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EXACT DISTANCE COLORING IN TREES

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Abstract. For an integer \( q \geq 2 \) and an even integer \( d \), consider the graph obtained from a large complete \( q \)-ary tree by connecting with an edge any two vertices at distance exactly \( d \) in the tree. This graph has clique number \( q + 1 \), and the purpose of this short note is to prove that its chromatic number is \( \Theta\left(\frac{d \log q}{\log d}\right) \). It was not known that the chromatic number of this graph grows with \( d \). As a simple corollary of our result, we give a negative answer to a problem of Van den Heuvel and Naserasr, asking whether there is a constant \( C \) such that for any odd integer \( d \), any planar graph can be colored with at most \( C \) colors such that any pair of vertices at distance exactly \( d \) have distinct colors. Finally, we study interval coloring of trees (where vertices at distance at least \( d \) and at most \( cd \), for some real \( c > 1 \), must be assigned distinct colors), giving a sharp upper bound in the case of bounded degree trees.

1. Introduction

Given a metric space \( X \) and some real \( d > 0 \), let \( \chi(X, d) \) be the minimum number of colors in a coloring of the elements of \( X \) such that any two elements at distance exactly \( d \) in \( X \) are assigned distinct colors. The classical Hadwiger-Nelson problem asks for the value of \( \chi(\mathbb{R}^2, 1) \), where \( \mathbb{R}^2 \) is the Euclidean plane. It is known that \( 5 \leq \chi(\mathbb{R}^2, 1) \leq 7 \) \([1]\) and since the Euclidean plane \( \mathbb{R}^2 \) is invariant under homothety, \( \chi(\mathbb{R}^2, 1) = \chi(\mathbb{R}^2, d) \) for any real \( d > 0 \). Let \( \mathbb{H}^2 \) denote the hyperbolic plane. Kloeckner \([3]\) proved that \( \chi(\mathbb{H}^2, d) \) is at most linear in \( d \) (the multiplicative constant was recently improved by Parlier and Petit \([6]\)), and observed that \( \chi(\mathbb{H}^2, d) \geq 4 \) for any \( d > 0 \). He raised the question of determining whether \( \chi(\mathbb{H}^2, d) \) grows with \( d \) or can be bounded independently of \( d \). As noticed by Kahle (see \([3]\)), it is not known whether \( \chi(\mathbb{H}^2, d) \geq 5 \) for some real \( d > 0 \). Parlier and Petit \([6]\) recently suggested to study infinite regular trees as a discrete analog of the hyperbolic plane. Note that any graph \( G \) can be considered as a metric space (whose elements are the vertices of \( G \) and whose metric is the graph distance in \( G \)), and in this context \( \chi(G, d) \) is precisely the minimum number of colors in a vertex coloring of \( G \) such that vertices at distance \( d \) apart are assigned different colors. Note that \( \chi(G, d) \) can be equivalently defined as the chromatic number of the exact \( d \)-th power of \( G \), that is, the graph with the same vertex-set as \( G \) in which two vertices are adjacent if and only if they are at distance exactly \( d \) in \( G \).

Let \( T_q \) denote the infinite \( q \)-regular tree. Parlier and Petit \([6]\) observed that when \( d \) is odd, \( \chi(T_q, d) = 2 \) and proved that when \( d \) is even, \( q \leq \chi(T_q, d) \leq (d + 1)(q - 1) \). A
similar upper bound can also be deduced from the results of Van den Heuvel, Kierstead, and Quiroz [2], while the lower bound is a direct consequence of the fact that when $d$ is even, the clique number of the exact $d$-th power of $T_q$ is $q$ (note that it does not depend on $d$). In this short note, we prove that when $q \geq 3$ is fixed,
\[
\frac{d\log(q-1)}{4\log(d/2)+4\log(q-1)} \leq \chi(T_q, d) \leq (2 + o(1))\frac{d\log(q-1)}{\log d},
\]
where the asymptotic $o(1)$ is in terms of $d$. A simple consequence of our main result is that for any even integer $d$, the exact $d$-th power of a complete binary tree of depth $d$ is of order $\Theta(d/\log d)$ (while its clique number is equal to 3).

