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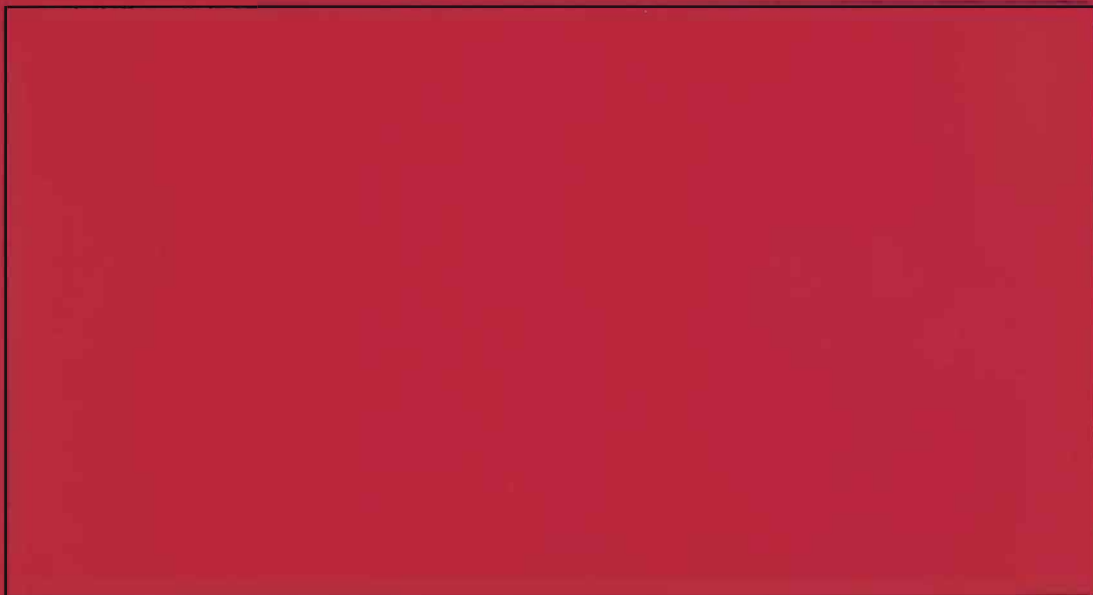
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GROUP DECISION THEORY WITH CONVEX COMBINATION  
OF FUZZY EVALUATIONS

Marc ROUBENS

May 1988

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ABSTRACT

A multiple criteria decision making problem involving a discrete set of alternatives  $A$  is investigated.

Each alternative is characterized by a fuzzy evaluation to a given number of criteria weighted by fuzzy numbers.

The classical mean aggregation procedure is exploited in order to

- (i) isolate the smallest subset (called set of "best" elements)  $A_0$  included in  $A$  liable to eliminate all alternatives belonging to  $A \setminus A_0$
- (ii) rank the alternatives in various classes (being or not exclusive) using an antisymmetric fuzzy preference graph.

Keywords : Multiple criteria decision making, Mean aggregation procedure, Valued preference relations, Choice functions.

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## 1. INTRODUCTION

Suppose that a set of actions  $A : \{a,b,c,\dots\}$  is described and each of the alternatives is characterized by a number of criteria  $i=1,\dots,I$ .

Let  $\tilde{\mu}_i(a)$  be the fuzzy evaluation of action  $a$  for a given criterion  $i$ . This evaluation is a fuzzy number which membership function is denoted  $\mu_i(a,x)$ ,  $x$  in  $R$  such  $\bigvee_x \mu_i(a,x)=1$ , for all  $i$ . Convexity property is assumed :

$$x \leq y \leq z \rightarrow \mu_i(a,y) \geq \mu_i(a,x) \wedge \mu_i(a,z), \text{ for all } x,y,z \text{ in } R$$

We define as usual :  $u \wedge v = \min(u,v)$ ,  $u \vee v = \max(u,v)$ ,  $u,v \in R$ .

The criteria are weighted by fuzzy values  $\{\tilde{\omega}_i\}$  which reflect their relative importance. The membership function related to  $\tilde{\omega}_i$  is denoted  $\omega_i(x)$ ,  $x$  in  $R$ .

If  $\{1,\dots,I\}$  represents a set of experts,  $\tilde{\mu}_i(a)$  can be considered as a fuzzy rating (say between 0 and 10) of action  $a$  by expert  $i$ , the higher the evaluation, the better the alternative satisfies the expert in question.

