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EXACT DISTANCE COLORING IN TREES

NICOLAS BOUSQUET, LOUIS ESPERET, ARARAT HARUTYUNYAN,
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Abstract. For an integer \( q \geq 2 \) and an even integer \( d \), consider the graph obtained from a large complete \( q \)-ary tree by connecting with an edge any two vertices at distance exactly \( d \) in the tree. This graph has clique number \( q + 1 \), and the purpose of this short note is to prove that its chromatic number is \( \Theta\left( \frac{d \log q}{\log d} \right) \). It was not known that the chromatic number of this graph grows with \( d \). As a simple corollary of our result, we give a negative answer to a problem of Van den Heuvel and Naserasr, asking whether there is a constant \( C \) such that for any odd integer \( d \), any planar graph can be colored with at most \( C \) colors such that any pair of vertices at distance exactly \( d \) have distinct colors. Finally, we study interval coloring of trees (where vertices at distance at least \( d \) and at most \( cd \), for some real \( c > 1 \), must be assigned distinct colors), giving a sharp upper bound in the case of bounded degree trees.

1. Introduction

Given a metric space \( X \) and some real \( d > 0 \), let \( \chi(X,d) \) be the minimum number of colors in a coloring of the elements of \( X \) such that any two elements at distance exactly \( d \) in \( X \) are assigned distinct colors. The classical Hadwiger-Nelson problem asks for the value of \( \chi(\mathbb{R}^2,1) \), where \( \mathbb{R}^2 \) is the Euclidean plane. It is known that \( 5 \leq \chi(\mathbb{R}^2,1) \leq 7 \) [1] and since the Euclidean plane \( \mathbb{R}^2 \) is invariant under homothety, \( \chi(\mathbb{R}^2,1) = \chi(\mathbb{R}^2,d) \) for any real \( d > 0 \). Let \( \mathbb{H}^2 \) denote the hyperbolic plane. Kloeckner [3] proved that \( \chi(\mathbb{H}^2,d) \) is at most linear in \( d \) (the multiplicative constant was recently improved by Parlier and Petit [6]), and observed that \( \chi(\mathbb{H}^2,d) \geq 4 \) for any \( d > 0 \). He raised the question of determining whether \( \chi(\mathbb{H}^2,d) \) grows with \( d \) or can be bounded independently of \( d \). As noticed by Kahle (see [3]), it is not known whether \( \chi(\mathbb{H}^2,d) \geq 5 \) for some real \( d > 0 \). Parlier and Petit [6] recently suggested to study infinite regular trees as a discrete analog of the hyperbolic plane. Note that any graph \( G \) can be considered as a metric space (whose elements are the vertices of \( G \) and whose metric is the graph distance in \( G \)), and in this context \( \chi(G,d) \) is precisely the minimum number of colors in a vertex coloring of \( G \) such that vertices at distance \( d \) apart are assigned different colors. Note that \( \chi(G,d) \) can be equivalently defined as the chromatic number of the exact \( d \)-th power of \( G \), that is, the graph with the same vertex-set as \( G \) in which two vertices are adjacent if and only if they are at distance exactly \( d \) in \( G \).

Let \( T_q \) denote the infinite \( q \)-regular tree. Parlier and Petit [6] observed that when \( d \) is odd, \( \chi(T_q,d) = 2 \) and proved that when \( d \) is even, \( q \leq \chi(T_q,d) \leq (d+1)(q-1) \). A
similar upper bound can also be deduced from the results of Van den Heuvel, Kierstead, and Quiroz [2], while the lower bound is a direct consequence of the fact that when \( d \) is even, the clique number of the exact \( d \)-th power of \( T_q \) is \( q \) (note that it does not depend on \( d \)). In this short note, we prove that when \( q \geq 3 \) is fixed,

\[
\frac{d \log(q-1)}{4 \log(d/2)+4 \log(q-1)} \leq \chi(T_q, d) \leq (2+o(1)) \frac{d \log(q-1)}{\log d},
\]

where the asymptotic \( o(1) \) is in terms of \( d \). A simple consequence of our main result is that for any even integer \( d \), the exact \( d \)-th power of a complete binary tree of depth \( d \) is of order \( \Theta(d/\log d) \) (while its clique number is equal to 3).