The following problem (attributed to Van den Heuvel and Naserasr) was raised in [4] (see also [2] and [5]).

**Problem 1.1** (Problem 11.1 in [4]). *Is there a constant $C$ such that for every odd integer $d$ and every planar graph $G$ we have $\chi(G, d) \leq C$?*

We will show that our result on large complete binary trees easily implies a negative answer to Problem 1.1. More precisely, we will prove that the graph $U_3^d$ obtained from a complete binary tree of depth $d$ by adding an edge between any two vertices with the same parent gives a negative answer to Problem 1.1 (in particular, for odd $d$, the chromatic number of the exact $d$-th power of $U_3^d$ grows as $\Theta(d/\log d)$). We will also prove that the exact $d$-th power of a specific subgraph $Q_{3}^{d}$ of $U_3^d$ grows as $\Omega(\log d)$. Note that $U_3^d$ and $Q_{3}^{d}$ are outerplanar (and thus, planar) and chordal (see Figure 2).

Kloeckner [3] proposed the following variant of the original problem: For a metric space $X$, an integer $d$ and a real $c > 1$, we denote by $\chi(X, [d, cd])$ the smallest number of colors in a coloring of the elements of $X$ such that any two elements of $X$ at distance at least $d$ and at most $cd$ apart have distinct colors. Considering as above the natural metric space defined by the infinite $q$-regular tree $T_q$, Parlier and Petit [6] proved that
\[
q(q - 1)^{|cd/2| - |d/2|} \leq \chi(T_q, [d, cd]) \leq (q - 1)^{|cd/2| + 1}(|cd| + 1).
\]

We will show that $\chi(T_q, [d, cd]) \leq \frac{q}{q-2}(q - 1)^{|cd/2| - d/2 + 1} + cd + 1$, which implies that the lower bound of Parlier and Petit [6] (which directly follows from a clique size argument) is asymptotically sharp.

2. Exact distance coloring

Throughout the paper, we assume that the infinite $q$-regular tree $T_q$ is rooted in some vertex $r$. This naturally defines the children and descendants of a vertex and the parent and ancestors of a vertex distinct from $r$. In particular, given a vertex $u$, we define the ancestors $u^0, u^1, \ldots$ of $u$ inductively as follows: $u^0 = u$ and for any $i$ such that $u^i$ is not the root, $u^{i+1}$ is the parent of $u^i$. With this notation, $u^d$ can be equivalently defined as the ancestor of $u$ at distance $d$ from $u$ (if such a vertex exists). For a given vertex $u$ in $T_q$, the *depth* of $u$, denoted by $\text{depth}(u)$, is the distance between $u$ and $r$ in $T_q$. For a vertex $v$ and an integer $\ell$, we define $L(v, \ell)$ as the set of descendants of $v$ at distance exactly $\ell$ from $v$ in $T_q$. 
We first prove an upper bound on $\chi(T_q, d)$.

**Theorem 2.1.** For any integer $q \geq 3$, any even integer $d$, and any integer $k \geq 1$ such that $k(q - 1)^{k-1} \leq d$, we have $\chi(T_q, d) \leq (q-1)^k + (q-1)^{\lceil k/2 \rceil} + \frac{d}{k} + 1$. In particular, $\chi(T_q, d) \leq d+q+1$, and when $q$ is fixed and $d$ tends to infinity, $\chi(T_q, d) \leq (2+o(1))\frac{d \log(q-1)}{\log d}$.

**Proof.** A vertex of $T_q$ distinct from $r$ and whose depth is a multiple of $k$ is said to be a special vertex. Let $v$ be a special vertex. Every special vertex $u$ distinct from $v$ such that $u^k = v^k$ is called a cousin of $v$. Note that $v$ has at most $q(q-1)^{k-1} - 1$ cousins (at most $(q-1)^k - 1$ if $v^k \neq r$). A special vertex $u$ is said to be a relative of $v$ if $u$ is either a cousin of $v$, or $u$ has the property that $u$ and $v$ have the same depth and are at distance at most $k$ apart in $T_q$. Two vertices $a, b$ at distance at most $k$ apart and at the same depth must satisfy $a^{\lceil k/2 \rceil} = b^{\lceil k/2 \rceil}$, and so the number of vertices $u$ such that $u$ and $v$ have the same depth and are at distance at most $k$ apart in $T_q$ is $(q-1)^{\lceil k/2 \rceil}$. It follows that if $v^k = r$, then $v$ has at most $q(q-1)^{k-1} - 1$ relatives and otherwise $v$ has at most $(q-1)^k + (q-1)^{\lceil k/2 \rceil} - 1$ relatives.

