lu and B. Ou and take $p = 2$, according to our case.

Theorem 6 *Let $n \geq 3$ and Ω satisfy the conditions of Theorem 3. There exists $c := c(n, \Omega) > 0$ so that for all $u \in W(\Omega)$ there exists $(u)_\infty \in \mathbb{R}$ such that*

$$\left(\int_{\Omega} |u - (u)_\infty|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \leq c(n, \Omega) \|\nabla u\|_{L_2(\Omega)}. \quad (6)$$

Moreover, it holds

1. *The space $W(\Omega)$ is a Hilbert space, corresponding the inner product*

$$(u, v) := \int_{\Omega} \nabla u \cdot \nabla v dx + (u)_\infty (v)_\infty.$$

The associated norm is denoted by $\|u\|_{W(\Omega)}$.

2. *The following norms are equivalent to $\|\cdot\|_{W(\Omega)}$:*

$$\|u\|_{\Gamma, \Omega} = (\|\nabla u\|_{L_2(\Omega)}^2 + \|\text{Tr} u\|_{L_2(\Gamma)}^2)^{1/2}, \quad \|u\|_{A, \Omega} = (\|\nabla u\|_{L_2(\Omega)}^2 + \|u\|_{L_2(A)}^2)^{1/2},$$

where $A \subset \Omega$ be a bounded measurable set with $\text{Vol}(A) > 0$.

3. *There exists a continuous extension operator $E : W(\Omega) \rightarrow W(\mathbb{R}^d)$.*
4. *The map $\text{Tr} : W(\Omega) \rightarrow L_2(\Gamma)$ is compact.*

Proof. The real number $(u)_\infty$ in the inequality (6) is merely the 'average' of an extension of u to \mathbb{R}^n , as defined in Theorem 5.

Thanks to Theorem 2, we update Theorem 5.2 and 5.3 [31] to obtain the inequality (6). Completeness of $W(\Omega)$ follows from Ref. [31] by updating the proof of Theorem 2.1. Let us notice the importance of the Sobolev embedding $H^1(\mathbb{R}^n) \subset L_{\frac{2n}{n-2}}(\mathbb{R}^n)$ which holds for $n \geq 3$, but false for $n = 2$. The equivalence of norms follows from Proposition 2.5 [6] and Theorems 2 and 3.

To prove point 3, we notice that the extension operator E is continuous if and only if Ω is such that the extension $E_\Omega : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ is a linear continuous operator. what is true in our case .

In addition, the continuity of E_Ω ensures that, independently on the geometric properties of the truncated boundary S ($S \cap \Gamma = \emptyset$), for all (bounded) truncated domains Ω_S the extension operator $E_0 : H^1(\Omega_S) \rightarrow H^1(\Omega_S \cup \overline{\Omega}_0)$ is continuous. Indeed, if $E_\Omega : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ is continuous, then $H_{loc}^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ is also continuous and hence, we can consider only functions with a support on Ω_S and extend them to $\Omega_S \cup \overline{\Omega}_0 = \mathbb{R}^n \cap \Omega_1$ to obtain the continuity of E_0 .

In our case, since Ω satisfies the conditions of Theorem 3, E_Ω is continuous. To prove the continuity of the extension operator $E : W(\Omega) \rightarrow W(\mathbb{R}^d)$, as in [6] Proposition 2.5, we use point 2 for A equal to a truncated domain Ω_S and also the obvious inclusions (see Fig. 1) $\mathbb{R}^n \setminus (\Omega_S \cup \overline{\Omega}_0) \subset \Omega$ and $\Omega_S \subset \Omega$. We find for all $u \in W(\Omega)$ that, independently

on the form of S ,

$$\begin{aligned}
\|u\|_{\Omega_S, \mathbb{R}^n}^2 &= \int_{\mathbb{R}^n} |\nabla Eu|^2 dx + \int_{\Omega_S} |Eu|^2 dx \\
&= \int_{\mathbb{R}^n \setminus (\Omega_S \cup \bar{\Omega}_0)} |\nabla Eu|^2 dx + \int_{\Omega_S \cup \bar{\Omega}_0} |\nabla Eu|^2 dx + \int_{\Omega_S} |Eu|^2 dx \\
&\leq \int_{\mathbb{R}^n \setminus (\Omega_S \cup \bar{\Omega}_0)} |\nabla u|^2 dx + \|Eu\|_{H^1(\Omega_S \cup \bar{\Omega}_0)}^2 \\
&\leq \int_{\Omega} |\nabla u|^2 dx + \|E_0\|^2 \|u\|_{H^1(\Omega_S)}^2 \\
&\leq (1 + \|E_0\|^2) \|u\|_{\Omega_S, \Omega}^2,
\end{aligned}$$

which ensures, with the equivalence of the norms, that E is continuous.

To prove point 4, we notice that $\text{Tr} : W(\Omega) \rightarrow L_2(\Gamma)$ is a composition of two traces operators:

$$\text{Tr} = \text{Tr}_\Gamma \circ \text{Tr}_{W \rightarrow H^1}, \quad \text{Tr}_{W \rightarrow H^1} : W(\Omega) \rightarrow H^1(\Omega_S), \quad \text{Tr}_\Gamma : H^1(\Omega_S) \rightarrow L_2(\Gamma).$$

Thus it is important to have Γ , S , Ω and Ω_S satisfying the conditions of Theorem 3, which ensures that $\text{Tr}_{W \rightarrow H^1}$ is continuous, *i.e.*

$$\|u\|_{H^1(\Omega_S)}^2 \leq C(\|\nabla u\|_{L_2(\Omega)}^2 + \|u\|_{L_2(\Omega_S)}^2),$$

and Tr_Γ is compact. \square

To have an analogy in the unbounded case with \tilde{H}^1 for a truncated domain, let us introduce, as in [6], the space $W^D(\Omega)$ as the closure of the space $\{u|_\Omega : u \in \mathcal{D}(\mathbb{R}^n) \ n \geq 3\}$ with respect to the norm $u \mapsto (\int_\Omega |\nabla u|^2)^{1/2}$. Therefore, for the inner product $(u, v)_{W^D(\Omega)} = \int_\Omega \nabla u \cdot \nabla v$, the space $(W^D(\Omega), (\cdot, \cdot)_{W^D(\Omega)})$ is a Hilbert space (see a discussion about it on p.8 of Ref. [31]).

