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# Extension of the product of a post-Lie algebra and application to the SISO feedback transformation group

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## Abstract

We describe the both post- and pre-Lie algebra  $\mathfrak{g}_{SISO}$  associated to the affine SISO feedback transformation group. We show that it is a member of a family of post-Lie algebras associated to representations of a particular solvable Lie algebra. We first construct the extension of the magmatic product of a post-Lie algebra to its enveloping algebra, which allows to describe free post-Lie algebras and is widely used to obtain the enveloping of  $\mathfrak{g}_{SISO}$  and its dual.

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*Keywords.* Post-Lie algebras; feedback transformation group; solvable Lie algebras.

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## Introduction

The affine SISO feedback transformation group  $G_{SISO}$  [4], which appears in Control Theory, can be seen as the character group of a Hopf algebra  $\mathcal{H}_{SISO}$ ; let us start by a short presentation of this object (we slightly modify the notations of [4]).

1. First, let us recall some algebraic structures on noncommutative polynomials.
  - (a) Let  $x_1, x_2$  be two indeterminates. We consider the algebra of noncommutative polynomials  $\mathbb{K}\langle x_1, x_2 \rangle$ . As a vector space, it is generated by words in letters  $x_1, x_2$ ; its product is the concatenation of words; its unit, the empty word, is denoted by  $\emptyset$ .
  - (b)  $\mathbb{K}\langle x_1, x_2 \rangle$  is a Hopf algebra with the concatenation product and the deshuffling coproduct  $\Delta_\Delta$ , defined by  $\Delta_\Delta(x_i) = x_i \otimes \emptyset + \emptyset \otimes x_i$ , for  $i \in \{1, 2\}$ .
  - (c)  $\mathbb{K}\langle x_1, x_2 \rangle$  is also a commutative, associative algebra with the shuffle product  $\Delta$ : for example, if  $i, j, k, l \in \{1, 2\}$ ,

$$\begin{aligned} x_i \Delta x_j &= x_i x_j + x_j x_i, \\ x_i x_j \Delta x_k &= x_i x_j x_k + x_i x_k x_j + x_k x_i x_j, \\ x_i \Delta x_j x_k &= x_i x_j x_k + x_j x_i x_k + x_j x_k x_i, \\ x_i x_j \Delta x_k x_l &= x_i x_j x_k x_l + x_i x_k x_j x_l + x_i x_k x_l x_j + x_k x_i x_j x_l + x_k x_i x_l x_j + x_k x_l x_i x_j. \end{aligned}$$

2. The vector space  $\mathbb{K}\langle x_1, x_2 \rangle^2$  is generated by words  $x_{i_1} \dots x_{i_k} \epsilon_j$ , where  $k \geq 0$ ,  $i_1, \dots, i_k, j \in \{1, 2\}$ , and  $(\epsilon_1, \epsilon_2)$  denotes the canonical basis of  $\mathbb{K}^2$ .
3. As an algebra,  $\mathcal{H}_{SISO}$  is equal to the symmetric algebra  $S(\mathbb{K}\langle x_1, x_2 \rangle^2)$ ; its product is denoted by  $\mu$  and its unit by 1. Two coproducts  $\overline{\Delta}_*$  and  $\overline{\Delta}_\bullet$  are defined on  $\mathcal{H}_{SISO}$ . For all  $h \in \mathcal{H}_{SISO}$ , we put  $\overline{\Delta}_*(h) = \Delta_*(h) - 1 \otimes h$  and  $\overline{\Delta}_\bullet(h) = \Delta_\bullet(h) - 1 \otimes h$ . Then:
  - For all  $i \in \{1, 2\}$ ,  $\overline{\Delta}_*(\emptyset \epsilon_i) = \emptyset \epsilon_i \otimes 1$ .
  - For all  $g \in \mathbb{K}\langle x_1, x_2 \rangle$ , for all  $i \in \{1, 2\}$ :

$$\begin{aligned} \overline{\Delta}_* \circ \theta_{x_1}(g \epsilon_i) &= (\theta_{x_1} \otimes Id) \circ \overline{\Delta}_*(g \epsilon_i) + (\theta_{x_2} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_\Delta(g) \epsilon_i \otimes \epsilon_2), \\ \overline{\Delta}_* \circ \theta_{x_2}(g \epsilon_i) &= (\theta_{x_2} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_\Delta(g) \epsilon_i \otimes \epsilon_1), \end{aligned}$$

where  $\theta_x(h \epsilon_i) = x h \epsilon_i$  for all  $x \in \{x_1, x_2\}$ ,  $h \in \mathbb{K}\langle x_1, x_2 \rangle$ ,  $i \in \{1, 2\}$ . These are formulas of Lemma 4.1 of [4], with the notations  $a_w = w \epsilon_2$ ,  $b_w = w \epsilon_1$ ,  $\theta_0 = \theta_{x_1}$ ,  $\theta_1 = \theta_{x_2}$  and  $\tilde{\Delta} = \overline{\Delta}_*$ .

- for all  $g \in \mathbb{K}\langle x_1, x_2 \rangle$ :

$$\begin{aligned} \overline{\Delta}_\bullet(g \epsilon_1) &= (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_\Delta(g)(\epsilon_1 \otimes \epsilon_1)), \\ \overline{\Delta}_\bullet(g \epsilon_2) &= \overline{\Delta}_*(g \epsilon_2) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_\Delta(g)(\epsilon_2 \otimes \epsilon_1)). \end{aligned}$$

This coproduct  $\overline{\Delta}_\bullet$  makes  $\mathcal{H}_{SISO}$  a Hopf algebra, and  $\overline{\Delta}_*$  is a right coaction on this coproduct, that is to say:

$$(\overline{\Delta}_\bullet \otimes Id) \circ \overline{\Delta}_\bullet = (Id \otimes \overline{\Delta}_\bullet) \circ \overline{\Delta}_\bullet, \quad (\overline{\Delta}_* \otimes Id) \circ \overline{\Delta}_* = (Id \otimes \overline{\Delta}_\bullet) \circ \overline{\Delta}_*.$$

4. After the identification of  $\emptyset \epsilon_1$  with the unit of  $\mathcal{H}_{SISO}$ , we obtain a commutative, graded and connected Hopf algebra, in other words the dual of an enveloping algebra  $\mathcal{U}(\mathfrak{g}_{SISO})$ .

Our aim is to give a description of the underlying Lie algebra  $\mathfrak{g}_{SISO}$ . It turns out that it is both a pre-Lie algebra (or a Vinberg algebra [1], see [5] for a survey on these objects) and a post-Lie

algebra [6, 10]: it has a Lie bracket  ${}_a[-, -]$  and two nonassociative products  $*$  and  $\bullet$ , such that for all  $x, y, z \in \mathfrak{g}_{SISO}$ :

$$\begin{aligned} x * {}_a[y, z] &= (x * y) * z - x * (y * z) - (x * z) * y + x * (z * y), \\ {}_a[x, y] * z &= {}_a[x * z, y] + {}_a[x, y * z]; \end{aligned}$$

$$(x \bullet y) \bullet z - x \bullet (y \bullet z) = (x \bullet z) \bullet y - x \bullet (z \bullet y).$$

The Lie bracket on  $\mathfrak{g}_{SISO}$  corresponding to  $G_{SISO}$  is  ${}_a[-, -]_*$ :

$$\forall x, y \in \mathfrak{g}_{SISO}, \quad {}_a[x, y]_* = {}_a[x, y] + x * y - y * x = x \bullet y - y \bullet x.$$

Let us be more precise on these structures. As a vector space,  $\mathfrak{g}_{SISO} = \mathbb{K}\langle x_1, x_2 \rangle^2$ , and:

$$\forall f, g \in \mathbb{K}\langle x_1, x_2 \rangle, \quad \forall i, j \in \{1, 2\}, \quad {}_a[f\epsilon_i, g\epsilon_j] = \begin{cases} 0 & \text{if } i = j, \\ -f \Delta g\epsilon_2 & \text{if } i = 2 \text{ and } j = 1, \\ f \Delta g\epsilon_2 & \text{if } i = 1 \text{ and } j = 2. \end{cases}$$

The magmatic product  $*$  is inductively defined. If  $f, g \in \mathbb{K}\langle x_1, x_2 \rangle$  and  $i, j \in \{1, 2\}$ :

$$\begin{aligned} \emptyset\epsilon_i * g\epsilon_j &= 0, & x_2 f\epsilon_i * g\epsilon_1 &= x_2(f\epsilon_i * g\epsilon_1) + x_2(f \Delta g)\epsilon_i, \\ x_1 f\epsilon_i * g\epsilon_j &= x_1(f\epsilon_i * g\epsilon_j), & x_2 f\epsilon_i * g\epsilon_2 &= x_2(f\epsilon_i * g\epsilon_2) + x_1(f \Delta g)\epsilon_i. \end{aligned}$$

The pre-Lie product  $\bullet$ , first determined in [4], is given by:

$$\forall f, g \in \mathbb{K}\langle x_1, x_2 \rangle, \quad \forall i, j \in \{1, 2\}, \quad f\epsilon_i \bullet g\epsilon_j = (f \Delta g)\delta_{i,1}\epsilon_j + f\epsilon_i * g\epsilon_j.$$

We shall show here that this is a special case of a family of post-Lie algebras, associated to modules over certain solvable Lie algebras.

We start with general preliminary results on post-Lie algebras. We extend the now classical Oudom-Guin construction on prelie algebras [7, 8] to the post-Lie context in the first section: this is a result of [2] (Proposition 3.1), which we prove here in a different, less direct way; our proof allows also to obtain a description of free post-Lie algebras. Recall that if  $(V, *)$  is a pre-Lie algebra, the pre-Lie product  $*$  can be extended to  $S(V)$  in such a way that the product defined by:

$$\forall f, g \in S(V), \quad f \circledast g = \sum f * g^{(1)}g^{(2)}$$

is associative, and makes  $S(V)$  a Hopf algebra, isomorphic to  $\mathcal{U}(V)$ . For any magmatic algebra  $(V, *)$ , we construct in a similar way an extension of  $*$  to  $T(V)$  in Proposition 1. We prove in Theorem 1 that the product  $\circledast$  defined by:

$$\forall f, g \in T(V), \quad f \circledast g = \sum f * g^{(1)}g^{(2)}$$

makes  $T(V)$  a Hopf algebra. The Lie algebra of its primitive elements, which is the free Lie algebra  $\mathcal{L}ie(V)$  generated by  $V$ , is stable under  $*$  and turns out to be a post-Lie algebra (Proposition 2) satisfying a universal property (Theorem 2). In particular, if  $V$  is, as a magmatic algebra, freely generated by a subspace  $W$ ,  $\mathcal{L}ie(V)$  is the free post-Lie algebra generated by  $W$  (Corollary 1). Moreover, if  $V = ([-, -], *)$  is a post-Lie algebra, this construction goes through the quotient defining  $\mathcal{U}(V, [-, -])$ , defining a new product  $\circledast$  on it, making it isomorphic to the enveloping algebra of  $V$  with the Lie bracket defined by:

$$\forall x, y \in V, \quad [x, y]_* = [x, y] + x * y - y * x.$$

For example, if  $x_1, x_2, x_3 \in V$ :

$$\begin{aligned} x_1 \circledast x_2 &= x_1 x_2 + x_1 * x_2 \\ x_1 \circledast x_2 x_3 &= x_1 x_2 x_3 + (x_1 * x_2) x_3 + (x_1 * x_3) x_2 + (x_1 * x_2) * x_3 - x_1 * (x_2 * x_3) \\ x_1 x_2 \circledast x_3 &= x_1 x_2 x_3 + (x_1 * x_3) x_2 + x_1 (x_2 * x_3). \end{aligned}$$

In the particular case where  $[-, -] = 0$ , we recover the Oudom-Guin construction.

The second section is devoted to the study of a particular solvable Lie algebra  $\mathfrak{g}_a$  associated to an element  $a \in \mathbb{K}^N$ . As the Lie bracket of  $\mathfrak{g}_a$  comes from an associative product, the construction of the first section holds, with many simplifications: we obtain an explicit description of  $\mathcal{U}(\mathfrak{g}_a)$  with the help of a product  $\blacktriangleleft$  on  $S(\mathfrak{g}_a)$  (Proposition 6). A short study of  $\mathfrak{g}_a$ -modules when  $a = (1, 0, \dots, 0)$  (which is a generic case) is done in Proposition 8, considering  $\mathfrak{g}_a$  as an associative algebra, and in Proposition 9, considering it as a Lie algebra. In particular, if  $\mathbb{K}$  is algebraically closed, any  $\mathfrak{g}_a$  modules inherits a natural decomposition in characteristic subspaces.

Our family of post-Lie algebras is introduced in the third section; it is reminiscent of the construction of [3]. Let us fix a vector space  $V$ ,  $(a_1, \dots, a_N) \in \mathbb{K}^N$  and a family  $F_1, \dots, F_N$  of endomorphisms of  $V$ . We define a product  $*$  on  $T(V)^N$ , such that for all  $f, g \in T(V)$ ,  $x \in V$ ,  $i, j \in \{1, \dots, N\}$ :

$$\begin{aligned} \emptyset \epsilon_i * g \epsilon_j &= 0, \\ x f \epsilon_i * g \epsilon_j &= x (f \epsilon_i * g \epsilon_j) + F_j(x) (f \Delta g) \epsilon_i, \end{aligned}$$

where  $(\epsilon_1, \dots, \epsilon_N)$  is the canonical basis of  $\mathbb{K}^N$  and  $\Delta$  is the shuffle product of  $T(V)$ . The Lie bracket of  $T(V)^N$  that we shall use here is:

$$\forall f, g \in T(V), \forall i, j \in \{1, \dots, N\}, {}_a [f \epsilon_i, g \epsilon_j] = (f \Delta g) (a_i \epsilon_j - a_j \epsilon_i).$$

This Lie bracket comes from an associative product  ${}_a \Delta$  defined by:

$$\forall f, g \in T(V), \forall i, j \in \{1, \dots, N\}, f \epsilon_i {}_a \Delta g \epsilon_j = a_i (f \Delta g) \epsilon_j.$$

We put  $\bullet = * + {}_a \Delta$ . We prove in Theorem 3 the equivalence of the three following conditions:

- $(T(V)^N, \bullet)$  is a pre-Lie algebra.
- $(T(V)^N, {}_a [-, -], *)$  is a post-Lie algebra.
- $F_1, \dots, F_N$  defines a structure of  $\mathfrak{g}_a$ -module on  $V$ .

If this holds, the construction of the first section allows to obtain two descriptions of the enveloping algebra of  $\mathcal{U}(T(V)^N)$ , respectively coming from the post-Lie product  $*$  and from the pre-Lie product  $\bullet$ : the extensions of  $*$  and of  $\bullet$  are respectively described in Propositions 15 and 16. It is shown in Proposition 17 that the two associated descriptions of  $\mathcal{U}(T(V)^N)$  are equal. For  $\mathfrak{g}_{SISO}$ , we take  $a = (1, 0)$ ,  $V = Vect(x_1, x_2)$  and:

$$F_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which indeed define a  $\mathfrak{g}_{(1,0)}$ -module. In order to relate this to the Hopf algebra  $\mathcal{H}_{SISO}$  of [4], we need to consider the dual of the enveloping of  $T(V)^N$ . First, if  $a = (1, 0, \dots, 0)$ , we observe that the decomposition of  $V$  as a  $\mathfrak{g}_a$ -module of the second section induces a graduation of the post-Lie algebra  $T(V)^N$  (Proposition 18), unfortunately not connected: the component of degree 0 is 1-dimensional, generated by  $\emptyset \epsilon_1$ . Forgetting this element, that is, considering the augmentation ideal of the graded post-Lie algebra  $T(V)^N$ , we can dualize the product  $\circledast$  of  $S(T(V)^N)$  in order to obtain the coproduct of the dual Hopf algebra in an inductive way. For  $\mathfrak{g}_{SISO}$ , we indeed obtain the inductive formulas of  $\mathcal{H}_{SISO}$ , finally proving that the dual Lie algebra of this

Hopf algebra, which in some sense can be exponentiated to  $G_{SISO}$ , is indeed post-Lie and pre-Lie.

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### Notations.

1. Let  $\mathbb{K}$  be a commutative field. The canonical basis of  $\mathbb{K}^n$  is denoted by  $(\epsilon_1, \dots, \epsilon_n)$ .
2. For all  $n \geq 1$ , we denote by  $[n]$  the set  $\{1, \dots, n\}$ .
3. We shall use Sweeder's notations: if  $C$  is a coalgebra and  $x \in C$ ,

$$\begin{aligned}\Delta^{(1)}(x) &= \Delta(x) = \sum x^{(1)} \otimes x^{(2)}, \\ \Delta^{(2)}(x) &= (\Delta \otimes Id) \circ \Delta(x) = \sum x^{(1)} \otimes x^{(2)} \otimes x^{(3)}, \\ \Delta^{(3)}(x) &= (\Delta \otimes Id \otimes Id) \circ (\Delta \otimes Id) \circ \Delta(x) = \sum x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)}, \\ &\vdots\end{aligned}$$

## 1 Extension of a post-Lie product

We first generalize the Oudom-Guin extension of a pre-Lie product in a post-Lie algebraic context, as done in [2]. Let us first recall what a post-Lie algebra is.

**Definition 1** 1. A (right) post-Lie algebra is a family  $(\mathfrak{g}, \{-, -\}, *)$ , where  $\mathfrak{g}$  is a vector space,  $\{-, -\}$  and  $*$  are bilinear products on  $\mathfrak{g}$  such that:

- $(\mathfrak{g}, \{-, -\})$  is a Lie algebra.
- For all  $x, y, z \in \mathfrak{g}$ :

$$x * \{y, z\} = (x * y) * z - x * (y * z) - (x * z) * y + x * (z * y), \quad (1)$$

$$\{x, y\} * z = \{x * z, y\} + \{x, y * z\}. \quad (2)$$

2. If  $(\mathfrak{g}, \{-, -\}, *)$  is post-Lie, we define a second Lie bracket on  $\mathfrak{g}$ :

$$\forall x, y \in \mathfrak{g}, \{x, y\}_* = \{x, y\} + x * y - y * x.$$

Note that if  $\{-, -\}$  is 0, then  $(\mathfrak{g}, *)$  is a (right) pre-Lie algebra, that is to say:

$$\forall x, y, z \in \mathfrak{g}, (x * y) * z - x * (y * z) = (x * z) * y - x * (z * y). \quad (3)$$

### 1.1 Extension of a magmatic product

Let  $V$  be a vector space. We here use the tensor Hopf algebra  $T(V)$ . In this section, we shall denote the unit of  $T(V)$  by 1. Its product is the concatenation of words, and its coproduct  $\Delta_\Delta$  is the cocommutative deshuffling coproduct. For example, if  $x_1, x_2, x_3 \in V$ :

$$\begin{aligned}\Delta_\Delta(x_1) &= x_1 \otimes 1 + 1 \otimes x_1, \\ \Delta_\Delta(x_1 x_2) &= x_1 x_2 \otimes 1 + x_1 \otimes x_2 + x_2 \otimes x_1 + 1 \otimes x_1 x_2, \\ \Delta_\Delta(x_1 x_2 x_3) &= x_1 x_2 x_3 \otimes 1 + x_1 x_2 \otimes x_3 + x_1 x_3 \otimes x_2 + x_2 x_3 \otimes x_1 \\ &\quad + x_1 \otimes x_2 x_3 + x_2 \otimes x_1 x_3 + x_3 \otimes x_1 x_2 + 1 \otimes x_1 x_2 x_3.\end{aligned}$$

Its counit is denoted by  $\varepsilon$ :  $\varepsilon(1) = 1$  and if  $k \geq 1$  and  $x_1, \dots, x_k \in V$ ,  $\varepsilon(x_1 \dots x_k) = 0$ .