The problem lies in making rational choices or ranking of alternatives from the given vectors of performances and the vector of weights :

$$(\tilde{\mu}_1(\cdot), \tilde{\mu}_2(\cdot), \dots, \tilde{\mu}_I(\cdot))$$

$$(\tilde{\omega}_1, \tilde{\omega}_2, \dots, \tilde{\omega}_I)$$

To achieve these goals the "mean" aggregation procedure is used to :

- (i) isolate the smallest subset  $A_0 \subset A$  liable to eliminate all actions belonging to  $A \setminus A_0$ ,
- (ii) build a preference structure  $(P,I)$  on the set  $A$  which permits to declare either  $aIb$  :  $a$  indifferent to  $b$ 
  - $aPb$  :  $a$  strictly preferred to  $b$
  - $bPa$  :  $b$  strictly preferred to  $a$ , for all  $a,b$ , in  $A$

Two different approaches can be considered in order to aggregate the fuzzy evaluations.

(A1) : consider the convex combination of the elements of the vector of performances related to alternative  $a$  to obtain the global performances  $\tilde{\mu}(a)$ ,  $a$  in  $A$ , where

$$\tilde{\mu}(a) = [\tilde{\omega}_1 \otimes \tilde{\mu}_1(a)] \oplus \dots \oplus [\tilde{\omega}_I \otimes \tilde{\mu}_I(a)] \quad (1.1)$$

In (1.1),  $\oplus$  and  $\otimes$  represent respectively the fuzzy summation and product operators :

$$\tilde{\mu}'_i(a) = \tilde{\omega}_i \otimes \mu_i(a) \text{ with } \mu'_i(a,x) = \bigvee_u [\omega(i,x) \wedge \mu_i(a,x/u)]$$

$$\tilde{\mu}'_{i+k}(a) = \tilde{\mu}'_i(a) \oplus \tilde{\mu}'_j(a) \text{ with } \mu'_{i+k}(a,x) = \bigvee_u [\mu'_i(a,x) \wedge \mu'_k(a,x-u)]$$

Taking the global performances related to all alternatives in  $A$  a degree of preference of  $a$  over  $b$  can be defined for each pair  $(a,b)$  in  $A$  :

$$R(a,b) = \bigvee_{x \succ y} [\mu(a,x) \wedge \mu(b,y)] \quad (1.2)$$

The degree of preference of  $a$  over  $b$ ,  $R(a,b)$  can be seen as the induced fuzzy ordering

$$R(a,b) = \bigvee_{x,y} [\mu(a,x) \wedge L(x,y) \wedge \mu(b,y)]$$

where  $L(x,y)$  is the characteristic function of the natural ordering  $>$  on real numbers (see Ovchinnikov [10]).

This procedure is called the "pooling then pairing" approach.

(A2) : transform the elements of the vector of individual performances into a matrix of degrees of preference for each criterion  $i$  :

$$\{R_i(a,b), a \text{ and } b \text{ in } A\}, i=1, \dots, I,$$

$$\text{where } R_i(a,b) = \bigvee_{x>y} [\mu_i(a,x) \wedge \mu_i(b,y)] \quad (1.3)$$

The elements of the matrices  $\{R_i(a,b)\}$  are aggregated using the convex combination to give a fuzzy degree of preference of  $a$  over  $b$

$$\tilde{R}(a,b) = [R_1(a,b) \otimes \tilde{\omega}_1] \oplus \dots \oplus [R_I(a,b) \otimes \tilde{\omega}_I] \quad (1.4)$$

This procedure is called the "pairing then pooling" approach.

In Section 2, we compare the "pairing then pooling" and the "pooling then pairing" approaches.

In Section 3, different ways to obtain a subset of best actions among the set of alternatives are proposed.

In Section 4, we analyse the question of "ordering" the alternatives.

All the propositions are applied an one example defined at the end of Section 2.

## 2. COMPARAISON OF THE "POOLING THEN PAIRING" AND "PAIRING THEN POOLING" APPROACHES

2.1. Buckley introduces the "pooling then pairing" approach in [4] considering that  $\tilde{\mu}_i(a)$  and  $\tilde{\omega}_i$  are trapezoidal fuzzy numbers.  $\tilde{\mu}_i(a)$  is defined with parameters  $(m_i^-(a), m_i^+(a), \sigma_i^-(a), \sigma_i^+(a))$  if the membership function  $\mu_i(a, x)$  corresponds to

$$\mu_i(a, x) = \begin{cases} 1 - \frac{m_i^-(a) - x}{\sigma_i^-(a)} & \text{if } m_i^-(a) - \sigma_i^-(a) \leq x \leq m_i^-(a) \\ 1 & \text{if } m_i^-(a) \leq x \leq m_i^+(a) \\ 1 - \frac{x - m_i^+(a)}{\sigma_i^+(a)} & \text{if } m_i^+(a) \leq x \leq m_i^+(a) + \sigma_i^+(a) \\ 0 & \text{otherwise} \end{cases}$$

$\tilde{\omega}_i$  is described by  $(w_i^-, w_i^+, w_i^-, \sigma_i^-, \sigma_i^+)$  in the same way.