It turns out that $W^D(\Omega)$ is the space of all $u \in W(\Omega)$ with average zero:

Proposition 3 *Let Ω be a unbounded (actually, exterior) domain in \mathbb{R}^n with $n \geq 3$. The space $W^D(\Omega)$ has co-dimension 1 in $W(\Omega)$. Moreover*

$$W^D(\Omega) = W(\Omega) \cap L_{\frac{2n}{n-2}}(\Omega) = \{u \in W(\Omega) : (u)_\infty = 0\}.$$

Proof. See [6] Proposition 2.6 and references therein. \square

Remark 3 *Note that, as $n \geq 3$, $H^1(\Omega) \subset W(\Omega) \cap L_{\frac{2n}{n-2}}(\Omega) = W^D(\Omega)$, which is false for $n = 2$.*

3.3 Well-posedness of the exterior problem and its approximation

Given $\psi \in L_2(\Gamma)$ and $\lambda \geq 0$, we consider the Dirichlet problem on the exterior domain Ω with Robin boundary conditions on the boundary Γ in \mathbb{R}^n , $n \geq 3$:

$$\begin{aligned}
-\Delta u &= 0 & x \in \Omega, \\
\lambda \text{Tr} u + \partial_\nu u &= \psi & x \in \Gamma.
\end{aligned} \tag{7}$$

At the infinity we consider Dirichlet boundary conditions. In [6] W. Arendt and A.F.M ter Elst also consider Neumann boundary conditions at infinity. Those results apply as well in our setting but we chose to focus on the Dirichlet boundary conditions at infinity in order not to clutter the presentation. It is worth emphasizing that in the following we only consider weak formulations that we describe below.

Since (see Subsection 3.2) $H^1(\Omega) \subset W^D(\Omega) \subset W(\Omega)$ by their definitions, therefore we need to update the definition of the normal derivative given by Eq. 3 in Section 2 to be able to work with elements of $W(\Omega)$.

Definition 7 Let $u \in W(\Omega)$ and $\Delta u \in L_2(\Omega)$. We say that u has a normal derivative ψ on Γ , denoted by $\partial_\nu u = \psi$, if $\psi \in L_2(\Gamma)$ and for all $v \in \mathcal{D}(\mathbb{R}^n)$

$$\int_{\Omega} (\Delta u)v dx + \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Gamma} \psi \text{Tr} v dm_d. \quad (8)$$

Remark 4 Definition 8 defines a weak notion of normal derivative of a function in $W(\Omega)$ in the distributional sense, if it exists. If it exists, it is unique. In addition, thanks to the definition of the space $W^D(\Omega)$, functions $v \in \mathcal{D}(\mathbb{R}^n)$ considered on Ω are dense in $W^D(\Omega)$, thus, by the density argument, Eq. (8) holds for all $v \in W^D(\Omega)$ (see [6] p. 321).

Next we define the associated variational formulation for the exterior problem 7:

Definition 8 Let $\psi \in L_2(\Gamma)$ and $\lambda \geq 0$, we say that $u \in W^D(\Omega)$ is a weak solution to the Robin problem with Dirichlet boundary conditions at infinity if

$$\forall v \in W^D(\Omega) \quad \int_{\Omega} \nabla u \nabla v dx + \lambda \int_{\Gamma} \text{Tr} u \text{Tr} v dm_d = \int_{\Gamma} \psi \text{Tr} v dm_d. \quad (9)$$

The variational formulation (9) is well-posed:

Theorem 7 Let Ω be the exterior of a bounded domain Ω_0 satisfying the conditions of Theorem 2 with a d -set boundary Γ ($n - 1 \leq d < n$, $n \geq 3$). For all $\lambda \in [0, \infty[$ and for all $\psi \in L_2(\Gamma)$ there exists a unique weak solution $u \in W^D(\Omega)$ to the Robin problem with Dirichlet boundary conditions at infinity in the sense of Definition 8. Moreover, if the operator B_λ is defined by

$$B_\lambda : \psi \in L_2(\Gamma) \mapsto u \in W^D(\Omega)$$

with u , the solution of Eq. (9), then it satisfies the same properties as the operator $B_\lambda(S)$ introduced in Theorem 4 for the truncated domains (see points 1-4): B_λ is a linear compact, positive and monotone operator with $0 \leq \lambda B_\lambda \mathbf{1}_\Gamma \leq \mathbf{1}_\Omega$ for all $\lambda \in [0, \infty[$.

Proof. As the trace operator Tr is continuous from $W^D(\Omega)$ to $L_2(\Gamma)$ thanks to Theorem 6, the well-posedness of Eq. (9) and the continuity of B_λ follow from the application of the Lax-Milgram theorem in the Hilbert space $W^D(\Omega)$. To prove the compactness of B_λ we follow Ref. [6] Proposition 3.9. Indeed, let $\lambda \in [0, \infty[$ and $(\psi_k)_{k \in \mathbb{N}}$ be a bounded sequence in $L_2(\Gamma)$. Then there exists $\psi \in L_2(\Gamma)$ such that, up to a sub-sequence, $\psi_k \xrightarrow{L_2(\Gamma)} \psi$. For all $k \in \mathbb{N}$ we set $u_k = B_\lambda \psi_k$ and $u = B_\lambda \psi$. From the continuity of

B_λ it follows that $u_k \xrightarrow{W^D(\Omega)} u$. Therefore, $\text{Tr}u_k \xrightarrow{L_2(\Gamma)} \text{Tr}u$, since the trace operator Tr is compact from $W^D(\Omega)$ to $L_2(\Gamma)$ (see Theorem 6 point 4).