**Proposition 1** Let  $V$  be a vector space and  $*$  :  $V \otimes V \rightarrow V$  be a magmatic product on  $V$ . Then  $*$  can be uniquely extended as a map from  $T(V) \otimes T(V)$  to  $T(V)$  such that for all  $f, g, h \in T(V)$ ,  $x, y \in V$ :

- $f * 1 = f$ .
- $1 * f = \varepsilon(f)1$ .
- $x * (fy) = (x * f) * y - x * (f * y)$ .
- $(fg) * h = \sum (f * h^{(1)}) (g * h^{(2)})$ .

**Proof.** *Existence.* We first inductively extend  $*$  from  $V \otimes T(V)$  to  $V$ . If  $n \geq 0$ ,  $x, y_1, \dots, y_n \in V$ , we put:

$$x * y_1 \dots y_n = \begin{cases} x & \text{if } n = 0, \\ x * y_1 & \text{if } n = 1, \\ \underbrace{\underbrace{(x * (y_1 \dots y_{n-1}))}_{\in V} * \underbrace{y_n}_{\in V}}_{\in V} - \sum_{i=1}^{n-1} \underbrace{x * (y_1 \dots \underbrace{(y_i * y_n)}_{\in V} \dots y_{n-1})}_{\in V} & \text{if } n \geq 2. \end{cases}$$

This product is then extended from  $T(V) \otimes T(V)$  to  $T(V)$  in the following way:

- For all  $f \in T(V)$ ,  $1 * f = \varepsilon(f)1$ .
- For all  $n \geq 1$ , for all  $x_1, \dots, x_n \in V$ ,  $f \in T(V)$ :

$$(x_1 \dots x_n) * f = \sum \underbrace{(x_1 * f^{(1)})}_{\in V} \dots \underbrace{(x_n * f^{(n)})}_{\in V} \in V^{\otimes n}.$$

Note that for all  $n \geq 0$ ,  $V^{\otimes n} * T(V) \subseteq V^{\otimes n}$ , which induces the second point. Let us prove the first point with  $f = x_1 \dots x_n \in V^{\otimes n}$ . If  $n = 0$ ,  $f * 1 = 1 * 1 = \varepsilon(1)1 = 1 = f$ . If  $n = 1$ ,  $f \in V$ , so  $f * 1 = f$  by definition of the extension of  $*$  on  $V \otimes T(V)$ . If  $n \geq 2$ :

$$f * 1 = (x_1 \dots x_n) * 1 = (x_1 * 1) \dots (x_n * 1) = x_1 \dots x_n = f.$$

Let us prove the third point for  $f = y_1 \dots y_n$ . Then:

$$x * (fy) = (x * f) * y - \sum x * (y_1 \dots (y_i * y) \dots y_n).$$

Moreover, as  $\Delta\Delta(y) = y \otimes 1 + 1 \otimes y$ :

$$f * y = \sum_{i=1}^n (y_1 * 1) \dots (y_i * y) \dots (y_n * 1) = \sum_{i=1}^n y_1 \dots (y_i * y) \dots y_n.$$

So  $x * (fy) = (x * f) * y - x * (f * y)$ . Let us finally prove the last point for  $f = x_1 \dots x_k$  and  $g = x_{k+1} \dots x_{k+l}$ . Then:

$$\begin{aligned} (fg) * h &= \sum \left( x_1 * h^{(1)} \right) \dots \left( x_{k+l} * h^{(k+l)} \right) \\ &= \sum \left( x_1 * (h^{(1)})^{(1)} \right) \dots \left( x_1 * (h^{(1)})^{(k)} \right) \left( x_{k+1} * (h^{(2)})^{(1)} \right) \dots \left( x_{k+l} * (h^{(2)})^{(l)} \right) \\ &= \sum \left( (x_1 \dots x_k) * h^{(1)} \right) \left( (x_{k+1} \dots x_{k+l}) * h^{(2)} \right) \\ &= \sum \left( f * h^{(1)} \right) \left( g * h^{(2)} \right). \end{aligned}$$

We used the coassociativity of  $\Delta_\Delta$  for the second equality.

*Unicity.* The first and third points uniquely determine  $x * (x_1 \dots x_n)$  for  $x, x_1, \dots, x_n \in V$ , by induction on  $n$ ; the second and fourth points then uniquely determine  $f * (x_1 \dots x_n)$  for all  $f \in T(V)$  by induction on the length of  $f$ .  $\square$

**Examples.** If  $x_1, x_2, x_3, x_4 \in V$ :

$$\begin{aligned} (x_1 x_2) * x_3 &= (x_1 * x_3) x_2 + x_1 (x_2 * x_3), \\ x_1 * (x_2 x_3) &= (x_1 * x_2) * x_3 - x_1 * (x_2 * x_3), \\ (x_1 x_2 x_3) * x_4 &= (x_1 * x_4) x_2 x_3 + x_1 (x_2 * x_4) x_3 + x_1 x_2 (x_3 * x_4), \\ (x_1 x_2) * (x_3 x_4) &= ((x_1 * x_3) * x_4) x_2 - (x_1 * (x_3 * x_4)) x_2 + x_1 ((x_2 * x_3) * x_4), \\ &\quad - x_1 (x_2 * (x_3 * x_4)) + (x_1 * x_3) (x_2 * x_4) + (x_1 * x_4) (x_2 * x_3), \\ x_1 * (x_2 x_3 x_4) &= ((x_1 * x_2) * x_3) * x_4 - (x_1 * (x_2 * x_3)) * x_4 - (x_1 * (x_2 * x_4)) * x_3 \\ &\quad + x_1 * ((x_2 * x_4) * x_3) - (x_1 * x_2) * (x_3 * x_4) + x_1 * (x_2 * (x_3 * x_4)). \end{aligned}$$

**Lemma 1** 1. For all  $k \in \mathbb{N}$ ,  $V^{\otimes k} * T(V) \subseteq V^{\otimes k}$ .

2. For all  $f, g \in T(V)$ ,  $\varepsilon(f * g) = \varepsilon(f) \varepsilon(g)$ .
3. For all  $f, g \in T(V)$ ,  $\Delta_\Delta(f * g) = \Delta_\Delta(f) * \Delta_\Delta(g)$ .
4. For all  $f, g \in T(V)$ ,  $y \in V$ ,  $f * (gy) = (f * g) * y - f * (g * y)$ .
5. For all  $f, g, h \in T(V)$ ,  $(f * g) * h = \sum f * ((g * h^{(1)}) h^{(2)})$ .

**Proof.** 1. This was observed in the proof of Proposition 1.

2. From the first point,  $\text{Ker}(\varepsilon) * T(V) + T(V) * \text{Ker}(\varepsilon) \subseteq \text{Ker}(\varepsilon)$ , so if  $\varepsilon(f) = 0$  or  $\varepsilon(g) = 0$ , then  $\varepsilon(f * g) = 0$ . As  $\varepsilon(1 * 1) = 1$ , the second point holds for all  $f, g$ .

3. We prove it for  $f = x_1 \dots x_n$ , by induction on  $n$ . If  $n = 0$ , then  $f = 1$ . Moreover,  $\Delta_\Delta(1 * g) = \varepsilon(g) \Delta_\Delta(1) = \varepsilon(g) 1 \otimes 1$ , and:

$$\Delta_\Delta(f) * \Delta_\Delta(g) = \sum 1 * g^{(1)} \otimes 1 * g^{(2)} = \varepsilon(g^{(1)}) \varepsilon(g^{(2)}) 1 \otimes 1 = \varepsilon(g) 1 \otimes 1.$$

If  $n = 1$ , then  $f \in V$ . In this case, from the second point,  $f * g \in V$ , so  $\Delta_\Delta(f * g) = f * g \otimes 1 + 1 \otimes f * g$ . Moreover:

$$\begin{aligned} \Delta_\Delta(f) * \Delta_\Delta(g) &= (f \otimes 1 + 1 \otimes f) * \Delta_\Delta(g) \\ &= \sum f * g^{(1)} \otimes 1 * g^{(2)} + \sum 1 * g^{(1)} \otimes f * g^{(2)} \\ &= \sum f * g^{(1)} \otimes \varepsilon(g^{(2)}) 1 + \sum \varepsilon(g^{(1)}) 1 \otimes f * g^{(2)} \\ &= f * g \otimes 1 + 1 \otimes f * g. \end{aligned}$$

If  $n \geq 2$ , we put  $f_1 = x_1 \dots x_{n-1}$  and  $f_2 = x_n$ . By the induction hypothesis applied to  $f_1$ :

$$\begin{aligned} \Delta_\Delta(f * g) &= \sum \Delta_\Delta\left(\left(f_1 * g^{(1)}\right) \left(f_2 * g^{(2)}\right)\right) \\ &= \Delta_\Delta\left(f_1 * g^{(1)}\right) \Delta_\Delta\left(f_2 * g^{(2)}\right) \\ &= \sum \left(f_1^{(1)} * (g^{(1)})^{(1)}\right) \left(f_2^{(1)} * (g^{(2)})^{(1)}\right) \otimes \left(f_1^{(2)} * (g^{(1)})^{(2)}\right) \left(f_2^{(2)} * (g^{(2)})^{(2)}\right) \\ &= \sum (f_1 f_2)^{(1)} * g^{(1)} \otimes (f_1 f_2)^{(2)} * g^{(2)} \\ &= \Delta_\Delta(f) * \Delta_\Delta(g). \end{aligned}$$



We used the cocommutativity of  $\Delta\Delta$  for the fourth equality.

4. We prove it for  $f = x_1 \dots x_n$ , by induction on  $n$ . If  $n = 0$ , then  $f = 1$  and:

$$1 * (gy) = (1 * g) * y - 1 * (g * y) = \varepsilon(g)\varepsilon(y) - \varepsilon(g * y) = 0.$$

For  $n = 1$ , this comes immediately from Proposition 1-3. If  $n \geq 2$ , we put  $f_1 = x_1 \dots x_{n-1}$  and  $f_2 = x_n$ . The induction hypothesis holds for  $f_1$ . Moreover:

$$\begin{aligned} f * (gy) &= \sum (f_1 * g^{(1)}) (f_2 * (g^{(2)}y)) + \sum (f_1 * (g^{(1)}y)) (f_2 * g^{(2)}) \\ &= \sum (f_1 * g^{(1)}) ((f_2 * g^{(2)}) * y) - \sum (f_1 * g^{(1)}) (f_2 * (g^{(2)} * y)) \\ &\quad + \sum ((f_1 * g^{(1)}) * y) (f_2 * g^{(2)}) - \sum (f_1 * (g^{(1)} * y)) (f_2 * g^{(2)}), \\ (f * g) * y &= \sum ((f_1 * g^{(1)}) (f_2 * g^{(2)})) * y \\ &= \sum ((f_1 * g^{(1)}) * y) (f_2 * g^{(2)}) + \sum (f_1 * g^{(1)}) ((f_2 * g^{(2)}) * y), \\ f * (g * y) &= \sum (f_1 * (g * y)^{(1)}) (f_2 * (g * y)^{(2)}) \\ &= \sum (f_1 * (g^{(1)} * y)) (f_2 * g^{(2)}) + \sum (f_1 * g^{(1)}) (f_2 * (g^{(2)} * y)). \end{aligned}$$

We use the third point for the third computation. So the result holds for all  $f$ .

5. We prove this for  $h = z_1 \dots z_n$  and we proceed by induction on  $n$ . If  $n = 0$ , then  $h = 1$  and  $(f * g) * 1 = f * g$ . Moreover,  $\sum f * ((g * h^{(1)}) h^{(2)}) = f * ((g * 1)1) = (f * g)1 = f * g$ . If  $n = 1$ , then  $h \in V$ , so  $\Delta\Delta(h) = h \otimes 1 + 1 \otimes h$ . So:

$$\begin{aligned} \sum f * ((g * h^{(1)}) h^{(2)}) &= f * ((g * h)1) + f * ((g * 1)h) \\ &= f * (g * h) + f * (gh) \\ &= f * (g * h) + (f * g) * h - f * (g * h) \\ &= (f * g) * h. \end{aligned}$$

We use Proposition 1-3 for the third equality. If  $n \geq 2$ , we put  $h_1 = z_1 \dots z_{n-1}$  and  $h_2 = z_n$ . From the fourth point:

$$\begin{aligned} (f * g) * h &= ((f * g) * h_1) * h_2 - (f * g) * (h_1 * h_2) \\ &= \sum (f * ((g * h_1^{(1)}) h_1^{(2)})) * h_2 - \sum f * ((g * (h_1 * h_2)^{(1)}) (h_1 * h_2)^{(2)}) \\ &= \sum f * (((g * h_1^{(1)}) h_1^{(2)}) * h_2) + \sum f * ((g * h_1^{(1)}) h_1^{(2)} h_2) \\ &\quad - \sum f * ((g * (h_1^{(1)} * h_2^{(1)})) (h_1^{(2)} * h_2^{(2)})) \\ &= \sum f * (((g * h_1^{(1)}) * h_2) h_1^{(2)}) + \sum f * ((g * h_1^{(1)}) (h_1^{(2)} * h_2)) \\ &\quad + \sum f * ((g * h_1^{(1)}) h_1^{(2)} h_2) - \sum f * ((g * (h_1^{(1)} * h_2)) h_1^{(2)}) \\ &\quad - \sum f * ((g * h_1^{(1)}) (h_1^{(2)} * h_2)) \\ &= \sum f * ((g * (h_1^{(1)} * h_2)) h_1^{(2)}) + \sum f * ((g * (h_1^{(1)} h_2)) h_1^{(2)}) \\ &\quad + \sum f * ((g * h_1^{(1)}) (h_1^{(2)} * h_2)) + \sum f * ((g * h_1^{(1)}) h_1^{(2)} h_2) \\ &\quad - \sum f * ((g * (h_1^{(1)} * h_2)) h_1^{(2)}) - \sum f * ((g * h_1^{(1)}) (h_1^{(2)} * h_2)) \\ &= \sum f * ((g * (h_1^{(1)} h_2)) h_1^{(2)}) + \sum f * ((g * h_1^{(1)}) h_1^{(2)} h_2). \end{aligned}$$

For the second equality, we used the induction hypothesis on  $h_1$  and  $h_1 * h_2 \in V^{\otimes(k-1)}$  by the first point; we used the third point for the third equality. As  $\Delta_\Delta(h_2) = h_2 \otimes 1 + 1 \otimes h_2$ ,  $\Delta_\Delta(h) = \sum h_1^{(1)} h_2 \otimes h_1^{(2)} + \sum h_1^{(1)} \otimes h_1^{(2)} h_2$ , so the result holds for  $h$ .  $\square$

## 1.2 Associated Hopf algebra and post-Lie algebra

**Theorem 1** *Let  $*$  be a magmatic product on  $V$ . This product is extended to  $T(V)$  by Proposition 1. We define a product  $\otimes$  on  $T(V)$  by:*

$$\forall f, g \in T(V), f \otimes g = \sum \left( f * g^{(1)} \right) g^{(2)}.$$

Then  $(T(V), \otimes, \Delta_\Delta)$  is a Hopf algebra.

**Proof.** For all  $f \in T(V)$ :

$$1 \otimes f \sum \left( 1 * f^{(1)} \right) f^{(2)} = \sum \varepsilon \left( f^{(1)} \right) f^{(2)} = f; \quad \otimes 1 = (f * 1)1 = f.$$

For all  $f, g, h \in T(V)$ , by Lemma 1-5:

$$\begin{aligned} (f \otimes g) \otimes h &= \sum \left( \left( f * g^{(1)} \right) g^{(2)} \right) \otimes h \\ &= \sum \left( \left( \left( f * g^{(1)} \right) g^{(2)} \right) * h^{(1)} \right) h^{(2)} \\ &= \sum \left( \left( f * g^{(1)} \right) * h^{(1)} \right) \left( g^{(2)} * h^{(2)} \right) h^{(3)} \\ &= \sum \left( f * \left( \left( g^{(1)} * h^{(1)} \right) h^{(2)} \right) \right) \left( g^{(2)} * h^{(3)} \right) h^{(4)}; \\ f \otimes (g \otimes h) &= \sum f \otimes \left( \left( g * h^{(1)} \right) h^{(2)} \right) \\ &= \sum \left( f * \left( \left( g^{(1)} * h^{(1)} \right) h^{(3)} \right) \right) \left( g^{(2)} * h^{(2)} \right) h^{(4)}. \end{aligned}$$

As  $\Delta_\Delta$  is cocommutative,  $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ , so  $(T(V), \otimes)$  is a unitary, associative algebra.

For all  $f, g \in T(V)$ , by Lemma 1-3:

$$\begin{aligned} \Delta_\Delta(f \otimes g) &= \sum \Delta_\Delta \left( \left( f * g^{(1)} \right) g^{(2)} \right) \\ &= \sum \left( f^{(1)} * \left( g^{(1)} \right)^{(1)} \right) \left( g^{(2)} \right)^{(1)} \otimes \left( f^{(2)} * \left( g^{(1)} \right)^{(2)} \right) \left( g^{(2)} \right)^{(2)} \\ &= \sum \left( f^{(1)} * \left( g^{(1)} \right)^{(1)} \right) \left( g^{(1)} \right)^{(2)} \otimes \left( f^{(2)} * \left( g^{(2)} \right)^{(1)} \right) \left( g^{(2)} \right)^{(2)} \\ &= \sum f^{(1)} \otimes g^{(1)} \otimes f^{(2)} \otimes g^{(2)}. \end{aligned}$$

Note that we used the cocommutativity of  $\Delta_\Delta$  for the third equality. Hence,  $(T(V), \otimes, \Delta_\Delta)$  is a Hopf algebra.  $\square$

**Remark.** By Lemma 1:

- For all  $f, g, h \in T(V)$ ,  $(f * g) * h = f * (g \otimes h)$ :  $(T(V), *)$  is a right  $(T(V), \otimes)$ -module.
- By restriction, for all  $n \geq 0$ ,  $(V^{\otimes n}, *)$  is a right  $(T(V), \otimes)$ -module. Moreover, for all  $n \geq 0$ ,  $(V^{\otimes n}, *) = (V, *)^{\otimes n}$  as a right module over the Hopf algebra  $(T(V), \otimes, \Delta_\Delta)$ .