Buckley showed that  $\tilde{\mu}(a)$  corresponding to relation (1.1) is a flat fuzzy number with parameters  $(m^-(a), m^+(a), \sigma^-(a), \sigma^+(a))$  and with left and right restrictions of  $\tilde{\mu}(a)$  to  $(-\infty, m^-(a)]$  and to  $[m^+(a), \infty)$  being second order curves.

If we assume that  $\tilde{\omega}_i$  are non-fuzzy numbers  $w_i$  and that the membership functions  $\mu_i(a, x)$  present a trapezoidal shape, we easily obtain for  $\tilde{\mu}(a)$  in (1.1) a flat trapezoidal fuzzy number with parameters  $(m^-(a), m^+(a), \sigma^-(a), \sigma^+(a))$  :

$$(\sum_i w_i m_i^-(a), \sum_i w_i m_i^+(a), \sum_i w_i \sigma_i^-(a), \sum_i w_i \sigma_i^+(a)) \quad (2.1)$$



Baas and Kwakernaak [1] consider that the evaluations  $\tilde{\mu}_i(a)$  and weights  $\tilde{w}_i$  correspond to linguistic variables and propose an aggregation procedure different from (1.1). They do not pair the aggregated evaluations  $\tilde{\mu}(a)$  and propose to extend the ordering of real numbers to fuzzy numbers with the use of an index which defines a nontransitive fuzzy relation.

The comparison of fuzzy subsets of the real line was extensively studied by many authors during the past years (for a good survey of these methods, see Bortolan and Degani [3]).

2.2. Siskos and Hubert [19] and Siskos, Lochard and Lombard [20] propose the use of the "pairing then pooling" approach.

Starting with  $\tilde{\mu}_i(a)$ , an individual degree of preference of a over b is calculated according to relation (1.3).

Using non-fuzzy weights  $w_i$ , the authors consider the weighted summation

$$C(a,b) = \sum_i w_i R_i(a,b) , \sum_i w_i = 1 \quad (2.2)$$

called "concordance relation" as in the ELECTRE methods [18].

Once more, if  $\tilde{\mu}_i(a)$  are restricted to trapezoidal fuzzy numbers, we obtain

$$R_i(a,b) = \begin{cases} 0 & \text{if } Q_i(a,b) \leq 0 \\ 1 & \text{if } Q_i(a,b) \leq 1 \\ Q_i(a,b) & \text{otherwise} \end{cases} \quad (2.3)$$

$$\text{where } Q_i(a,b) = 1 + \frac{m_i^+(a) - m_i^-(b)}{\sigma_i^+(a) + \sigma_i^-(b)}$$

The comparison of formula (1.2) and (1.4) shows that both pathes illustrated in Fig.1 give different results, even if  $\tilde{w}_i$  correspond to crisp (non-fuzzy) values.

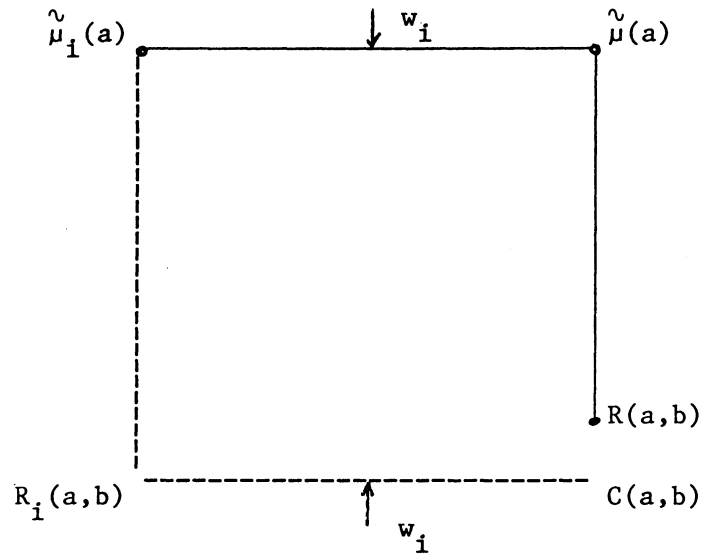


Fig. 1.

Let us consider the following example =  $A=\{a,b,c\}$ ,  $I=2$  with the crisp weights  $w_1, w_2 (w_1+w_2=1)$  and the input data :

	$m^-$	$m^+$	$\sigma^-$	$\sigma^+$
$\tilde{\mu}_1(a)$	5	5	1	1
$\tilde{\mu}_2(a)$	4	4	0	2
$\tilde{\mu}_1(b)$	6	8	2	0
$\tilde{\mu}_2(b)$	6	8	0	0
$\tilde{\mu}_1(c)$	3	3	0	2
$\tilde{\mu}_2(c)$	9	9	3	0

These figures might be considered as the translation of the

following judgements : for expert 1, a is approximately equal to 5, b is between 6 and 8 and c is at least equal to 3,...

