# Exact distance coloring in trees

Nicolas Bousquet, Louis Esperet, Ararat Harutyunyan, Rémi de Joannis de Verclos

# EXACT DISTANCE COLORING IN TREES

NICOLAS BOUSQUET, LOUIS ESPERET, ARARAT HARUTYUNYAN,
AND RÉMI DE JOANNIS DE VERCLOS

Abstract. For an integer $q \geq 2$ and an even integer $d$, consider the graph obtained from a large complete $q$-ary tree by connecting with an edge any two vertices at distance exactly $d$ in the tree. This graph has clique number $q+1$, and the purpose of this short note is to prove that its chromatic number is $\Theta\left(\frac{d \log q}{\log d}\right)$. It was not known that the chromatic number of this graph grows with $d$. As a simple corollary of our result, we give a negative answer to a problem of Van den Heuvel and Naserasr, asking whether there is a constant $C$ such that for any odd integer $d$, any planar graph can be colored with at most $C$ colors such that any pair of vertices at distance exactly $d$ have distinct colors. Finally, we study interval coloring of trees (where vertices at distance at least $d$ and at most $cd$, for some real $c > 1$, must be assigned distinct colors), giving a sharp upper bound in the case of bounded degree trees.

## 1. Introduction

Given a metric space $X$ and some real $d > 0$, let $\chi(X, d)$ be the minimum number of colors in a coloring of the elements of $X$ such that any two elements at distance exactly $d$ in $X$ are assigned distinct colors. The classical Hadwiger-Nelson problem asks for the value of $\chi(\mathbb{R}^2, 1)$, where $\mathbb{R}^2$ is the Euclidean plane. It is known that $5 \leq \chi(\mathbb{R}^2, 1) \leq 7$ [1] and since the Euclidean plane $\mathbb{R}^2$ is invariant under homothety, $\chi(\mathbb{R}^2, 1) = \chi(\mathbb{R}^2, d)$ for any real $d > 0$. Let $\mathbb{H}^2$ denote the hyperbolic plane. Kloeckner [3] proved that $\chi(\mathbb{H}^2, d)$ is at most linear in $d$ (the multiplicative constant was recently improved by Parlier and Petit [6]), and observed that $\chi(\mathbb{H}^2, d) \geq 4$ for any $d > 0$. He raised the question of determining whether $\chi(\mathbb{H}^2, d)$ grows with $d$ or can be bounded independently of $d$. As noticed by Kahle (see [3]), it is not known whether $\chi(\mathbb{H}^2, d) \geq 5$ for some real $d > 0$. Parlier and Petit [6] recently suggested to study infinite regular trees as a discrete analog of the hyperbolic plane. Note that any graph $G$ can be considered as a metric space (whose elements are the vertices of $G$ and whose metric is the graph distance in $G$), and in this context $\chi(G, d)$ is precisely the minimum number of colors in a vertex coloring of $G$ such that vertices at distance $d$ apart are assigned different colors. Note that $\chi(G, d)$ can be equivalently defined as the chromatic number of the *exact $d$-th power* of $G$, that is, the graph with the same vertex-set as $G$ in which two vertices are adjacent if and only if they are at distance exactly $d$ in $G$.

Let $T_q$ denote the infinite $q$-regular tree. Parlier and Petit [6] observed that when $d$ is odd, $\chi(T_q, d) = 2$ and proved that when $d$ is even, $q \leq \chi(T_q, d) \leq (d+1)(q-1)$. A

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similar upper bound can also be deduced from the results of Van den Heuvel, Kierstead, and Quiroz [2], while the lower bound is a direct consequence of the fact that when $d$ is even, the clique number of the exact $d$-th power of $T_q$ is $q$ (note that it does not depend on $d$). In this short note, we prove that when $q \geq 3$ is fixed,

$$\tfrac{d\log(q-1)}{4\log(d/2)+4\log(q-1)} \leq \chi(T_q, d) \leq (2+o(1))\tfrac{d\log(q-1)}{\log d},$$

where the asymptotic $o(1)$ is in terms of $d$. A simple consequence of our main result is that for any even integer $d$, the exact $d$-th power of a complete binary tree of depth $d$ is of order $\Theta(d/\log d)$ (while its clique number is equal to 3).