Let $k \in \mathbb{N}$, choosing $v = u_k$ in Eq. (9), we obtain

$$\|u_k\|_{W^D(\Omega)}^2 = \int_{\Omega} |\nabla u_k|^2 dx = \int_{\Gamma} \psi \text{Tr}u_k dm_d - \lambda \int_{\Gamma} |\text{Tr}u_k|^2 dm_d.$$

Consequently, using Eq. (9) with $v = u$, we have

$$\lim_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 dx = \int_{\Gamma} \psi \text{Tr}u dm_d - \lambda \int_{\Gamma} |\text{Tr}u|^2 dm_d = \int_{\Omega} |\nabla u|^2 dx = \|u\|_{W^D(\Omega)}^2.$$

Hence $\|u_k\|_{W^D(\Omega)} \rightarrow \|u\|_{W^D(\Omega)}$ and consequently B_λ is compact.

The positivity and the monotone property of B_λ follow respectively from Ref. [6] Proposition 3.5 and Proposition 3.7 a). The equality $0 \leq \lambda B_\lambda \mathbb{1}_\Gamma \leq \mathbb{1}_\Omega$ follows from Ref. [6] Proposition 3.6 and Corollary 3.8 b). \square

Now, let us show that the truncated problem, studied in Subsection 3.1, independently of the form of the boundary S , is an approximation of the exterior problem in \mathbb{R}^n with $n \geq 3$. We denote by Ω_S the exterior domain Ω , truncated by the boundary S . In this framework, we also truncate [6] the space $W^D(\Omega)$, introducing a subspace

$$W_S^D(\Omega) := \{u \in W^D(\Omega) : u|_{\mathbb{R}^n \setminus \Omega_S} = 0\}$$

which is closed and thus is a Hilbert space for the inner product $(\cdot, \cdot)_{W^D(\Omega)}$. Since

$$H_0^1(\Omega_1) = \{u|_{\Omega_1} : u \in H^1(\mathbb{R}^n) \text{ and } u|_{\mathbb{R}^n \setminus \Omega_1} = 0\},$$

we notice that the map $\Psi : u \in W_S^D(\Omega) \mapsto u|_{\Omega_S} \in \tilde{H}^1(\Omega_S)$ is a bi-continuous bijection. Consequently, problem (5) is also well-posed in $W_S^D(\Omega)$ with the same properties described in Theorem 4.

In what follows, we will also suppose that the boundary S is far enough from the boundary Γ . Precisely, we suppose that $\Omega_0 \subset B_r$ is a domain (all time satisfying the conditions of Theorem 2) included in a ball of a radius $r_0 > 0$ (which exists as Ω_0 is bounded) and Ω_{S_r} with $r \geq r_0$ is such that $(\mathbb{R}^n \setminus \Omega_0) \cap B_r \subset \Omega_{S_r}$ with $\partial B_r \cap S_r = \emptyset$. If $r \rightarrow +\infty$ the boundaries S_r (for each $r \geq r_0$ the domains Ω_{S_r} satisfy the conditions of Theorem 2) will be more and more far from Γ and in the limit Ω_{S_r} gives Ω . Let us precise the properties of solutions $u \in W_S^D(\Omega)$ for the truncated problem to compare to the solutions of the exterior domain:

Lemma 1 *Let Ω_0 , Ω and Ω_S (or Ω_{S_r} for all $r \geq r_0 > 0$) satisfy conditions of Theorem 3 in \mathbb{R}^n with $n \geq 3$. Let $B_\lambda(S) : \psi \in L_2(\Gamma) \mapsto u \in W_S^D(\Omega)$ be the operator for the truncated problem and $B_\lambda : \psi \in L_2(\Gamma) \mapsto u \in W^D(\Omega)$ be the operator for the exterior problem.*

Then for all $\lambda \in [0, \infty[$ and $\psi \in L_2(\Gamma)$, if $\psi \geq 0$ in $L_2(\Gamma)$ and $r_2 \geq r_1 \geq r_0$, then

$$0 \leq u_{S_{r_1}} = B_\lambda(S_{r_1})\psi \leq u_{S_{r_2}} = B_\lambda(S_{r_2})\psi \leq B_\lambda\psi = u.$$

Proof.

The proof follows the analogous proof as in Ref. [6] Propositions 3.5 and 3.6 (see also [6] Proposition 4.4). \square We can now state the approximation result, ensuring that a solution in any admissible truncated domain, even with a fractal boundary, but which is sufficiently far from Γ is an approximation of the solution of the exterior problem:

Theorem 8 *Let $\lambda \in [0, \infty)$, $\psi \in L_2(\Gamma)$ and $(S_m)_{m \in \mathbb{N}}$ be a fixed sequence of the boundaries of the truncated domains Ω_{S_m} , satisfying for all $m \in \mathbb{N}$ the conditions of Theorem 3 and such that $(\Omega_{S_m} \cup \overline{\Omega}_0) \supset B_m \supset \Omega_0$. Let $u_{S_m} = B_\lambda(S_m)\psi$ and $u = B_\lambda\psi$. Then for all $\varepsilon > 0$ there exists $m_0(\varepsilon) > 0$, independent on the chosen sequence of the boundaries (S_m) , such that*

$$\forall m \geq m_0 \quad \|u_{S_m} - u\|_{W^D(\Omega)} < \varepsilon.$$

Equivalently, for all described sequences $(S_m)_{m \in \mathbb{N}}$, it holds

$$\|B_\lambda(S_m) - B_\lambda\|_{\mathcal{L}(L_2(\Gamma), W^D(\Omega))} \rightarrow 0 \quad \text{as } m \rightarrow +\infty.$$

Proof. It is a simple generation using our previous results of Theorem 4.3 [6]. \square

4 Spectral properties of the Poincaré-Steklov operator defined by the interior and by the exterior problems

The Poincaré-Steklov operator, also named the Dirichlet-to-Neumann operator, was originally introduced by V.A. Steklov and usually defined by a map $A : u|_\Gamma \mapsto \frac{\partial u}{\partial \nu}|_\Gamma$ for a solution u of the elliptic Dirichlet problem: $-\Delta u = 0$ in a domain Ω and $u|_\Gamma = f$ ($\partial\Omega = \Gamma$).