With the use of relations (2.1) and (1.2) we obtain for procedure (A.1) :

	$m^-$	$m^+$	$\sigma^-$	$\sigma^+$
$\tilde{\mu}(a)$	$w_1+4$	$w_1+4$	$w_1$	$2-w_1$
$\tilde{\mu}(b)$	6	8	$2w_1$	0
$\tilde{\mu}(c)$	$9-6w_1$	$9-6w_1$	$3-3w_1$	$2w_1$

$R(.,.)$	a	b	c
a	1	$\frac{2w_1}{w_1+2}$	$\begin{cases} \frac{3w_1}{5-4w_1} & \text{if } w_1 \leq \frac{5}{7} \\ 1 & \text{otherwise} \end{cases}$
b	1	1	$\begin{cases} \frac{3w_1+2}{3-3w_1} & \text{if } w_1 \leq \frac{1}{6} \\ 1 & \text{otherwise} \end{cases}$
c	$\begin{cases} \frac{5-4w_1}{3w_1} & \text{if } w_1 > \frac{5}{7} \\ 1 & \text{otherwise} \end{cases}$	$\begin{cases} \frac{3-2w_1}{4w_1} & \text{if } w_1 > \frac{1}{2} \\ 1 & \text{otherwise} \end{cases}$	1

The application of formula (2.3) and (2.2) gives for procedure (A.2) :

$R_1(.,.)$	a	b	c
a	1	$\frac{2}{3}$	1
b	1	1	1
c	$\frac{1}{3}$	$\frac{1}{4}$	1

$R_2(.,.)$	a	b	c
a	1	1	0
b	1	1	$\frac{2}{3}$
c	1	1	1

$C(.,.)$	a	b	c
a	1	$\frac{3-w_1}{3}$	$w_1$
b	1	1	$\frac{w_1+2}{3}$
c	$\frac{3-2w_1}{3}$	$\frac{4-3w_1}{4}$	1

If we consider the "pairing then pooling" approach with trapezoidal fuzzy numbers  $\omega_i : (w_i^-, w_i^+, \sigma_i^-, \sigma_i^+)$ , formula (1.4) gives trapezoidal fuzzy degrees of preference

$$\hat{R}(a,b) : (\sum_i w_i^- R_i(a,b), \sum_i w_i^+ R_i(a,b), \sum_i \sigma_i^- R_i(a,b), \sum_i \sigma_i^+ R_i(a,b))$$

The treatment of such fuzzy degrees constitutes an open problem.

On the other end, if  $\tilde{\mu}_i(a)$ , for all  $a$  in  $A$ , are restricted to crisp evaluations  $g_i(a)$  and if  $\tilde{\omega}_i$  are crisp weights  $w_i$ ,

- the "pooling then pairing" approach gives

$$\tilde{\mu}(a) = g(a) = \sum_i w_i g_i(a) \quad \text{and}$$

$$R(a,b) = \begin{cases} 1 & \text{if } g(a) \geq g(b) \\ 0 & \text{otherwise} \end{cases}$$

which corresponds to the matricial representation of a complete preorder (see Roubens and Vincke [15]).

- the "pairing then pooling" approach gives

$$R_i(a,b) = \begin{cases} 1 & \text{if } g_i(a) \geq g_i(b) \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$C(a,b) = \sum_i w_i$$

$$i \text{ such that } g_i(a) \geq g_i(b)$$

which corresponds to the "Condorcet" procedure.

### 3. SUBSET OF "BEST" ACTIONS AMONG A SET OF ALTERNATIVES

The purpose of this analysis is to provide the best decision alternative or a selection of a small number of nondominated alternatives when a complete ranking of the set of alternatives is not needed.

3.1. Let us first reconsider the "pooling then pairing" approach, the input data being given by  $\{\mu_i(a,x), \omega_i(x), i=1, \dots, I, a \text{ in } A, x \text{ in } R\}$ . One output consists in the degree of preference  $R$  given by (1.2). In order to find the subset of best actions, Orlovsky [9] introduced the concept of fuzzy dominance.

Starting with a fuzzy relation  $R$ , Orlovsky considers a fuzzy strict preference relation  $P$ ,

$$P = 0 \vee (R - R^{-1}) \quad (3.1)$$

where  $R^{-1}$  is the inverse relation of  $R$ .

$P$  is, according to the crisp corresponding properties for a strict preference relation, irreflexive ( $P(a,a)=0$ , for all  $a$  in  $A$ ) and antisymmetric ( $P \wedge P^{-1}=0$ ).

Orlovsky then defined a *choice function* related to each alternative  $a$  in  $A$ :

$$C_{ND}(a) = 1 - \bigvee_{b \in A} P(b,a) = 1 - \bigvee_{b \in A} \{R(b,a) - R(a,b)\} \quad (3.2)$$

where  $C_{ND}(a)$  represents the degree to which the element  $a$  is strictly dominated by no one of the elements of the set  $A$ .