**Examples.** Let  $x_1, x_2, x_3 \in V$ .

$$\begin{aligned} x_1 \otimes x_2 &= x_1 x_2 + x_1 * x_2 \\ x_1 \otimes x_2 x_3 &= x_1 x_2 x_3 + (x_1 * x_2) x_3 + (x_1 * x_3) x_2 + (x_1 * x_2) * x_3 - x_1 * (x_2 * x_3) \\ x_1 x_2 \otimes x_3 &= x_1 x_2 x_3 + (x_1 * x_3) x_2 + x_1 (x_2 * x_3). \end{aligned}$$

The vector space of primitive elements of  $(T(V), \otimes, \Delta_\Delta)$  is  $\mathcal{L}ie(V)$ . Let us now describe the Lie bracket induced on  $\mathcal{L}ie(V)$  by  $\otimes$ .

**Proposition 2** 1. Let  $*$  be a magmatic product on  $V$ . The Hopf algebras  $(T(V), \otimes, \Delta_\Delta)$  and  $(T(V), \cdot, \Delta_\Delta)$  are isomorphic, via the following algebra morphism:

$$\phi_* : \begin{cases} (T(V), \cdot, \Delta_\Delta) & \longrightarrow & (T(V), \otimes, \Delta_\Delta) \\ x_1 \dots x_k \in V^{\otimes k} & \longrightarrow & x_1 \otimes \dots \otimes x_k. \end{cases}$$

2.  $\mathcal{L}ie(V) * T(V) \subseteq \mathcal{L}ie(V)$ . Moreover,  $(\mathcal{L}ie(V), [-, -], *)$  is a post-Lie algebra. The induced Lie bracket on  $\mathcal{L}ie(V)$  is denoted by  $\{-, -\}_*$ :

$$\forall f, g \in \mathcal{L}ie(V), \{f, g\}_* = [f, g] + f * g - g * f = fg - gf + f * g - g * f.$$

The Lie algebra  $(\mathcal{L}ie(V), \{-, -\}_*)$  is isomorphic to  $\mathcal{L}ie(V)$ .

**Proof.** 1. There exists a unique algebra morphism  $\phi_* : (T(V), \cdot) \longrightarrow (T(V), \otimes)$ , sending any  $x \in V$  on itself. As the elements of  $V$  are primitive in both Hopf algebras,  $\phi_*$  is a Hopf algebra morphism. As  $V^{\otimes k} * T(V) \subseteq V^{\otimes k}$  for all  $k \geq 0$ , we deduce that for all  $x_1, \dots, x_{k+l} \in V$ :

$$x_1 \dots x_k \otimes x_{k+1} \dots x_{k+l} = x_1 \dots x_{k+l} + \text{a sum of words of length } < k + l.$$

Hence, if  $x_1, \dots, x_k \in V$ :

$$\phi_*(x_1 \dots x_k) = x_1 \otimes \dots \otimes x_k = x_1 \dots x_k + \text{a sum of words of length } < k.$$

Consequently:

- If  $k \geq 0$  and  $x_1, \dots, x_k \in V$ , an induction on  $k$  proves that  $x_1 \dots x_k \in \phi_*(T(V))$ , so  $\phi_*$  is surjective.
- If  $f$  is a nonzero element of  $T(V)$ , let us write  $f = f_0 + \dots + f_k$ , with  $f_i \in V^{\otimes i}$  for all  $i$  and  $f_k \neq 0$ . Then:

$$\phi_*(f) = f_k + \text{terms in } \mathbb{K} \oplus \dots \oplus V^{\otimes(k-1)},$$

so  $\phi_*(f) \neq 0$ :  $\phi_*$  is injective.

Hence,  $\phi_*$  is an isomorphism.

2. We consider  $A = \{f \in \mathcal{L}ie(V) \mid f * T(V) \subseteq \mathcal{L}ie(V)\}$ . By Lemma 1-3,  $V \subseteq A$ . Let  $f, g \in A$ . For all  $h \in T(V)$ :

$$\begin{aligned} [f, g] * h &= (fg) * h - (gf) * h \\ &= \sum (f * h^{(1)}) (g * h^{(2)}) - \sum (g * h^{(1)}) (f * h^{(2)}) \\ &= \sum (f * h^{(1)}) (g * h^{(2)}) - \sum (g * h^{(2)}) (f * h^{(1)}) \\ &= \sum [f * h^{(1)}, g * h^{(2)}]. \end{aligned}$$

We used the cocommutativity for the third equality. By hypothesis,  $f * h^{(1)}, g * h^{(2)} \in \mathcal{L}ie(V)$ , so  $[f, g] \in A$ . As  $A$  is a Lie subalgebra of  $\mathcal{L}ie(V)$  containing  $V$ , it is equal to  $\mathcal{L}ie(V)$ .

Let  $f, g, h \in \mathcal{L}ie(V)$ . Then  $g \circledast h = \sum (g * h^{(1)}) h^{(2)} = gh + g * h$ . Similarly,  $\sum (h * g^{(1)}) g^{(2)} = hg + h * g$ , so, by Lemma 1-5:

$$\begin{aligned} f * [g, h] &= f * (gh) - f * (hg) \\ &= \sum f * \left( (g * h^{(1)}) h^{(2)} \right) - f * (g * h) - \sum f * \left( (h * g^{(1)}) g^{(2)} \right) + f * (h * g) \\ &= (f * g) * h - f * (g * h) - (f * h) * g + f * (g * h). \end{aligned}$$

Moreover:

$$\begin{aligned} [f, g] * h &= (fg) * h - (gf) * h \\ &= (f * h)g + f(g * h) - (g * h)f - g(f * h) \\ &= [f * h, g] + [f, g * h]. \end{aligned}$$

So  $\mathcal{L}ie(V)$  is a post-Lie algebra.

Consequently,  $\{-, -\}_*$  is a second Lie bracket on  $\mathcal{L}ie(V)$ . In  $(T(V), \circledast)$ , if  $f$  and  $g$  are primitive:

$$f \circledast g - g \circledast f = fg + f * g - gf - g * f = \{f, g\}_*.$$

So, by the Cartier-Quillen-Milnor-Moore's theorem,  $(T(V), \circledast, \Delta_\Delta)$  is the enveloping algebra of  $(\mathcal{L}ie(V), \{-, -\}_*)$ . As it is isomorphic to the enveloping algebra of  $\mathcal{L}ie(V)$ , namely  $(T(V), \cdot, \Delta_\Delta)$ , these two Lie algebras are isomorphic.  $\square$

Let us give a combinatorial description of  $\phi_*$ .

**Proposition 3** *Let  $(V, *)$  be a magmatic algebra, and  $x_1, \dots, x_k \in V$ .*

- *Let  $I = \{i_1, \dots, i_p\} \subseteq [k]$ , with  $i_1 < \dots < i_p$ . We put:*

$$x_I^* = (\dots ((x_{i_1} * x_{i_2}) * x_{i_3}) * \dots) * x_{i_p} \in V.$$

- *Let  $P$  be a partition of  $[p]$ . We denote it by  $P = \{P_1, \dots, P_p\}$ , with the convention  $\min(P_1) < \dots < \min(P_p)$ . We put:*

$$x_P^* = x_{P_1}^* \dots x_{P_p}^* \in V^{\otimes p}.$$

Then:

$$\phi^*(x_1 \dots x_k) = \sum_{P \text{ partition of } [k]} x_P^*.$$

**Proof.** By induction on  $k$ . As  $\phi_*(x) = x$  for all  $x \in V$ , it is obvious if  $k = 1$ . Let us assume the result at rank  $k$ .

$$\begin{aligned} \phi_*(x_1 \dots x_{k+1}) &= \phi_*(x_1 \dots x_k) \circledast x_{k+1} \\ &= \phi_*(x_1 \dots x_k)x_{k+1} + \phi_*(x_1 \dots x_k) * x_{k+1} \\ &= \sum_{P \text{ partition of } [k]} x_P^* x_{k+1} + \sum_{\substack{P = \{P_1, \dots, P_p\} \\ \text{partition of } [k]}} \sum_{i=1}^p x_{P_1}^* \dots (x_{P_i}^* * x_{k+1}) \dots x_{P_p}^* \\ &= \sum_{\substack{P = \{P_1, \dots, P_p\} \\ \text{partition of } [k]}} x_{\{P_1, \dots, P_p, \{k+1\}\}}^* + \sum_{\substack{P = \{P_1, \dots, P_p\} \\ \text{partition of } [k]}} \sum_{i=1}^p x_{\{P_1, \dots, P_i \cup \{k+1\}, \dots, P_p\}}^* \\ &= \sum_{P \text{ partition of } [k+1]} x_P^*. \end{aligned}$$

So the result holds for all  $k$ . □

**Examples.** Let  $x_1, x_2, x_3 \in V$ .

$$\begin{aligned}\phi_*(x_1) &= x_1, \\ \phi_*(x_1x_2) &= x_1x_2 + x_1 * x_2, \\ \phi_*(x_1x_2x_3) &= x_1x_2x_3 + (x_1 * x_2)x_3 + (x_1 * x_3)x_2 + x_1(x_2 * x_3) + (x_1 * x_2) * x_3.\end{aligned}$$

**Theorem 2** *Let  $(V, *)$  be a magmatic algebra and let  $(L, \{-, -\}, \star)$  be a post-Lie algebra. Let  $\phi : (V, *) \rightarrow (L, \star)$  be a morphism of magmatic algebras. There exists a unique morphism of post-Lie algebras  $\bar{\phi} : \mathcal{L}ie(V) \rightarrow L$  extending  $\phi$ .*

**Proof.** Let  $\psi : \mathcal{L}ie(V) \rightarrow L$  be the unique Lie algebra morphism extending  $\phi$ . Let us fix  $h \in \mathcal{L}ie(V)$ . We consider:

$$A_h = \{h \in \mathcal{L}ie(V) \mid \forall f \in \mathcal{L}ie(V), \psi(f * h) = \psi(f) \star \psi(h)\}.$$

If  $f, g \in A_h$ , then:

$$\begin{aligned}\psi([f, g] * h) &= \psi([f * h, g] + [f, g * h]) \\ &= \{\psi(f * h), \psi(g)\} + \{\psi(f), \psi(g * h)\} \\ &= \{\psi(f) \star \psi(h), \psi(g)\} + \{\psi(f), \psi(g) \star \psi(h)\} \\ &= \{\psi(f), \psi(g)\} \star \psi(h) \\ &= \psi([f, g]) \star \psi(h).\end{aligned}$$

So  $[f, g] \in A_h$ : for all  $h \in \mathcal{L}ie(V)$ ,  $A_h$  is a Lie subalgebra of  $\mathcal{L}ie(V)$ . Moreover, if  $h \in V$ , as  $\psi|_V = \phi$  is a morphism of magmatic algebras,  $V \subseteq A_h$ ; as a consequence, if  $h \in V$ ,  $A_h = \mathcal{L}ie(V)$ .

Let  $A = \{h \in \mathcal{L}ie(V) \mid A_h = \mathcal{L}ie(V)\}$ . We put  $\mathcal{L}ie(V)_n = \mathcal{L}ie(V) \cap V^{\otimes n}$ ; let us prove inductively that  $\mathcal{L}ie(V)_n \subseteq A$  for all  $n$ . We already proved that  $V \subseteq A$ , so this is true for  $n = 1$ . Let us assume the result at all rank  $k < n$ . Let  $h \in \mathcal{L}ie(V)_n$ . We can assume that  $h = [h_1, h_2]$ , with  $h_1 \in \mathcal{L}ie(V)_k$ ,  $h_2 \in \mathcal{L}ie(V)_{n-k}$ ,  $1 \leq k \leq n - 1$ . From Lemma 1 and Proposition 2,  $1f * h_2 \in \mathcal{L}ie(V)_k$  and  $h_2 * h_1 \in \mathcal{L}ie(V)_{n-k}$ , so the induction hypothesis holds for  $h_1, h_2, h_1 * h_2$  and  $h_2 * h_1$ . Hence, for all  $f \in T(V)$ :

$$\begin{aligned}\psi(f * h) &= \psi(f * [h_1, h_2]) \\ &= \psi((f * h_1) * h_2 - f * (h_1 * h_2) - (f * h_2) * h_1 + f * (h_2 * h_1)) \\ &= (\psi(f) \star \psi(h_1)) \star \psi(h_2) - \psi(f) \star (\psi(h_1) \star \psi(h_2)) \\ &\quad - (\psi(f) \star \psi(h_2)) \star \psi(h_1) + \psi(f) \star (\psi(h_2) \star \psi(h_1)) \\ &= \psi(f) \star \{\psi(h_1), \psi(h_2)\} \\ &= \psi(f) \star \psi(h).\end{aligned}$$

As a consequence,  $\mathcal{L}ie(V)_n \subseteq A$ . Finally,  $A = \mathcal{L}ie(V)$ , so for all  $f, g \in \mathcal{L}ie(V)$ ,  $\psi(f * g) = \psi(f) \star \psi(g)$ . □

**Corollary 1** *Let  $V$  be a vector space. The free magmatic algebra generated by  $V$  is denoted by  $\mathcal{M}ag(V)$ . Then  $\mathcal{L}ie(\mathcal{M}ag(V))$  is the free post-Lie algebra generated by  $V$ .*

**Proof.** Let  $L$  be a post-Lie algebra and let  $\phi$  be a linear map from  $V$  to  $L$ . From the universal property of  $\mathcal{M}ag(V)$ , there exists a unique morphism of magmatic algebras from  $\mathcal{M}ag(V)$  to  $L$  extending  $\phi$ ; from the universal property of  $\mathcal{L}ie(\mathcal{M}ag(V))$ , this morphism can be uniquely extended as a morphism of post-Lie algebras from  $\mathcal{L}ie(\mathcal{M}ag(V))$  to  $V$ . So  $\mathcal{L}ie(\mathcal{M}ag(V))$  satisfies the required universal property to be a post-Lie algebra generated by  $V$ . □

**Remark.** Describing the free magmatic algebra generated by  $V$  in terms of planar rooted trees with a grafting operation, we get back the construction of free post-Lie algebras of [6].

### 1.3 Enveloping algebra of a post-Lie algebra

Let  $(V, \{-, -\}, *)$  be a post-Lie algebra. We extend  $*$  onto  $T(V)$  as previously in Proposition 1. The usual bracket of  $\mathcal{L}ie(V) \subseteq T(V)$  is denoted by  $[f, g] = fg - gf$ , and should not be confused with the bracket  $\{-, -\}$  of the post-Lie algebra  $V$ .

**Lemma 2** *Let  $I$  be the two-sided ideal of  $T(V)$  generated by the elements  $xy - yx - \{x, y\}$ ,  $x, y \in V$ . Then  $I * T(V) \subseteq I$  and  $T(V) * I = (0)$ .*

**Proof.** *First step.* Let us prove that for all  $x, y \in V$ , for all  $h \in T(V)$ :

$$\{x, y\} * h = \sum \left\{ x * h^{(1)}, y * h^{(2)} \right\}.$$

Note that the second member of this formula makes sense, as  $V * T(V) \subseteq V$  by Lemma 1. We assume that  $h = z_1 \dots z_n$  and we work by induction on  $n$ . If  $n = 0$ , then  $h = 1$  and  $\{x, y\} * 1 = \{x, y\} = \{x * 1, y * 1\}$ . If  $n = 1$ , then  $h \in V$ , so  $\Delta_\Delta(h) = h \otimes 1 + 1 \otimes h$ .

$$\{x, y\} * h = \{x * h, y\} + \{x, y * h\} = \{x * h, y * 1\} + \{x * 1, y * h\} = \sum \{x * h^{(1)}, y * h^{(2)}\}.$$

If  $n \geq 2$ , we put  $h_1 = z_1 \dots z_{n-1}$  and  $h_2 = z_n$ . The induction hypothesis holds for  $h_1, h_2$  and  $h_1 * h_2$ :

$$\begin{aligned} \{x, y\} * h &= (\{x, y\} * h_1) * h_2 - \{x, y\} * (h_1 * h_2) \\ &= \sum \left\{ x * h_1^{(1)}, y * h_1^{(2)} \right\} * h_2 - \sum \left\{ x * (h_1 * h_2)^{(1)}, y * (h_1 * h_2)^{(2)} \right\} \\ &= \sum \left\{ (x * h_1^{(1)}) * h_2^{(1)}, (y * h_1^{(2)}) * h_2^{(2)} \right\} - \sum \left\{ x * (h_1^{(1)} * h_2^{(1)}), y * (h_1^{(2)} * h_2^{(2)}) \right\} \\ &= \sum \left\{ (x * h_1^{(1)}) * h_2, y * h_1^{(2)} \right\} + \sum \left\{ x * h_1^{(1)}, (y * h_1^{(2)}) * h_2 \right\} \\ &\quad - \sum \left\{ x * (h_1^{(1)} * h_2), y * h_1^{(2)} \right\} - \sum \left\{ x * h_1^{(1)}, y * (h_1^{(2)} * h_2) \right\} \\ &= \sum \left\{ (x * h_1^{(1)}) * h_2 - x * (h_1^{(1)} * h_2), y * h_1^{(2)} \right\} \\ &\quad + \sum \left\{ x * h_1^{(1)}, (y * h_1^{(2)}) * h_2 - y * (h_1^{(2)} * h_2) \right\} \\ &= \sum \left\{ x * (h_1^{(1)} h_2), y * h_1^{(2)} \right\} + \sum \left\{ x * h_1^{(1)}, y * (h_1^{(2)} h_2) \right\} \\ &= \sum \left\{ x * h^{(1)}, y * h^{(2)} \right\}. \end{aligned}$$

Consequently, the result holds for all  $h \in T(V)$ .