The first step is to define a coloring $C$ of the special vertices of $T_q$. This will be used later to define the desired coloring of $T_q$, i.e. a coloring such that vertices of $T_q$ at distance $d$ apart are assigned distinct colors (in this second coloring, the special vertices will not retain their color from $C$).

We greedily assign a color $C(v)$ to each special vertex $v$ of $T_q$ as follows: we consider the vertices of $T_q$ in a breadth-first search starting at $r$, and for each special vertex $v$ we encounter, we assign to $v$ a color distinct from the colors already assigned to its relatives, and from the set of ancestors $v^i$ of $v$, where $2 \leq i \leq \frac{d}{k}+1$ (there are at most $\frac{d}{k}$ such vertices). Note that if $v^k = r$, the number of colors forbidden for $v$ is at most $q(q-1)^{k-1} - 1$ and if $v^k \neq r$ the number of colors forbidden for $v$ is at most $(q-1)^k + (q-1)^{\lceil k/2 \rceil} + \frac{d}{k} - 1$. Since $k(q-1)^{k-1} \leq d$, in both cases $v$ has at most $(q-1)^k + (q-1)^{\lceil k/2 \rceil} + \frac{d}{k} - 1$ forbidden colors, therefore we can obtain the coloring $C$ by using at most $(q-1)^k + (q-1)^{\lceil k/2 \rceil} + \frac{d}{k}$ colors.

For any special vertex $v$, the set of descendants of $v$ at distance at least $d/2 - k$ and at most $d/2 - 1$ from $v$ is denoted by $K(v, k)$. We now define the desired coloring of $T_q$ as follows: for each special vertex $v$, all the vertices of $K(v, k)$ are assigned the color $C(v)$. Finally, all the vertices at distance at most $d/2 - 1$ from $r$ are colored with a single new color (note that any two vertices in this set lie at distance less than $d$ apart). The resulting vertex-coloring of $T_q$ is called $c$. Note that $c$ uses at most $(q-1)^k + (q-1)^{\lceil k/2 \rceil} + \frac{d}{k} + 1$ colors, and indeed every vertex of $T_q$ gets exactly one color.

We now prove that vertices at distance $d$ apart in $T_q$ are assigned distinct colors in $c$. Assume for the sake of contradiction that two vertices $x$ and $y$ at distance $d$ apart were assigned the same color. Then the depth of both $x$ and $y$ is at least $d/2$. We can assume by symmetry that the difference $t$ between the depth of $x$ and the depth of $y$ is such that $0 \leq t \leq d$ since otherwise they would be at distance more than $d$. Let $u$ be the unique (special) vertex of $T_q$ such that $x \in K(u, k)$ and $v$ be the unique (special) vertex such that $y \in K(v, k)$. By the definition of our coloring $c$, we have $C(u) = C(v)$. Note that $u$ and $v$ are distinct; indeed, otherwise $x$ and $y$ would not be at distance $d$ in $T_q$. Assume first that
u and v have the same depth. Then since u and x (resp. v and y) are distance at least \(d/2 - k\) apart, u and v are cousins (and thus, relatives), which contradicts the definition of the vertex-coloring \(C\). We may, therefore, assume that the depths of u and v are distinct. Moreover, since u and v are special vertices, we may assume that their depths differ by at least \(k\). In particular, \(u\) lies deeper than \(v\) in \(T_q\).