The following problem (attributed to Van den Heuvel and Naserasr) was raised in [4] (see also [2] and [5]).

**Problem 1.1** (Problem 11.1 in [4])**.** *Is there a constant $C$ such that for every odd integer $d$ and every planar graph $G$ we have $\chi(G, d) \leq C$?*

We will show that our result on large complete binary trees easily implies a negative answer to Problem 1.1. More precisely, we will prove that the graph $U_3^d$ obtained from a complete binary tree of depth $d$ by adding an edge between any two vertices with the same parent gives a negative answer to Problem 1.1 (in particular, for odd $d$, the chromatic number of the exact $d$-th power of $U_3^d$ grows as $\Theta(d/\log d)$). We will also prove that the exact $d$-th power of a specific subgraph $Q_3^d$ of $U_3^d$ grows as $\Omega(\log d)$. Note that $U_3^d$ and $Q_3^d$ are outerplanar (and thus, planar) and chordal (see Figure 2).

Kloeckner [3] proposed the following variant of the original problem: For a metric space $X$, an integer $d$ and a real $c > 1$, we denote by $\chi(X, [d, cd])$ the smallest number of colors in a coloring of the elements of $X$ such that any two elements of $X$ at distance at least $d$ and at most $cd$ apart have distinct colors. Considering as above the natural metric space defined by the infinite $q$-regular tree $T_q$, Parlier and Petit [6] proved that

$$q(q-1)^{\lfloor cd/2\rfloor - \lfloor d/2\rfloor} \leq \chi(T_q, [d, cd]) \leq (q-1)^{\lfloor cd/2+1\rfloor}(\lfloor cd\rfloor + 1).$$

We will show that $\chi(T_q, [d, cd]) \leq \frac{q}{q-2}(q-1)^{\lfloor cd/2\rfloor - d/2+1} + cd + 1$, which implies that the lower bound of Parlier and Petit [6] (which directly follows from a clique size argument) is asymptotically sharp.

## 2. Exact distance coloring

Throughout the paper, we assume that the infinite $q$-regular tree $T_q$ is rooted in some vertex $r$. This naturally defines the children and descendants of a vertex and the parent and ancestors of a vertex distinct from $r$. In particular, given a vertex $u$, we define the ancestors $u^0, u^1, \ldots$ of $u$ inductively as follows: $u^0 = u$ and for any $i$ such that $u^i$ is not the root, $u^{i+1}$ is the parent of $u^i$. With this notation, $u^d$ can be equivalently defined as the ancestor of $u$ at distance $d$ from $u$ (if such a vertex exists). For a given vertex $u$ in $T_q$, the *depth* of $u$, denoted by $\mathrm{depth}(u)$, is the distance between $u$ and $r$ in $T_q$. For a vertex $v$ and an integer $\ell$, we define $L(v, \ell)$ as the set of descendants of $v$ at distance exactly $\ell$ from $v$ in $T_q$.

We first prove an upper bound on $\chi(T_q, d)$.

**Theorem 2.1.** *For any integer $q \geq 3$, any even integer $d$, and any integer $k \geq 1$ such that $k(q-1)^{k-1} \leq d$, we have $\chi(T_q, d) \leq (q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} + 1$. In particular, $\chi(T_q, d) \leq d+q+1$, and when $q$ is fixed and $d$ tends to infinity, $\chi(T_q, d) \leq (2+o(1))\frac{d \log(q-1)}{\log d}$.*

*Proof.* A vertex of $T_q$ distinct from $r$ and whose depth is a multiple of $k$ is said to be a *special vertex*. Let $v$ be a special vertex. Every special vertex $u$ distinct from $v$ such that $u^k = v^k$ is called a *cousin* of $v$. Note that $v$ has at most $q(q-1)^{k-1} - 1$ cousins (at most $(q-1)^k - 1$ if $v^k \neq r$). A special vertex $u$ is said to be a *relative* of $v$ if $u$ is either a cousin of $v$, or $u$ has the property that $u$ and $v^k$ have the same depth and are at distance at most $k$ apart in $T_q$. Two vertices $a, b$ at distance at most $k$ apart and at the same depth must satisfy $a^{\lfloor k/2 \rfloor} = b^{\lfloor k/2 \rfloor}$, and so the number of vertices $u$ such that $u$ and $v^k$ have the same depth and are at distance at most $k$ apart in $T_q$ is $(q-1)^{\lfloor k/2 \rfloor}$. It follows that if $v^k = r$, then $v$ has at most $q(q-1)^{k-1} - 1$ relatives and otherwise $v$ has at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} - 1$ relatives.