*Second step.* Let  $J = Vect(xy - yx - \{x, y\} \mid x, y \in V)$ . For all  $x, y \in V$ , for all  $h \in T(V)$ , by the first step:

$$(xy - yx - \{x, y\}) * h = \sum \left( x * h^{(1)} \right) \left( y * h^{(2)} \right) - \left( y * h^{(1)} \right) \left( y * h^{(2)} \right) - \left\{ x * h^{(1)}, y * h^{(2)} \right\} \in J.$$

So  $J * T(V) \subseteq J$ . If  $g \in J, f_1, f_2, h \in T(V)$ :

$$(f_1 g f_2) * h = \sum \left( f_1 * d^{(1)} \right) \underbrace{\left( g * h^{(2)} \right)}_{\in J} \left( f_2 * h^{(3)} \right) \in I.$$

So  $I * T(V) \subseteq I$ .

Let us prove that  $T(V) * (T(V)JV^{\otimes n}) = (0)$  for all  $n \geq 0$ . We start with  $n = 0$ . First,  $1 * (T(V)J) = \varepsilon(T(V)J) = (0)$ . Let  $x, y, z \in V, g \in T(V)$ . Then:

$$\begin{aligned}
& x * (gyz - gzy - g\{y, z\}) \\
&= (x * (gy)) * z - x * ((gy) * z) - (x * (gz)) * y + x * ((gz) * y) \\
&\quad - (x * g) * \{y, z\} + x * (g * \{y, z\}) \\
&= ((x * g) * y) * z - (x * (g * y)) * z - x * ((g * z) * y) \\
&\quad - x * (g(y * z)) - ((x * g) * z) * y - (x * (g * z)) * y \\
&\quad + x * ((g * y) * z) + x * (g(z * y)) - (x * g) * \{y, z\} + x * (g * \{y, z\}) \\
&= ((x * g) * y) * z - (x * (g * y)) * z - (x * (g * z)) * y + x * ((g * z) * y) \\
&\quad - (x * g) * (y * z) + x * (g * (y * z)) - ((x * g) * z) * y + (x * (g * z)) * y \\
&\quad (x * (g * y)) * z - x * ((g * y) * z) + (x * g) * (z * y) - x * (g * (z * y)) \\
&\quad - (x * g) * \{y, z\} + x * (g * \{y, z\}) \\
&= x * ((g * z) * y) + x * (g * (y * z)) - x * ((g * y) * z) - x * (g * (z * y)) + x * (g * \{y, z\}) \\
&\quad + ((x * g) * y) * z - (x * g) * (y * z) - ((x * g) * z) * y + (x * g) * (z * y) - (x * g) * \{y, z\} \\
&= 0 + 0.
\end{aligned}$$

So  $V * (T(V)J) = (0)$ . As the elements of  $J$  are primitive,  $T(V)J$  is a coideal. If  $n \geq 1$ ,  $x_1, \dots, x_n \in V$  and  $g \in T(V)J$ , we put  $\Delta_{\Delta}^{(n-1)}(g) = \sum g^{(1)} \otimes \dots \otimes g^{(n)}$ , with at least one  $g_i \in T(V)J$ . Then  $(x_1 \dots x_n) * g = \sum (x_1 * g^{(1)}) \dots (x_n * g^{(n)}) = 0$ , so  $T(V) * (T(V)J) = (0)$ .

If  $n \geq 1$ , we take  $f \in T(V)$ ,  $g \in T(V)JV^{\otimes(n-1)}$  and  $y \in V$ . We put  $g = g_1 g_2 g_3$ , with  $g_1 \in T(V)$ ,  $g_2 \in J$ ,  $g_3 \in V^{\otimes(n-1)}$ . Then:

$$g * y = (g_1 * y)g_2 g_3 + g_1 \underbrace{(g_2 * y)}_{\in J * T(V) \subseteq J} g_3 + g_1 g_2 \underbrace{(g_3 * y)}_{\in V^{\otimes n}} \in T(V)JV^{\otimes n}.$$

So the induction hypothesis holds for  $g$  and for  $g * y$ . Then  $f * (gy) = (f * g) * y - f * (g * y) = 0$ . So  $T(V) * I = (0)$ .  $\square$

As a consequence, the quotient  $T(V)/I$  inherits a magmatic product  $*$ . Moreover,  $I$  is a Hopf ideal, and this implies that it is also a two-sided ideal for  $\otimes$ . As  $T(V)/I$  is the enveloping algebra  $\mathcal{U}(V, \{-, -\})$ , we obtain Proposition 3.1 of [2]:

**Proposition 4** *Let  $(\mathfrak{g}, \{-, -\}, *)$  be a post-Lie algebra. Its magmatic product can be uniquely extended to  $\mathcal{U}(\mathfrak{g})$  such that for all  $f, g, h \in \mathcal{U}(\mathfrak{g})$ ,  $x, y \in \mathfrak{g}$ :*

- $f * 1 = f$ .
- $1 * f = \varepsilon(f)1$ .
- $f * (gy) = (f * g) * y - f * (g * y)$ .
- $(fg) * h = \sum (f * h^{(1)}) (g * h^{(2)})$ , where  $\Delta(h) = \sum h^{(1)} \otimes h^{(2)}$  is the usual coproduct of  $\mathcal{U}(\mathfrak{g})$ .

We define a product  $\otimes$  on  $\mathcal{U}(\mathfrak{g})$  by  $f * g = \sum (f * g^{(1)}) g^{(2)}$ . Then  $(\mathcal{U}(\mathfrak{g}), \otimes, \Delta)$  is a Hopf algebra, isomorphic to  $\mathcal{U}(\mathfrak{g}, \{-, -\}_*)$ .

**Proof.** By Cartier-Quillen-Milnor-Moore's theorem,  $(\mathcal{U}(\mathfrak{g}), \otimes, \Delta)$  is an enveloping algebra; the underlying Lie algebra is  $\text{Prim}(\mathcal{U}(\mathfrak{g})) = \mathfrak{g}$ , with the Lie bracket defined by:

$$\{x, y\}_{\otimes} = x \otimes y - y \otimes x = xy + x * y - yx - y * x.$$

This is the bracket  $\{-, -\}_*$ .  $\square$

**Remarks.**

1. If  $\mathfrak{g}$  is a post-Lie algebra with  $\{-, -\} = 0$ , it is a pre-Lie algebra, and  $\mathcal{U}(\mathfrak{g}) = S(\mathfrak{g})$ . We obtain again the Oudom-Guin construction [7, 8].
2. By Lemma 1,  $(\mathcal{U}(\mathfrak{g}), *)$  is a right  $(\mathcal{U}(\mathfrak{g}), \otimes)$ -module. By restriction,  $(\mathfrak{g}, *)$  is also a right  $(\mathcal{U}(\mathfrak{g}), \otimes)$ -module.

#### 1.4 The particular case of associative algebras

Let  $(V, \triangleleft)$  be an associative algebra. The associated Lie bracket is denoted by  $[-, -]_{\triangleleft}$ . As  $(V, 0, \triangleleft)$  is post-Lie, the construction of the enveloping algebra of  $(V, [-, -]_{\triangleleft})$  can be done: we obtain a product  $\triangleleft$  defined on  $S(V)$  and an associative product  $\blacktriangleleft$  making  $(S(V), \blacktriangleleft, \Delta)$  a Hopf algebra, isomorphic to the enveloping algebra of  $(V, [-, -]_{\triangleleft})$ .

**Lemma 3** *If  $x_1, \dots, x_k, y_1, \dots, y_l \in V$ :*

$$x_1 \dots x_k \triangleleft y_1 \dots y_l = \sum_{\theta: [l] \rightarrow [k]} \left( \prod_{i \notin \text{Im}(\theta)} x_i \right) \left( \prod_{i=1}^k x_{\theta(i)} \triangleleft y_i \right),$$

$$x_1 \dots x_k \blacktriangleleft y_1 \dots y_l = \sum_{I \subseteq [l]} \sum_{\theta: I \rightarrow [k]} \left( \prod_{i \notin \text{Im}(\theta)} x_i \right) \left( \prod_{j \notin I} y_j \right) \left( \prod_{i \in I} x_{\theta(i)} \triangleleft y_i \right).$$

**Proof.** We first prove that for all  $k \geq 2$ ,  $x, y_1, \dots, y_k \in V$ ,  $x \triangleleft y_1 \dots y_k = 0$ . We proceed by induction on  $k$ . For  $k = 2$ ,  $x \triangleleft y_1 y_2 = (x \triangleleft y_1) \triangleleft y_2 - x \triangleleft (y_1 \triangleleft y_2) = 0$ , as  $\triangleleft$  is associative. Let us assume the result at rank  $k$ . Then:

$$x \triangleleft y_1 \dots y_{k+1} = (x \triangleleft y_1 \dots y_k) \triangleleft y_{k+1} - \sum_{i=1}^k x \triangleleft (y_1 \dots (y_i \triangleleft y_{k+1}) \dots y_k) = 0.$$

Let us now prove the formula for  $\triangleleft$ .

$$x_1 \dots x_k \triangleleft y_1 \dots y_l = \sum_{[l] = I_1 \sqcup \dots \sqcup I_k} \left( x_1 \triangleleft \prod_{i \in I_1} y_i \right) \dots \left( x_k \triangleleft \prod_{i \in I_k} y_i \right).$$

Moreover, for all  $j$ :

$$x_j \triangleleft \prod_{i \in I_j} y_i = \begin{cases} x_j & \text{if } I_j = \emptyset, \\ x_j \triangleleft y_p & \text{if } I_j = \{p\}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence:

$$\begin{aligned} x_1 \dots x_k \triangleleft y_1 \dots y_l &= \sum_{\substack{[l] = I_1 \sqcup \dots \sqcup I_k \\ \forall p, |I_p| \leq 1}} \left( x_1 \triangleleft \prod_{i \in I_1} y_i \right) \dots \left( x_k \triangleleft \prod_{i \in I_k} y_i \right) \\ &= \sum_{\theta: [l] \rightarrow [k]} \left( \prod_{i \notin \text{Im}(\theta)} x_i \right) \left( \prod_{i=1}^k x_{\theta(i)} \triangleleft y_i \right). \end{aligned}$$

Finally:

$$x_1 \dots x_k \blacktriangleleft y_1 \dots y_l = \sum_{I \subseteq [l]} \left( \prod_{i \notin I} y_i \right) x_1 \dots x_k \triangleleft \left( \prod_{i \in I} y_i \right),$$

as announced. □



**Examples.** Let  $x_1, x_2, y_1, y_2 \in V$ .

$$\begin{aligned} x_1 \blacktriangleleft y_1 &= x_1 y_1 + x_1 \triangleleft y_1, \\ x_1 x_2 \blacktriangleleft y_1 &= x_1 x_2 y_1 + (x_1 \triangleleft y_1) x_2 + x_1 (x_2 \triangleleft y_1), \\ x_1 \blacktriangleleft y_1 y_2 &= x_1 y_1 y_2 + (x_1 \triangleleft y_1) y_2 + (x_1 \triangleleft y_2) y_1, \\ x_1 x_2 \blacktriangleleft y_1 y_2 &= x_1 x_2 y_1 y_2 + (x_1 \triangleleft y_1) x_2 y_2 + (x_1 \triangleleft y_2) x_2 y_1 + x_1 (x_2 \triangleleft y_1) y_2 \\ &\quad + x_1 (x_2 \triangleleft y_2) y_1 + (x_1 \triangleleft y_1) (x_2 \triangleleft y_2) + (x_1 \triangleleft y_2) (x_2 \triangleleft y_1). \end{aligned}$$

**Remark.** The number of terms in  $x_1 \dots x_k \triangleleft y_1 \dots y_l$  is:

$$\sum_{i=0}^{\min(k,l)} \binom{l}{i} \binom{k}{i} i!,$$

see sequences A086885 and A176120 of [9].

## 2 A family of solvable Lie algebras

### 2.1 Definition

**Definition 2** Let us fix  $a = (a_1, \dots, a_N) \in \mathbb{K}^N$ . We define an associative product  $\triangleleft$  on  $\mathbb{K}^N$ :

$$\forall i, j \in [N], \epsilon_i \triangleleft \epsilon_j = a_j \epsilon_i.$$

The associated Lie bracket is denoted by  $[-, -]_a$ :

$$\forall i, j \in [N], [\epsilon_i, \epsilon_j]_a = a_j \epsilon_i - a_i \epsilon_j.$$

This Lie algebra is denoted by  $\mathfrak{g}_a$ .

**Remarks.**

1. Let  $A \in M_{N,M}(\mathbb{K})$ , and  $a \in \mathbb{K}^N$ . The following map is a Lie algebra morphism:

$$\begin{cases} \mathfrak{g}_a \cdot {}^t A & \longrightarrow \mathfrak{g}_a \\ x & \longrightarrow Ax. \end{cases}$$

Consequently, if  $a \neq (0, \dots, 0)$ ,  $\mathfrak{g}_a$  is isomorphic to  $\mathfrak{g}_{(1,0,\dots,0)}$ .

2. These Lie algebras  $\mathfrak{g}_a$  are characterized by the following property: if  $\mathfrak{g}$  is a  $n$ -dimensional Lie algebra such that any 2-dimensional subspace of  $\mathfrak{g}$  is a Lie subalgebra, there exists  $a \in \mathbb{K}^n$  such that  $\mathfrak{g}$  and  $\mathfrak{g}_a$  are isomorphic.

**Definition 3** Let  $A = T(V)^N$ . The elements of  $A$  will be denoted by:

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} = f_1 \epsilon_1 + \dots + f_N \epsilon_N.$$

For all  $i, j \in [N]$ , we define bilinear products  ${}_i \Delta$  and  $\Delta_j$ :

$$\forall f, g \in T(V)^N, \quad f {}_i \Delta g = \begin{pmatrix} f_i \Delta g_1 \\ \vdots \\ f_i \Delta g_N \end{pmatrix}, \quad f \Delta_j g = \begin{pmatrix} f_1 \Delta g_j \\ \vdots \\ f_N \Delta g_j \end{pmatrix}.$$

In other words, if  $f, g \in T(V)$ , for all  $k, l \in [N]$ :

$$f \epsilon_k {}_i \Delta g \epsilon_l = \delta_{i,k} (f \Delta g) \epsilon_l, \quad f \epsilon_k \Delta_j g \epsilon_l = \delta_{j,l} (f \Delta g) \epsilon_k.$$

If  $a = (a_1, \dots, a_N) \in \mathbb{K}^N$ , we put  ${}_a \Delta = a_1 {}_1 \Delta + \dots + a_N {}_N \Delta$  and  $\Delta_a = a_1 \Delta_1 + \dots + a_N \Delta_N$ .

**Proposition 5** Let  $f, g \in \mathbb{K}^N$ . For all  $f, g, h \in A$ :

$$\begin{aligned} (f \Delta_a g) \Delta_b h &= f \Delta_a (g \Delta_b h), & (f \Delta_a g)_b \Delta h &= f \Delta_a (g_b \Delta h), \\ (f_a \Delta g) \Delta_b h &= f_a \Delta (g \Delta_b h), & (f_a \Delta g)_b \Delta h &= f_a \Delta (g_b \Delta h), \\ f \Delta_a g &= g_a \Delta f. \end{aligned}$$

**Proof.** Direct verifications, using the associativity and the commutativity of  $\Delta$ . □

**Definition 4** Let  $a \in \mathbb{K}^N$ . We define a Lie bracket on  $A$ :

$$\forall f, g \in A, \quad {}_a[f, g] = f_a \Delta g - g_a \Delta f = g \Delta_a f - f \Delta_a g.$$

This Lie algebra is denoted by  $\mathfrak{g}'_a$ .

**Remark.** If  $A$  is an associative commutative algebra and  $\mathfrak{g}$  is a Lie algebra, then  $A \otimes \mathfrak{g}$  is a Lie algebra, with the following Lie bracket:

$$\forall f, g \in A, x, y \in \mathfrak{g}, \quad [f \otimes x, g \otimes y] = fg \otimes [x, y].$$

Then, as a Lie algebra,  $\mathfrak{g}'_a$  is isomorphic to the tensor product of the associative commutative algebra  $(T(V), \Delta)$ , and of the Lie algebra  $\mathfrak{g}_{-a}$ . Consequently, if  $a \neq (0, \dots, 0)$ ,  $\mathfrak{g}'_a$  is isomorphic to  $\mathfrak{g}'_{(1,0,\dots,0)}$ .

## 2.2 Enveloping algebra of $\mathfrak{g}_a$

Let us apply Lemma 3 to the Lie algebra  $\mathfrak{g}_a$ :

**Proposition 6** The symmetric algebra  $S(\mathfrak{g}_a)$  is given an associative product  $\blacktriangleleft$  such that for all  $i_1, \dots, i_k, j_1, \dots, j_l \in [N]$ :

$$\epsilon_{i_1} \dots \epsilon_{i_k} \blacktriangleleft \epsilon_{j_1} \dots \epsilon_{j_l} = \sum_{I \subseteq [l]} k(k-1) \dots (k-|I|+1) \left( \prod_{q \in I} a_{j_q} \right) \left( \prod_{p \notin I} \epsilon_{j_p} \right) \epsilon_{i_1} \dots \epsilon_{i_k}.$$

The Hopf algebra  $(S(\mathfrak{g}_a), \blacktriangleleft, \Delta)$  is isomorphic to the enveloping algebra of  $\mathfrak{g}_a$ .

The enveloping algebra of  $\mathfrak{g}_a$  has two distinguished bases, the Poincaré-Birkhoff-Witt basis and the monomial basis:

$$(\epsilon_{i_1} \blacktriangleleft \dots \blacktriangleleft \epsilon_{i_k})_{k \geq 0, 1 \leq i_1 \leq \dots \leq i_k \leq N}, \quad (\epsilon_{i_1} \dots \epsilon_{i_k})_{k \geq 0, 1 \leq i_1 \leq \dots \leq i_k \leq N}.$$

Here is the passage between them.