First assume that the depths of \(u\) and \(v\) differ by at least \(2k\). Then \(v\) is not an ancestor of \(u\) in \(T_q\). Indeed, for otherwise we would have \(v = u^k\) for some integer \(2 \leq i \leq \frac{d}{k} + 1\), which would contradict the definition of \(C\). This implies that the distance between \(x\) and \(y\) is at least \(d/2 - k + d/2 - k + 2k + 2 = d + 2\), which is a contradiction. Therefore, we can assume that the depths of \(u\) and \(v\) differ by precisely \(k\). Since \(v\) is not a relative of \(u\), we have that \(v \neq u^k\) and the distance between \(u^k\) and \(v\) is more than \(k\). Moreover, since \(u\) and \(x\) (resp. \(v\) and \(y\)) are at distance at least \(d/2 - k\) apart, this implies that the distance between \(x\) and \(y\) is more than \(d/2 - k + k + d/2 - k = d\), a contradiction.

Thus, \(c\) is a proper coloring.

By taking \(k = 1\) we obtain a coloring \(c\) using at most \((q - 1)^1 + (q - 1)^{1/2} + \frac{d}{2} + 1 = q + d + 1\) colors, and by taking \(k = \lceil \frac{d\log(q-1)}{\log d} \rceil\), we obtain a coloring \(c\) using at most

\[
\frac{d\log(q-1)}{\log d} + \sqrt{\frac{d\log(q-1)}{\log d}} + \frac{d\log(q-1)}{\log d\log d + \log\log(q-1)} + 1 = (2 + o(1))\frac{d\log(q-1)}{\log d}
\]

colors.

For \(k = 1\), the proof above can be optimized to show that \(\chi(T_q, d) \leq q + \frac{d}{2}\) (by simply noting that vertices at even depth and vertices at odd depth can be colored independently). Since we are mostly interested in the asymptotic behaviour of \(\chi(T_q, d)\) (which is of order \(O\left(\frac{d}{\log d}\right)\)), we omit the details.

We now prove a simple lower bound on \(\chi(T_q, d)\). Let \(T_q^d\) be the rooted complete \((q - 1)\)-ary tree of depth \(d\), with root \(v\). Note that each node has \(q - 1\) children, so this graph is a subtree of \(T_q\).

**Theorem 2.2.** For any integer \(q \geq 3\) and any even \(d\), \(\chi(T_q^d, d) \geq \log_2\left(\frac{d}{4} + q - 1\right)\).

**Proof.** Consider any coloring of \(T_q^d\) with colors \(1, 2, \ldots, C\), such that vertices at distance precisely \(d\) apart have distinct colors. For any vertex \(v\) at depth at most \(\frac{d}{2} + 1\) in \(T_q^d\), the set of colors appearing in \(L(v, \frac{d}{2} - 1)\) is denoted by \(S_v\). Observe that if \(v\) and \(w\) have the same parent, then \(S_v\) and \(S_w\) are disjoint since for any \(x \in L(v, \frac{d}{2} - 1)\) and \(y \in L(w, \frac{d}{2} - 1)\), \(x\) and \(y\) are at distance \(d\).

Fix some vertex \(u\) at depth at most \(\frac{d}{2}\) in \(T_q^d\) and some child \(v\) of \(u\). We claim that:

**Claim 2.3.** For any integer \(1 \leq k \leq \frac{\text{depth}(u)}{2}\), there is a color of \(S_{u^{2k-1}}\) that does not appear in \(S_v\).

To see that Claim 2.3 holds, observe that in the subtree of \(T_q^d\) rooted in \(u^k\), there is a vertex of \(L(u^{2k-1}, \frac{d}{2} - 1)\) at distance \(d\) from all the elements of \(L(v, \frac{d}{2} - 1)\). The color of such a vertex does not appear in \(S_v\), therefore Claim 2.3 holds.
In particular, Claim 2.3 implies that all the sets \( \{ S_{u2k-1} \mid 1 \leq k \leq d/4 \} \) and \( \{ S_w \mid w \text{ is a child of } u \} \) are pairwise distinct. Since there are \( \frac{d}{4} + q - 1 \) such sets, we have \( \frac{d}{4} + q - 1 \leq 2^C \) and therefore \( C \geq \log_2(\frac{d}{4} + q - 1) \), as desired. \( \square \)

It was observed by Stéphan Thomassé that the proof of Theorem 2.2 only uses a small fraction of the graph \( T_d^q \). Consider for simplicity the case \( q = 3 \), and define \( P_d^3 \) as the graph obtained from a path \( P = v_0, v_1, \ldots, v_d \) on \( d \) edges, by adding, for each \( 1 \leq i \leq d \), a path on \( i \) edges ending at \( v_i \) (see Figure 1). This graph is an induced subgraph of \( T_d^q \) and the proof of Theorem 2.2 directly shows the following 1.

