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# EXACT DISTANCE COLORING IN TREES

NICOLAS BOUSQUET, LOUIS ESPERET, ARARAT HARUTYUNYAN,  
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ABSTRACT. For an integer  $q \geq 2$  and an even integer  $d$ , consider the graph obtained from a large complete  $q$ -ary tree by connecting with an edge any two vertices at distance exactly  $d$  in the tree. This graph has clique number  $q + 1$ , and the purpose of this short note is to prove that its chromatic number is  $\Theta\left(\frac{d \log q}{\log d}\right)$ . It was not known that the chromatic number of this graph grows with  $d$ . As a simple corollary of our result, we give a negative answer to a problem of Van den Heuvel and Naserasr, asking whether there is a constant  $C$  such that for any odd integer  $d$ , any planar graph can be colored with at most  $C$  colors such that any pair of vertices at distance exactly  $d$  have distinct colors. Finally, we study interval coloring of trees (where vertices at distance at least  $d$  and at most  $cd$ , for some real  $c > 1$ , must be assigned distinct colors), giving a sharp upper bound in the case of bounded degree trees.

## 1. INTRODUCTION

Given a metric space  $X$  and some real  $d > 0$ , let  $\chi(X, d)$  be the minimum number of colors in a coloring of the elements of  $X$  such that any two elements at distance exactly  $d$  in  $X$  are assigned distinct colors. The classical Hadwiger-Nelson problem asks for the value of  $\chi(\mathbb{R}^2, 1)$ , where  $\mathbb{R}^2$  is the Euclidean plane. It is known that  $5 \leq \chi(\mathbb{R}^2, 1) \leq 7$  [1] and since the Euclidean plane  $\mathbb{R}^2$  is invariant under homothety,  $\chi(\mathbb{R}^2, 1) = \chi(\mathbb{R}^2, d)$  for any real  $d > 0$ . Let  $\mathbb{H}^2$  denote the hyperbolic plane. Kloeckner [3] proved that  $\chi(\mathbb{H}^2, d)$  is at most linear in  $d$  (the multiplicative constant was recently improved by Parlier and Petit [6]), and observed that  $\chi(\mathbb{H}^2, d) \geq 4$  for any  $d > 0$ . He raised the question of determining whether  $\chi(\mathbb{H}^2, d)$  grows with  $d$  or can be bounded independently of  $d$ . As noticed by Kahle (see [3]), it is not known whether  $\chi(\mathbb{H}^2, d) \geq 5$  for some real  $d > 0$ . Parlier and Petit [6] recently suggested to study infinite regular trees as a discrete analog of the hyperbolic plane. Note that any graph  $G$  can be considered as a metric space (whose elements are the vertices of  $G$  and whose metric is the graph distance in  $G$ ), and in this context  $\chi(G, d)$  is precisely the minimum number of colors in a vertex coloring of  $G$  such that vertices at distance  $d$  apart are assigned different colors. Note that  $\chi(G, d)$  can be equivalently defined as the chromatic number of the *exact  $d$ -th power* of  $G$ , that is, the graph with the same vertex-set as  $G$  in which two vertices are adjacent if and only if they are at distance exactly  $d$  in  $G$ .

Let  $T_q$  denote the infinite  $q$ -regular tree. Parlier and Petit [6] observed that when  $d$  is odd,  $\chi(T_q, d) = 2$  and proved that when  $d$  is even,  $q \leq \chi(T_q, d) \leq (d + 1)(q - 1)$ . A

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similar upper bound can also be deduced from the results of Van den Heuvel, Kierstead, and Quiroz [2], while the lower bound is a direct consequence of the fact that when  $d$  is even, the clique number of the exact  $d$ -th power of  $T_q$  is  $q$  (note that it does not depend on  $d$ ). In this short note, we prove that when  $q \geq 3$  is fixed,

$$\frac{d \log(q-1)}{4 \log(d/2) + 4 \log(q-1)} \leq \chi(T_q, d) \leq (2 + o(1)) \frac{d \log(q-1)}{\log d},$$

where the asymptotic  $o(1)$  is in terms of  $d$ . A simple consequence of our main result is that for any even integer  $d$ , the exact  $d$ -th power of a complete binary tree of depth  $d$  is of order  $\Theta(d/\log d)$  (while its clique number is equal to 3).

The following problem (attributed to Van den Heuvel and Naserasr) was raised in [4] (see also [2] and [5]).