**Proposition 7** Let us fix  $n \geq 1$ . For all  $I = \{i_1 < \dots < i_k\} \subseteq [n]$ , we put:

$$\lambda(I) = (i_1 - 1) \dots (i_k - k), \quad \mu(I) = (-1)^k (i_1 - 1) i_2 (i_3 + 1) \dots (i_k + k - 2).$$

We use the following notation: if  $[n] \setminus I = \{q_1 < \dots < q_l\}$ ,  $\overset{\blacktriangleleft}{\prod}_{q \notin I} \epsilon_{i_q} = \epsilon_{i_{q_1}} \blacktriangleleft \dots \blacktriangleleft \epsilon_{i_{q_l}}$ . Then:

$$\begin{aligned} \epsilon_{i_1} \blacktriangleleft \dots \blacktriangleleft \epsilon_{i_n} &= \sum_{I \subseteq [n]} \lambda(I) \left( \prod_{p \in I} a_{i_p} \right) \left( \prod_{q \notin I} \epsilon_{i_q} \right), \\ \epsilon_{i_1} \dots \epsilon_{i_n} &= \sum_{I \subseteq [n]} \mu(I) \left( \prod_{p \in I} a_{i_p} \right) \left( \overset{\blacktriangleleft}{\prod}_{q \notin I} \epsilon_{i_q} \right). \end{aligned}$$

**Proof.** *First step.* Let us prove the first formula by induction on  $n$ . It is obvious if  $n = 1$ , as  $\lambda(\emptyset) = 1$  and  $\lambda(\{1\}) = 0$ . Let us assume the result at rank  $n$ .

$$\begin{aligned}
\epsilon_{i_1} \blacktriangleleft \dots \blacktriangleleft \epsilon_{i_{n+1}} &= \sum_{I \subseteq [n]} \lambda(I) \left( \prod_{p \in I} a_{i_p} \right) \left( \prod_{q \notin I} \epsilon_{i_q} \right) \blacktriangleleft \epsilon_{i_{n+1}} \\
&= \sum_{I \subseteq [n]} \lambda(I) \left( \prod_{p \in I} a_{i_p} \right) \left( \prod_{q \notin I} \epsilon_{i_q} \epsilon_{i_{n+1}} + (k - |I|) a_{i_{n+1}} \prod_{q \notin I} \epsilon_{i_q} \right) \\
&= \sum_{\substack{I \subseteq [n+1], \\ n+1 \notin I}} \lambda(I) \left( \prod_{p \in I} a_{i_p} \right) \left( \prod_{q \notin I} \epsilon_{i_p} \right) + \sum_{\substack{I \subseteq [n+1], \\ n+1 \in I}} \lambda(I) \left( \prod_{p \in I} a_{i_p} \right) \left( \prod_{q \notin I} \epsilon_{i_p} \right) \\
&= \sum_{I \subseteq [n+1]} \lambda(I) \left( \prod_{p \in I} a_{i_p} \right) \left( \prod_{q \notin I} \epsilon_{i_p} \right).
\end{aligned}$$

*Second step.* Let us prove that for all  $I \subseteq [n]$ ,  $\sum_{J \subseteq I} \lambda(J) \mu(I \setminus J) = \delta_{I, \emptyset}$ .

We put  $I = \{i_1 < \dots < i_k\}$  and we proceed by induction on  $k$ . As  $\lambda(\emptyset) = \mu(\emptyset) = 1$ , the result is obvious at rank  $k = 0$  and  $k = 1$ . Let us assume the result at rank  $k - 1$ , with  $k \geq 2$ .

$$\begin{aligned}
\sum_{J \subseteq I} \lambda(J) \mu(I \setminus J) &= \sum_{\substack{J \subseteq I, \\ i_k \in J}} \lambda(J) \mu(I \setminus J) + \sum_{\substack{J \subseteq I, \\ i_k \notin J}} \lambda(J) \mu(I \setminus J) \\
&= \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J \cup \{i_k\}) \mu(I \setminus \{i_k\} \setminus J) + \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J) \mu(I \setminus J) \\
&= \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J) (i_k - |J|) \mu(I \setminus \{i_k\} \setminus J) \\
&\quad - \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J) \mu(I \setminus \{i_k\} \setminus J) (i_k + |I \setminus \{i_k\} \setminus J| + 1) \\
&= \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J) \mu(I \setminus \{i_k\} \setminus J) (i_k - |J| - i_k - |I| + 1 + |J| - 1) \\
&= -|I| \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J) \mu(I \setminus \{i_k\} \setminus J) \\
&= 0.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\sum_{I \subseteq [n]} \mu(I) \left( \prod_{p \in I} a_{i_p} \right) \left( \prod_{q \notin I} \epsilon_{i_q} \right) &= \sum_{I \subseteq [n]} \sum_{J \subseteq [n] \setminus I} \mu(I) \lambda(J) \left( \prod_{p \in I} a_{i_p} \right) \left( \prod_{p \in J} a_{i_p} \right) \left( \prod_{q \in [n] \setminus I \setminus J} \epsilon_{i_q} \right) \\
&= \sum_{A \sqcup B \sqcup C = [n]} \mu(A) \lambda(B) \left( \prod_{p \in A \sqcup B} a_{i_p} \right) \left( \prod_{q \in C} \epsilon_{i_q} \right) \\
&= \sum_{I \sqcup J = [n]} \left( \underbrace{\sum_{I' \subseteq I} \lambda(I') \mu(I \setminus I')}_{=\delta_{I, \emptyset}} \right) \left( \prod_{p \in I} a_{i_p} \right) \left( \prod_{q \in J} \epsilon_{i_q} \right) \\
&= \epsilon_{i_1} \dots \epsilon_{i_n},
\end{aligned}$$

which ends the proof.  $\square$

### 2.3 Modules over $\mathfrak{g}_{(1,0,\dots,0)}$

**Proposition 8** 1. Let  $V$  be a module over the associative (non unitary) algebra  $(\mathfrak{g}_{(1,0,\dots,0)}, \triangleleft)$ . Then  $V = V^{(0)} \oplus V^{(1)}$ , with:

- $\epsilon_1.v = v$  if  $v \in V^{(1)}$  and  $\epsilon_1.v = 0$  if  $v \in V^{(0)}$ .
- For all  $i \geq 2$ ,  $\epsilon_i.v \in V^{(0)}$  if  $v \in V^{(1)}$  and  $\epsilon_i.v = 0$  if  $v \in V^{(0)}$ .

2. Conversely, let  $V = V^{(1)} \oplus V^{(0)}$  be a vector space and let  $f_i : V^{(1)} \rightarrow V^{(0)}$  for all  $2 \leq i \leq N$ . One defines a structure of  $(\mathfrak{g}_{(1,0,\dots,0)}, \triangleleft)$ -module over  $V$ :

$$\epsilon_1.v = \begin{cases} v & \text{if } v \in V^{(1)}, \\ 0 & \text{if } v \in V^{(0)}; \end{cases} \quad \text{if } i \geq 2, \epsilon_i.v = \begin{cases} f_i(v) & \text{if } v \in V^{(1)}, \\ 0 & \text{if } v \in V^{(0)}. \end{cases}$$

Shortly:

$$\epsilon_1 : \begin{bmatrix} 0 & 0 \\ 0 & Id \end{bmatrix}, \quad \forall i \geq 2, \epsilon_i : \begin{bmatrix} 0 & f_i \\ 0 & 0 \end{bmatrix}.$$

**Proof.** Note that in  $\mathfrak{g}_{(1,0,\dots,0)}$ ,  $\epsilon_i \triangleleft \epsilon_j = \delta_{1,j} \epsilon_i$ .

1. In particular,  $\epsilon_1 \triangleleft \epsilon_1 = \epsilon_1$ . If  $F_1 : V \rightarrow V$  is defined by  $F_1(v) = \epsilon_1.v$ , then:

$$F_1 \circ F_1(v) = \epsilon_1.(\epsilon_1.v) = (\epsilon_1 \triangleleft \epsilon_1).v = \epsilon.v = F_1(v),$$

so  $F_1$  is a projection, which implies the decomposition of  $V$  as  $V^{(0)} \oplus V^{(1)}$ . Let  $x \in V^{(1)}$  and  $i \geq 2$ . Then  $F_1(\epsilon_i.v) = \epsilon_i.(\epsilon_i.v) = (\epsilon_1 \triangleleft \epsilon_i).v = 0$ , so  $\epsilon_i.v \in V^{(0)}$ . Let  $x \in V^{(0)}$ . Then  $\epsilon_i.v = (\epsilon_i \triangleleft \epsilon_1).v = \epsilon_i.F_1(v) = 0$ , so  $\epsilon_i.v = 0$ .

2. Let  $i \geq 2$  and  $j \in [N]$ . If  $v \in V^{(1)}$ :

$$\epsilon_1.(\epsilon_1.v) = v = \epsilon_1.v, \quad \epsilon_i.(\epsilon_1.v) = f_i(v) = \epsilon_i.v, \quad \epsilon_j.(\epsilon_i.v) = \epsilon_j.f_i(v) = 0.v.$$

If  $v \in V^{(0)}$ :

$$\epsilon_1.(\epsilon_1.v) = 0 = \epsilon_1.v, \quad \epsilon_i.(\epsilon_1.v) = 0 = \epsilon_i.v, \quad \epsilon_j.(\epsilon_i.v) = 0 = 0.v.$$

So  $V$  is indeed a  $(\mathfrak{g}_{(1,0,\dots,0)}, \triangleleft)$ -module. □

**Example.** There are, up to an isomorphism, three indecomposable  $(\mathfrak{g}_{(1,0)}, \triangleleft)$ -modules:

$$\begin{array}{c|c|c|c} \epsilon_1 & (0) & (1) & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \hline \epsilon_2 & (0) & (0) & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array}$$

**Proposition 9** (We assume  $\mathbb{K}$  algebraically closed). Let  $V$  be an indecomposable finite-dimensional module over the Lie algebra  $\mathfrak{g}_{(1,0,\dots,0)}$ . There exists a scalar  $\lambda$  and a decomposition:

$$V = V^{(0)} \oplus \dots \oplus V^{(k)}$$

such that, for all  $0 \leq p \leq k$ :

- $\epsilon_1(V^{(p)}) \subseteq V^{(p)}$  and there exists  $n \geq 1$  such that  $(\epsilon_1 - (\lambda + p)Id)_{|V^{(p)}}^n = (0)$ .
- If  $i \geq 2$ ,  $\epsilon_i(V^{(p)}) \subseteq V^{(p-1)}$ , with the convention  $V^{(-1)} = (0)$ .

**Proof.** First, observe that in the enveloping algebra of  $\mathfrak{g}_{(1,0,\dots,0)}$ , if  $i \geq 2$  and  $\lambda \in \mathbb{K}$ :

$$\epsilon_i \blacktriangleleft (\epsilon_1 - \lambda) = \epsilon_i \epsilon_1 + \epsilon_i - \lambda \epsilon_i = \epsilon_i \epsilon_1 + (1 - \lambda) \epsilon_i = (\epsilon_1 - \lambda + 1) \blacktriangleleft \epsilon_i.$$

Therefore, for all  $i \geq 2$ , for all  $n \in \mathbb{N}$ , for all  $\lambda \in \mathbb{K}$ :

$$\epsilon_i \blacktriangleleft (\epsilon_1 - \lambda) \blacktriangleleft^n = (\epsilon_1 - \lambda + 1) \blacktriangleleft^n \blacktriangleleft \epsilon_i.$$

Let  $V$  be a finite-dimensional module over the Lie algebra  $\mathfrak{g}_{(1,0,\dots,0)}$ . We denote by  $E_\lambda$  the characteristic subspace of eigenvalue  $\lambda$  for the action of  $\epsilon_1$ . Let us prove that for all  $\lambda \in \mathbb{K}$ , if  $i \geq 2$ ,  $\epsilon_i(E_\lambda) \subseteq E_{\lambda-1}$ . If  $x \in E_\lambda$ , there exists  $n \geq 1$ , such that  $(\epsilon_1 - \lambda Id) \blacktriangleleft^n . x = 0$ . Hence:

$$0 = \epsilon_i . ((\epsilon_1 - \lambda Id)^n . x) = (\epsilon_1 - (\lambda - 1) Id)^n . (\epsilon_i . x),$$

so  $\epsilon_i \in E_{\lambda-1}$ .

Let us take now  $V$  an indecomposable module, and let  $\Lambda$  be the spectrum of the action of  $\epsilon_1$ . The group  $\mathbb{Z}$  acts on  $\mathbb{K}$  by translation. We consider  $\Lambda' = \Lambda + \mathbb{Z}$  and let  $\Lambda''$  be a system of representants of the orbits of  $\Lambda'$ . Then:

$$V = \bigoplus_{\lambda \in \Lambda''} \underbrace{\left( \bigoplus_{n \in \mathbb{Z}} E_{\lambda+n} \right)}_{V_\lambda}.$$

By the preceding remarks,  $V_\lambda$  is a module. As  $V$  is indecomposable,  $\Lambda''$  is reduced to a single element. As the spectrum of  $\epsilon_1$  is finite, it is included in a set of the form  $\{\lambda, \lambda + 1, \dots, \lambda + k\}$ . We then take  $V^{(p)} = E_{\lambda+p}$  for all  $p$ .  $\square$

**Example.** Let us give the indecomposable modules of  $\mathfrak{g}_{(1,0)}$  of dimension  $\leq 3$ . For any  $\lambda \in \mathbb{K}$ :

$\epsilon_1$	$\epsilon_2$	$\epsilon_1$	$\epsilon_2$
$(\lambda)$	$(0)$	$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda+1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda+1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda+1 & 1 \\ 0 & 0 & \lambda+1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & \lambda+2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

**Definition 5** Let  $V$  be a module over the Lie algebra  $\mathfrak{g}_a$ . The associated algebra morphism is:

$$\phi_V : \begin{cases} \mathcal{U}(\mathfrak{g}_a) = (S(\mathfrak{g}_a), \blacktriangleleft) & \longrightarrow & \text{End}(V) \\ \epsilon_i & \longrightarrow & \begin{cases} V & \longrightarrow & V \\ v & \longrightarrow & \epsilon_i . v. \end{cases} \end{cases}$$

For all  $i_1, \dots, i_k \in [N]$ , we put  $F_{i_1, \dots, i_k} = \phi_V(\epsilon_{i_1} \dots \epsilon_{i_k})$ ; this does not depend on the order on the indices  $i_p$ .

By Proposition 7:

**Proposition 10** For all  $i_1, \dots, i_n \in [N]$ :

$$F_{i_1} \circ \dots \circ F_{i_n} = \sum_{\substack{I \subseteq [n], \\ I \setminus J = \{j_1 < \dots < j_l\}}} \lambda(I) \left( \prod_{p \in I} a_{i_p} \right) F_{i_{j_1}, \dots, i_{j_l}},$$

$$F_{i_1, \dots, i_n} = \sum_{\substack{I \subseteq [n], \\ I \setminus J = \{j_1 < \dots < j_l\}}} \mu(I) \left( \prod_{p \in I} a_{i_p} \right) F_{i_{j_1}} \circ \dots \circ F_{i_{j_l}}.$$

When  $V$  is a module over the associative algebra  $(\mathfrak{g}_A, \triangleleft)$ , these morphisms are easy to describe:

**Proposition 11** Let  $V$  be a module over the associative algebra  $(\mathfrak{g}_a, \triangleleft)$ ; it is also a module over the Lie algebra  $(\mathfrak{g}_a, [-, -]_a)$ . For all  $k \geq 2$ ,  $i_1, \dots, i_k \in [N]$ ,  $F_{i_1, \dots, i_k} = 0$ .

**Proof.** As  $V$  is a module over the associative algebra  $(\mathfrak{g}_a, \triangleleft)$ , for any  $i_1, i_2 \in [N]$ :

$$F_{i_1} \circ F_{i_2} = a_{i_2} F_{i_1}.$$

We proceed by induction on  $k$ . If  $k = 2$ ,  $\epsilon_{i_1} \epsilon_{i_2} = \epsilon_{i_1} \triangleleft \epsilon_{i_2} - a_{i_2} \epsilon_{i_1}$ , so:

$$F_{i_1, i_2} = F_{i_1} \circ F_{i_2} - a_{i_2} F_{i_1} = a_{i_2} F_{i_1} - a_{i_2} F_{i_1} = 0.$$

Let us assume the result at rank  $k$ . Then  $\epsilon_1 \dots \epsilon_{i_{k+1}} = \epsilon_{i_1} \dots \epsilon_{i_k} \triangleleft \epsilon_{i_{k+1}} - k a_{i_{k+1}} \epsilon_{i_1} \dots \epsilon_{i_k}$ , and  $F_{i_1, \dots, i_{k+1}} = F_{i_1, \dots, i_k} \circ F_{i_{k+1}} - k a_{i_{k+1}} F_{i_1, \dots, i_k} = 0$ .  $\square$

### 3 A family of post-Lie algebras

#### 3.1 Reminders

We defined in [3] a family of pre-Lie algebras, associated to endomorphisms. Let us briefly recall this construction.

**Proposition 12** Let  $V$  be a vector space and  $F : V \rightarrow V$  be an endomorphism. We define a product  $*$  on  $T(V)$ : for all  $f, g \in T(V)$ , for all  $x \in V$ ,

$$\emptyset * g = 0, \quad x f * g = x(f * g) + F(x)(f \Delta g).$$

This product is pre-Lie. The pre-Lie algebra  $(T(V), *)$  is denoted by  $T(V, F)$ . Moreover, for all  $f, g, h \in T(V, F)$ :

$$(f \Delta g) * h = (f * h) \Delta g + f \Delta (g * h).$$

We also proved the following result:

**Proposition 13** Let  $k, l \geq 0$ .

- The set  $Sh(k, l)$  of  $(k, l)$ -shuffles is the set of permutation  $\sigma \in \mathfrak{S}_{k+l}$  such that  $\sigma(1) < \dots < \sigma(k)$  and  $\sigma(k+1) < \dots < \sigma(k+l)$ .
- If  $\sigma \in Sh(k, l)$ , we put  $m_k(\sigma) = \max\{i \in [k] \mid \sigma(1) = 1, \dots, \sigma(i) = i\}$ . In particular, if  $\sigma(1) \neq 1$ ,  $m_k(\sigma) = 0$ .