The first step is to define a coloring $C$ of the special vertices of $T_q$. This will be used later to define the desired coloring of $T_q$, i.e. a coloring such that vertices of $T_q$ at distance $d$ apart are assigned distinct colors (in this second coloring, the special vertices will not retain their color from $C$).

We greedily assign a color $C(v)$ to each special vertex $v$ of $T_q$ as follows: we consider the vertices of $T_q$ in a breadth-first search starting at $r$, and for each special vertex $v$ we encounter, we assign to $v$ a color distinct from the colors already assigned to its relatives, and from the set of ancestors $v^{ik}$ of $v$, where $2 \leq i \leq \frac{d}{k}+1$ (there are at most $\frac{d}{k}$ such vertices). Note that if $v^k = r$, the number of colors forbidden for $v$ is at most $q(q-1)^{k-1} - 1$ and if $v^k \neq r$ the number of colors forbidden for $v$ is at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} - 1$. Since $k(q-1)^{k-1} \leq d$, in both cases $v$ has at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} - 1$ forbidden colors, therefore we can obtain the coloring $C$ by using at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k}$ colors.

For any special vertex $v$, the set of descendants of $v$ at distance at least $d/2 - k$ and at most $d/2 - 1$ from $v$ is denoted by $K(v, k)$. We now define the desired coloring of $T_q$ as follows: for each special vertex $v$, all the vertices of $K(v, k)$ are assigned the color $C(v)$. Finally, all the vertices at distance at most $d/2 - 1$ from $r$ are colored with a single new color (note that any two vertices in this set lie at distance less than $d$ apart). The resulting vertex-coloring of $T_q$ is called $c$. Note that $c$ uses at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} + 1$ colors, and indeed every vertex of $T_q$ gets exactly one color.

We now prove that vertices at distance $d$ apart in $T_q$ are assigned distinct colors in $c$. Assume for the sake of contradiction that two vertices $x$ and $y$ at distance $d$ apart were assigned the same color. Then the depth of both $x$ and $y$ is at least $d/2$. We can assume by symmetry that the difference $t$ between the depth of $x$ and the depth of $y$ is such that $0 \leq t \leq d$ since otherwise they would be at distance more than $d$. Let $u$ be the unique (special) vertex of $T_q$ such that $x \in K(u, k)$ and $v$ be the unique (special) vertex such that $y \in K(v, k)$. By the definition of our coloring $c$, we have $C(u) = C(v)$. Note that $u$ and $v$ are distinct; indeed, otherwise $x$ and $y$ would not be at distance $d$ in $T_q$. Assume first that

$u$ and $v$ have the same depth. Then since $u$ and $x$ (resp. $v$ and $y$) are distance at least $d/2-k$ apart, $u$ and $v$ are cousins (and thus, relatives), which contradicts the definition of the vertex-coloring $C$. We may, therefore, assume that the depths of $u$ and $v$ are distinct. Moreover, since $u$ and $v$ are special vertices, we may assume that their depths differ by at least $k$. In particular, $u$ lies deeper than $v$ in $T_q$.

First assume that the depths of $u$ and $v$ differ by at least $2k$. Then $v$ is not an ancestor of $u$ in $T_q$. Indeed, for otherwise we would have $v = u^{ik}$ for some integer $2 \le i \le \frac{d}{k}+1$, which would contradict the definition of $C$. This implies that the distance between $x$ and $y$ is at least $d/2-k+d/2-k+2k+2 = d+2$, which is a contradiction. Therefore, we can assume that the depths of $u$ and $v$ differ by precisely $k$. Since $v$ is not a relative of $u$, we have that $v \neq u^k$ and the distance between $u^k$ and $v$ is more than $k$. Moreover, since $u$ and $x$ (resp. $v$ and $y$) are at distance at least $d/2-k$ apart, this implies that the distance between $x$ and $y$ is more than $d/2-k+k+k+d/2-k = d$, a contradiction.