**Problem 1.1** (Problem 11.1 in [4]). *Is there a constant  $C$  such that for every odd integer  $d$  and every planar graph  $G$  we have  $\chi(G, d) \leq C$ ?*

We will show that our result on large complete binary trees easily implies a negative answer to Problem 1.1. More precisely, we will prove that the graph  $U_3^d$  obtained from a complete binary tree of depth  $d$  by adding an edge between any two vertices with the same parent gives a negative answer to Problem 1.1 (in particular, for odd  $d$ , the chromatic number of the exact  $d$ -th power of  $U_3^d$  grows as  $\Theta(d/\log d)$ ). We will also prove that the exact  $d$ -th power of a specific subgraph  $Q_3^d$  of  $U_3^d$  grows as  $\Omega(\log d)$ . Note that  $U_3^d$  and  $Q_3^d$  are outerplanar (and thus, planar) and chordal (see Figure 2).

Kloeckner [3] proposed the following variant of the original problem: For a metric space  $X$ , an integer  $d$  and a real  $c > 1$ , we denote by  $\chi(X, [d, cd])$  the smallest number of colors in a coloring of the elements of  $X$  such that any two elements of  $X$  at distance at least  $d$  and at most  $cd$  apart have distinct colors. Considering as above the natural metric space defined by the infinite  $q$ -regular tree  $T_q$ , Parlier and Petit [6] proved that

$$q(q-1)^{\lfloor cd/2 \rfloor - \lfloor d/2 \rfloor} \leq \chi(T_q, [d, cd]) \leq (q-1)^{\lfloor cd/2 + 1 \rfloor} (\lfloor cd \rfloor + 1).$$

We will show that  $\chi(T_q, [d, cd]) \leq \frac{q}{q-2}(q-1)^{\lfloor cd/2 \rfloor - d/2 + 1} + cd + 1$ , which implies that the lower bound of Parlier and Petit [6] (which directly follows from a clique size argument) is asymptotically sharp.

## 2. EXACT DISTANCE COLORING

Throughout the paper, we assume that the infinite  $q$ -regular tree  $T_q$  is rooted in some vertex  $r$ . This naturally defines the children and descendants of a vertex and the parent and ancestors of a vertex distinct from  $r$ . In particular, given a vertex  $u$ , we define the ancestors  $u^0, u^1, \dots$  of  $u$  inductively as follows:  $u^0 = u$  and for any  $i$  such that  $u^i$  is not the root,  $u^{i+1}$  is the parent of  $u^i$ . With this notation,  $u^d$  can be equivalently defined as the ancestor of  $u$  at distance  $d$  from  $u$  (if such a vertex exists). For a given vertex  $u$  in  $T_q$ , the *depth* of  $u$ , denoted by  $\text{depth}(u)$ , is the distance between  $u$  and  $r$  in  $T_q$ . For a vertex  $v$  and an integer  $\ell$ , we define  $L(v, \ell)$  as the set of descendants of  $v$  at distance exactly  $\ell$  from  $v$  in  $T_q$ .

We first prove an upper bound on  $\chi(T_q, d)$ .

**Theorem 2.1.** *For any integer  $q \geq 3$ , any even integer  $d$ , and any integer  $k \geq 1$  such that  $k(q-1)^{k-1} \leq d$ , we have  $\chi(T_q, d) \leq (q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} + 1$ . In particular,  $\chi(T_q, d) \leq d+q+1$ , and when  $q$  is fixed and  $d$  tends to infinity,  $\chi(T_q, d) \leq (2+o(1)) \frac{d \log(q-1)}{\log d}$ .*

*Proof.* A vertex of  $T_q$  distinct from  $r$  and whose depth is a multiple of  $k$  is said to be a *special vertex*. Let  $v$  be a special vertex. Every special vertex  $u$  distinct from  $v$  such that  $u^k = v^k$  is called a *cousin* of  $v$ . Note that  $v$  has at most  $q(q-1)^{k-1} - 1$  cousins (at most  $(q-1)^k - 1$  if  $v^k \neq r$ ). A special vertex  $u$  is said to be a *relative* of  $v$  if  $u$  is either a cousin of  $v$ , or  $u$  has the property that  $u$  and  $v^k$  have the same depth and are at distance at most  $k$  apart in  $T_q$ . Two vertices  $a, b$  at distance at most  $k$  apart and at the same depth must satisfy  $a^{\lfloor k/2 \rfloor} = b^{\lfloor k/2 \rfloor}$ , and so the number of vertices  $u$  such that  $u$  and  $v^k$  have the same depth and are at distance at most  $k$  apart in  $T_q$  is  $(q-1)^{\lfloor k/2 \rfloor}$ . It follows that if  $v^k = r$ , then  $v$  has at most  $q(q-1)^{k-1} - 1$  relatives and otherwise  $v$  has at most  $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} - 1$  relatives.