For all  $x_1, \dots, x_k, y_1, \dots, y_l \in V$ :

$$x_1 \dots x_k * y_1 \dots y_l = \sum_{\sigma \in Sh(k, l)} \sum_{p=1}^{m_k(\sigma)} \left( Id^{\otimes(p)} \otimes F \otimes Id^{\otimes(k+l-p)} \right) \sigma.(x_1 \dots x_k y_1 \dots y_l).$$

### 3.2 Construction

Let us fix a vector space  $V$ , a family of  $N$  endomorphisms  $(F_1, \dots, F_N)$  of  $V$  and  $a = (a_1, \dots, a_N) \in \mathbb{K}^N$ . We define inductively a product  $*$  on  $T(V)^N$ : for all  $f, g \in T(V)^N$ ,  $x \in V$ ,  $i \in [N]$ ,

$$\emptyset \epsilon_i * g = 0, \quad xf * g = x(f * g) + F_1(x)(f \Delta_1 g) + \dots + F_N(x)(f \Delta_N g).$$

We define a second product  $\bullet$  on  $T(V)^N$ :

$$\forall f, g \in T(V)^N, f \bullet g = f * g + f a \Delta g.$$

**Examples.** Let  $x, y, z \in V$ ,  $g \in T(V)$ ,  $i, j \in [N]$ . Then:

$$\begin{aligned} x \epsilon_i * g \epsilon_j &= F_j(x) g \epsilon_j, \\ xy \epsilon_i * g \epsilon_j &= (xF_j(y)g + F_j(x)(y \Delta g)) \epsilon_i, \\ xyz \epsilon_i * g \epsilon_j &= (xyF_j(z)g + xF_j(y)(z \Delta g) + F_j(x)(yz \Delta g)) \epsilon_i. \end{aligned}$$

**Proposition 14** *Let  $x_1, \dots, x_k, y_1, \dots, y_l \in V$ ,  $i, j \in [N]$ .*

$$x_1 \dots x_k \epsilon_i * y_1 \dots y_l \epsilon_j = \sum_{\sigma \in Sh(k,l)} \sum_{p=1}^{m_k(\sigma)} \left( Id^{\otimes(p_1)} \otimes F_j \otimes Id^{\otimes(k+l-p)} \right) \sigma.(x_1 \dots x_k y_1 \dots y_l) \epsilon_i.$$

**Proof.** By induction on  $k$ . It is immediate if  $k = 0$ , as both sides are equal to 0. Let us assume the result at rank  $k - 1$ .

$$\begin{aligned} x_1 \dots x_k \epsilon_i * y_1 \dots y_l \epsilon_j &= x_1(x_2 \dots x_k \epsilon_i * y_1 \dots y_l \epsilon_j) + F_j(x_1)(x_2 \dots x_k \Delta y_1 \dots y_l) \epsilon_i \\ &= \sum_{\substack{\sigma \in Sh(k,l), \\ \sigma(1)=1}} \sum_{p=2}^{m_k(\sigma)} (Id^{\otimes(p_1)} \otimes F_j \otimes Id^{\otimes(k+l-p)}) \sigma.(x_1 \dots x_k y_1 \dots y_l) \epsilon_i \\ &\quad + \sum_{\substack{\sigma \in Sh(k,l), \\ \sigma(1)=1}} (F_j \otimes Id^{\otimes(k+l-1)}) \sigma.(x_1 \dots x_k y_1 \dots y_l) \epsilon_i \\ &= \sum_{\substack{\sigma \in Sh(k,l), \\ \sigma(1)=1}} \sum_{p=1}^{m_k(\sigma)} (Id^{\otimes(p_1)} \otimes F_j \otimes Id^{\otimes(k+l-p)}) \sigma.(x_1 \dots x_k y_1 \dots y_l) \epsilon_i \\ &= \sum_{\sigma \in Sh(k,l)} \sum_{p=1}^{m_k(\sigma)} (Id^{\otimes(p_1)} \otimes F_j \otimes Id^{\otimes(k+l-p)}) \sigma.(x_1 \dots x_k y_1 \dots y_l) \epsilon_i, \end{aligned}$$

so the result holds for all  $k$ . □

**Remark.** Let  $*_j$  be the pre-Lie product of  $T(V, F_j)$ , described in [3]. For all  $f, g \in T(V)$ , for all  $i, j \in [N]$ :

$$f \epsilon_i * g \epsilon_j = (f *_j g) \epsilon_i.$$

**Corollary 2** *For all  $f, g, h \in T(V)^N$ , for all  $i \in [N]$ :*

$$\begin{aligned} (f \epsilon_i \Delta g) * h &= (f * h) \epsilon_i \Delta g + f \epsilon_i \Delta (g * h), \\ (f \Delta_i g) * h &= (f * h) \Delta_i g + f \Delta_i (g * h), \\ (f \Delta g) * h &= (f * h) \Delta g + f \Delta (g * h). \end{aligned}$$

**Proof.** It is enough to prove these assertions for  $f = f'\epsilon_k$ ,  $g = g'\epsilon_l$  and  $h = h'\epsilon_m$ , with  $f', g', h' \in T(V)$ . For the first assertion:

$$\begin{aligned}
(f \Delta_i g) * h &= \delta_{i,k}(f' \Delta g' \epsilon_l) * h' \epsilon_m \\
&= \delta_{i,k}(f' \Delta g') *_m h' \epsilon_l \\
&= \delta_{i,k}((f' *_m h') \Delta g' + f' \Delta (g' *_m h')) \epsilon_l \\
&= (f * h) \Delta_i g + f \Delta_i (g * h).
\end{aligned}$$

The second point is deduced from the first one, as  $\Delta_i = \Delta_i^{op}$ . Finally:

$$\begin{aligned}
(f \Delta_i g) * h &= \delta_{k,l}(f' \Delta g' \epsilon_l) * h' \epsilon_m \\
&= \delta_{k,l}(f' \Delta g') *_m h' \epsilon_l \\
&= \delta_{k,l}((f' *_m h') \Delta g' + f' \Delta (g' *_m h')) \epsilon_l \\
&= (f * h) \Delta g + f \Delta (g * h).
\end{aligned}$$

So the last point holds. □

**Theorem 3** *The following conditions are equivalent:*

1.  $(T(V)^N, \bullet)$  is a pre-Lie algebra.
2.  $\mathfrak{g}'_a = (T(V)^N, {}_a[-, -], *)$  is a post-Lie algebra.
3.  $V$  is a module over the Lie algebra  $\mathfrak{g}_a$ , with the action given by  $\epsilon_i.v = F_i(v)$ .

**Proof.** By Corollary 2, for all  $f, g, h \in \mathfrak{g}'_a$ ,  ${}_a[f, g] * h = {}_a[f * h, g] + {}_a[f, g * h]$ .

1.  $\iff$  2. Let  $f, g, h \in \mathfrak{g}$ .

$$\begin{aligned}
&(f \bullet g) \bullet h - f \bullet (g \bullet h) - (f \bullet h) \bullet g + f \bullet (h \bullet g) \\
&= (f * g) * h - f * (g * h) - (f * h) * g + f * (h * g) \\
&+ (f \Delta_a g) * h - f \Delta_a (g * h) - (f \Delta_a h) * g + f \Delta_a (h * g) \\
&+ (f * g) \Delta_a h - f * (g \Delta_a h) - (f * h) \Delta_a g + f * (h \Delta_a g) \\
&+ (f \Delta_a g) \Delta_a h - f \Delta_a (g \Delta_a h) - (f \Delta_a g) \Delta_a h + f \Delta_a (g \Delta_a h) \\
&= (f * g) * h - f * (g * h) - (f * h) * g + f * (h * g) \\
&+ f * (g \Delta_a h) - f * (h \Delta_a g) \\
&+ [(f \Delta_a g) * h - f \Delta_a (g * h) - (f * h) \Delta_a g] \\
&+ [(f \Delta_a h) * g - f \Delta_a (h * g) - (f * g) \Delta_a h] \\
&- [(f * h) \Delta_a g - f \Delta_a (h * g) - (f * g) \Delta_a h] \\
&+ [(f \Delta_a g) \Delta_a h - f \Delta_a (g \Delta_a h)] - [(f \Delta_a g) \Delta_a h - f \Delta_a (g \Delta_a h)] \\
&= (f * g) * h - f * (g * h) - (f * h) * g + f * (h * g) - f * {}_a[g, h].
\end{aligned}$$

So  $(\mathfrak{g}'_a, \bullet)$  is pre-Lie if, and only if,  $(\mathfrak{g}'_a, {}_a[-, -], *)$  is post-Lie.

2.  $\implies$  3. Let  $x, y, v \in V$  and  $i, j, k \in [N]$ . Then:

$$\begin{aligned}
x\epsilon_i * y\epsilon_j &= F_j(x)y\epsilon_i, & xy\epsilon_i * z\epsilon_k &= xF_k(y)z\epsilon_i + F_k(x)(y \Delta z)\epsilon_i. \\
x\epsilon_i * yz\epsilon_k &= F_k(x)yz\epsilon_i,
\end{aligned}$$



Hence:

$$\begin{aligned}
(x\epsilon_i * y\epsilon_j) * z\epsilon_k &= F_j(x)y\epsilon_i * z\epsilon_k \\
&= F_j(x)F_k(y)z\epsilon_i + F_k \circ F_j(x)y \Delta z\epsilon_i, \\
x\epsilon_i * (y\epsilon_j * z\epsilon_k) &= x\epsilon_i * F_k(y)z\epsilon_j \\
&= F_j(x)F_k(y)z\epsilon_i, \\
x\epsilon_i \circ [y\epsilon_j, z\epsilon_k] &= a_j x\epsilon_i * (y \Delta z)\epsilon_k - a_k x\epsilon_i * (y \Delta z)\epsilon_j \\
&= (a_j F_k(x)(y \Delta z) - a_k F_j(x)(y \Delta z))\epsilon_i.
\end{aligned}$$

The post-Lie relation (2) gives:

$$\begin{aligned}
&(a_j F_k(x) - a_k F_j(x))(y \Delta z) \\
&= F_j(x)F_k(y)z + F_k \circ F_j(x)(y \Delta z) - F_j(x)F_k(y)z - F_j \circ F_k(x)(y \Delta z) \\
&= (F_j \circ F_k - F_k \circ F_j)(x)(y \Delta z).
\end{aligned}$$

Let  $y = z$  be a nonzero element of  $V$ . Then  $y \Delta z \neq 0$ , and we obtain that for all  $x \in V$ ,  $a_j F_k(x) - a_k F_j(x) = (F_j \circ F_k - F_k \circ F_j)(x)$ :  $V$  is a  $\mathfrak{g}_a$ -module.

3.  $\implies$  2. Let us prove the post-Lie relation (2) for  $f\epsilon_i$ ,  $g$  and  $h$ , with  $f \in T(V)$ ,  $i \in [N]$ ,  $g, h \in \mathfrak{g}'_a$ . We assume that  $f$  is a word and we proceed by induction on the length  $n$  of  $f$ . If  $n = 0$ , then  $f = \emptyset$  and every term is 0 in the relation. Let us assume the result at rank  $n - 1$ . We put  $f = xf'$ , with  $x \in V$ , and  $f'$  a word of length  $n - 1$ .

$$\begin{aligned}
(f * g) * h &= x((f'\epsilon_i * g) * h) + \sum_{p=1}^N F_p(x)((f'\epsilon_i * g) \Delta_p h) \\
&\quad + \sum_{p=1}^N F_p(x)((f'\epsilon_i \Delta_p g) * h) + \sum_{p,q=1}^N F_q \circ F_p(x)(f'\epsilon_i \Delta_p g \Delta_q h), \\
f * (g * h) &= x(f'\epsilon_i * (g * h)) + \sum_{p=1}^N F_p(x)(f'\epsilon_i \Delta_p (g * h)), \\
\sum_{p=1}^N a_p f * (g_p \Delta h) &= \sum_{p=1}^N a_p x(f'\epsilon_i * (g_p \Delta h)) + \sum_{p,q=1}^N a_p F_q(x)(f'\epsilon_i \Delta_q (g_p \Delta h)).
\end{aligned}$$

We put:

$$P(f, g, h) = f * (g * h) - (f * g) * h + \sum_{p=1}^N a_p f * (g_p \Delta h).$$

In order to prove the post-Lie relation (2), we have to prove that  $P(f, g, h) = P(f, h, g)$ . First:

$$\begin{aligned}
(f * g) * h &= x((f'\epsilon_i * g) * h) + \sum_{p=1}^N F_p(x)((f'\epsilon_i * g) \Delta_p h) \\
&\quad + \sum_{p=1}^N F_p(x)((f'\epsilon_i \Delta_p g) * h) + \sum_{p,q=1}^N F_q \circ F_p(x)(f'\epsilon_i \Delta_p g \Delta_q h), \\
f * (g * h) &= x(f'\epsilon_i * (g * h)) + \sum_{p=1}^N F_p(x)(f'\epsilon_i \Delta_p (g * h)), \\
\sum_{p=1}^N a_p f * (g_p \Delta h) &= \sum_{p=1}^N a_p x(f'\epsilon_i * (g_p \Delta h)) + \sum_{p,q=1}^N a_p F_q(x)(f'\epsilon_i \Delta_q (g_p \Delta h)).
\end{aligned}$$

Consequently:

$$\begin{aligned}
P(f, g, h) &= xP(f', g, h) \\
&+ \sum_{p=1}^N F_p(x) (-(f' \epsilon_i * g) \Delta_p h - ((f' \epsilon_i \Delta_p g) * h + f' \epsilon_i \Delta_p (g * h))) \\
&+ \sum_{p,q=1}^N a_p F_q(x) (f' \epsilon_i \Delta_q g_p \Delta h) - F_q \circ F_p(x) (f' \epsilon_i \Delta_p g \Delta_q h) \\
&= xP(f', g, h) - \sum_{p=1}^N F_p(x) ((f' \epsilon_i * g) \Delta_p h + (f' \epsilon_i * h) \Delta_p g) \\
&+ \sum_{p,q=1}^N a_p F_q(x) (f' \epsilon_i \Delta_q h \Delta_p g) - F_q \circ F_p(x) (f' \epsilon_i \Delta_p g \Delta_q h) \\
&= xP(f', g, h) - \sum_{p=1}^N F_p(x) (f' \epsilon_i * g) \Delta_p h + (f' \epsilon_i * h) \Delta_p g) \\
&+ \sum_{p,q=1}^N (a_p F_q(x) - F_q \circ F_p(x)) (f' \epsilon_i \Delta_p g \Delta_q h).
\end{aligned}$$

By the induction hypothesis,  $P(f', g, h) = P(f', h, g)$ , so the first row is symmetric in  $g, h$ . As  $V$  is a  $\mathfrak{g}_a$ -module,  $a_p F_q - F_q \circ F_p = a_q F_p - F_p \circ F_q$ , so the second row is symmetric in  $g, h$ , and finally  $P(f, g, h) = P(f, h, g)$ :  $\mathfrak{g}'_a$  is a post-Lie algebra.  $\square$

**Example.** The post-Lie algebra  $\mathfrak{g}_{SISO}$  is associated to  $a = (1, 0)$ ,  $V = Vect(x_1, x_2)$  and:

$$F_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

As  $F_1$  and  $F_2$  define a module over the Lie algebra  $\mathfrak{g}_{(1,0)}$ , even in fact over the associative algebra  $(\mathfrak{g}_{(1,0)}, \triangleleft)$ , we obtain indeed a post-Lie algebra. For all  $f, g \in T(V)$ , for all  $i, j \in \{1, 2\}$ :

$$\begin{aligned}
\emptyset \epsilon_i * g \epsilon_j &= 0, & x_2 f \epsilon_i * g \epsilon_1 &= x_2 (f \epsilon_i * g \epsilon_1) + x_2 (f \Delta g) \epsilon_i, \\
x_1 f \epsilon_i * g \epsilon_j &= x_1 (f \epsilon_i * g \epsilon_j), & x_2 f \epsilon_i * g \epsilon_2 &= x_2 (f \epsilon_i * g \epsilon_2) + x_1 (f \Delta g) \epsilon_i.
\end{aligned}$$

### 3.3 Extension of the post-Lie product

We now extend the post-Lie product of  $\mathfrak{g}'_a$  to the enveloping algebra  $\mathcal{U}(\mathfrak{g}'_a)$ . As this Lie bracket is obtained from an associative product  $\triangleleft = {}_a \Delta$ , we can see  $\mathcal{U}(\mathfrak{g}'_a)$  as  $(S(\mathfrak{g}'_a), \triangleleft, \Delta)$ . The post-Lie product  $*$  is extended to  $\mathcal{U}(\mathfrak{g}'_a)$ , and we obtain a Hopf algebra  $(\mathcal{U}(\mathfrak{g}), \otimes, \Delta)$ , isomorphic to  $\mathcal{U}(\mathfrak{g}'_a, {}_a[-, -]_*)$ , with:

$$\forall f, g \in \mathfrak{g}, \quad {}_a[f, g]_* = {}_a[f, g] + f * g - g * f = f {}_a \Delta g + f * g - g {}_a \Delta f - g * f.$$

As  $\bullet$  is a pre-Lie product, it can also be extended to  $S(\mathfrak{g})$  and gives a product  $\odot$ , making  $S(\mathfrak{g}'_a)$  a Hopf algebra isomorphic to  $\mathcal{U}(\mathfrak{g}'_a, [-, -]_\bullet)$ .

**Remark.** Let  $f, g \in \mathfrak{g}'_a$ .

$$\begin{aligned}
[f, g]_\bullet &= f \bullet g - g \bullet f \\
&= f {}_a \Delta g + f * g - g {}_a \Delta f - g * f \\
&= {}_a[f, g] + f * g - g * f \\
&= {}_a[f, g]_*.
\end{aligned}$$

So  $[-, -]_\bullet = {}_a[-, -]_*$ .