Orlovsky finally proved that under the conditions of reflexivity ( $R(a,a)=1$  for all  $a$  in  $A$ ) and transitivity ( $R(a,b) \geq R(a,c) \wedge R(c,b)$ , for all  $a,b,c$  in  $A$ ) of relation  $R$ , the subset of "unfuzzily undominated elements" (UND) is not empty, i.e.

$$UND(A) = \{a \mid C_{ND}(a) = 1\} \neq \emptyset$$

Let us apply these results to the fuzzy binary relation  $R$  given by (1.2).

From results given in Roubens and Vincke [17], we know that if  $\tilde{\mu}(a)$  are convex fuzzy numbers (i.e. they have convex  $\alpha$ -cuts),  $R(a,b) = \bigvee_{x \succcurlyeq y} [\mu(a,x) \wedge \mu(b,y)]$  presents the following properties : for all  $a,b,c,d$  in  $A$ ,

$$R(a,b) \wedge R(b,a) = \bigvee_x [\mu(a,x) \wedge \mu(b,x)] = \text{hgt}(\tilde{\mu}(a) \cap \tilde{\mu}(b))$$

$$R(a,a) = 1 \quad (\text{reflexivity})$$

$$R(a,b) \vee R(b,a) = 1 \quad (\text{completeness})$$

$$R(a,b) \wedge R(c,d) \leq R(a,d) \vee R(c,b) \quad (\text{Ferrers property})$$

which immediately implies that negative transitivity is satisfied ( $R(a,b) \leq R(a,c) \vee R(c,b)$ ).

$$\begin{aligned} R \text{ being complete, } P(a,b) &= 0 \vee (R(a,b) - R(b,a)) \\ &= 1 - R(b,a) \\ &= R^d(a,b) \end{aligned}$$

where  $R^d$  in the dual relation for  $R$ .

It is then straightforward that the degree of strict preference  $P$  presents the Ferrers property and is transitive. Following the proof given in Orlovsky [9] it is easily seen that the subset of unfuzzily undominated elements is non empty.

Let us reconsider the example given in Section 2.

We immediately obtain ( $P=R^d$ ) :

$P(.,.)$	a	b	c
a	0	0	$\begin{cases} 0 & \text{if } w_1 \leq \frac{5}{7} \\ \frac{7w_1-5}{3w_1} & \text{otherwise} \end{cases}$
b	$\frac{2-w_1}{2+w_1}$	0	$\begin{cases} 0 & \text{if } w_1 \leq \frac{1}{2} \\ \frac{6w_1-3}{4w_1} & \text{otherwise} \end{cases}$
c	$\begin{cases} 0 & \text{if } w_1 > \frac{5}{7} \\ \frac{5-7w_1}{5-w_1} & \text{otherwise} \end{cases}$	$\begin{cases} 0 & \text{if } w_1 > \frac{1}{6} \\ \frac{1-6w_1}{3-3w_1} & \text{otherwise} \end{cases}$	0

$$C_{ND}(a) = 1 - \bigvee_{b \in A} P(b,a) = \bigwedge_{b \in A} R(a,b), \text{ for all } a \text{ in } A.$$

$$C_{ND}(a) = \begin{cases} \frac{3w_1}{5-4w_1} & \text{if } w_1 \leq \frac{4}{11} \\ \frac{2w_1}{5-4w_1} & \text{otherwise} \end{cases}$$

$$C_{ND}(b) = \begin{cases} \frac{3w_1+2}{3-3w_1} & \text{if } w_1 \leq \frac{1}{6} \\ 1 & \text{otherwise} \end{cases}$$

$$C_{ND}(c) = \begin{cases} 1 & \text{if } w_1 \leq \frac{1}{2} \\ \frac{3-2w_1}{4w_1} & \text{otherwise} \end{cases}$$



$$\text{Finally, } \text{UND}(A) = \begin{cases} \{c\} & \text{if } w_1 < \frac{1}{6}, \\ \{b,c\} & \text{if } \frac{1}{6} \leq w_1 \leq \frac{1}{2}, \\ \{b\} & \text{if } w_1 > \frac{1}{2}. \end{cases} \quad (3.3)$$

Ovchinnikov [10] has strengthened the results of Roubens and Vincke [17] by introducing the following induced fuzzy ordering  $S$  :

$$S(a,b) = \bigvee_{x \geq y} T[\mu(a,x), \mu(b,y)] \quad (3.4)$$

where the min-operator is replaced in (1.2) by the t-norm  $T$ .

The induced fuzzy ordering  $S$  presents the following properties :  
for all  $a,b,c,d$  in  $A$ ,

$$S(a,b) \wedge S(b,a) = \bigvee_x T[\mu(a,x), \mu(b,x)]$$

$$S(a,a) = 1 \quad (\text{reflexivity})$$

$$S(a,b) \vee S(b,a) = 1 \quad (\text{completeness})$$

$$T[S(a,b), S(c,d)] \leq T^*[S(a,d), S(c,b)] \quad (\text{T-Ferrers})$$

where  $T^*$  is the t-conorm related to  $T$  :  $T^*(u,v) = 1 - T(1-u, 1-v)$ .