The first step is to define a coloring  $C$  of the special vertices of  $T_q$ . This will be used later to define the desired coloring of  $T_q$ , i.e. a coloring such that vertices of  $T_q$  at distance  $d$  apart are assigned distinct colors (in this second coloring, the special vertices will not retain their color from  $C$ ).

We greedily assign a color  $C(v)$  to each special vertex  $v$  of  $T_q$  as follows: we consider the vertices of  $T_q$  in a breadth-first search starting at  $r$ , and for each special vertex  $v$  we encounter, we assign to  $v$  a color distinct from the colors already assigned to its relatives, and from the set of ancestors  $v^{ik}$  of  $v$ , where  $2 \leq i \leq \frac{d}{k} + 1$  (there are at most  $\frac{d}{k}$  such vertices). Note that if  $v^k = r$ , the number of colors forbidden for  $v$  is at most  $q(q-1)^{k-1} - 1$  and if  $v^k \neq r$  the number of colors forbidden for  $v$  is at most  $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} - 1$ . Since  $k(q-1)^{k-1} \leq d$ , in both cases  $v$  has at most  $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} - 1$  forbidden colors, therefore we can obtain the coloring  $C$  by using at most  $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k}$  colors.

For any special vertex  $v$ , the set of descendants of  $v$  at distance at least  $d/2 - k$  and at most  $d/2 - 1$  from  $v$  is denoted by  $K(v, k)$ . We now define the desired coloring of  $T_q$  as follows: for each special vertex  $v$ , all the vertices of  $K(v, k)$  are assigned the color  $C(v)$ . Finally, all the vertices at distance at most  $d/2 - 1$  from  $r$  are colored with a single new color (note that any two vertices in this set lie at distance less than  $d$  apart). The resulting vertex-coloring of  $T_q$  is called  $c$ . Note that  $c$  uses at most  $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} + 1$  colors, and indeed every vertex of  $T_q$  gets exactly one color.

We now prove that vertices at distance  $d$  apart in  $T_q$  are assigned distinct colors in  $c$ . Assume for the sake of contradiction that two vertices  $x$  and  $y$  at distance  $d$  apart were assigned the same color. Then the depth of both  $x$  and  $y$  is at least  $d/2$ . We can assume by symmetry that the difference  $t$  between the depth of  $x$  and the depth of  $y$  is such that  $0 \leq t \leq d$  since otherwise they would be at distance more than  $d$ . Let  $u$  be the unique (special) vertex of  $T_q$  such that  $x \in K(u, k)$  and  $v$  be the unique (special) vertex such that  $y \in K(v, k)$ . By the definition of our coloring  $c$ , we have  $C(u) = C(v)$ . Note that  $u$  and  $v$  are distinct; indeed, otherwise  $x$  and  $y$  would not be at distance  $d$  in  $T_q$ . Assume first that

$u$  and  $v$  have the same depth. Then since  $u$  and  $x$  (resp.  $v$  and  $y$ ) are distance at least  $d/2 - k$  apart,  $u$  and  $v$  are cousins (and thus, relatives), which contradicts the definition of the vertex-coloring  $C$ . We may, therefore, assume that the depths of  $u$  and  $v$  are distinct. Moreover, since  $u$  and  $v$  are special vertices, we may assume that their depths differ by at least  $k$ . In particular,  $u$  lies deeper than  $v$  in  $T_q$ .

First assume that the depths of  $u$  and  $v$  differ by at least  $2k$ . Then  $v$  is not an ancestor of  $u$  in  $T_q$ . Indeed, for otherwise we would have  $v = u^{ik}$  for some integer  $2 \leq i \leq \frac{d}{k} + 1$ , which would contradict the definition of  $C$ . This implies that the distance between  $x$  and  $y$  is at least  $d/2 - k + d/2 - k + 2k + 2 = d + 2$ , which is a contradiction. Therefore, we can assume that the depths of  $u$  and  $v$  differ by precisely  $k$ . Since  $v$  is not a relative of  $u$ , we have that  $v \neq u^k$  and the distance between  $u^k$  and  $v$  is more than  $k$ . Moreover, since  $u$  and  $x$  (resp.  $v$  and  $y$ ) are at distance at least  $d/2 - k$  apart, this implies that the distance between  $x$  and  $y$  is more than  $d/2 - k + k + k + d/2 - k = d$ , a contradiction.

Thus,  $c$  is a proper coloring.