**Lemma 4** Let  $f_1, \dots, f_k, g \in \mathfrak{g}'_a$ ,  $k \geq 1$ .

$$(f_1 \blacktriangleleft \dots \blacktriangleleft f_k) * g = \sum_{p=1}^k f_1 \blacktriangleleft \dots \blacktriangleleft (f_p * g) \blacktriangleleft \dots \blacktriangleleft f_k,$$

$$(f_1 \dots f_k) * g = \sum_{p=1}^k f_1 \dots (f_p * g) \dots f_k.$$

**Proof.** The first point comes by the very definition of  $*$ . For the second point, we proceed by induction on  $k$ . This is obvious if  $k = 1$ . Let us assume the result at rank  $k$ ,  $k \geq 1$ . Observe that:

$$f_1 \dots f_{k+1} = f_1 \dots f_k \blacktriangleleft f_{k+1} - \sum_{p=1}^k f_1 \dots (f_p \triangleleft f_{k+1}) \dots f_k,$$

so:

$$\begin{aligned} f_1 \dots f_{k+1} * g &= (f_1 \dots f_k * g) \blacktriangleleft f_{k+1} + f_1 \dots f_k \blacktriangleleft (f_{k+1} * g) - \sum_{p=1}^k f_1 \dots (f_p \triangleleft f_{k+1}) \dots f_k * g \\ &= \sum_{p=1}^k f_1 \dots (f_p * g) \dots f_k \blacktriangleleft f_{k+1} + f_1 \dots f_k \blacktriangleleft (f_{k+1} * g) \\ &\quad - \sum_{p \neq q} f_1 \dots (f_p \triangleleft f_{k+1}) \dots (f_q * g) \dots f_k - \sum_{p=1}^k f_1 \dots ((f_p \triangleleft f_{k+1}) * g) \dots f_k \\ &= \sum_{p=1}^k f_1 \dots (f_p * g) \dots f_k f_{k+1} + \sum_{p \neq q} f_1 \dots (f_p \triangleleft f_{k+1}) \dots (f_q * g) \dots f_k \\ &\quad + \sum_{p=1}^k f_1 \dots ((f_p * g) \triangleleft f_{k+1}) \dots f_k - \sum_{p \neq q} f_1 \dots (f_p \triangleleft f_{k+1}) \dots (f_q * g) \dots f_k \\ &\quad - \sum_{p=1}^k f_1 \dots ((f_p * g) \triangleleft f_{k+1}) \dots f_k - \sum_{p=1}^k f_1 \dots (f_p \triangleleft (f_{k+1} * g)) \dots f_k \\ &\quad + f_1 \dots f_k (f_{k+1} * g) + \sum_{p=1}^k f_1 \dots (f_p \triangleleft (f_{k+1} * g)) \dots f_k \\ &= \sum_{p=1}^{k+1} f_1 \dots (f_p * g) \dots f_{k+1}. \end{aligned}$$

Finally, the result holds for all  $k \geq 1$ . □

The following result allows to compute  $f * g_1 \dots g_k$  by induction on the length of  $f$ :

**Proposition 15** Let  $x \in V$ ,  $k \geq 1$ ,  $f, g_1, \dots, g_k \in T(V)^N$ ,  $i \in [N]$ .

$$\emptyset \epsilon_i * (g_1 \blacktriangleleft \dots \blacktriangleleft g_k) = 0,$$

$$x f * (g_1 \blacktriangleleft \dots \blacktriangleleft g_k) = \sum_{\substack{I=\{i_1 < \dots < i_l\} \subseteq [k], \\ j_1, \dots, j_l \in [N]}} F_{j_1} \circ \dots \circ F_{j_l}(x) \left( \left( f * \prod_{i \notin I}^{\blacktriangleleft} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right);$$

$$\emptyset \epsilon_i * (g_1 \dots g_k) = 0,$$

$$x f * (g_1 \dots g_k) = \sum_{\substack{I=\{i_1 < \dots < i_l\} \subseteq [k], \\ j_1, \dots, j_l \in [N]}} F_{j_1, \dots, j_l}(x) \left( \left( f * \prod_{i \notin I} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right).$$

**Proof.** In order to ease the redaction, we put:

$$I_k = \{(I, j_1, \dots, j_l) \mid I = \{i_1 < \dots < i_l\} \subseteq [k], j_1, \dots, j_l \in [N]\}.$$

We proceed by induction on  $k$ . It is immediate if  $k = 1$ . Let us assume the result at rank  $k$ ,  $k \geq 1$ . Then:

$$\begin{aligned} \emptyset \epsilon_i * (g_1 \blacktriangleleft \dots \blacktriangleleft g_{k+1}) &= (\emptyset \epsilon_i * (g_1 \blacktriangleleft \dots \blacktriangleleft g_k)) * g_{k+1} \\ &\quad - \sum_{p=1}^k \emptyset \epsilon_i * (g_1 \blacktriangleleft \dots \blacktriangleleft (g_p * g_{k+1}) \blacktriangleleft \dots \blacktriangleleft g_k) \\ &= 0. \end{aligned}$$

Moreover:

$$\begin{aligned} &xf * (g_1 \blacktriangleleft \dots \blacktriangleleft g_{k+1}) \\ &= (xf * (g_1 \blacktriangleleft \dots \blacktriangleleft g_k)) * g_{k+1} - \sum_{p=1}^k xf * (g_1 \blacktriangleleft \dots \blacktriangleleft (g_p * g_{k+1}) \blacktriangleleft \dots \blacktriangleleft g_k) \\ &= \sum_{I_k} F_{j_l} \circ \dots \circ F_{j_1}(x) \left( \left( f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) * g_{k+1} \\ &\quad - \sum_{I_k} F_{j_l} \circ \dots \circ F_{j_1}(x) \left( f * \left( \left( \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\ &\quad - \sum_{p=1}^k \sum_{I_k} F_{j_l} \circ \dots \circ F_{j_1}(x) \left( \left( f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_p} (g_{i_p} * g_{k+1}) \dots \Delta_{j_l} g_{i_l} \right) \\ &= \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_l} \circ \dots \circ F_{j_1}(x) \left( \left( f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \Delta_{j_{l+1}} g_{k+1} \right) \\ &\quad + \sum_{I_k} F_{j_l} \circ \dots \circ F_{j_1}(x) \\ &\quad \left( \left( \left( f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} - f * \left( \left( \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} \right) \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\ &= \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_l} \circ \dots \circ F_{j_1}(x) \left( \left( f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \Delta_{j_{l+1}} g_{k+1} \right) \\ &\quad + \sum_{I_k} F_{j_l} \circ \dots \circ F_{j_1}(x) \left( \left( f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \blacktriangleleft g_{k+1} \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \Delta_{j_l} g_{k+1} \right) \\ &= \sum_{I_{k+1}, k+1 \in I} F_{j_l} \circ \dots \circ F_{j_1}(x) \left( \left( f * \overset{\blacktriangleleft}{\prod}_{i \notin I} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\ &\quad + \sum_{I_{k+1}, k+1 \notin I} F_{j_l} \circ \dots \circ F_{j_1}(x) \left( \left( f * \overset{\blacktriangleleft}{\prod}_{i \notin I} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\ &= \sum_{I_{k+1}} F_{j_l} \circ \dots \circ F_{j_1}(x) \left( \left( f * \overset{\blacktriangleleft}{\prod}_{i \notin I} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right). \end{aligned}$$

So, for all  $F \in \mathcal{U}(\mathfrak{g})_+$ ,  $\emptyset \epsilon_i * F = 0$ . As  $g_1 \dots g_k \in \mathcal{U}(\mathfrak{g})_+$ , the first point holds. Let us prove the second point by induction on  $k$ . The result is immediate if  $k = 1$ . Let us assume the result at rank  $k \geq 1$ .

$$\begin{aligned}
& xf * g_1 \dots g_{k+1} \\
&= xf * (g_1 \dots g_k * g_{k+1}) - \sum_{p=1}^k xf * (g_1 \dots (g_p \Delta g_{k+1}) \dots g_k) \\
&= (xf * g_1 \dots g_k) * g_{k+1} - \sum_{p=1}^k xf * (g_1 \dots (g_p \Delta g_{k+1}) \dots g_k) - \sum_{p=1}^k xf * (g_1 \dots (g_p * g_{k+1}) \dots g_k) \\
&= \sum_{I_k} F_{j_1, \dots, j_l}(x) \left( \left( \left( f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{p=1}^k F_{j_1, \dots, j_l}(x) \left( \left( f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_p} (g_{i_p} * g_{k+1}) \dots \Delta_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_{l+1}} \circ F_{j_1, \dots, j_l}(x) \left( \left( f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} F_{j_1, \dots, j_l}(x) \left( f * \left( \left( \prod_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} \sum_{p=1}^k F_{j_1, \dots, j_l}(x) \left( \left( f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_p} (g_{i_p} * g_{k+1}) \dots \Delta_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} \sum_{p=1}^k F_{j_1, \dots, j_l}(x) \left( \left( f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_p} (g_{i_p} \Delta g_{k+1}) \dots \Delta_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} F_{j_1, \dots, j_l}(x) \left( f * \left( \left( \prod_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} - \left( \prod_{i \notin J \sqcup \{k+1\}} g_i \right) g_{k+1} \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\
&= \sum_{I_k} F_{j_1, \dots, j_l}(x) \left( f * \left( \left( \prod_{i \notin J \sqcup \{k+1\}} g_i \right) g_{k+1} \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_{l+1}} \circ F_{j_1, \dots, j_l}(x) \left( \left( f * \left( \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \Delta_{j_{l+1}} g_{i_{l+1}} \right) \\
&- \sum_{I_k} \sum_{p=1}^k \sum_{j \in [N]} a_j F_{j_1, \dots, j_l}(x) \left( \left( f * \left( \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_p} (g_{k+1} \Delta_j g_p) \dots \Delta_{j_l} g_{i_l} \right) \\
&= \sum_{I_k} F_{j_1, \dots, j_l}(x) \left( f * \left( \left( \prod_{i \notin J \sqcup \{k+1\}} g_i \right) g_{k+1} \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{j_{l+1} \in [N]} \left( F_{j_{l+1}} \circ F_{j_1, \dots, j_l} - \sum_{p=1}^l a_{j_p} F_{j_1, \dots, \widehat{j_p}, \dots, j_{k+1}} \right) (x) \\
&\quad \left( \left( f * \left( \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \Delta_{j_{l+1}} g_{i_{l+1}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{I_{k+1}, k+1 \notin J} F_{j_1, \dots, j_l}(x) \left( \left( f * \prod_{i \notin I} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\
&+ \sum_{I_{k+1}, k+1 \in J} F_{j_1, \dots, j_l}(x) \left( \left( f * \prod_{i \notin I} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right) \\
&= \sum_{I_{k+1}} F_{j_1, \dots, j_l}(x) \left( \left( f * \prod_{i \notin I} g_i \right) \Delta_{j_1} g_{i_1} \dots \Delta_{j_l} g_{i_l} \right).
\end{aligned}$$

Note that we used  $F_{j_{l+1}} \circ F_{j_1, \dots, j_l} = F_{j_1, \dots, j_{l+1}} + \sum_{p=1}^l a_{j_p} F_{j_1, \dots, \widehat{j_p}, \dots, j_{k+1}}$ .  $\square$

**Proposition 16** *Let  $k \geq 1$ ,  $f, g_1, \dots, g_k \in T(V)^N$ . Then:*

$$f \bullet g_1 \dots g_k = f * g_1 \dots g_k + \sum_{p=1}^k (f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) \Delta g_p.$$

**Proof.** We proceed by induction on  $k$ . This is obvious if  $k = 1$ . Let us assume the result at rank  $k$ ,  $k \geq 1$ .

$$f \bullet g_1 \dots g_{k+1}$$

$$\begin{aligned}
&= (f \bullet g_1 \dots g_k) \bullet g_{k+1} - \sum_{p=1}^k f \bullet (g_1 \dots (g_p \bullet g_{k+1}) \dots g_k) \\
&= (f * g_1 \dots g_k) * g_{k+1} + (f * g_1 \dots g_k) \Delta g_{k+1} \\
&+ \sum_{p=1}^k ((f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) \Delta g_p) * g_{k+1} + \sum_{p=1}^k (f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) \Delta g_p \Delta g_{k+1} \\
&- \sum_{p=1}^k f * (g_1 \dots (g_p \bullet g_{k+1}) \dots g_k) - \sum_{p=1}^k (f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) \Delta (g_p \bullet g_{k+1}) \\
&- \sum_{p \neq q} f * (g_1 \dots (g_p \bullet g_{k+1}) \dots \widehat{g}_q \dots g_k) \Delta g_q \\
&= (f * g_1 \dots g_k) * g_{k+1} + (f * g_1 \dots g_k) \Delta g_{k+1} + \sum_{p=1}^k ((f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) * g_{k+1}) \Delta g_p \\
&+ \sum_{p=1}^k (f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) \Delta (g_p * g_{k+1} - g_p \bullet g_{k+1} + g_p \Delta g_{k+1}) \\
&- \sum_{p \neq q} f * (g_1 \dots (g_p \bullet g_{k+1}) \dots \widehat{g}_q \dots g_k) \Delta g_q - \sum_{p=1}^k f * (g_1 \dots (g_p \bullet g_{k+1}) \dots g_k) \\
&= (f * g_1 \dots g_k) * g_{k+1} + (f * g_1 \dots g_k) \Delta g_{k+1} \\
&+ \sum_{p=1}^k \left( (f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) * g_{k+1} - \sum_{q \neq p} f * g_1 \dots g_{p-1} g_{p+1} \dots (g_k \bullet g_{k+1}) \dots g_k \right) \Delta g_p \\
&- \sum_{p=1}^k f * g_1 \dots (g_p * g_{k+1}) \dots g_k - \sum_{p=1}^k f * g_1 \dots (g_p \Delta g_{k+1}) \dots g_k \\
&= f * \left( g_1 \dots g_k \blacktriangleleft g_{k+1} - \sum_{p=1}^k g_1 \dots (g_p \Delta g_{k+1}) \dots g_k \right) + \sum_{p=1}^{k+1} (f * g_1 \dots g_{p-1} g_{p+1} \dots g_{k+1}) \Delta g_p
\end{aligned}$$

$$= f * g_1 \dots g_{k+1} + \sum_{p=1}^{k+1} (f * g_1 \dots g_{p-1} g_{p+1} \dots g_{k+1}) \triangleleft_a g_p.$$

So the result holds for all  $k \geq 1$ . □

**Proposition 17** *On  $S(\mathfrak{g}'_a)$ ,  $\otimes = \odot$ .*

**Proof.** Let  $f, g \in S(\mathfrak{g}'_a)$ ; let us prove that  $f \otimes g = f \odot g$ . We assume that  $f = f_1 \dots f_k$ ,  $g = g_1, \dots, g_l$ , with  $f_1, \dots, f_k, g_1, \dots, g_l \in \mathfrak{g}'_a$ , and we proceed by induction on  $k$ . If  $k = 0$ , then  $f = 1$  and  $f \otimes g = f \odot g = g$ . Let us assume the result at all ranks  $< k$ . We proceed by induction on  $l$ . If  $l = 0$ , then  $g = 1$  and  $f \otimes g = f \odot g = f$ . Let us assume the result at all ranks  $< l$ . We put:

$$\Delta(f) = f \otimes 1 + 1 \otimes f + f' \otimes f'', \quad \Delta(g) = g \otimes 1 + 1 \otimes g + g' \otimes g''.$$

The induction hypothesis on  $k$  holds for  $f'$  and  $f''$  and the induction hypothesis on  $l$  holds for  $g'$  and  $g''$ . From:

$$\Delta(f \otimes g - f \odot g) = f^{(1)} \otimes g^{(1)} \otimes f^{(2)} \otimes g^{(2)} - f^{(1)} \odot g^{(1)} \otimes f^{(2)} \odot g^{(2)},$$

these two induction hypotheses give:

$$\Delta(f \otimes g - f \odot g) = (f \otimes g - f \odot g) \otimes 1 + 1 \otimes (f \otimes g - f \odot g).$$

So  $f \otimes g - f \odot g \in \text{Prim}(S(\mathfrak{g}'_a)) = \mathfrak{g}'_a$ . Let  $\pi$  be the canonical projection on  $\mathfrak{g}'_a$  in  $S(\mathfrak{g}'_a)$ . We obtain:

$$\begin{aligned} \pi(f \otimes g) &= \pi \left( \sum_{I \subseteq [l]} \left( f * \prod_{i \in I} g_i \right) \triangleleft \prod_{j \notin I} g_j \right) \\ &= \pi \left( \sum_{[l] = I_0 \sqcup \dots \sqcup I_k} \left( f_1 * \prod_{i \in I_1} g_i \right) \dots \left( f_k * \prod_{i \in I_k} g_i \right) \triangleleft \prod_{i \in I_0} g_i \right) \\ &= \pi \left( \sum_{[l] = J_1 \sqcup \dots \sqcup J_k} \prod_{p=1}^k \left( f_p * \prod_{i \in J_k} g_i + \sum_{j_p \in J_p} \left( f_p * \prod_{i \in J_p \setminus \{j_p\}} g_i \right) \triangleleft_a g_{j_p} \right) \right) \\ &= \pi \left( \sum_{[l] = J_1 \sqcup \dots \sqcup J_k} \left( \prod_{p=1}^k f_p \bullet \prod_{i \in J_p} g_i \right) \right) \\ &= \pi \left( (f_1 \bullet g^{(1)}) \dots (f_k \bullet g^{(k)}) \right) \\ &= \pi(f \bullet g) \\ &= \pi(f \odot g). \end{aligned}$$

As  $f \otimes g - f \odot g \in \mathfrak{g}'_a$ ,  $f \otimes g = f \odot g$ . □

### 3.4 Graduation

We assume in this whole paragraph that  $a = (1, 0, \dots, 0)$  and  $V$  is finite-dimensional. We decompose the  $\mathfrak{g}_a$ -module  $V$  as a direct sum of indecomposables. By Proposition 9, decomposing each indecomposables, we obtain a decomposition of  $V$  of the form:

$$V = V^{(0)} \oplus \dots \oplus V^{(k)},$$

with  $F_1(V^{(p)}) \subseteq V^{(p)}$  and  $F_i(V^{(p)}) \subseteq V^{(p-1)}$  for all  $i \geq 2$ , for all  $p \in [k]$ . We put  $V_p = V^{(k+1-p)}$  for all  $p \in [k+1]$ . This defines a graduation of  $V$ , which induces a connected graduation of  $T(V)$ . For this graduation of  $V$ ,  $F_1$  is homogeneous of degree 0 and  $F_i$  is homogeneous of degree 1 for all  $i \geq 2$ . We define a graduation of  $\mathfrak{g}'_a = T(V)^N$ :

$$\forall n \geq 0, (\mathfrak{g}'_a)_n = T(V)_n \epsilon_1 \oplus \bigoplus_{i=2}^N T(V)_{n-1} \epsilon_i.$$

Let  $v, w \in T(V)$ , homogeneous of respective degree  $k$  and  $l$ . Let  $i, j \geq 2$ . Then:

- $v\epsilon_1$  is homogeneous of degree  $k$ .
- $v\epsilon_i$  is homogeneous of degree  $k+1$ .
- $w\epsilon_1$  is homogeneous of degree  $l$ .
- $w\epsilon_j$  is homogeneous of degree  $l+1$ .

As  $v \Delta w$  is homogeneous of degree  $k+l$ :

- $v\epsilon_1 \Delta w\epsilon_1 = v \Delta w\epsilon_1$  is homogeneous of degree  $k+l$ .
- $v\epsilon_1 \Delta w\epsilon_j = v \Delta w\epsilon_j$  is homogeneous of degree  $k+l+1$ .
- $v\epsilon_i \Delta w\epsilon_1 = 0$  is homogeneous of degree  $k+l+1$ .
- $v\epsilon_i \Delta w\epsilon_j = 0$  is homogeneous of degree  $k+l+2$ .

Consequently, the product  $(1,0,\dots,0)\Delta$  is homogeneous of degree 0. Proposition 14 implies that  $*$  is homogeneous of degree 0; summing,  $\bullet$  is also homogeneous of degree 0. Hence:

**Proposition 18** *The decomposition of  $V$  in indecomposable  $\mathfrak{g}_{(1,0,\dots,0)}$ -modules induces a graduation of the post-Lie algebra  $\mathfrak{g}'_{(1,0,\dots,0)}$ .*

We put:

$$P(X) = \sum_{i=1}^{k+1} \dim(V_p) X^p \in \mathbb{K}[X].$$

the formal series of  $\mathfrak{g}'_{(1,0,\dots,0)}$  is:

$$\begin{aligned} R(X) &= \sum_{p=1}^{\infty} \dim((\mathfrak{g}'_{(1,0,\dots,0)})_p) X^p \\ &= \frac{1}{1-P(X)} + (N-1) \frac{X}{1-P(X)} = \frac{1+(N-1)X}{1-P(X)}. \end{aligned}$$

Note that  $R(0) = 1$ : indeed,  $(\mathfrak{g}'_{(1,0,\dots,0)})_0 = \text{Vect}(\emptyset\epsilon_1)$ . The augmentation ideal of  $\mathfrak{g}'_{(1,0,\dots,0)}$  is:

$$(\mathfrak{g}'_{(1,0,\dots,0)})_+ = T(V)_+ \times T(V)^{N-1}.$$

This is a graded, connected post-Lie algebra.