The dual of  $S$  is antisymmetric and  $T$ -transitive ( $S^d(a,b) \geq T[S^d(a,c), S^d(c,b)]$ , for all  $a,b,c$  in  $A$ ).

3.2. Let us now consider the "pairing then pooling" approach. In this case, if  $C(a,b)$  is defined with the relation (2.2), every  $R_i, i=1, \dots, I$ , is a reflexive, complete and Ferrers relation but no particular property- except reflexivity- can be evoked for  $C$ .

The same remark holds for the procedures developed in Blin [2], Kacprzyk [7], Nurmi [8] which all start with the input matrices  $\{R_i(a,b)\}$ ;  $0 \leq R_i(a,b) \leq 1$  are such that the higher  $R_i(a,b)$  the higher the preference of individual  $i$  of  $a$  over  $b$ .

Tanino [22] deals with the same type of relations and carefully studies different types of transitivity and aggregation procedures.

In order to cover these general cases, Ovchinnikov and Roubens [12] reexamined all the definitions given by Orlovsky [9].

According to them, a valued relation  $P$  is said to be a strict preference relation if it satisfies the following conditions :

(c1) : for any given  $a$  and  $b$  in  $A$ ,  $P(a,b)$  depends only on the values of the binary relations  $C(a,b)$  and  $C(b,a)$ . Therefore there exists a function  $f : [0,1] \rightarrow [0,1]$  such that

$$P(a,b) = f[C(a,b), C(b,a)], \text{ for all } a, b \text{ in } A \quad (3.5)$$

(c2) :  $f$  in (3.5) is a nondecreasing function with respect to the first argument and a nonincreasing function with respect to the second argument,

(c3) :  $f(y,z) > 0$  implies  $y > z$  for all  $y, z$  in  $[0,1]$  (3.6)

There are many ways to define a strict preference satisfying (c1), (c2) and (c3). Let us take three examples :

$$\text{Orlovsky [9] : } P(a,b) = \max[0, C(a,b) - C(b,a)]$$

$$\text{Ovchinnikov [11]} : P(a,b) = \begin{cases} C(a,b), & \text{if } C(a,b) > C(b,a) \\ 0 & \text{otherwise} \end{cases}$$

Roubens [14] : Let T be a t-norm satisfying condition :

$y+z \leq 1$  implies  $T(y,z)=0$ , for all  $y,z$  in  $[0,1]$ .

Then  $P(a,b)=T[C(a,b),1-C(b,a)]=T[C(a,b),C^d(a,b)]$

Ovchinnikov and Roubens [12] have proved the following statements :

Proposition 3.1.

P satisfying (c1) and (c2) is antisymmetric if and only if (c3) holds.

Proposition 3.2.

A valued strict preference relation P associated with a transitive valued relation C is a transitive valued relation.

The question of determining the best choices by pairwise comparison can be solved with the introduction of a choice function  $C_{ND}$  defined in Section 3.1 :  $C_{ND}(a) = \bigwedge_{b \in A/\{a\}} [1-P(b,a)]$ , for all a in A.

We now prove the following result :

Proposition 3.3.

If C is a transitive binary relation and if P satisfies (c1), (c2) and (c3),  $UND(a) = \{a | C_{ND}(a) = 1\}$  is non empty.

Proof

C transitive implies that P is transitive (proposition 3.2). Let us consider the unfuzzy relation  $\hat{P}(a,b) = \begin{cases} 1 & \text{if } P(a,b) > 0 \\ 0 & \text{otherwise} \end{cases}$

$\hat{P}$  is transitive :  $P(a,b)=1$  and  $\hat{P}(b,c)=1$  implies that  $P(a,b) > 0$ ,  $P(b,c) > 0$  and  $P(a,c) > 0$  or  $\hat{P}(a,c)=1$ , for all  $a,b,c$  in  $A$ .

$\hat{P}$  is asymmetric :  $\hat{P}(a,b)=1$  implies  $\hat{P}(b,a)=0$ .

The graph  $G(A, \hat{P})$ , where  $A$  is the set of nodes and when  $\hat{P}(a,b)=1$  indicates that the arc  $(a,b)$  exists, clearly contains no circuit. There exists some element  $a^*$  such that  $\hat{P}(b, a^*)=0$  for any  $b$  in  $A$ .

It immediately follows that  $P(b, a^*)=0$ , for any  $b$  in  $A$  and  $b \neq a$ .

$$C_{ND}(a^*) = \bigwedge_{b \in A / \{a^*\}} [1 - P(b, a^*)] = 1$$

It has also been proved in Roubens [14] that :

Proposition 3.4.