By taking  $k = 1$  we obtain a coloring  $c$  using at most  $(q-1)^1 + (q-1)^{\lfloor 1/2 \rfloor} + \frac{d}{1} + 1 = q + d + 1$  colors, and by taking  $k = \lfloor \frac{\log d - \log \log d + \log \log(q-1)}{\log(q-1)} \rfloor$ , we obtain a coloring  $c$  using at most

$$\frac{d \log(q-1)}{\log d} + \sqrt{\frac{d \log(q-1)}{\log d}} + \frac{d \log(q-1)}{\log d - \log \log d + \log \log(q-1) - \log(q-1)} + 1 = (2 + o(1)) \frac{d \log(q-1)}{\log d}$$

colors. □

For  $k = 1$ , the proof above can be optimized to show that  $\chi(T_q, d) \leq q + \frac{d}{2}$  (by simply noting that vertices at even depth and vertices at odd depth can be colored independently). Since we are mostly interested in the asymptotic behaviour of  $\chi(T_q, d)$  (which is of order  $O(\frac{d}{\log d})$ ), we omit the details.

We now prove a simple lower bound on  $\chi(T_q, d)$ . Let  $T_q^d$  be the rooted complete  $(q-1)$ -ary tree of depth  $d$ , with root  $r$ . Note that each node has  $q-1$  children, so this graph is a subtree of  $T_q$ .

**Theorem 2.2.** *For any integer  $q \geq 3$  and any even  $d$ ,  $\chi(T_q^d, d) \geq \log_2(\frac{d}{4} + q - 1)$ .*

*Proof.* Consider any coloring of  $T_q^d$  with colors  $1, 2, \dots, C$ , such that vertices at distance precisely  $d$  apart have distinct colors. For any vertex  $v$  at depth at most  $\frac{d}{2} + 1$  in  $T_q^d$ , the set of colors appearing in  $L(v, \frac{d}{2} - 1)$  is denoted by  $S_v$ . Observe that if  $v$  and  $w$  have the same parent, then  $S_v$  and  $S_w$  are disjoint since for any  $x \in L(v, \frac{d}{2} - 1)$  and  $y \in L(w, \frac{d}{2} - 1)$ ,  $x$  and  $y$  are at distance  $d$ .

Fix some vertex  $u$  at depth at most  $\frac{d}{2}$  in  $T_q^d$  and some child  $v$  of  $u$ . We claim that:

**Claim 2.3.** *For any integer  $1 \leq k \leq \frac{\text{depth}(u)}{2}$ , there is a color of  $S_{u^{2k-1}}$  that does not appear in  $S_v$ .*

To see that Claim 2.3 holds, observe that in the subtree of  $T_q^d$  rooted in  $u^k$ , there is a vertex of  $L(u^{2k-1}, \frac{d}{2} - 1)$  at distance  $d$  from all the elements of  $L(v, \frac{d}{2} - 1)$ . The color of such a vertex does not appear in  $S_v$ , therefore Claim 2.3 holds.

In particular, Claim 2.3 implies that all the sets  $\{S_{u^{2k-1}} \mid 1 \leq k \leq d/4\}$  and  $\{S_w \mid w \text{ is a child of } u\}$  are pairwise distinct. Since there are  $\frac{d}{4} + q - 1$  such sets, we have  $\frac{d}{4} + q - 1 \leq 2^C$  and therefore  $C \geq \log_2(\frac{d}{4} + q - 1)$ , as desired.  $\square$

It was observed by Stéphan Thomassé that the proof of Theorem 2.2 only uses a small fraction of the graph  $T_q^d$ . Consider for simplicity the case  $q = 3$ , and define  $P_3^d$  as the graph obtained from a path  $P = v_0, v_1, \dots, v_d$  on  $d$  edges, by adding, for each  $1 \leq i \leq d$ , a path on  $i$  edges ending at  $v_i$  (see Figure 1). This graph is an induced subgraph of  $T_q^d$  and the proof of Theorem 2.2 directly shows the following<sup>1</sup>.

**Corollary 2.4.** *For any even integer  $d$ ,  $\chi(P_3^d, d) \geq \log_2(d + 8) - 2$ .*

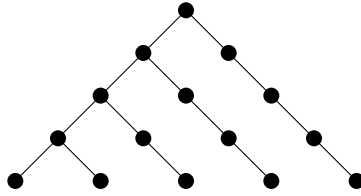


FIGURE 1. The graph  $P_3^4$ .

The proof of Theorem 2.2 can be refined to prove the following better estimate for  $T_q^d$ , showing that the upper bound of Theorem 2.1 is (asymptotically) tight within a constant multiplicative factor of 8.