It is well-known that if Ω is a bounded domain with C^∞ -regular boundary (a regular manifolds with boundary), then the operator $A : C^\infty(\Gamma) \rightarrow C^\infty(\Gamma)$ is an elliptic self-adjoint pseudo-differentiable operator of the first order (see [40] §11 and 12 of Chapter 7) with a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \text{with } \lambda_k \rightarrow +\infty \text{ as } k \rightarrow +\infty.$$

If A is considered as an operator $H^1(\Gamma) \rightarrow L_2(\Gamma)$, then its eigenfunctions form a basis in $L_2(\Gamma)$. For any Lipschitz boundary Γ of a bounded domain Ω , the Dirichlet-to-Neumann operator $A : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$ is well-defined and it is a linear continuous self-adjoint operator. Thanks to [4], we also know that the Dirichlet-to-Neumann operator A has compact resolvent, and hence discrete spectrum, as long as the trace operator $\text{Tr} : H^1(\Omega) \rightarrow L_2(\Gamma)$ is compact (see also [5] and [41] for abstract definition of the elliptic operators on a d -set). Thus, thanks to Theorem 3, the property of the compact resolvent also holds for bounded n -set Ω with a d -set boundary Γ (see the conditions on Ω in Theorem 3). We will discuss this method in details in the next section. From [7], we also have that $\text{Ker } A \neq \{0\}$ since 0 is the eigenvalue of the Neumann eigenvalue problem for the Laplacian. For the Weil asymptotic formulas for the distribution of the eigenvalues of the Dirichlet-to-Neumann operator there are results for bounded smooth compact Riemannian manifolds with C^∞ boundaries [16], for polygons [17] and more general class of plane domains [15] and also for a bounded domain with a fractal boundary [37].

In the aim to relate these spectral results obtained for the Dirichlet-to-Neumann operator for a bounded domain with the case of the exterior domain, we prove the following Theorem:

Theorem 9 *Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded domain with a boundary Γ and let $\Omega = \mathbb{R}^n \setminus \overline{\Omega_0}$ be its complement in \mathbb{R}^n satisfying conditions of Theorem 3. Then the Dirichlet-to-Neumann operator $A : L_2(\Gamma) \rightarrow L_2(\Gamma)$, associated with the Laplacian on Ω_0 or Ω , has a compact resolvent and a discrete positive spectrum, denoted by σ_{int} and $\sigma_{ext} \subset [0, +\infty[$ for the interior and the exterior problem respectively. Moreover, all non-zero eigenvalue of the Dirichlet-to-Neumann operator of the interior problem is also a eigenvalue of the Dirichlet-to-Neumann operator of the exterior problem and converse. Hence the eigenfunctions form the same basis in $L_2(\Gamma)$. More precisely,*

- *For $n = 2$ the spectrum $\sigma_{int} = \sigma_{ext} \subset \mathbb{R}^+$ and $0 \in \sigma_{ext}$. In addition, let $\lambda_k(r) \in \sigma_S(r)$, where $\sigma_S(r) \subset \mathbb{R}_*^+$ is the spectrum of the Dirichlet-to-Neumann operator associated with the truncated domain Ω_{S_r} , satisfying for all $r > 0$ the conditions of Theorem 3 and such that $(\Omega_{S_r} \cup \overline{\Omega_0}) \supset B_r \supset \Omega_0$. If $\lambda_0(r) = \min_{k \in \mathbb{N}} \lambda_k(r)$, then $\lambda_1(r) \rightarrow 0$ for $r \rightarrow +\infty$ independently on the form of S_r .*
- *For $n > 2$ the spectrum $\sigma_{int} = \{0\} \cup \sigma_{ext}$ with $\sigma_{ext} \subset]0, +\infty[$, i.e. the Dirichlet-to-Neumann operator of the exterior problem, also as of the truncated problem, is a injective operator with the compact inverse.*

To prove Theorem 9 we need to define the Dirichlet-to-Neumann operator on a d -set Γ in $L_2(\Gamma)$. Hence, we firstly do it in Section 5 and then give the proof in Section 6.

5 Poincaré-Steklov operator on d -set

5.1 On a bounded domain

Let Ω be a closed (ε, δ) -domain with a d -set boundary Γ ($n - 1 \leq d < n$, $n \geq 2$). Knowing the well-posedness results for the Dirichlet problem (Theorem 7 [28]) and the definition of the normal derivative by the Green formula (3), thanks to [27], we notice that the general setting of [7] p. 5904 for Lipschitz domains (see also [36] Theorem 4.10) still holds in the case for a d -set boundary by replacing $H^{\frac{1}{2}}(\Gamma)$ by $B_\beta^{2,2}(\Gamma)$ with $\beta = 1 - \frac{n-d}{2} > 0$ and $H^{-\frac{1}{2}}(\Gamma)$ by $B_{-\beta}^{2,2}(\Gamma)$. Precisely, we have that for all $\lambda \in \mathbb{C}$ the Dirichlet problem

$$-\Delta u = \lambda u, \quad u|_\Gamma = \phi \tag{10}$$

is solvable if $\phi \in B_\beta^{2,2}(\Gamma)$ satisfies

$$\langle \partial_\nu \psi, \phi \rangle_{B_{-\beta}^{2,2}(\Gamma) \times B_\beta^{2,2}(\Gamma)} = 0$$

for all solutions $\psi \in H^1(\Omega)$ of the corresponding homogeneous problem

$$-\Delta \psi = \lambda \psi, \quad \psi|_\Gamma = 0. \tag{11}$$

We are especially interesting in the case $\lambda = 0$. Thus, we directly conclude that problem (11) has only the trivial solution $\psi = 0$ ($\lambda = 0$ is not an eigenvalue of the Dirichlet Laplacian), and consequently the Poincaré-Steklov operator $A : B_{\beta}^{2,2}(\Gamma) \rightarrow B_{-\beta}^{2,2}(\Gamma)$ mapping $u|_{\Gamma}$ to $\partial_{\nu}u|_{\Gamma}$ is well-defined on $B_{\beta}^{2,2}(\Gamma)$.