Thus, $c$ is a proper coloring.

By taking $k=1$ we obtain a coloring $c$ using at most $(q-1)^1+(q-1)^{\lfloor 1/2\rfloor}+\frac{d}{1}+1 = q+d+1$ colors, and by taking $k = \lfloor \frac{\log d - \log\log d + \log\log(q-1)}{\log(q-1)} \rfloor$, we obtain a coloring $c$ using at most

$$\tfrac{d\log(q-1)}{\log d} + \sqrt{\tfrac{d\log(q-1)}{\log d}} + \tfrac{d\log(q-1)}{\log d - \log\log d + \log\log(q-1) - \log(q-1)} + 1 = (2+o(1))\tfrac{d\log(q-1)}{\log d}$$

colors. $\square$

For $k=1$, the proof above can be optimized to show that $\chi(T_q, d) \le q + \frac{d}{2}$ (by simply noting that vertices at even depth and vertices at odd depth can be colored independently). Since we are mostly interested in the asymptotic behaviour of $\chi(T_q, d)$ (which is of order $O\big(\frac{d}{\log d}\big)$), we omit the details.

We now prove a simple lower bound on $\chi(T_q, d)$. Let $T_q^d$ be the rooted complete $(q-1)$-ary tree of depth $d$, with root $r$. Note that each node has $q-1$ children, so this graph is a subtree of $T_q$.

**Theorem 2.2.** *For any integer $q \ge 3$ and any even $d$, $\chi(T_q^d, d) \ge \log_2(\frac{d}{4}+q-1)$.*

*Proof.* Consider any coloring of $T_q^d$ with colors $1, 2, \ldots, C$, such that vertices at distance precisely $d$ apart have distinct colors. For any vertex $v$ at depth at most $\frac{d}{2}+1$ in $T_q^d$, the set of colors appearing in $L(v, \frac{d}{2}-1)$ is denoted by $S_v$. Observe that if $v$ and $w$ have the same parent, then $S_v$ and $S_w$ are disjoint since for any $x \in L(v, \frac{d}{2}-1)$ and $y \in L(w, \frac{d}{2}-1)$, $x$ and $y$ are at distance $d$.

Fix some vertex $u$ at depth at most $\frac{d}{2}$ in $T_q^d$ and some child $v$ of $u$. We claim that:

**Claim 2.3.** *For any integer $1 \le k \le \frac{\mathrm{depth}(u)}{2}$, there is a color of $S_{u^{2k-1}}$ that does not appear in $S_v$.*

To see that Claim 2.3 holds, observe that in the subtree of $T_q^d$ rooted in $u^k$, there is a vertex of $L(u^{2k-1}, \frac{d}{2}-1)$ at distance $d$ from all the elements of $L(v, \frac{d}{2}-1)$. The color of such a vertex does not appear in $S_v$, therefore Claim 2.3 holds.

In particular, Claim 2.3 implies that all the sets $\{S_{u^{2k-1}} \mid 1 \leq k \leq d/4\}$ and $\{S_w \mid w \text{ is a child of } u\}$ are pairwise distinct. Since there are $\frac{d}{4} + q - 1$ such sets, we have $\frac{d}{4} + q - 1 \leq 2^C$ and therefore $C \geq \log_2(\frac{d}{4} + q - 1)$, as desired. □

It was observed by Stéphan Thomassé that the proof of Theorem 2.2 only uses a small fraction of the graph $T_q^d$. Consider for simplicity the case $q = 3$, and define $P_3^d$ as the graph obtained from a path $P = v_0, v_1, \ldots, v_d$ on $d$ edges, by adding, for each $1 \leq i \leq d$, a path on $i$ edges ending at $v_i$ (see Figure 1). This graph is an induced subgraph of $T_q^d$ and the proof of Theorem 2.2 directly shows the following[1].

**Corollary 2.4.** *For any even integer $d$, $\chi(P_3^d, d) \geq \log_2(d + 8) - 2$.*

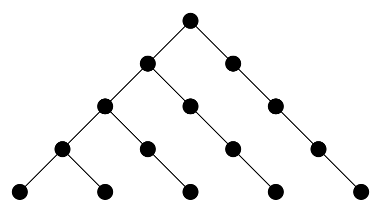

FIGURE 1. The graph $P_3^4$.