**Theorem 2.5.** *For any integer  $q \geq 3$  and every even integer  $d \geq 2$ ,  $\chi(T_q^d, d) \geq \frac{d \log(q-1)}{4 \log(d/2) + 4 \log(q-1)}$ .*

*Proof.* Consider any coloring of  $T_q^d$  with colors  $1, 2, \dots, C$ , such that vertices at distance precisely  $d$  apart have distinct colors. We perform a random walk  $v_0, v_1, \dots, v_d$  in  $T_q^d$  as follows: we start with  $v_0 = r$ , and for each  $i \geq 1$ , we choose a child of  $v_i$  uniformly at random and set it as  $v_{i+1}$ . Note that the depth of each vertex  $v_i$  is precisely  $i$ .

From now on we fix a color  $c \in \{1, \dots, C\}$ . For any vertex  $v$  of  $T_q^d$ , the set of vertices contained in the subtree of  $T_q^d$  rooted in  $v$  is denoted by  $V_v$ , and we set  $A_v = \{\text{depth}(u) \mid u \in V_v \text{ and } u \text{ has color } c\}$ . When  $v = v_i$ , for some integer  $0 \leq i \leq d$ , we write  $A_i$  instead of  $A_{v_i}$ .

**Claim 2.6.** *Assume that for some even integers  $i$  and  $j$  with  $2 \leq i < j \leq d$ , and for some vertex  $v$  at depth  $\frac{i+j-d}{2}$ , the set  $A_v$  contains both  $i$  and  $j$ . Then  $v$  has precisely one child  $u$  such that  $A_u$  contains  $i$  and  $j$ , and moreover all the children  $w$  of  $v$  distinct from  $u$  are such that  $A_w$  contains neither  $i$  nor  $j$ .*

<sup>1</sup>Stéphan Thomassé noticed that this can also be deduced from the fact that the vertices at depth at least  $\frac{d}{2}$  and at most  $d$  in the exact  $d$ -th power of  $P_3^d$  induce a *shift graph*.

To see that Claim 2.6 holds, simply note that  $\frac{i+j-d}{2} < i < j$  and if two vertices  $u_1, u_2$  colored  $c$  are respectively at depths  $i$  and  $j$ , and their common ancestor is  $v$ , then they are at distance  $d$  in  $T_q^d$  (which contradicts the fact that they were assigned the same color). Indeed, the distance of  $u_1$  to  $v$  is  $i - \frac{i+j-d}{2}$  and the distance of  $u_2$  to  $v$  is  $j - \frac{i+j-d}{2}$ . This proves the claim.

We now define a family of graphs  $(G_k)_{0 \leq k \leq d/2}$  as follows. For any  $0 \leq k \leq \frac{d}{2}$ , the vertex-set  $V(G_k)$  of  $G_k$  is the set  $A_k \cap 2\mathbb{N} \cap (d/2, d]$ , and two (distinct) even integers  $i, j \in A_k$  are adjacent in  $G_k$  if and only if  $\frac{i+j-d}{2} < k$ . For each  $0 \leq k \leq \frac{d}{2}$  we define the *energy*  $\mathcal{E}_k$  of  $G_k$  as follows:  $\mathcal{E}_k = \sum_{i \in V(G_k)} (q-1)^{\deg(i)}$ , where  $\deg(i)$  denotes the degree of the vertex  $i$  in  $G_k$ .

Note that each graph  $G_k$  depends on the (random) choice of  $v_1, v_2, \dots, v_k$ .

**Claim 2.7.** *For any  $0 \leq k \leq \frac{d}{2} - 1$ ,  $\mathbb{E}(\mathcal{E}_{k+1}) \leq \mathbb{E}(\mathcal{E}_k)$ .*