**Corollary 2.4.** For any even integer \( d \), \( \chi(P_d^3, d) \geq \log_2(d + 8) - 2 \).

![Figure 1. The graph P_3^4.](image)

The proof of Theorem 2.2 can be refined to prove the following better estimate for \( T_d^q \), showing that the upper bound of Theorem 2.1 is (asymptotically) tight within a constant multiplicative factor of 8.

**Theorem 2.5.** For any integer \( q \geq 3 \) and every even integer \( d \geq 2 \), \( \chi(T_d^q, d) \geq \frac{d}{4\log(d/2) + 4\log(q-1)} \).

*Proof.* Consider any coloring of \( T_d^q \) with colors \( 1, 2, \ldots, C \), such that vertices at distance precisely \( d \) apart have distinct colors. We perform a random walk \( v_0, v_1, \ldots, v_d \) in \( T_d^q \) as follows: we start with \( v_0 = r \), and for each \( i \geq 1 \), we choose a child of \( v_i \) uniformly at random and set it as \( v_{i+1} \). Note that the depth of each vertex \( v_i \) is precisely \( i \).

From now on we fix a color \( c \in \{1, \ldots, C\} \). For any vertex \( v \) of \( T_d^q \), the set of vertices contained in the subtree of \( T_d^q \) rooted in \( v \) is denoted by \( V_v \), and we set \( A_v = \{ \text{depth}(u) \mid u \in V_v \text{ and } u \text{ has color } c \} \). When \( v = v_i \), for some integer \( 0 \leq i \leq d \), we write \( A_i \) instead of \( A_{v_i} \).

**Claim 2.6.** Assume that for some even integers \( i \) and \( j \) with \( 2 \leq i < j \leq d \), and for some vertex \( v \) at depth \( \frac{i+j-d}{2} \), the set \( A_v \) contains both \( i \) and \( j \). Then \( v \) has precisely one child \( u \) such that \( A_u \) contains \( i \) and \( j \), and moreover all the children \( w \) of \( v \) distinct from \( u \) are such that \( A_w \) contains neither \( i \) nor \( j \).

1Stéphan Thomassé noticed that this can also be deduced from the fact that the vertices at depth at least \( \frac{d}{2} \) and at most \( d \) in the exact \( d \)-th power of \( P_3^d \) induce a shift graph.
To see that Claim 2.6 holds, simply note that \( \frac{i+j-d}{2} < i < j \) and if two vertices \( u_1, u_2 \) colored \( c \) are respectively at depths \( i \) and \( j \), and their common ancestor is \( v \), then they are at distance \( d \) in \( T_q^d \) (which contradicts the fact that they were assigned the same color). Indeed, the distance of \( u_1 \) to \( v \) is \( i - \frac{i+j-d}{2} \) and the distance of \( u_2 \) to \( v \) is \( j - \frac{i+j-d}{2} \). This proves the claim.

We now define a family of graphs \((G_k)_{0 \leq k \leq d/2}\) as follows. For any \( 0 \leq k \leq \frac{d}{2} \), the vertex-set \( V(G_k) \) of \( G_k \) is the set \( A_k \cap 2\mathbb{N} \cap (d/2, d] \), and two (distinct) even integers \( i, j \in A_k \) are adjacent in \( G_k \) if and only if \( \frac{i+j-d}{2} < k \). For each \( 0 \leq k \leq \frac{d}{2} \) we define the energy \( \mathcal{E}_k \) of \( G_k \) as follows: \( \mathcal{E}_k = \sum_{i \in V(G_k)} (q - 1)^{\deg(i)} \), where \( \deg(i) \) denotes the degree of the vertex \( i \) in \( G_k \).