If  $P(a,b) = T[C(a,b), C^d(a,b)]$  and if  $P$  is acyclic (for any sequence  $a_0, \dots, a_n$ ,  $P(a_i, a_{i+1}) > 0$  for all  $0 \leq i \leq n-1$ , implies  $P(a_n, a_0) = 0$ ) then  $UND(A)$  is non empty.

Once more we reconsider the example given in Section 2 and we define

$$P = T[C, C^d] \text{ with } T(u,v) = \max[u+v-1, 0]$$

$$P(a,b) = \max(0, C(a,b) - C(b,a))$$

$P(.,.)$	a	b	a
a	0	0	$\begin{cases} 0 & \text{if } w_1 \leq \frac{3}{5} \\ \frac{5w_1-3}{3} & \text{otherwise} \end{cases}$
b	$\frac{w_1}{3}$	0	$\begin{cases} 0 & \text{if } w_1 \leq \frac{4}{13} \\ \frac{13w_1-4}{12} & \text{otherwise} \end{cases}$
c	$\begin{cases} 0 & \text{if } w_1 > \frac{3}{5} \\ \frac{3-5w_1}{3} & \text{otherwise} \end{cases}$	$\begin{cases} 0 & \text{if } w_1 > \frac{4}{13} \\ \frac{4-13w_1}{12} & \text{otherwise} \end{cases}$	0

If  $C_{ND}(a) = \bigwedge_{b \in A} [1 - P(b, a)]$ , for all  $a$  in  $A$ ,

$$C_{ND}(a) = \begin{cases} \frac{5w_1}{3} & \text{if } w_1 \leq \frac{1}{2} \\ \frac{3-w_1}{3} & \text{if } w_1 \geq \frac{1}{2} \end{cases}$$

$$C_{ND}(b) = \begin{cases} \frac{8+13w_1}{12} & \text{if } w_1 \leq \frac{4}{13} \\ 1 & \text{otherwise} \end{cases}$$

$$C_{ND}(c) = \begin{cases} 1 & \text{if } w_1 \leq \frac{4}{13} \\ \frac{16-13w_1}{12} & \text{otherwise} \end{cases}$$

$$\text{Finally, } \text{UND}(A) = \begin{cases} \{c\} & \text{if } w_1 < \frac{4}{13} \\ \{b,c\} & \text{if } w_1 = \frac{4}{13} \\ \{b\} & \text{if } w_1 > \frac{4}{13} \end{cases} \quad (3.7)$$

The fact that UND is not empty derives from the acyclicity of P. Comparison of results (3.3) and (3.7) clearly shows that the "pooling then pairing" and the "pairing then pooling" approaches give different answers.

## 4. PREFERENCE STRUCTURE (P,I) ON THE SET A

The problem of ordering the elements of A is clearly related to the comparison of fuzzy numbers.

4.1. In the "pooling then pairing" procedure, we can restrict ourselves at the first step. We obtain, with relation (1.1),  $\mu(a,x)$ , for every  $a$  in A.

We have shown in [13] that one possible answer consists in constructing the (.5)- level set  $I^{.5}(a)$  related to  $\tilde{\mu}(a)$ , and considering the unfuzzy preference  $\hat{R}^{.5}$  :

$$\left[ \begin{array}{l} a\hat{R}^{.5}b, \text{ iff } I^{.5}(a) > I^{.5}(b) \text{ or } I^{.5}(a) \cap I^{.5}(b) \neq \emptyset \\ \text{not } a\hat{R}^{.5}b, \text{ otherwise.} \end{array} \right.$$

where  $I^{.5}(a) > I^{.5}(b)$  iff  $x > y$ , for all  $x$  in  $I^{.5}(a)$  and all  $y$  in  $I^{.5}(b)$ .

$\hat{R}^{.5}$  presents a total interval order structure and this structure can be interpreted as follows :

$$\left[ \begin{array}{l} aPb \text{ iff } a\hat{R}^{.5}b \text{ and not } b\hat{R}^{.5}a \\ aIb \text{ iff } a\hat{R}^{.5}b \text{ and } b\hat{R}^{.5}a \end{array} \right.$$

or in an equivalent form,

$$\left[ \begin{array}{l} aPb \text{ iff } R(a,b) = 1 \text{ and } R(b,a) \leq .5 \\ aIb \text{ iff } R(a,b) > .5 \text{ or } R(b,a) > .5 \end{array} \right.$$

where R corresponds to (1.2).

$\hat{R}^{.5}$  minimizes the Hamming distance between the fuzzy binary relation S and all possible unfuzzy binary relation  $\hat{T}$  on A, i.e.

$$\hat{R}^{.5} \text{ minimizes } \sum_{a,b \in A} |R(a,b) - \hat{T}(a,b)|$$

where  $\hat{T}(a,b) = 1$  if  $a\hat{T}b$

= 0 otherwise.