Assume that  $v_1, v_2, \dots, v_k$  (and therefore also  $G_k$ ) are fixed. Observe that  $G_{k+1}$  is obtained from  $G_k$  by possibly removing some vertices and adding some edges. Thus,  $\mathcal{E}_{k+1}$  can be larger than  $\mathcal{E}_k$  only if  $G_{k+1}$  contains edges that are not in  $G_k$ . Therefore, it suffices to consider the contributions of those pairs of nonadjacent vertices in  $G_k$  which could become adjacent in  $G_{k+1}$  (since these correspond to pairs  $i, j$  with  $k = \frac{i+j-d}{2}$ , these pairs are pairwise disjoint), and prove that these contributions are, in expectation, equal to 0. Fix a pair of even integers  $i < j$  in  $V(G_k)$  with  $k = \frac{i+j-d}{2}$  (and note that  $i$  and  $j$  are not adjacent in  $G_k$ ). By Claim 2.6, either  $v_{k+1}$  is such that  $A_{k+1}$  contains  $i$  and  $j$  (this event occurs with probability  $\frac{1}{q-1}$ ), or  $A_{k+1}$  contains neither  $i$  nor  $j$  (with probability  $1 - \frac{1}{q-1}$ ). As a consequence, for any  $i < j$  in  $V(G_k)$  with  $k = \frac{i+j-d}{2}$ , with probability  $\frac{1}{q-1}$  we add the edge  $ij$  in  $G_{k+1}$  and with probability  $1 - \frac{1}{q-1}$  we remove vertices  $i$  and  $j$  from  $G_{k+1}$ . This implies that for any  $i, j \in V(G_k)$ ,  $i < j$ , with  $k = \frac{i+j-d}{2}$ , with probability  $\frac{1}{q-1}$  we have contribution at most  $(q-1)^{\deg(i)+1} + (q-1)^{\deg(j)+1} - (q-1)^{\deg(i)} - (q-1)^{\deg(j)} = (q-2)((q-1)^{\deg(i)} + (q-1)^{\deg(j)})$  to  $\mathcal{E}_{k+1}$  (where  $\deg$  refers to the degree in  $G_k$ ) and with probability  $1 - \frac{1}{q-1}$  we have a contribution of at most  $-(q-1)^{\deg(i)} - (q-1)^{\deg(j)}$  to  $\mathcal{E}_{k+1}$ . Thus, the expected contribution of such a pair  $i, j$  is at most  $\frac{1}{q-1}(q-2)((q-1)^{\deg(i)} + (q-1)^{\deg(j)}) - \frac{q-2}{q-1}((q-1)^{\deg(i)} + (q-1)^{\deg(j)}) = 0$ .

Summing over all such pairs  $i, j$ , we obtain  $\mathbb{E}(\mathcal{E}_{k+1}) \leq \mathbb{E}(\mathcal{E}_k)$ . This proves Claim 2.7.

Since  $2 \leq i < j \leq d$ , we have  $\frac{i+j-d}{2} \leq \frac{d}{2} - 1$ , and in particular it follows that  $G_{d/2}$  is a (possibly empty) complete graph, whose number of vertices is denoted by  $\omega \geq 0$ . Note that the energy  $\mathcal{E}$  of a complete graph on  $\omega$  vertices is equal to  $\omega(q-1)^{\omega-1}$ , while the energy  $\mathcal{E}_0$  of  $G_0$  is equal to  $|A_0 \cap 2\mathbb{N} \cap (d/2, d]| \leq \frac{d}{4}$ . For a vertex  $u \in L(r, \frac{d}{2})$ , let  $\omega_u = |A_u \cap 2\mathbb{N} \cap (d/2, d]|$  (this is the number of distinct even depths at which a vertex colored  $c$  appears in the subtree of height  $\frac{d}{2}$  rooted in  $u$ ). It follows from Claim 2.7 that the average of  $\omega_u(q-1)^{\omega_u-1}$ , over all vertices  $u \in L(r, \frac{d}{2})$ , is at most  $\frac{d}{4}$ . Let  $a$  be the average of  $\omega_u$ , over all vertices  $u \in L(r, \frac{d}{2})$ . By Jensen's inequality and the convexity of the function  $x \mapsto x(q-1)^{x-1}$  for  $x \geq 0$ , we have that  $a(q-1)^{a-1} \leq \frac{d}{4}$ , and thus  $a \leq \frac{\log(d/2)}{\log(q-1)} + 1$ .

Note that  $a$  depends on the color  $c$  under consideration (to make this more explicit, let us now write  $a_c$  instead of  $a$ ). Since there are  $\frac{d}{4}$  even depths between depth  $\frac{d}{2}$  and depth  $d$ , there is a color  $c \in \{1, \dots, C\}$  such that  $a_c \cdot C \geq \frac{d}{4}$  and thus,  $C \geq \frac{d}{4a_c} \geq \frac{d \log(q-1)}{4 \log(d/2) + 4 \log(q-1)}$ , as desired.  $\square$

We now explain how the results proved above give a negative answer to Problem 1.1. Let  $U_3^d$  (resp.  $Q_3^d$ ) be obtained from  $T_3^d$  (resp.  $P_3^d$ ) by adding an edge  $uv$  for any pair of vertices  $u, v$  having the same parent. Note that for any  $d$ ,  $U_3^d$  and  $Q_3^d$  are outerplanar (and thus, planar) and chordal, and  $Q_3^d$  has pathwidth 2 ( $U_3^3$  and  $Q_3^5$  are depicted in Figure 2) and the original copies of  $T_3^d$  and  $P_3^d$  are spanning trees of  $U_3^d$  and  $Q_3^d$ , respectively. In the remainder of this section, whenever we write  $T_3^d$ , we mean *the original copy of  $T_3^d$  in  $U_3^d$* .