The following problem (attributed to Van den Heuvel and Naserasr) was raised in [4] (see also [2] and [5]).

**Problem 1.1** (Problem 11.1 in [4]). Is there a constant \( C \) such that for every odd integer \( d \) and every planar graph \( G \) we have \( \chi(G, d) \leq C \)?

We will show that our result on large complete binary trees easily implies a negative answer to Problem 1.1. More precisely, we will prove that the graph \( U_3^d \) obtained from a complete binary tree of depth \( d \) by adding an edge between any two vertices with the same parent gives a negative answer to Problem 1.1 (in particular, for odd \( d \), the chromatic number of the exact \( d \)-th power of \( U_3^d \) grows as \( \Theta(d/\log d) \)). We will also prove that the exact \( d \)-th power of a specific subgraph \( Q_3^d \) of \( U_3^d \) grows as \( \Omega(\log d) \). Note that \( U_3^d \) and \( Q_3^d \) are outerplanar (and thus, planar) and chordal (see Figure 2).

Kloeckner [3] proposed the following variant of the original problem: For a metric space \( X \), an integer \( d \) and a real \( c > 1 \), we denote by \( \chi(X, [d, cd]) \) the smallest number of colors in a coloring of the elements of \( X \) such that any two elements of \( X \) at distance at least \( d \) and at most \( cd \) apart have distinct colors. Considering as above the natural metric space defined by the infinite \( q \)-regular tree \( T_q \), Parlier and Petit [6] proved that

\[
q(q-1)^{[cd/2] d/2} \leq \chi(T_q, [d, cd]) \leq (q-1)^{[cd/2]+1}([cd]+1).
\]

We will show that \( \chi(T_q, [d, cd]) \leq \frac{c}{q-2} (q-1)^{[cd/2]-d/2+1} + cd + 1 \), which implies that the lower bound of Parlier and Petit [6] (which directly follows from a clique size argument) is asymptotically sharp.

## 2. Exact distance coloring

Throughout the paper, we assume that the infinite \( q \)-regular tree \( T_q \) is rooted in some vertex \( r \). This naturally defines the children and descendants of a vertex and the parent and ancestors of a vertex distinct from \( r \). In particular, given a vertex \( u \), we define the ancestors \( u^0, u^1, \ldots \) of \( u \) inductively as follows: \( u^0 = u \) and for any \( i \) such that \( u^i \) is not the root, \( u^{i+1} \) is the parent of \( u^i \). With this notation, \( u^d \) can be equivalently defined as the ancestor of \( u \) at distance \( d \) from \( u \) (if such a vertex exists). For a given vertex \( u \) in \( T_q \), the depth of \( u \), denoted by \( \text{depth}(u) \), is the distance between \( u \) and \( r \) in \( T_q \). For a vertex \( v \) and an integer \( \ell \), we define \( L(v, \ell) \) as the set of descendants of \( v \) at distance exactly \( \ell \) from \( v \) in \( T_q \).
We first prove an upper bound on $\chi(T_q, d)$.

**Theorem 2.1.** For any integer $q \geq 3$, any even integer $d$, and any integer $k \geq 1$ such that $k(q - 1)^{k-1} \leq d$, we have $\chi(T_q, d) \leq (q - 1)^k + (q - 1)^{\lfloor k/2 \rfloor} + \frac{d}{k} + 1$. In particular, $\chi(T_q, d) \leq d + q + 1$, and when $q$ is fixed and $d$ tends to infinity, $\chi(T_q, d) \leq 2 + o(1)\frac{d \log(q-1)}{\log d}$.