Note that each graph \( G_k \) depends on the (random) choice of \( v_1, v_2, \ldots, v_k \).

**Claim 2.7.** For any \( 0 \leq k \leq \frac{d}{2} - 1 \), \( \mathbb{E}(\mathcal{E}_{k+1}) \leq \mathbb{E}(\mathcal{E}_k) \).

Assume that \( v_1, v_2, \ldots, v_k \) (and therefore also \( G_k \)) are fixed. Observe that \( G_{k+1} \) is obtained from \( G_k \) by possibly removing some vertices and adding some edges. Thus, \( \mathcal{E}_{k+1} \) can be larger than \( \mathcal{E}_k \) only if \( G_{k+1} \) contains edges that are not in \( G_k \). Therefore, it suffices to consider the contributions of those pairs of nonadjacent vertices in \( G_k \) which could become adjacent in \( G_{k+1} \) (since these correspond to pairs \( i, j \) with \( k = \frac{i+j-d}{2} \), these pairs are pairwise disjoint), and prove that these contributions are, in expectation, equal to 0. Fix a pair of even integers \( i < j \) in \( V(G_k) \) with \( k = \frac{i+j-d}{2} \) (and note that \( i \) and \( j \) are not adjacent in \( G_k \)). By Claim 2.6, either \( v_{k+1} \) is such that \( A_{k+1} \) contains \( i \) and \( j \) (this event occurs with probability \( \frac{1}{q-1} \)), or \( A_{k+1} \) contains neither \( i \) nor \( j \) (with probability \( 1 - \frac{1}{q-1} \)). As a consequence, for any \( i < j \) in \( V(G_k) \) with \( k = \frac{i+j-d}{2} \), with probability \( \frac{1}{q-1} \) we add the edge \( ij \) in \( G_{k+1} \) and with probability \( 1 - \frac{1}{q-1} \) we remove vertices \( i \) and \( j \) from \( G_{k+1} \). This implies that for any \( i, j \in V(G_k) \), \( i < j \), with \( k = \frac{i+j-d}{2} \), with probability \( \frac{1}{q-1} \) we have contribution at most \((q-1)^{\deg(i)+1} + (q-1)^{\deg(j)+1} - (q-1)^{\deg(i)} - (q-1)^{\deg(j)} = (q-2)((q-1)^{\deg(i)} + (q-1)^{\deg(j)}) \) to \( \mathcal{E}_{k+1} \) (where \( \deg \) refers to the degree in \( G_k \)) and with probability \( 1 - \frac{1}{q-1} \) we have a contribution of at most \(-((q-1)^{\deg(i)} - (q-1)^{\deg(j)}) \) to \( \mathcal{E}_{k+1} \). Thus, the expected contribution of such a pair \( i, j \) is at most \( \frac{1}{q-1}((q-1)^{\deg(i)} + (q-1)^{\deg(j)}) - \frac{q-2}{q-1}((q-1)^{\deg(i)} + (q-1)^{\deg(j)}) = 0 \).

Summing over all such pairs \( i, j \), we obtain \( \mathbb{E}(\mathcal{E}_{k+1}) \leq \mathbb{E}(\mathcal{E}_k) \). This proves Claim 2.7.