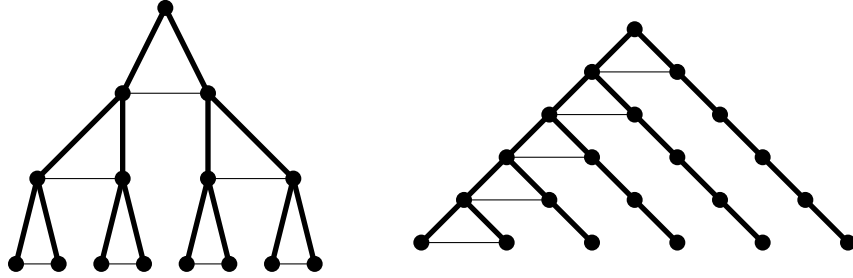


FIGURE 2. The graphs  $U_3^3$  (left) and  $Q_3^5$  (right). The bold edges represent the original copies of  $T_3^3$  and  $P_3^5$ , respectively.

Observe that for any two vertices  $u$  and  $v$  distinct from the root of  $T_3^d$ ,  $u$  and  $v$  are at distance  $d$  in  $T_3^d$  if and only if they are at distance  $d - 1$  in  $U_3^d$  (since the depth of  $T_3^d$  is  $d$ , the fact that  $u$  and  $v$  differ from the root and are at distance  $d$  apart implies that none of the two vertices is an ancestor of the other). The same property holds for  $Q_3^d$  and  $P_3^d$ . As a consequence, for any odd integer  $d$ ,  $\chi(U_3^{d+1}, d)$  and  $\chi(T_3^{d+1}, d + 1)$  differ by at most one, and  $\chi(Q_3^{d+1}, d)$  and  $\chi(P_3^{d+1}, d + 1)$  also differ by at most one. Using this observation, we immediately obtain the following corollary of Theorem 2.5 and Corollary 2.4, which gives a negative answer to Problem 1.1.

**Corollary 2.8.** *For any odd integer  $d$ ,*

$$\chi(U_3^{d+1}, d) \geq \frac{(d+1) \log(2)}{4 \log((d+1)/2) + 4 \log(2)} - 1 \text{ and } \chi(Q_3^{d+1}, d) \geq \log_2(d + 8) - 3.$$

The graphs  $U_3^{d+1}$  and its exact  $d$ -th power have  $n = 2^{d+2}$  vertices, and thus the chromatic number of the exact  $d$ -th power of  $U_3^{d+1}$  grows as  $\Omega\left(\frac{\log n}{\log \log n}\right)$ . The graphs  $Q_3^{d+1}$  and its exact  $d$ -th power have  $n = \binom{d+2}{2}$  vertices, and thus the chromatic number of the exact  $d$ -th power of  $Q_3^{d+1}$  grows as  $\Omega(\log n)$ . It is not difficult (using Theorem 2.1 for  $U_3^{d+1}$ ) to show that these bounds are asymptotically tight.

It was recently proved by Quiroz [8] that if  $G$  is a chordal graph of clique number at most  $t \geq 2$ , and  $d$  is an odd number, then  $\chi(G, d) \leq \binom{t}{2}(d + 1)$ . By Corollary 2.8, the



graph  $U_3^d$  shows that this is asymptotically best possible (as  $d$  tends to infinity), up to a  $\log d$  factor.

### 3. INTERVAL COLORING

For an integer  $d$  and a real  $c > 1$ , recall that  $\chi(T_q, [d, cd])$  denotes the smallest number of colors in a coloring of the vertices of  $T_q$  such that any two vertices of  $T_q$  at distance at least  $d$  and at most  $cd$  apart have distinct colors. Parlier and Petit [6] proved that

$$q(q-1)^{\lfloor cd/2 \rfloor - \lfloor d/2 \rfloor} \leq \chi(T_q, [d, cd]) \leq (q-1)^{\lfloor cd/2+1 \rfloor} (\lfloor cd \rfloor + 1).$$

In this final section, we prove that their lower bound (which is proved by finding a set of vertices of this cardinality that are pairwise at distance at least  $d$  and at most  $cd$  apart in  $T_q$ ) is asymptotically tight.