**Proof.** A vertex of $T_q$ distinct from $r$ and whose depth is a multiple of $k$ is said to be a special vertex. Let $v$ be a special vertex. Every special vertex $u$ distinct from $v$ such that $u^k = v^k$ is called a cousin of $v$. Note that $v$ has at most $q(q - 1)^{k-1} - 1$ cousins (at most $(q - 1)^k - 1$ if $v^k \neq r$). A special vertex $u$ is said to be a relative of $v$ if $u$ is either a cousin of $v$, or $u$ has the property that $u$ and $v^k$ have the same depth and are at distance at most $k$ apart in $T_q$. Two vertices $a, b$ at distance at most $k$ apart and at the same depth must satisfy $a^{[k/2]} = b^{[k/2]}$, and so the number of vertices $u$ such that $u$ and $v^k$ have the same depth and are at distance at most $k$ apart in $T_q$ is $(q - 1)^{\lfloor k/2 \rfloor}$. It follows that if $v^k = r$, then $v$ has at most $q(q - 1)^{k-1} - 1$ relatives and otherwise $v$ has at most $(q - 1)^k + (q - 1)^{\lfloor k/2 \rfloor} - 1$ relatives.

The first step is to define a coloring $C$ of the special vertices of $T_q$. This will be used later to define the desired coloring of $T_q$, i.e. a coloring such that vertices of $T_q$ at distance $d$ apart are assigned distinct colors (in this second coloring, the special vertices will not retain their color from $C$).

We greedily assign a color $C(v)$ to each special vertex $v$ of $T_q$ as follows: we consider the vertices of $T_q$ in a breadth-first search starting at $r$, and for each special vertex $v$ we encounter, we assign to $v$ a color distinct from the colors already assigned to its relatives, and from the set of ancestors $v^{i^k}$ of $v$, where $2 \leq i \leq \frac{d}{k} + 1$ (there are at most $\frac{d}{k}$ such vertices). Note that if $v^k = r$, the number of colors forbidden for $v$ is at most $q(q - 1)^{k-1} - 1$ and if $v^k \neq r$ the number of colors forbidden for $v$ is at most $(q - 1)^k + (q - 1)^{\lfloor k/2 \rfloor} + \frac{d}{k} - 1$. Since $k(q - 1)^{k-1} \leq d$, in both cases $v$ has at most $(q - 1)^k + (q - 1)^{\lfloor k/2 \rfloor} + \frac{d}{k} - 1$ forbidden colors, therefore we can obtain the coloring $C$ by using at most $(q - 1)^k + (q - 1)^{\lfloor k/2 \rfloor} + \frac{d}{k}$ colors.

For any special vertex $v$, the set of descendants of $v$ at distance at least $d/2 - k$ and at most $d/2 - 1$ from $v$ is denoted by $K(v, k)$. We now define the desired coloring of $T_q$ as follows: for each special vertex $v$, all the vertices of $K(v, k)$ are assigned the color $C(v)$. Finally, all the vertices at distance at most $d/2 - 1$ from $r$ are colored with a single new color (note that any two vertices in this set lie at distance less than $d$ apart). The resulting vertex-coloring of $T_q$ is called $c$. Note that $c$ uses at most $(q - 1)^k + (q - 1)^{\lfloor k/2 \rfloor} + \frac{d}{k} + 1$ colors, and indeed every vertex of $T_q$ gets exactly one color.

We now prove that vertices at distance $d$ apart in $T_q$ are assigned distinct colors in $c$. Assume for the sake of contradiction that two vertices $x$ and $y$ at distance $d$ apart were assigned the same color. Then the depth of both $x$ and $y$ is at least $d/2$. We can assume by symmetry that the difference $t$ between the depth of $x$ and the depth of $y$ is such that $0 \leq t \leq d$ since otherwise they would be at distance more than $d$. Let $u$ be the unique (special) vertex of $T_q$ such that $x \in K(u, k)$ and $y$ be the unique (special) vertex such that $y \in K(v, k)$. By the definition of our coloring $c$, we have $C(u) = C(v)$. Note that $u$ and $v$ are distinct; indeed, otherwise $x$ and $y$ would not be at distance $d$ in $T_q$. Assume first that
and \( v \) have the same depth. Then since \( u \) and \( x \) (resp. \( v \) and \( y \)) are distance at least \( d/2 - k \) apart, \( u \) and \( v \) are cousins (and thus, relatives), which contradicts the definition of the vertex-coloring \( C \). We may, therefore, assume that the depths of \( u \) and \( v \) are distinct. Moreover, since \( u \) and \( v \) are special vertices, we may assume that their depths differ by at least \( k \). In particular, \( u \) lies deeper than \( v \) in \( T_q \).