In the other hand, as it was done in [5] for bounded domains with $(n-1)$ -dimensional boundaries, it is also possible to consider A as operator from $L_2(\Gamma) \rightarrow L_2(\Gamma)$, if we consider the trace map $\text{Tr} : H^1(\Omega) \rightarrow L_2(\Gamma)$ (note that $B_{\beta}^{2,2}(\Gamma) \subset L_2(\Gamma)$) and update the definition of the normal derivative by analogy with Definition 7:

Definition 9 *Let $u \in H^1(\Omega)$ and $\Delta u \in L_2(\Omega)$. If there exists $\psi \in L_2(\Gamma)$ such that for all $v \in H^1(\Omega)$ it holds Eq. (8), then ψ is called a L_2 -normal derivative of u , $\partial_{\nu}u = \psi$.*

Definition 9 restricts the normal derivative of u , which is naturally in $B_{-\beta}^{2,2}(\Gamma)$, to a consideration of only the normal derivative from its dense subspace. Thus, the L_2 -normal derivative can does not exist, but if it exists, it is unique.

Therefore, to define the Dirichlet-to-Neumann operator on $L_2(\Gamma)$, we use the following Theorem from [5] (see Theorem 3.4)

Theorem 10 *Let $D(a)$ be a real vector space and let $a : D(a) \times D(a) \rightarrow \mathbb{R}$ be bilinear symmetric such that $a(u, u) \geq 0$ for all $u \in D(a)$. Let H be a real Hilbert space and let $j : D(a) \rightarrow H$ be linear operator with dense image. Then there exists an operator A on H such that for all $\phi, \psi \in H$, one has $\phi \in D(A)$ and $A\phi = \psi$ if and only if there exists a sequence $(u_k)_{k \in \mathbb{N}}$ in $D(a)$ such that:*

1. $\lim_{n,m \rightarrow \infty} a(u_n - u_m, u_n - u_m) = 0$,
2. $\lim_{n \rightarrow \infty} j(u_n) = \phi$ in H ,
3. for all $v \in D(a)$ $\lim_{k \rightarrow \infty} a(u_k, v) = (\psi, j(v))_H$.

Moreover, A is positive and self-adjoint. The operator A is called the operator associated with (a, j) .

Consequently we state

Theorem 11 *Let Ω be a closed (ε, δ) -domain with a d -set boundary Γ ($n-1 \leq d < n$, $n \geq 2$). Then the Poncaré-Steklov operator $A : B_{\beta}^{2,2}(\Gamma) \rightarrow B_{-\beta}^{2,2}(\Gamma)$ mapping $u|_{\Gamma}$ to $\partial_{\nu}u|_{\Gamma}$ is linear bounded self-adjoint operator with $\text{Ker } A \neq \{0\}$. In addition, the Poncaré-Steklov operator A , considered from $L_2(\Gamma)$ to $L_2(\Gamma)$, is self-adjoint positive operator with a compact resolvent. Therefore, there exists a discrete spectrum of A with eigenvalues*

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \text{with } \lambda_k \rightarrow +\infty \text{ as } k \rightarrow +\infty$$

and the corresponding eigenfunctions forms an orthonormal basis in $L_2(\Gamma)$.

Proof. We have already noticed that the domain of A is exactly $B_{\beta}^{2,2}(\Gamma)$. As 0 is an eigenvalue of the Neumann Laplacian, $\text{Ker } A \neq \{0\}$. From the following Green formula

for $u, v \in H^1(\Omega)$ with $\Delta u, \Delta v \in L_2(\Omega)$

$$\begin{aligned} \int_{\Omega} \Delta uv dx - \int_{\Omega} u \Delta v dx \\ = \left\langle \frac{\partial u}{\partial \nu}, \text{Tr} v \right\rangle_{B_{-\beta}^{2,2}(\Gamma), B_{\beta}^{2,2}(\Gamma)} - \left\langle \text{Tr} u, \frac{\partial v}{\partial \nu} \right\rangle_{B_{\beta}^{2,2}(\Gamma), B_{-\beta}^{2,2}(\Gamma)}, \end{aligned}$$

we directly find that for all $u, v \in B_{\beta}^{2,2}(\Gamma)$

$$\langle Au, v \rangle_{B_{-\beta}^{2,2}(\Gamma), B_{\beta}^{2,2}(\Gamma)} = \langle u, Av \rangle_{B_{\beta}^{2,2}(\Gamma), B_{-\beta}^{2,2}(\Gamma)},$$

i.e. the operator A is self-adjoint and closed. Since $B_{-\beta}^{2,2}(\Gamma)$ is a Banach space, by the closed graph Theorem, A is continuous.

To define A as an operator on $L_2(\Gamma)$ we use [5, 4, 2]. As Ω is such that the trace operator Tr is compact from $H^1(\Omega)$ to $L_2(\Gamma)$, then the embedding of its image $\text{Tr}(H^1(\Omega)) = B_{\beta}^{2,2}(\Gamma)$ into $L_2(\Gamma)$ is compact. Now, as it was noticed in [4], the space $\{v|_{\Gamma} : v \in \mathcal{D}(\mathbb{R}^n)\}$ is dense in $C(\Gamma)$ by the Stone-Weierstrass theorem for the uniform norm and, therefore, it is also dense in $L_2(\Gamma)$, since we endowed Γ with the d -dimensional Hausdorff measure which is Borel regular. Hence, $B_{\beta}^{2,2}(\Gamma)$ is dense in $L_2(\Gamma)$. It allows us to apply Theorem 2.2 and follow Section 4.4 of Ref. [2]. We notice that, thanks to [42] Theorem 3, $\text{Ker Tr} = H_0^1(\Omega)$ and, thanks to Lemma 2.2 [4], $H^1(\Omega) = H_0^1(\Omega) \oplus H$ with $H = \{u \in H^1(\Omega) \mid \Delta u = 0 \text{ weakly}\}$. Hence, $\text{Tr}(H) = B_{\beta}^{2,2}(\Gamma)$ and $\text{Tr} : H \rightarrow B_{\beta}^{2,2}(\Gamma)$ is a linear bijection. Therefore, the bilinear map $a_0 : B_{\beta}^{2,2}(\Gamma) \times B_{\beta}^{2,2}(\Gamma) \rightarrow \mathbb{R}$, given by

$$a_0(\phi, \psi) = \int_{\Omega} \nabla u \nabla v \quad \text{for } u, v \in H \quad \text{Tr} u = \phi, \quad \text{Tr} v = \psi, \quad (12)$$

is symmetric, continuous and elliptic [3] (see Proposition 3.3, based on the compactness of the embedding $H \rightarrow L^2(\Omega)$ and on the injective property of the trace from $H \rightarrow L_2(\Gamma)$):

$$\exists \omega \geq 0 \text{ such that } \forall u \in H \quad a_0(\text{Tr} u, \text{Tr} u) + \omega \int_{\Gamma} |u|^2 dm_d \geq \frac{1}{2} \|u\|_{H^1(\Omega)}^2.$$