The proof of Theorem 2.2 can be refined to prove the following better estimate for $T_q^d$, showing that the upper bound of Theorem 2.1 is (asymptotically) tight within a constant multiplicative factor of 8.

**Theorem 2.5.** *For any integer $q \geq 3$ and every even integer $d \geq 2$, $\chi(T_q^d, d) \geq \frac{d \log(q-1)}{4 \log(d/2) + 4 \log(q-1)}$.*

*Proof.* Consider any coloring of $T_q^d$ with colors $1, 2, \ldots, C$, such that vertices at distance precisely $d$ apart have distinct colors. We perform a random walk $v_0, v_1, \ldots, v_d$ in $T_q^d$ as follows: we start with $v_0 = r$, and for each $i \geq 1$, we choose a child of $v_i$ uniformly at random and set it as $v_{i+1}$. Note that the depth of each vertex $v_i$ is precisely $i$.

From now on we fix a color $c \in \{1, \ldots, C\}$. For any vertex $v$ of $T_q^d$, the set of vertices contained in the subtree of $T_q^d$ rooted in $v$ is denoted by $V_v$, and we set $A_v = \{\text{depth}(u) \mid u \in V_v \text{ and } u \text{ has color } c\}$. When $v = v_i$, for some integer $0 \leq i \leq d$, we write $A_i$ instead of $A_{v_i}$.

**Claim 2.6.** *Assume that for some even integers $i$ and $j$ with $2 \leq i < j \leq d$, and for some vertex $v$ at depth $\frac{i+j-d}{2}$, the set $A_v$ contains both $i$ and $j$. Then $v$ has precisely one child $u$ such that $A_u$ contains $i$ and $j$, and moreover all the children $w$ of $v$ distinct from $u$ are such that $A_w$ contains neither $i$ nor $j$.*

[1]Stéphan Thomassé noticed that this can also be deduced from the fact that the vertices at depth at least $\frac{d}{2}$ and at most $d$ in the exact $d$-th power of $P_3^d$ induce a *shift graph*.

To see that Claim 2.6 holds, simply note that $\frac{i+j-d}{2} < i < j$ and if two vertices $u_1, u_2$ colored $c$ are respectively at depths $i$ and $j$, and their common ancestor is $v$, then they are at distance $d$ in $T_q^d$ (which contradicts the fact that they were assigned the same color). Indeed, the distance of $u_1$ to $v$ is $i - \frac{i+j-d}{2}$ and the distance of $u_2$ to $v$ is $j - \frac{i+j-d}{2}$. This proves the claim.

We now define a family of graphs $(G_k)_{0\le k\le d/2}$ as follows. For any $0 \le k \le \frac{d}{2}$, the vertex-set $V(G_k)$ of $G_k$ is the set $A_k \cap 2\mathbb{N} \cap (d/2, d]$, and two (distinct) even integers $i, j \in A_k$ are adjacent in $G_k$ if and only if $\frac{i+j-d}{2} < k$. For each $0 \le k \le \frac{d}{2}$ we define the *energy* $\mathcal{E}_k$ of $G_k$ as follows: $\mathcal{E}_k = \sum_{i\in V(G_k)} (q-1)^{\deg(i)}$, where $\deg(i)$ denotes the degree of the vertex $i$ in $G_k$.

Note that each graph $G_k$ depends on the (random) choice of $v_1, v_2, \ldots, v_k$.