Since \( 2 \leq i < j \leq d \), we have \( \frac{i+j-d}{2} \leq \frac{d}{2} - 1 \), and in particular it follows that \( G_d/2 \) is a (possibly empty) complete graph, whose number of vertices is denoted by \( \omega \geq 0 \). Note that the energy \( \mathcal{E} \) of a complete graph on \( \omega \) vertices is equal to \( \omega(q-1)^{\omega-1} \), while the energy \( \mathcal{E}_0 \) of \( G_0 \) is equal to \( |A_0 \cap 2\mathbb{N} \cap (d/2, d]| \leq \frac{d}{4} \). For a vertex \( u \in L(r, d/2) \), let \( \omega_u = |A_u \cap 2\mathbb{N} \cap (d/2, d]| \) (this is the number of distinct even depths at which a vertex colored \( c \) appears in the subtree of height \( d/2 \) rooted in \( u \)). It follows from Claim 2.7 that the average of \( \omega_u (q-1)^{\omega_u-1} \), over all vertices \( u \in L(r, d/2) \), is at most \( \frac{d}{4} \). Let \( a \) be the average of \( \omega_u \), over all vertices \( u \in L(r, d/2) \). By Jensen’s inequality and the convexity of the function \( x \mapsto x(q-1)^{x-1} \) for \( x \geq 0 \), we have that \( a(q-1)^{a-1} \leq \frac{d}{4} \), and thus \( a \leq \frac{\log(d/2)}{\log(q-1)} + 1 \).
Note that $a$ depends on the color $c$ under consideration (to make this more explicit, let us now write $a_c$ instead of $a$). Since there are $\frac{d}{2}$ even depths between depth $\frac{d}{2}$ and depth $d$, there is a color $c \in \{1, \ldots, C\}$ such that $a_c \cdot C \geq \frac{d}{2}$ and thus, $C \geq \frac{d}{4a_c} \geq \frac{d \log(q-1)}{4 \log(d/2) + 4 \log(q-1)}$, as desired.

We now explain how the results proved above give a negative answer to Problem 1.1. Let $U^d_3$ (resp. $Q^d_3$) be obtained from $T^d_3$ (resp. $P^d_3$) by adding an edge $uv$ for any pair of vertices $u, v$ having the same parent. Note that for any $d$, $U^d_3$ and $Q^d_3$ are outerplanar (and thus, planar) and chordal, and $Q^d_3$ has pathwidth 2 ($U^d_3$ and $Q^d_3$ are depicted in Figure 2) and the original copies of $T^d_3$ and $P^d_3$ are spanning trees of $U^d_3$ and $Q^d_3$, respectively. In the remainder of this section, whenever we write $T^d_3$, we mean the original copy of $T^d_3$ in $U^d_3$.

![Figure 2](image_url). The graphs $U^3_3$ (left) and $Q^5_3$ (right). The bold edges represent the original copies of $T^3_3$ and $P^5_3$, respectively.

Observe that for any two vertices $u$ and $v$ distinct from the root of $T^d_3$, $u$ and $v$ are at distance $d$ in $T^d_3$ if and only if they are at distance $d - 1$ in $U^d_3$ (since the depth of $T^d_3$ is $d$), the fact that $u$ and $v$ differ from the root and are at distance $d$ apart implies that none of the two vertices is an ancestor of the other). The same property holds for $Q^d_3$ and $P^d_3$. As a consequence, for any odd integer $d$, $\chi(U^{d+1}_3, d)$ and $\chi(T^{d+1}_3, d + 1)$ differ by at most one, and $\chi(Q^{d+1}_3, d)$ and $\chi(P^{d+1}_3, d + 1)$ also differ by at most one. Using this observation, we immediately obtain the following corollary of Theorem 2.5 and Corollary 2.4, which gives a negative answer to Problem 1.1.

**Corollary 2.8.** For any odd integer $d$,

$$\chi(U^{d+1}_3, d) \geq \frac{(d+1) \log(2)}{4 \log((d+1)/2) + 4 \log(2)} - 1 \quad \text{and} \quad \chi(Q^{d+1}_3, d) \geq \log_2(d + 8) - 3.$$ 

The graphs $U^{d+1}_3$ and its exact $d$-th power have $n = 2^{d+2}$ vertices, and thus the chromatic number of the exact $d$-th power of $U^{d+1}_3$ grows as $\Omega\left(\frac{\log n}{\log \log n}\right)$. The graphs $Q^{d+1}_3$ and its exact $d$-th power have $n = \binom{d+2}{2}$ vertices, and thus the chromatic number of the exact $d$-th power of $Q^{d+1}_3$ grows as $\Omega(\log n)$. It is not difficult (using Theorem 2.1 for $U^{d+1}_3$) to show that these bounds are asymptotically tight.