If the operator $N : L_2(\Gamma) \rightarrow L_2(\Gamma)$ is the operator associated with a_0 , then it is the Dirichlet-to-Neumann operator A on $L_2(\Gamma)$, *i.e.* $A\phi = \partial_{\nu} u$ in $L_2(\Gamma)$ with $\mathcal{D}(A) = \{\phi \in L_2(\Gamma) \mid \exists u \in H^1(\Omega) \text{ such that } \text{Tr} u = \phi, \Delta u = 0 \text{ and } \exists \partial_{\nu} u \in L_2(\Gamma)\}$. Moreover, we have that for all $\phi \in L_2(\Gamma)$, $\phi \in D(A)$ and there exists an element $\psi = A\phi$ of $L_2(\Gamma)$ if and only if there exists $u \in H^1(\Omega)$ such that $\text{Tr} u = \phi$ and

$$\forall v \in H^1(\Omega) \quad \int_{\Omega} \nabla u \nabla v = \int_{\Gamma} \psi \text{Tr} v dm_d.$$

In other hand, we also can directly use Theorem 3.3 in Ref. [5], by applying Theorem 10. Let now $D(a) = H^1(\Omega) \cap C(\overline{\Omega})$, which is dense in $H^1(\Omega)$ (see the discussion of Ref. [5]). Then $\text{Tr}(D(a))$ is dense in $L_2(\Gamma)$. Therefore, taking in Theorem 10 $a(u, v) = \int_{\Omega} \nabla u \nabla v dx : D(a) \times D(a) \rightarrow \mathbb{R}$, $H = L_2(\Gamma)$ and $j = \text{Tr} : D(a) \rightarrow L_2(\Gamma)$, which is compact, we conclude that the operator associated to (a, Tr) is the Dirichlet-to-Neumann operator A , positive and self-adjoint in $L_2(\Gamma)$ (see the proof of Theorem 3.3 in Ref. [5]). As the compactness of the trace implies that A has a compact resolvent, it is sufficient to apply the Hilbert-Schmidt Theorem to finish the proof. \square

5.2 On the exterior and truncated domains

In this subsection we generalize [6] and introduce the Dirichlet-to-Neumann operator A on $L_2(\Gamma)$ with respect to the exterior domain $\Omega \subset \mathbb{R}^n$ and A_S with respect to a truncated domain for $n \geq 2$ in the framework of d -sets using the Theorem 10.

Definition 10 (*Dirichlet-to-Neumann operator for the exterior domain $n \geq 3$*)
Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an exterior domain satisfying the conditions of Theorem 3. The operator $A : L_2(\Gamma) \rightarrow L_2(\Gamma)$, associated with the bilinear form $a^D : W^D(\Omega) \times W^D(\Omega) \rightarrow \mathbb{R}$ given by

$$a^D(u, v) = \int_{\Omega} \nabla u \nabla v dx = \langle u, v \rangle_{W^D(\Omega)},$$

and the trace operator $\text{Tr} : W^D(\Omega) \rightarrow L_2(\Gamma)$, is called the Dirichlet-to-Neumann operator with the Dirichlet boundary condition at infinity.

Remark 5 *Theorem 10 does not require to $D(a)$ the completeness, i.e. $a(\cdot, \cdot)$ can be equivalent to a semi-norm on $D(a)$, what is the case of $W^D(\Omega)$ with $a(u, u) = \int_{\Omega} |\nabla u|^2 dx$ for $n = 2$. Therefore, it allows us to define the Dirichlet-to-Neumann operator A of the exterior problem in \mathbb{R}^2 , which can be understood as the limit case for $r \rightarrow +\infty$ of the problem for a truncated domain well-posed in $\tilde{H}^1(\Omega_{S_r})$. In the case of $W^D(\Omega)$ in \mathbb{R}^n with $n \geq 3$, we have that $D(a) = W^D(\Omega)$ is the Hilbert space corresponding to the inner product $a(\cdot, \cdot)$.*

Let us notice that the trace on the Γ satisfies $\text{Tr}(\mathcal{D}(\mathbb{R}^n)) \subset \text{Tr}(W^D(\Omega)) \subset L_2(\Gamma)$ and, since $\text{Tr}(\mathcal{D}(\mathbb{R}^n))$ is dense in $L_2(\Gamma)$, $\text{Tr}(W^D(\Omega))$ is dense in $L_2(\Gamma)$. In addition, a^D is Tr -elliptic thanks to point 2 of Theorem 6, i.e. there exists $\alpha \in \mathbb{R}$ and $\delta > 0$ such that

$$\forall u \in W^D(\Omega) \quad \int_{\Omega} |\nabla u|^2 dx + \alpha \int_{\Gamma} |\text{Tr} u|^2 dm_d \geq \delta \int_{\Omega} |\nabla u|^2 dx.$$

Thus, for $n \geq 3$ we can also apply Theorem 2.2 and follow Section 4.4 of Ref. [2].