**Claim 2.7.** *For any* $0 \le k \le \frac{d}{2} - 1$, $\mathbb{E}(\mathcal{E}_{k+1}) \le \mathbb{E}(\mathcal{E}_k)$.

Assume that $v_1, v_2, \ldots, v_k$ (and therefore also $G_k$) are fixed. Observe that $G_{k+1}$ is obtained from $G_k$ by possibly removing some vertices and adding some edges. Thus, $\mathcal{E}_{k+1}$ can be larger than $\mathcal{E}_k$ only if $G_{k+1}$ contains edges that are not in $G_k$. Therefore, it suffices to consider the contributions of those pairs of nonadjacent vertices in $G_k$ which could become adjacent in $G_{k+1}$ (since these correspond to pairs $i, j$ with $k = \frac{i+j-d}{2}$, these pairs are pairwise disjoint), and prove that these contributions are, in expectation, equal to 0. Fix a pair of even integers $i < j$ in $V(G_k)$ with $k = \frac{i+j-d}{2}$ (and note that $i$ and $j$ are not adjacent in $G_k$). By Claim 2.6, either $v_{k+1}$ is such that $A_{k+1}$ contains $i$ and $j$ (this event occurs with probability $\frac{1}{q-1}$), or $A_{k+1}$ contains neither $i$ nor $j$ (with probability $1 - \frac{1}{q-1}$). As a consequence, for any $i < j$ in $V(G_k)$ with $k = \frac{i+j-d}{2}$, with probability $\frac{1}{q-1}$ we add the edge $ij$ in $G_{k+1}$ and with probability $1 - \frac{1}{q-1}$ we remove vertices $i$ and $j$ from $G_{k+1}$. This implies that for any $i, j \in V(G_k)$, $i < j$, with $k = \frac{i+j-d}{2}$, with probability $\frac{1}{q-1}$ we have contribution at most $(q-1)^{\deg(i)+1} + (q-1)^{\deg(j)+1} - (q-1)^{\deg(i)} - (q-1)^{\deg(j)} = (q-2)((q-1)^{\deg(i)} + (q-1)^{\deg(j)})$ to $\mathcal{E}_{k+1}$ (where deg refers to the degree in $G_k$) and with probability $1 - \frac{1}{q-1}$ we have a contribution of at most $-(q-1)^{\deg(i)} - (q-1)^{\deg(j)}$ to $\mathcal{E}_{k+1}$. Thus, the expected contribution of such a pair $i, j$ is at most $\frac{1}{q-1}(q-2)((q-1)^{\deg(i)} + (q-1)^{\deg(j)}) - \frac{q-2}{q-1}((q-1)^{\deg(i)} + (q-1)^{\deg(j)}) = 0$.

Summing over all such pairs $i, j$, we obtain $\mathbb{E}(\mathcal{E}_{k+1}) \le \mathbb{E}(\mathcal{E}_k)$. This proves Claim 2.7.

Since $2 \le i < j \le d$, we have $\frac{i+j-d}{2} \le \frac{d}{2} - 1$, and in particular it follows that $G_{d/2}$ is a (possibly empty) complete graph, whose number of vertices is denoted by $\omega \ge 0$. Note that the energy $\mathcal{E}$ of a complete graph on $\omega$ vertices is equal to $\omega(q-1)^{\omega-1}$, while the energy $\mathcal{E}_0$ of $G_0$ is equal to $|A_0 \cap 2\mathbb{N} \cap (d/2, d]| \le \frac{d}{4}$. For a vertex $u \in L(r, \frac{d}{2})$, let $\omega_u = |A_u \cap 2\mathbb{N} \cap (d/2, d]|$ (this is the number of distinct even depths at which a vertex colored $c$ appears in the subtree of height $\frac{d}{2}$ rooted in $u$). It follows from Claim 2.7 that the average of $\omega_u(q-1)^{\omega_u-1}$, over all vertices $u \in L(r, \frac{d}{2})$, is at most $\frac{d}{4}$. Let $a$ be the average of $\omega_u$, over all vertices $u \in L(r, \frac{d}{2})$. By Jensen's inequality and the convexity of the function $x \mapsto x(q-1)^{x-1}$ for $x \ge 0$, we have that $a(q-1)^{a-1} \le \frac{d}{4}$, and thus $a \le \frac{\log(d/2)}{\log(q-1)} + 1$.

Note that $a$ depends on the color $c$ under consideration (to make this more explicit, let us now write $a_c$ instead of $a$). Since there are $\frac{d}{4}$ even depths between depth $\frac{d}{2}$ and depth $d$, there is a color $c \in \{1, \ldots, C\}$ such that $a_c \cdot C \geq \frac{d}{4}$ and thus, $C \geq \frac{d}{4a_c} \geq \frac{d\log(q-1)}{4\log(d/2)+4\log(q-1)}$, as desired. □

We now explain how the results proved above give a negative answer to Problem 1.1. Let $U_3^d$ (resp. $Q_3^d$) be obtained from $T_3^d$ (resp. $P_3^d$) by adding an edge $uv$ for any pair of vertices $u, v$ having the same parent. Note that for any $d$, $U_3^d$ and $Q_3^d$ are outerplanar (and thus, planar) and chordal, and $Q_3^d$ has pathwidth 2 ($U_3^3$ and $Q_3^5$ are depicted in Figure 2) and the original copies of $T_3^d$ and $P_3^d$ are spanning trees of $U_3^d$ and $Q_3^d$, respectively. In the remainder of this section, whenever we write $T_3^d$, we mean *the original copy of* $T_3^d$ *in* $U_3^d$.