First assume that the depths of \( u \) and \( v \) differ by at least \( 2k \). Then \( v \) is not an ancestor of \( u \) in \( T_q \). Indeed, for otherwise we would have \( v = u^i \) for some integer \( 2 \leq i \leq \frac{d}{2} + 1 \), which would contradict the definition of \( C \). This implies that the distance between \( x \) and \( y \) is at least \( d/2 - k + d/2 - k + 2k = d + 2 \), which is a contradiction. Therefore, we can assume that the depths of \( u \) and \( v \) differ by precisely \( k \). Since \( v \) is not a relative of \( u \), we have that \( v \neq u^k \) and the distance between \( u^k \) and \( v \) is more than \( k \). Moreover, since \( u \) and \( x \) (resp. \( v \) and \( y \)) are at distance at least \( d/2 - k \) apart, this implies that the distance between \( x \) and \( y \) is more than \( d/2 - k + k + d/2 - k = d \), a contradiction.

Thus, \( c \) is a proper coloring.

By taking \( k = 1 \) we obtain a coloring \( c \) using at most \((q-1)^1+(q-1)^{[1/2]}+\frac{d}{2}+1 = q+d+1\) colors, and by taking \( k = \lceil \frac{\log d - \log d + \log \log (q-1)}{\log (q-1)} \rceil \), we obtain a coloring \( c \) using at most
\[
\frac{d \log (q-1)}{\log d} + \sqrt{\frac{d \log (q-1)}{\log d}} + \frac{d \log (q-1)}{\log d - \log d + \log \log (q-1) - \log \log (q-1)} + 1 = (2 + o(1)) \frac{d \log (q-1)}{\log d}
\]
colors.

For \( k = 1 \), the proof above can be optimized to show that \( \chi(T_q, d) \leq q + \frac{d}{2} \) (by simply noting that vertices at even depth and vertices at odd depth can be colored independently). Since we are mostly interested in the asymptotic behaviour of \( \chi(T_q, d) \) (which is of order \( O\left(\frac{d}{\log d}\right)\)), we omit the details.

We now prove a simple lower bound on \( \chi(T_q, d) \). Let \( T_q^d \) be the rooted complete \((q-1)\)-ary tree of depth \( d \), with root \( v \). Note that each node has \( q - 1 \) children, so this graph is a subtree of \( T_q \).

**Theorem 2.2.** For any integer \( q \geq 3 \) and any even \( d \), \( \chi(T_q^d, d) \geq \log_2(\frac{d}{4} + q - 1) \).

**Proof.** Consider any coloring of \( T_q^d \) with colors \( 1, 2, \ldots, C \), such that vertices at distance precisely \( d \) apart have distinct colors. For any vertex \( v \) at depth at most \( \frac{d}{2} + 1 \) in \( T_q^d \), the set of colors appearing in \( L(v, \frac{d}{2} - 1) \) is denoted by \( S_v \). Observe that if \( v \) and \( w \) have the same parent, then \( S_v \) and \( S_w \) are disjoint since for any \( x \in L(v, \frac{d}{2} - 1) \) and \( y \in L(w, \frac{d}{2} - 1) \), \( x \) and \( y \) are at distance \( d \).

Fix some vertex \( u \) at depth at most \( \frac{d}{2} \) in \( T_q^d \) and some child \( v \) of \( u \). We claim that:

**Claim 2.3.** For any integer \( 1 \leq k \leq \frac{\text{depth}(u)}{2} \), there is a color of \( S_{u^{2k-1}} \) that does not appear in \( S_v \).