For the two-dimensional case, we define A associated to the bilinear form a_0 from Eq. (12), initially given for the interior case:

Proposition 4 (*Dirichlet-to-Neumann operator for the exterior domain $n = 2$*)
Let $\Omega \subset \mathbb{R}^2$ be an exterior domain satisfying the conditions of Theorem 3. The operator $A : L_2(\Gamma) \rightarrow L_2(\Gamma)$, associated with the bilinear form a_0 , defined in Eq. (12), is the Dirichlet-to-Neumann operator with the Dirichlet boundary condition at infinity in the sense that for all $\phi \in L_2(\Gamma)$, $\phi \in D(A)$ and there exists an element $\psi = A\phi$ of $L_2(\Gamma)$ if and only if there exists $u \in H^1(\Omega)$ such that $\text{Tr} u = \phi$ and

$$\forall v \in H^1(\Omega) \quad \int_{\Omega} \nabla u \nabla v = \int_{\Gamma} \psi \text{Tr} v dm_d.$$

Proof. We use that $H^1(\mathbb{R}^2) = H_0^1(\mathbb{R}^2)$ and that the compactness of the embedding $H = \{u \in H^1(\Omega) \mid \Delta u = 0 \text{ weakly}\} \subset L_2(\Omega)$ and the injective property of the trace from H to $L_2(\Gamma)$ still hold for the exterior case. In addition 0 is not an eigenvalue of the

Dirichlet Laplacian on Ω . Thus we can follow the proof of Lemma 3.2 and Proposition 3.3 in [3], given for a Lipschitz bounded domain. \square

The following proposition legitimates Definition 10 in the framework of Theorem 10 for $n \geq 3$:

Proposition 5 *Let $\phi, \psi \in L_2(\Gamma)$. The following are equivalent:*

$\phi \in D(A)$ and $A\phi = \psi$ if and only if there exists a function $u \in W^D(\Omega)$ such that $\text{Tr}u = \phi$, $\Delta u = 0$ and $\partial_\nu u = \psi$.

Proof. Let $\phi, \psi \in L_2(\Gamma)$ such that $\phi \in D(A)$ and $A\phi = \psi$. Then, according to Theorem 10, there exists a sequence $(u_k)_{k \in \mathbb{N}}$ in $W^D(\Omega)$ such that

1. $\lim_{k,m \rightarrow \infty} \int_{\Omega} |\nabla(u_k - u_m)|^2 = 0$,
2. $\lim_{k \rightarrow \infty} \text{Tr}u_k = \phi$,
3. $\lim_{k \rightarrow \infty} \int_{\Omega} \nabla u_k \nabla v = \int_{\Gamma} \psi \text{Tr}v$ for all $v \in W^D(\Omega)$.

From item 1 it follows that $(u_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $W^D(\Omega)$. Therefore, by the completeness of $W^D(\Omega)$ (thanks to $n \geq 3$), there exists $u \in W^D(\Omega)$ such that $u_k \rightarrow u$ in $W^D(\Omega)$. Moreover, since $\text{Tr} : W^D(\Omega) \rightarrow L_2(\Gamma)$ is continuous by point 4 of Theorem 6, $\text{Tr}u = \phi$ according to item 2. From item 3 we deduce that for all $v \in W^D(\Omega)$

$$\int_{\Omega} \nabla u \nabla v = \int_{\Gamma} \psi \text{Tr}v, \quad (13)$$

and hence, in particular for all $v \in \mathcal{D}(\Omega)$. Therefore $\Delta u = 0$. This with Eq. (13) yields that u has a normal derivative in $L_2(\Gamma)$ and $\partial_\nu u = \psi$.

Conversely, let $\phi, \psi \in L_2(\Gamma)$ be such that there exists a function $u \in W^D(\Omega)$, so that $\text{Tr}u = \phi$, $\Delta u = 0$, $\partial_\nu u = \psi$. According to the definition of normal derivatives (see Definition 7 and Remark 4), since $\Delta u = 0$, we have for all $v \in W^D(\Omega)$:

$$\int_{\Omega} \nabla u \nabla v = \int_{\Gamma} \psi \text{Tr}v.$$

Therefore, for $n \geq 3$ we can apply the Theorem 10 to the sequence, defined by $u_k = u$ for all $k \in \mathbb{N}$, and the result follows. \square Let us notice that the Dirichlet-to-Neumann operator A_S for the truncated domain by a d_S set S ($n-1 \leq d_S < n$) can be defined absolutely in the same way as A for the exterior domain if we replace $W^D(\Omega)$ by $\tilde{H}^1(\Omega_S)$ or, equivalently, by $W_S^D(\Omega)$.

Consequently, for the exterior and truncated domain we have

Theorem 12 *Let Ω be the exterior domain in \mathbb{R}^n with $n \geq 3$ and Ω_S be a truncated domain in \mathbb{R}^n with $n \geq 2$ satisfying conditions of Theorem 3 and $\lambda \in [0, \infty[$. Then the Dirichlet-to-Neumann operator with the Dirichlet boundary condition at infinity $A : L_2(\Gamma) \rightarrow L_2(\Gamma)$ (see Definition 10) and the Dirichlet-to-Neumann operator A_S of the truncated domain are positive self-adjoint operators with a compact resolvent*

$$\forall \lambda \in [0, +\infty[\quad (\lambda I + A)^{-1} = \text{Tr}_\Gamma \circ B_\lambda, \quad (\lambda I + A_S)^{-1} = \text{Tr}_\Gamma \circ B_\lambda(S)$$

where $B_\lambda : \psi \in L_2(\Gamma) \mapsto u \in W^D(\Omega)$ with u , the solution of Eq. (9), and $B_\lambda(S) : \psi \in L_2(\Gamma) \mapsto u \in W_S^D(\Omega)$ with u , the solution of Eq. (5) are defined in Theorem 7 and Theorem 4 respectively. Moreover, $\text{Ker } A = \text{Ker } A_S = \{0\}$ and for $n \geq 3$, independently on a d -set S_r ($(\mathbb{R}^n \setminus \Omega_0) \cap B_r \subset \Omega_{S_r}$ with $\partial B_r \cap S_r = \emptyset$),

$$\forall \lambda \in [0, +\infty[\quad \|(\lambda I + A_{S_r})^{-1} - (\lambda I + A)^{-1}\|_{\mathcal{L}(L_2(\Gamma))} \rightarrow 0 \text{ as } r \rightarrow +\infty.$$

Therefore, the spectrum of A and A_S are discrete with all eigenvalues $(\lambda_k)_{k \in \mathbb{N}}$ strictly positive

$$0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \text{with } \lambda_k \rightarrow +\infty \text{ as } k \rightarrow +\infty,$$

and the corresponding eigenfunctions forms an orthonormal basis of $L_2(\Gamma)$.