It was recently proved by Quiroz [8] that if $G$ is a chordal graph of clique number at most $t \geq 2$, and $d$ is an odd number, then $\chi(G, d) \leq \binom{t}{2}(d + 1)$. By Corollary 2.8, the
Theorem 3.1. For any integers \( q \geq 3 \) and \( d \) and any real \( c > 1 \), \( \chi(T_q, [d, cd]) \) denotes the smallest number of colors in a coloring of the vertices of \( T_q \) such that any two vertices of \( T_q \) at distance at least \( d \) and at most \( cd \) apart have distinct colors. Parlier and Petit [6] proved that

\[
q(q - 1)^{[cd/2] - [d/2]} \leq \chi(T_q, [d, cd]) \leq (q - 1)^{[cd/2]+1}([cd] + 1).
\]

In this final section, we prove that their lower bound (which is proved by finding a set of \( q \) colors in which each pair of vertices at distance at least \( d \) and at most \( cd \) apart have distinct colors, as desired. □

Proof. The proof is similar to the proof of Theorem 2.1. We consider any ordering \( e_1, e_2, \ldots \) of the edges of \( T_q \) obtained from a breadth-first search starting at \( r \). Then, for any \( i = 1, 2, \ldots \) in order, we assign a color \( c(e_i) \) to the edge \( e_i \) as follows. Let \( e_i = uv \), with \( u \) being the parent of \( v \), and let \( \ell = \lfloor cd/2 \rfloor - d/2 \). We assign to \( uv \) a color \( c(uv) \) distinct from the colors of all the edges \( xy \) (with \( x \) being the parent of \( y \)) such that \( x \) is at distance at most \( \ell \) from \( u^k \) (where \( k \) is the minimum of \( \ell \) and the depth of \( u \)), or \( x \) is an ancestor of \( u \) at distance at most \( cd \) from \( u \) (and \( y \) lies on the path from \( u \) to \( x \)). There are at most \( cd + \sum_{j=0}^{\ell} q(q - 1)^j \leq \frac{q}{q-2}(q - 1)^{\ell+1} + d - 1 \) such edges, so we can color all the edges following this procedure by using a total of at most \( \frac{q}{q-2}(q - 1)^{\ell+1} + cd \) colors.

As in the proof of Theorem 2.1, we now define our coloring of the vertices of \( T_q \) as follows: first color all the vertices at distance at most \( \frac{d}{2} - 1 \) from \( r \) with a new color that does not appear on any edge of \( T_q \), then for each vertex \( v \) with parent \( u \), we color all the vertices of \( L(v, \frac{d}{2} - 1) \) with color \( c(uv) \). In this vertex-coloring, at most \( \frac{q}{q-2}(q - 1)^{\ell+1} + cd + 1 \) colors are used.

Assume that two vertices \( s \) and \( t \), at distance at least \( d \) and at most \( cd \) apart, were assigned the same color. This implies that \( c(s^{d/2-1}s^{d/2}) = c(t^{d/2-1}t^{d/2}) \). Assume without loss of generality that the depth of \( s \) is at least the depth of \( t \), and consider first the case where \( t^{d/2-1} \) is an ancestor of \( s \). Then \( t^{d/2} \) is an ancestor of \( s^{d/2} \) at distance at most \( cd \) from \( s^{d/2} \) (and \( t^{d/2-1} \) lies on the path from \( s^{d/2} \) to \( t^{d/2} \)), which contradicts the definition of our edge-coloring \( c \). Thus, we can assume that \( t^{d/2-1} \) is not an ancestor of \( s \). This implies that \( t^{d/2-1}t^{d/2} \) lies on the path between \( s \) and \( t \), and therefore \( t^{d/2} \) is at distance at most \( \ell = \lfloor cd/2 \rfloor - d/2 \) from the ancestor of \( s^{d/2} \) at distance \( \ell \) from \( s^{d/2} \) (or simply from \( r \), if the depth of \( s^{d/2} \) is at most \( \ell \)). Again, this contradicts the definition of our coloring \( c \). We obtained a coloring of the vertices of \( T_q \) with at most \( \frac{q}{q-2}(q - 1)^{\ell+1} + cd + 1 \) colors in which each pair of vertices at distance at least \( d \) and at most \( cd \) apart have distinct colors, as desired. □
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