To see that Claim 2.3 holds, observe that in the subtree of \( T_q^d \) rooted in \( u^k \), there is a vertex of \( L(u^{2k-1}, \frac{d}{2} - 1) \) at distance \( d \) from all the elements of \( L(v, \frac{d}{2} - 1) \). The color of such a vertex does not appear in \( S_v \), therefore Claim 2.3 holds.
In particular, Claim 2.3 implies that all the sets \( \{ S_{u2k-1} \mid 1 \leq k \leq d/4 \} \) and \( \{ S_w \mid w \text{ is a child of } u \} \) are pairwise distinct. Since there are \( \frac{d}{4} + q - 1 \) such sets, we have \( \frac{d}{4} + q - 1 \leq 2^C \) and therefore \( C \geq \log_2(\frac{d}{4} + q - 1) \), as desired. \( \square \)

It was observed by Stéphan Thomassé that the proof of Theorem 2.2 only uses a small fraction of the graph \( T_d^q \). Consider for simplicity the case \( q = 3 \), and define \( P_d^3 \) as the graph obtained from a path \( P = v_0, v_1, \ldots, v_d \) on \( d \) edges, by adding, for each \( 1 \leq i \leq d \), a path on \( i \) edges ending at \( v_i \) (see Figure 1). This graph is an induced subgraph of \( T_d^q \) and the proof of Theorem 2.2 directly shows the following.

**Corollary 2.4.** For any even integer \( d \), \( \chi(P_d^3, d) \geq \log_2(d + 8) - 2 \).

![Figure 1. The graph \( P_d^3 \).](image)

The proof of Theorem 2.2 can be refined to prove the following better estimate for \( T_d^q \), showing that the upper bound of Theorem 2.1 is (asymptotically) tight within a constant multiplicative factor of 8.

**Theorem 2.5.** For any integer \( q \geq 3 \) and every even integer \( d \geq 2 \), \( \chi(T_d^q, d) \geq \frac{d \log(q-1)}{4 \log(d/2) + 4 \log(q-1)} \).

**Proof.** Consider any coloring of \( T_d^q \) with colors \( 1, 2, \ldots, C \), such that vertices at distance precisely \( d \) apart have distinct colors. We perform a random walk \( v_0, v_1, \ldots, v_d \) in \( T_d^q \) as follows: we start with \( v_0 = r \), and for each \( i \geq 1 \), we choose a child of \( v_i \) uniformly at random and set it as \( v_{i+1} \). Note that the depth of each vertex \( v_i \) is precisely \( i \).

From now on we fix a color \( c \in \{1, \ldots, C\} \). For any vertex \( v \) of \( T_d^q \), the set of vertices contained in the subtree of \( T_d^q \) rooted in \( v \) is denoted by \( V_v \), and we set \( A_v = \{ \text{depth}(u) \mid u \in V_v \text{ and } u \text{ has color } c \} \). When \( v = v_i \), for some integer \( 0 \leq i \leq d \), we write \( A_i \) instead of \( A_{v_i} \).

**Claim 2.6.** Assume that for some even integers \( i \) and \( j \) with \( 2 \leq i < j \leq d \), and for some vertex \( v \) at depth \( \frac{i+j-d}{2} \), the set \( A_v \) contains both \( i \) and \( j \). Then \( v \) has precisely one child \( u \) such that \( A_u \) contains \( i \) and \( j \), and moreover all the children \( w \) of \( v \) distinct from \( u \) are such that \( A_w \) contains neither \( i \) nor \( j \).

\( ^1 \)Stéphan Thomassé noticed that this can also be deduced from the fact that the vertices at depth at least \( \frac{d}{2} \) and at most \( d \) in the exact \( d \)-th power of \( P_d^3 \) induce a shift graph.
To see that Claim 2.6 holds, simply note that $\frac{i+j-d}{2} < i < j$ and if two vertices $u_1, u_2$ colored $c$ are respectively at depths $i$ and $j$, and their common ancestor is $v$, then they are at distance $d$ in $T^d_q$ (which contradicts the fact that they were assigned the same color). Indeed, the distance of $u_1$ to $v$ is $i - \frac{i+j-d}{2}$ and the distance of $u_2$ to $v$ is $j - \frac{i+j-d}{2}$. This proves the claim.