Proof. The compactness of the resolvents $(\lambda I + A)^{-1}$ and $(\lambda I + A_S)^{-1}$ directly follows from the compactness properties of the operators Tr_Γ , B_λ , $B_\lambda(S)$. Using the previous results and the Hilbert-Schmidt Theorem for self-adjoint compact operators on a Hilbert space, we finish the proof. \square

6 Proof of Theorem 9 and final remarks

Now, we can prove Theorem 9: **Proof.** For $n > 2$ the statement of the theorem directly follows from Theorems 11 and 12 (see also [18] pp. 129-132 and 134 for the explicit calculus of the interior and exterior spectra of A for a ball).

For $n = 2$ we also have Theorem 11 and Proposition 4. Let us prove that if $n = 2$ the resolvent $(\lambda I + A(S_r))^{-1} \rightarrow (\lambda I + A^{int})^{-1}$ for $r \rightarrow +\infty$ in $\mathcal{L}(L_2(\Gamma))$, where by A^{int} is denoted the Dirichlet-to-Neumann operator corresponding to the bounded domain Ω_0 (the interior case).

It holds in the case of a ball for $n = 2$ by an explicit calculus (see [18] pp. 129-132). For a Lipschitz plane boundary, it is sufficient to apply a conform map to project Γ to a sphere and hence to obtain the same result. If Ω_0 is a (ε, δ) -domain in \mathbb{R}^2 , by [24], its boundary is a quasi-circle. Let us take a sequence of conformal to the circle boundaries, for example, if Ω_0 is a snowflake of the Von Koch, corresponding to the sequence of the fractal generations (of its complement), Γ_j , which for $j \rightarrow +\infty$ becomes Γ such that $\Omega_{j,S} \subset \Omega_{j+1,S}$ and $\Omega_S = \cup_{i=1}^{\infty} \Omega_{i,S}$.

Thus, for each Γ_j there exist $A_j(S_r)$ and A_j^{int} with compact resolvent on $L^2(\Gamma_j)$ and $(\lambda I + A_j(S_r))^{-1} \rightarrow (\lambda I + A_j^{int})^{-1}$ for $r \rightarrow +\infty$ in $\mathcal{L}(L_2(\Gamma_j))$. Thanks to Theorem 11, there also exist A^{int} on the d -set Γ , positive self-adjoint on $L_2(\Gamma)$ with a compact resolvent and $0 \in \sigma^{int}$.

Thanks to the relation $(\lambda I + A_j(S_r))^{-1} = \text{Tr} \circ B_j(S_r)$, with $B_j(S_r)$ defined in Theorem 4, to be able to pass to the limit for $j \rightarrow +\infty$ we consider the sequence of compact operators $\text{Ext}_{\Omega_{j,S} \rightarrow \Omega_S} \circ B_j(S_r)$ from $L_2(\Gamma_j) \rightarrow \tilde{H}^1(\Omega_S)$, which defines, for instance for $\mathbb{1}_{\Gamma_j} \in L_2(\Gamma_j)$, a monotone increasing sequence u_j in $\tilde{H}^1(\Omega_S)$ bounded for $j \rightarrow +\infty$ by its limit $u = B(S_r)\mathbb{1}_\Gamma$. Therefore, thanks to the continuity and compactness of the operators, we have that the spectrum σ_{ext} of A for the exterior of a quasi-disk is equal to the spectrum σ_{int} of A for the quasi-disk, which, as it is shown in Theorem 11, is discrete and containing in \mathbb{R}^+ with $\lambda_k \rightarrow +\infty$ for $k \rightarrow +\infty$ and with $\lambda_0 = 0$. \square

Let us also notice that for the convergence of the series (2) on the truncated or the exterior domain, we need to have $\mathbb{1}_\Gamma \in \mathcal{D}(A)$. For a Lipschitz boundary Γ it was proven in Proposition 5.7 of Ref. [6]. In this framework we state more generally

Proposition 6 *Let Ω and Ω_S be domains in \mathbb{R}^n with $n \geq 3$ and with $n \geq 2$ respectively with a d -set boundary Γ ($n - 1 \leq d < n$), satisfying Theorem 3. Then for all $\psi \in L_2(\Gamma)$ there exists $\phi = A\psi \in L_2(\Gamma)$. If Ω is a exterior domain in \mathbb{R}^2 , then for all $\psi \in B_{d/2}^{2,2}(\Gamma)$ there exist $\phi = A\psi \in L_2(\Gamma)$.*

Proof. It is a corollary of the fact that the operator $A : L_2(\Gamma) \rightarrow L_2(\Gamma)$, considered on Ω (for $n \geq 3$) and Ω_S (for $n \geq 2$) respectively, is invertible with a compact inverse operator A^{-1} (since $\lambda = 0$ is a regular point by Theorem 9).

For instance, for the exterior case with $n \geq 3$, $\mathbb{1}_\Gamma \in L_2(\Gamma)$, thus, for $\lambda = 0$, $B_0\mathbb{1}_\Gamma \in W^D(\Omega)$, by the well-posedness of the Robin Laplacian exterior problem, and hence $\text{Tr}(B_0\mathbb{1}_\Gamma) = A^{-1}\mathbb{1}_\Gamma \in L_2(\Gamma)$.

If Ω is a exterior domain in \mathbb{R}^2 , then for all $u \in H^1(\Omega)$, such that $\nabla u = 0$ weakly, there exists unique $\text{Tr}u \in B_\beta^{2,2}(\Gamma) \subset L_2(\Gamma)$ with $\beta = d/2$ (see Theorem 2), thus for all $\psi \in B_\beta^{2,2}(\Gamma)$ there exist $\phi = A\psi \in L_2(\Gamma)$. Consequently, as $\mathbb{1}_\Gamma \in B_\beta^{2,2}(\Gamma)$, thus $\mathbb{1}_\Gamma \in \mathcal{D}(A)$. \square

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