**Example.** For the SISO case,  $V_1 = \text{Vect}(x_2)$  and  $V_2 = \text{Vect}(x_1)$ . The formal series of  $\mathfrak{g}_{SISO}$  is:

$$R_{SISO}(X) = \frac{1+X}{1-X-X^2} = 1 + 2X + 3X^2 + 5X^3 + 8X^4 + 13X^5 + \dots$$

Hence,  $(\dim(\mathfrak{g}_{SISO})_n)_{n \geq 0}$  is the Fibonacci sequence A000045 [9]. For example:

$$\begin{aligned} (\mathfrak{g}_{SISO})_0 &= \text{Vect}(\emptyset\epsilon_1), \\ (\mathfrak{g}_{SISO})_1 &= \text{Vect}(x_2\epsilon_1, \emptyset\epsilon_2), \\ (\mathfrak{g}_{SISO})_2 &= \text{Vect}(x_1\epsilon_1, x_2x_2\epsilon_1, x_2\epsilon_2), \\ (\mathfrak{g}_{SISO})_3 &= \text{Vect}(x_1x_2\epsilon_1, x_2x_1\epsilon_1, x_2x_2x_2\epsilon_1, x_1\epsilon_2, x_2x_2\epsilon_2). \end{aligned}$$



## 4 Graded dual

We assume in this section that  $a = (1, 0, \dots, 0)$ . The augmentation ideal of  $\mathfrak{g}'_a$  is denoted by  $(\mathfrak{g}'_a)_+$ ; recall that  $(\mathfrak{g}'_a)_0 = Vect(\emptyset\epsilon_1)$ .

- As  $(\mathfrak{g}'_a)_+$  is a graded, connected Lie algebra, its enveloping algebra  $\mathcal{U}((\mathfrak{g}'_a)_+)$  is a graded, connected Hopf algebra, and its graded dual also is. We denote it by  $\mathcal{H}_V$ .
- As an algebra,  $\mathcal{H}_V$  is identified with  $S((\mathfrak{g}'_a)^*)/\langle\emptyset\epsilon_1\rangle$ . We identify  $(\mathfrak{g}'_a)^*$  with  $T(V^*)^N$  via the pairing:

$$\langle f_1 \dots f_k \epsilon_i, x_1 \dots x_l \epsilon_j \rangle = \delta_{i,j} \delta_{k,l} f_1(x_1) \dots f_k(x_k).$$

- The coproduct dual of  $\odot = \otimes$  is denoted by  $\Delta_\bullet$ .
- The dual of the product  $\Delta_j$  defined on  $\mathfrak{g}'_a$  is denoted by  $\Delta_{\Delta_j}$ , defined on  $(\mathfrak{g}'_a)^* = T(V^*)^N$ .
- We define a coproduct  $\Delta_*$  on  $S((\mathfrak{g}'_a)_+)^*$ , dual of the right action  $*$ . Therefore, this is right coaction of  $(\mathcal{H}_V, \Delta_\bullet)$  on itself:

$$(\Delta_* \otimes Id) \circ \Delta_* = (Id \otimes \Delta_\bullet) \circ \Delta_*.$$

**Notations.**

1. For all  $y \in V^*$ , we define  $\theta_y : (\mathfrak{g}'_a)^* \rightarrow (\mathfrak{g}'_a)^*$  by  $\theta_y(f) = yf$ .
2. For all  $x \in (\mathcal{H}_V)_+$ , we put  $\overline{\Delta}_\bullet(x) = \Delta_\bullet(x) - 1 \otimes x$  and  $\overline{\Delta}_*(x) = \Delta_*(x) - 1 \otimes x$ . For all  $g, f, f_1, \dots, f_k \in (\mathfrak{g}'_a)_+^*$ :

$$\langle \overline{\Delta}_*(g), f \otimes f_1 \dots f_k \rangle = \langle g, f * f_1 \dots f_k \rangle.$$

### 4.1 Deshuffling coproducts

**Proposition 19** For all  $g \in T(V)$ , for all  $i \in [N]$ ,  $\Delta_{\Delta_j}(g\epsilon_k) = \Delta_\Delta(g)(\epsilon_k \otimes \epsilon_j)$ .

**Proof.** Let  $f_1, f_2 \in T(V)$ ,  $i_1, i_2 \in [N]$ .

$$\begin{aligned} \langle \Delta_{\Delta_j}(g\epsilon_k), f_1 \epsilon_{i_1} \otimes f_2 \epsilon_{i_2} \rangle &= \langle g\epsilon_k, f_1 \epsilon_{i_1} \Delta_j f_2 \epsilon_{i_2} \rangle \\ &= \delta_{i_2, j} \langle g\epsilon_k, f_1 \Delta f_2 \epsilon_{i_1} \rangle \\ &= \delta_{i_2, j} \delta_{i_1, k} \langle g, f_1 \Delta f_2 \rangle \\ &= \delta_{i_2, j} \delta_{i_1, k} \langle \Delta_\Delta(g), f_1 \otimes f_2 \rangle \\ &= \langle \Delta_\Delta(g)(\epsilon_k \otimes \epsilon_j), f_1 \epsilon_{i_1} \otimes f_2 \epsilon_{i_2} \rangle. \end{aligned}$$

As the pairing is nondegenerate, we obtain the result.  $\square$

**Notations.** We define inductively, for  $l \geq 0$ ,  $j_1, \dots, j_l \in [N]$ :

$$\begin{cases} \Delta_{\Delta_\emptyset} = Id, \\ \Delta_{\Delta_{j_1, \dots, j_l}} = \left( \Delta_{\Delta_{j_1}} \otimes Id^{\otimes(l-1)} \right) \circ \Delta_{\Delta_{j_2, \dots, j_l}}. \end{cases}$$

For all  $g \in T(V^*)$ , for all  $i \in [N]$ :

$$\Delta_{\Delta_{j_1, \dots, j_l}}(g\epsilon_k) = \Delta_\Delta^{(l)}(g)(\epsilon_k \otimes \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_l});$$

for all  $f_1, \dots, f_l \in T(V)$ :

$$\langle \Delta_{\Delta_{j_1, \dots, j_l}}(g), f_1 \otimes \dots \otimes f_{l+1} \rangle = \langle g, f_1 \Delta_{j_1} \dots \Delta_{j_l} f_{l+1} \rangle.$$

## 4.2 Dual of the post-Lie product

**Proposition 20** In  $\mathcal{H}_V = S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1 \rangle$ :

- For all  $i \in [N]$ ,  $\Delta_*(\emptyset\epsilon_i) = \emptyset\epsilon_i \otimes 1 + 1 \otimes \emptyset\epsilon_i$ .
- For all  $y \in V^*$ ,  $g \in (\mathfrak{g}'_a)^*$ :

$$\bar{\Delta}_* \circ \theta_y(g) = \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l}^*(y)} \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\Delta_{j_1, \dots, j_l}}(g),$$

where we denote by  $\mu$  the sum of the iterated products of  $\mathcal{H}_V$ :

$$\mu : \begin{cases} T(\mathcal{H}_V) & \longrightarrow \mathcal{H}_V \\ g_1 \otimes \dots \otimes g_k & \longrightarrow g_1 \dots g_k. \end{cases}$$

**Proof.** The first point comes from  $\emptyset\epsilon_i * \mathcal{U}(\mathfrak{g}'_a)_+ = (0)$ .

In order to prove the formula, it is enough to prove that, for  $f, f_1, \dots, f_k \in \mathfrak{g}$ :

$$\langle \bar{\Delta}_* \circ \theta_y(g), f \otimes f_1 \dots f_k \rangle = \left\langle \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l}^*(y)} \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\Delta_{j_1, \dots, j_l}}(g), f \otimes f_1 \dots f_k \right\rangle,$$

or equivalently:

$$\langle \theta_y(g), f * f_1 \dots f_k \rangle = \left\langle \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l}^*(y)} \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\Delta_{j_1, \dots, j_l}}(g), f \otimes f_1 \dots f_k \right\rangle,$$

If  $f = \emptyset\epsilon_i$ , both sides are equal to 0. Otherwise, we can assume that  $f = x f'$ , with  $x \in V$  and  $f' \in \mathfrak{g}$ .

$$\begin{aligned} & \langle \theta_y(g), f * f_1 \dots f_k \rangle \\ &= \left\langle yg, \sum_{I=\{i_1 < \dots < i_l\} \subseteq [k]} \sum_{j_1, \dots, j_l \in [N]} F_{j_1, \dots, j_l}(x) \left( f' * \left( \prod_{i \notin I} f_i \right) \Delta_{j_1} f_{i_1} \dots \Delta_{j_l} f_{i_l} \right) \right\rangle \\ &= \sum_{I=\{i_1 < \dots < i_l\} \subseteq [k]} \sum_{j_1, \dots, j_l \in [N]} \langle y, F_{j_1, \dots, j_l}(x) \rangle \langle \Delta_{\Delta_{j_1, \dots, j_l}}(g), f' * \left( \prod_{i \notin I} f_i \right) \otimes f_{i_1} \dots \otimes f_{i_l} \rangle \\ &= \sum_{j_1, \dots, j_l \in [N]} \langle F_{j_1, \dots, j_l}^*(y), x \rangle \langle (Id \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\Delta_{j_1, \dots, j_l}}(g), f' \otimes f_1 \dots f_k \rangle \\ &= \sum_{j_1, \dots, j_l \in [N]} \langle (\theta_{F_{j_1, \dots, j_l}^*(y)} \otimes Id) \circ (Id \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\Delta_{j_1, \dots, j_l}}(g), x f' \otimes f_1 \dots f_k \rangle, \end{aligned}$$

which ends the proof.  $\square$

In order to obtain a better description of the coproduct  $\bar{\Delta}_*$ , we are going to identify the following three objects:

$$\begin{array}{ccc} & S((\mathfrak{g}'_a)_+^*) & \\ \sim \swarrow & & \searrow \sim \\ S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1 \rangle & & S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1 - 1 \rangle \end{array}$$

Both identification sends  $x \in (\mathfrak{g}'_a)_+^*$  to its class. Let us reformulate Proposition 20 in the vector space  $S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1 - 1 \rangle$ :

$$\begin{aligned} \bar{\Delta}_* \circ \theta_y(g\epsilon_k) &= \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l}^*}(y) \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\tilde{\Delta}}^{(l)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_l}) \\ &\quad - \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l, 1}^*}(y) \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\tilde{\Delta}}^{(l+1)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_l} \otimes \epsilon_1) \\ &= \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l}^*}(y) \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\tilde{\Delta}}^{(l)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_l}) \\ &\quad - \left( \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l, 1}^*}(y) \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\tilde{\Delta}}^{(l)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_l}) \right) (1 \otimes \emptyset\epsilon_1). \end{aligned}$$

Finally, identifying in  $S((\mathfrak{g}'_a)_+^*)$ :

**Proposition 21** *For all  $j_1, \dots, j_l \in [N]$ , we put:*

$$G_{j_1, \dots, j_l} = F_{j_1, \dots, j_l} - F_{j_1, \dots, j_l, 1}.$$

In  $S((\mathfrak{g}'_a)_+^*)/\langle \emptyset\epsilon_1 - 1 \rangle$ :

- For all  $i \in [N]$ ,  $\bar{\Delta}_*(\emptyset\epsilon_i) = \emptyset\epsilon_i \otimes 1$ .
- For all  $y \in V^*$ , for all  $g \in (\mathfrak{g}'_a)_+^*$ :

$$\bar{\Delta}_* \circ \theta_y(g) = \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{G_{j_1, \dots, j_l}^*}(y) \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\Delta_{j_1, \dots, j_l}}(g).$$

**Example.** For  $\mathfrak{g}_{SISO}$ , as  $V$  is a module over the associative algebra  $(\mathfrak{g}_{(1,0)}, \triangleleft)$ , if  $l \geq 2$ ,  $F_{j_1, \dots, j_l} = 0$  by Proposition 11, so  $G_{j_1, \dots, j_l} = 0$ . Moreover:

$$\begin{aligned} F_{\emptyset} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & F_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & F_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \\ G_{\emptyset} = F_{\emptyset} - F_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & G_1 = F_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & G_2 = F_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \\ G_{\emptyset}^* &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & G_1^* &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & G_2^* &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

The coproduct  $\bar{\Delta}_*$  on  $S((\mathfrak{g}_{SISO})_+^*)$  is given by:

- For all  $i \in [2]$ ,  $\bar{\Delta}_*(\emptyset\epsilon_i) = \emptyset\epsilon_i \otimes 1$ .
- For all  $g \in \mathbb{K}\langle x_1, x_2 \rangle$ , for all  $i \in [2]$ :

$$\begin{aligned} \bar{\Delta}_* \circ \theta_{x_1}(g\epsilon_i) &= (\theta_{x_1} \otimes Id) \circ \bar{\Delta}_*(g\epsilon_i) + (\theta_{x_2} \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\Delta}(g)\epsilon_i \otimes \epsilon_2), \\ \bar{\Delta}_* \circ \theta_{x_2}(g\epsilon_i) &= (\theta_{x_2} \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\Delta}(g)\epsilon_i \otimes \epsilon_1). \end{aligned}$$

These are formulas of Lemma 4.1 of [4], where  $a_w = w\epsilon_2$ ,  $b_w = w\epsilon_1$ ,  $\theta_0 = \theta_{x_1}$ ,  $\theta_1 = \theta_{x_2}$  and  $\tilde{\Delta} = \bar{\Delta}_*$ .

### 4.3 Dual of the pre-Lie product

**Notations.** We denote by  $\Delta_{\bullet}\Delta$  the coproduct on  $T_+(V^*) \otimes (V)^{N-1}$  dual to the product  ${}_{\bullet}\Delta$ . As  ${}_{\bullet}\Delta = \Delta_{\bullet}^{op}$ ,  $\Delta_{\bullet}\Delta = \Delta_{\Delta_{\bullet}}^{cop}$ , and for all  $g \in T(V)$ , for all  $i \in [N]$ :

$$\Delta_{\bullet}\Delta(g\epsilon_i) = \Delta_{\Delta}(g)(\epsilon_1 \otimes \epsilon_k).$$

**Proposition 22** *In  $S((\mathfrak{g}'_a)_+^*)/\langle \emptyset\epsilon_1 \rangle$ , for all  $g \in (\mathfrak{g}'_a)_+^*$ :*

$$\overline{\Delta}_{\bullet}(g) = \overline{\Delta}_*(g) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id) \circ \Delta_{\bullet}\Delta(g).$$

**Proof.** Let  $f, f_1, \dots, f_k \in (\mathfrak{g}'_a)_+$ .

$$\begin{aligned} \langle \overline{\Delta}_{\bullet}(g), f \otimes f_1 \dots f_k \rangle &= \langle g, f \bullet f_1 \dots f_k \rangle \\ &= \langle g, f * f_1 \dots f_k + \sum_{p=1}^k (f * f_1 \dots \widehat{f}_p \dots f_k) {}_{\bullet}\Delta f_p \rangle \\ &= \langle \overline{\Delta}_*(g), f \otimes f_1 \dots f_k \rangle + \langle \Delta_{\bullet}\Delta(g), \sum_{p=1}^k f * f_1 \dots \widehat{f}_p \dots f_k \otimes f_p \rangle \\ &= \langle \overline{\Delta}_*(g), f \otimes f_1 \dots f_k \rangle + \langle (\Delta_* \otimes Id) \circ \Delta_{\bullet}\Delta(g), \sum_{p=1}^k f \otimes f_1 \dots \widehat{f}_p \dots f_k \otimes f_p \rangle \\ &= \langle \overline{\Delta}_*(g), f \otimes f_1 \dots f_k \rangle + \langle (Id \otimes \mu) \circ (\Delta_* \otimes Id) \circ \Delta_{\bullet}\Delta(g), f \otimes f_1 \dots f_k \rangle. \end{aligned}$$

As  $(\mathfrak{g}'_a, *)$  is pre-Lie,  $\overline{\Delta}_{\bullet}(g) \in (\mathfrak{g}'_a)_+^* \otimes S((\mathfrak{g}'_a)_+^*)$  and the nondegeneracy of the pairing implies the formula.  $\square$

Rewriting this formula in  $S((\mathfrak{g}'_a)_+^*)/\langle \emptyset\epsilon_1 - 1 \rangle$ :

$$\begin{aligned} \overline{\Delta}_{\bullet}(g\epsilon_1) &= \overline{\Delta}_*(g\epsilon_1) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\Delta}(g)(\epsilon_1 \otimes \epsilon_1)) \\ &= \overline{\Delta}_*(g\epsilon_1) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)((\Delta_{\Delta}(g) - g \otimes \emptyset)(\epsilon_1 \otimes \epsilon_1)) \\ &= \overline{\Delta}_*(g\epsilon_1)(1 \otimes (1 - \emptyset\epsilon_1)) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\Delta}(g)(\epsilon_1 \otimes \epsilon_1)) \\ &= (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\Delta}(g)(\epsilon_1 \otimes \epsilon_1)). \end{aligned}$$

Identifying in  $S((\mathfrak{g}'_a)_+^*)$ :

**Proposition 23** *In  $S((\mathfrak{g}'_a)_+^*)/\langle \emptyset\epsilon_1 - 1 \rangle$ , if  $g \in T(V^*)$ :*

$$\begin{aligned} \overline{\Delta}_{\bullet}(g\epsilon_1) &= (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\Delta}(g)(\epsilon_1 \otimes \epsilon_1)), \\ \text{if } i \geq 2, \overline{\Delta}_{\bullet}(g\epsilon_i) &= \overline{\Delta}_*(g\epsilon_i) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\Delta}(g)(\epsilon_i \otimes \epsilon_1)), \end{aligned}$$

with the convention  $\emptyset\epsilon_1 = 1$ . We put  $\Delta_{\bullet}(g) = \overline{\Delta}_{\bullet}(g) + 1 \otimes g$  for all  $g \in (\mathfrak{g}'_a)_+^*$  and extend  $\Delta_{\bullet}$  to  $S((\mathfrak{g}'_a)_+^*)$  as an algebra morphism. This coproduct makes  $S((\mathfrak{g}'_a)_+^*)$  a Hopf algebra, isomorphic to the graded dual of the enveloping algebra of  $((\mathfrak{g}'_a)_+, [-, -]_*)$ .

**Remark.** These are *mutatis mutandis* the formulas of Lemma 4.3 in [4].

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