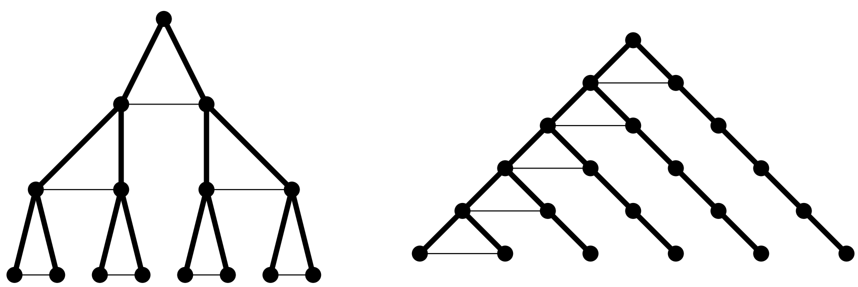

FIGURE 2. The graphs $U_3^3$ (left) and $Q_3^5$ (right). The bold edges represent the original copies of $T_3^3$ and $P_3^5$, respectively.

Observe that for any two vertices $u$ and $v$ distinct from the root of $T_3^d$, $u$ and $v$ are at distance $d$ in $T_3^d$ if and only if they are at distance $d-1$ in $U_3^d$ (since the depth of $T_3^d$ is $d$, the fact that $u$ and $v$ differ from the root and are at distance $d$ apart implies that none of the two vertices is an ancestor of the other). The same property holds for $Q_3^d$ and $P_3^d$. As a consequence, for any odd integer $d$, $\chi(U_3^{d+1}, d)$ and $\chi(T_3^{d+1}, d+1)$ differ by at most one, and $\chi(Q_3^{d+1}, d)$ and $\chi(P_3^{d+1}, d+1)$ also differ by at most one. Using this observation, we immediately obtain the following corollary of Theorem 2.5 and Corollary 2.4, which gives a negative answer to Problem 1.1.

**Corollary 2.8.** *For any odd integer $d$,*

$$\chi(U_3^{d+1}, d) \geq \frac{(d+1)\log(2)}{4\log((d+1)/2)+4\log(2)} - 1 \textit{ and } \chi(Q_3^{d+1}, d) \geq \log_2(d+8) - 3.$$

The graphs $U_3^{d+1}$ and its exact $d$-th power have $n = 2^{d+2}$ vertices, and thus the chromatic number of the exact $d$-th power of $U_3^{d+1}$ grows as $\Omega\left(\frac{\log n}{\log\log n}\right)$. The graphs $Q_3^{d+1}$ and its exact $d$-th power have $n = \binom{d+2}{2}$ vertices, and thus the chromatic number of the exact $d$-th power of $Q_3^{d+1}$ grows as $\Omega(\log n)$. It is not difficult (using Theorem 2.1 for $U_3^{d+1}$) to show that these bounds are asymptotically tight.

It was recently proved by Quiroz [8] that if $G$ is a chordal graph of clique number at most $t \geq 2$, and $d$ is an odd number, then $\chi(G, d) \leq \binom{t}{2}(d+1)$. By Corollary 2.8, the graph $U_3^d$ shows that this is asymptotically best possible (as $d$ tends to infinity), up to a $\log d$ factor.

## 3. Interval coloring

For an integer $d$ and a real $c > 1$, recall that $\chi(T_q, [d, cd])$ denotes the smallest number of colors in a coloring of the vertices of $T_q$ such that any two vertices of $T_q$ at distance at least $d$ and at most $cd$ apart have distinct colors. Parlier and Petit [6] proved that

$$q(q-1)^{\lfloor cd/2 \rfloor - \lfloor d/2 \rfloor} \le \chi(T_q, [d, cd]) \le (q-1)^{\lfloor cd/2+1 \rfloor}(\lfloor cd \rfloor + 1).$$

In this final section, we prove that their lower bound (which is proved by finding a set of vertices of this cardinality that are pairwise at distance at least $d$ and at most $cd$ apart in $T_q$) is asymptotically tight.