If we assume that  $\tilde{\omega}_i$  are unfuzzy numbers  $w_i$  and that the membership functions  $\mu_i(a,x)$  present a trapezoidal shape with parameters  $(m_i^-(a), m_i^+(a), \sigma_i^-(a), \sigma_i^+(a))$ ,

$$\left[ \begin{array}{l} aPb \text{ iff } \sum_i w_i [m_i^-(a) - \frac{1}{2} \sigma_i^-(a)] \geq \sum_i w_i [m_i^+(b) + \frac{1}{2} \sigma_i^+(b)] \\ bPa \text{ iff } \sum_i w_i [m_i^-(b) - \frac{1}{2} \sigma_i^-(b)] \geq \sum_i w_i [m_i^+(a) + \frac{1}{2} \sigma_i^+(a)] \\ aIb \text{ otherwise.} \end{array} \right.$$

4.2. We now consider both procedures (A1) and (A2) when  $\tilde{\omega}_i$  are unfuzzy numbers  $w_i$  ( $\sum_i w_i = 1$ ) and where formula (3.2) is used.

We first consider  $A_1 = \{a^{**} | C_{ND}(A)(a^{**}) \text{ is max}\}$  and we rediscover results (3.3) and (3.7). We define iteratively

$$A_{k+1} = \{a^{**} | C_{ND}(A \setminus \bigcup_{j=1}^k A_j)(a^{**}) \text{ is max}\}, k=1,2,\dots$$

and we obtain the subsets  $A_1, A_2, A_3, \dots$ , such that  $A = \bigcup_k A_k$ , giving the following crisp total preorder

$$A_1 > A_2 > A_3 > \dots$$

with the elements in each  $A_k, k=1,2,\dots$ , being taken as indifferent.



A solution close to this procedure was proposed in Blin [2], Tong and Bonissone [23], Dubois and Prade [6] and Buckley [4].

For the example from Section 2, we have, for procedure (A.1) :

$$I^{.5}(a) = [.5w_1+4, .5w_1+5]$$

$$I^{.5}(b) = [6-w_1, 8]$$

$$I^{.5}(c) = [7.5-4.5w_1, 9-5w_1]$$

It derives that

$$\text{if } w_1 < \frac{1}{2}, \quad cPa, bPa, bIc \text{ or } (b \approx c > a) \quad (4.1)$$

$$\frac{1}{2} < w_1 < \frac{2}{3}, \quad cPa, bPa, bIc \text{ or } (b \approx c, c \approx a, b > a)$$

$$\frac{2}{3} < w_1 < \frac{3}{4}, \quad bIa, cIa, bIc \text{ or } (a \approx b \approx c)$$

$$\frac{3}{4} < w_1 < \frac{10}{11}, \quad bIa, cIa, bPc \text{ or } (b \approx a, a \approx c, b > c)$$

$$\frac{10}{11} < w_1 < 1, \quad bIa, aPc, bPc \text{ or } (a \approx b > c)$$

If we reconsider procedure (A1) with  $w_1 < \frac{1}{6}$

$$\{a^{**} | C_{ND}(A)(a^{**}) \text{ is max}\} = \{c\} \text{ and}$$

$$C_{ND}(A \setminus \{c\})(a) = \frac{2w_1}{2+w_1},$$

$$C_{ND}(A \setminus \{c\})(b) = 1,$$

we obtain the complete order :  $\{c\} > \{b\} > \{a\}$  (4.2)

In a similar way,

$$\text{if } \frac{1}{6} \leq w_1 \leq \frac{1}{2}, \{b, c\} > \{a\} \quad (4.2\text{bis})$$

$$\frac{1}{2} < w_1 < \frac{5}{7}, \{b\} > \{c\} > \{a\}$$

$$w_1 = \frac{5}{7}, \{b\} > \{c, a\}$$

$$\frac{5}{7} < w_1 \leq 1, \{b\} > \{a\} > \{c\}$$

For procedure (A.2) :

with  $P(a, b) = \max(0, C(a, b) - C(b, a))$  and  $C_{ND}(a) = \min_{b \in A} [1 - P(b, a)]$

$$\text{if } w_1 = 0 : \{c\} > \{a, b\} \quad (4.3)$$

$$0 < w_1 < \frac{4}{13} : \{c\} > \{b\} > \{a\}$$

$$w_1 = \frac{4}{13} : \{b, c\} > \{a\}$$

$$\frac{4}{13} < w_1 < \frac{3}{5} : \{b\} > \{c\} > \{a\}$$

$$w_1 = \frac{3}{5} : \{b\} > \{a, c\}$$

$$\frac{3}{5} < w_1 \leq 1 : \{b\} > \{a\} > \{c\}$$

Results (4.2), (4.2bis) and (4.3) are quite similar but are rather different from (4.1).

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