**Theorem 3.1.** *For any integers  $q \geq 3$  and  $d$  and any real  $c > 1$ ,  $\chi(T_q, [d, cd]) \leq \frac{q}{q-2}(q-1)^{\lfloor cd/2 \rfloor - d/2 + 1} + cd + 1$ .*

*Proof.* The proof is similar to the proof of Theorem 2.1. We consider any ordering  $e_1, e_2, \dots$  of the edges of  $T_q$  obtained from a breadth-first search starting at  $r$ . Then, for any  $i = 1, 2, \dots$  in order, we assign a color  $c(e_i)$  to the edge  $e_i$  as follows. Let  $e_i = uv$ , with  $u$  being the parent of  $v$ , and let  $\ell = \lfloor cd/2 \rfloor - d/2$ . We assign to  $uv$  a color  $c(uv)$  distinct from the colors of all the edges  $xy$  (with  $x$  being the parent of  $y$ ) such that  $x$  is at distance at most  $\ell$  from  $u^k$  (where  $k$  is the minimum of  $\ell$  and the depth of  $u$ ), or  $x$  is an ancestor of  $u$  at distance at most  $cd$  from  $u$  (and  $y$  lies on the path from  $u$  to  $x$ ). There are at most  $cd + \sum_{j=0}^{\ell} q(q-1)^j \leq \frac{q}{q-2}(q-1)^{\ell+1} + d - 1$  such edges, so we can color all the edges following this procedure by using a total of at most  $\frac{q}{q-2}(q-1)^{\ell+1} + cd$  colors.

As in the proof of Theorem 2.1, we now define our coloring of the vertices of  $T_q$  as follows: first color all the vertices at distance at most  $\frac{d}{2} - 1$  from  $r$  with a new color that does not appear on any edge of  $T_q$ , then for each vertex  $v$  with parent  $u$ , we color all the vertices of  $L(v, \frac{d}{2} - 1)$  with color  $c(uv)$ . In this vertex-coloring, at most  $\frac{q}{q-2}(q-1)^{\ell+1} + cd + 1$  colors are used.

Assume that two vertices  $s$  and  $t$ , at distance at least  $d$  and at most  $cd$  apart, were assigned the same color. This implies that  $c(s^{d/2-1}s^{d/2}) = c(t^{d/2-1}t^{d/2})$ . Assume without loss of generality that the depth of  $s$  is at least the depth of  $t$ , and consider first the case where  $t^{d/2-1}$  is an ancestor of  $s$ . Then  $t^{d/2}$  is an ancestor of  $s^{d/2}$  at distance at most  $cd$  from  $s^{d/2}$  (and  $t^{d/2-1}$  lies on the path from  $s^{d/2}$  to  $t^{d/2}$ ), which contradicts the definition of our edge-coloring  $c$ . Thus, we can assume that  $t^{d/2-1}$  is not an ancestor of  $s$ . This implies that  $t^{d/2-1}t^{d/2}$  lies on the path between  $s$  and  $t$ , and therefore  $t^{d/2}$  is at distance at most  $\ell = \lfloor cd/2 \rfloor - d/2$  from the ancestor of  $s^{d/2}$  at distance  $\ell$  from  $s^{d/2}$  (or simply from  $r$ , if the depth of  $s^{d/2}$  is at most  $\ell$ ). Again, this contradicts the definition of our coloring  $c$ . We obtained a coloring of the vertices of  $T_q$  with at most  $\frac{q}{q-2}(q-1)^{\ell+1} + cd + 1$  colors in which each pair of vertices at distance at least  $d$  and at most  $cd$  apart have distinct colors, as desired.  $\square$

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## REFERENCES

- [1] A.D.N.J. de Grey, *The chromatic number of the plane is at least 5*, Manuscript, [arXiv:1804.02385](#), 2018.
- [2] J. van den Heuvel, H.A. Kierstead, and D. Quiroz, *Chromatic Numbers of Exact Distance Graphs*, Manuscript, [arXiv:1612.02160](#), 2016.
- [3] B.R. Kloeckner, *Coloring distance graphs: a few answers and many questions*, *Geombinatorics* **24(3)** (2015), 117–134.
- [4] J. Nešetřil and P. Ossona de Mendez, *Sparsity – Graphs, Structures, and Algorithms*, Springer-Verlag, Berlin, Heidelberg, 2012.
- [5] J. Nešetřil and P. Ossona de Mendez, *On low tree-depth decompositions*, *Graphs Combin.* **31** (2015), 1941–1963.
- [6] H. Parlier and C. Petit, *Chromatic numbers for the hyperbolic plane and discrete analogs*, Manuscript, [arXiv:1701.08648](#), 2017.
- [7] H. Parlier and C. Petit, *Chromatic numbers of hyperbolic surfaces*, *Indiana Univ. Math. J.* **65(4)**, 1401–1423.
- [8] D. Quiroz, *Colouring exact distance graphs of chordal graphs*, Manuscript, [arXiv:1703.07008](#), 2017.

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