**Theorem 3.1.** *For any integers $q \ge 3$ and $d$ and any real $c > 1$, $\chi(T_q, [d, cd]) \le \frac{q}{q-2}(q-1)^{\lfloor cd/2 \rfloor - d/2+1} + cd + 1$.*

*Proof.* The proof is similar to the proof of Theorem 2.1. We consider any ordering $e_1, e_2, \ldots$ of the edges of $T_q$ obtained from a breadth-first search starting at $r$. Then, for any $i = 1, 2, \ldots$ in order, we assign a color $c(e_i)$ to the edge $e_i$ as follows. Let $e_i = uv$, with $u$ being the parent of $v$, and let $\ell = \lfloor cd/2 \rfloor - d/2$. We assign to $uv$ a color $c(uv)$ distinct from the colors of all the edges $xy$ (with $x$ being the parent of $y$) such that $x$ is at distance at most $\ell$ from $u^k$ (where $k$ is the minimum of $\ell$ and the depth of $u$), or $x$ is an ancestor of $u$ at distance at most $cd$ from $u$ (and $y$ lies on the path from $u$ to $x$). There are at most $cd + \sum_{j=0}^{\ell} q(q-1)^j \le \frac{q}{q-2}(q-1)^{\ell+1} + d - 1$ such edges, so we can color all the edges following this procedure by using a total of at most $\frac{q}{q-2}(q-1)^{\ell+1} + cd$ colors.

As in the proof of Theorem 2.1, we now define our coloring of the vertices of $T_q$ as follows: first color all the vertices at distance at most $\frac{d}{2} - 1$ from $r$ with a new color that does not appear on any edge of $T_q$, then for each vertex $v$ with parent $u$, we color all the vertices of $L(v, \frac{d}{2} - 1)$ with color $c(uv)$. In this vertex-coloring, at most $\frac{q}{q-2}(q-1)^{\ell+1} + cd + 1$ colors are used.

Assume that two vertices $s$ and $t$, at distance at least $d$ and at most $cd$ apart, were assigned the same color. This implies that $c(s^{d/2-1}s^{d/2}) = c(t^{d/2-1}t^{d/2})$. Assume without loss of generality that the depth of $s$ is at least the depth of $t$, and consider first the case where $t^{d/2-1}$ is an ancestor of $s$. Then $t^{d/2}$ is an ancestor of $s^{d/2}$ at distance at most $cd$ from $s^{d/2}$ (and $t^{d/2-1}$ lies on the path from $s^{d/2}$ to $t^{d/2}$), which contradicts the definition of our edge-coloring $c$. Thus, we can assume that $t^{d/2-1}$ is not an ancestor of $s$. This implies that $t^{d/2-1}t^{d/2}$ lies on the path between $s$ and $t$, and therefore $t^{d/2}$ is at distance at most $\ell = \lfloor cd/2 \rfloor - d/2$ from the ancestor of $s^{d/2}$ at distance $\ell$ from $s^{d/2}$ (or simply from $r$, if the depth of $s^{d/2}$ is at most $\ell$). Again, this contradicts the definition of our coloring $c$. We obtained a coloring of the vertices of $T_q$ with at most $\frac{q}{q-2}(q-1)^{\ell+1} + cd + 1$ colors in which each pair of vertices at distance at least $d$ and at most $cd$ apart have distinct colors, as desired. $\square$

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Univ. Grenoble Alpes, CNRS, G-SCOP, Grenoble, France
*E-mail address*: `nicolas.bousquet@grenoble-inp.fr`

Univ. Grenoble Alpes, CNRS, G-SCOP, Grenoble, France
*E-mail address*: `louis.esperet@grenoble-inp.fr`

LAMSADE, University of Paris-Dauphine, Paris, France
*E-mail address*: `ararat.harutyunyan@dauphine.fr`

Radboud University Nijmegen, Netherlands
*E-mail address*: `r.deverclos@math.ru.nl`