We now define a family of graphs $(G_k)_{0 \leq k \leq d/2}$ as follows. For any $0 \leq k \leq \frac{d}{2}$, the vertex-set $V(G_k)$ of $G_k$ is the set $A_k \cap 2N \cap (d/2, d]$, and two (distinct) even integers $i, j \in A_k$ are adjacent in $G_k$ if and only if $\frac{i+j-d}{2} < k$. For each $0 \leq k \leq \frac{d}{2}$ we define the energy $\mathcal{E}_k$ of $G_k$ as follows: $\mathcal{E}_k = \sum_{i \in V(G_k)} (q - 1)^{\text{deg}(i)}$, where $\text{deg}(i)$ denotes the degree of the vertex $i$ in $G_k$.

Note that each graph $G_k$ depends on the (random) choice of $v_1, v_2, \ldots, v_k$.

**Claim 2.7.** For any $0 \leq k \leq \frac{d}{2} - 1$, $\mathbb{E}(\mathcal{E}_{k+1}) \leq \mathbb{E}(\mathcal{E}_k)$.

Assume that $v_1, v_2, \ldots, v_k$ (and therefore also $G_k$) are fixed. Observe that $G_{k+1}$ is obtained from $G_k$ by possibly removing some vertices and adding some edges. Thus, $\mathcal{E}_{k+1}$ can be larger than $\mathcal{E}_k$ only if $G_{k+1}$ contains edges that are not in $G_k$. Therefore, it suffices to consider the contributions of those pairs of nonadjacent vertices in $G_k$ which could become adjacent in $G_{k+1}$ (since these correspond to pairs $i, j$ with $k = \frac{i+j-d}{2}$, these pairs are pairwise disjoint), and prove that these contributions are, in expectation, equal to $0$. Fix a pair of even integers $i < j$ in $V(G_k)$ with $k = \frac{i+j-d}{2}$ (and note that $i$ and $j$ are not adjacent in $G_k$). By Claim 2.6, either $v_{k+1}$ is such that $A_{k+1}$ contains $i$ and $j$ (this event occurs with probability $\frac{1}{q-1}$), or $A_{k+1}$ contains neither $i$ nor $j$ (with probability $1 - \frac{1}{q-1}$). As a consequence, for any $i < j$ in $V(G_k)$ with $k = \frac{i+j-d}{2}$, with probability $\frac{1}{q-1}$ we add the edge $ij$ in $G_{k+1}$ and with probability $1 - \frac{1}{q-1}$ we remove vertices $i$ and $j$ from $G_{k+1}$. This implies that for any $i, j \in V(G_k), i < j$, with $k = \frac{i+j-d}{2}$, with probability $\frac{1}{q-1}^{-1}$ we have contribution at most $(q-1)^{\text{deg}(i)+1} + (q-1)^{\text{deg}(j)+1} - (q-1)^{\text{deg}(i)} - (q-1)^{\text{deg}(j)} = (q-2)((q-1)^{\text{deg}(i)} + (q-1)^{\text{deg}(j)})$ to $\mathcal{E}_{k+1}$ (where $\text{deg}$ refers to the degree in $G_k$) and with probability $1 - \frac{1}{q-1}$ we have a contribution of at most $-(q-1)^{\text{deg}(i)} - (q-1)^{\text{deg}(j)}$ to $\mathcal{E}_{k+1}$. Thus, the expected contribution of such a pair $i, j$ is at most $\frac{1}{q-1}^{-1}((q-1)^{\text{deg}(i)} + (q-1)^{\text{deg}(j)}) - \frac{q-2}{q-1} - ((q-1)^{\text{deg}(i)} + (q-1)^{\text{deg}(j)})$. Summing over all such pairs $i, j$, we obtain $\mathbb{E}(\mathcal{E}_{k+1}) \leq \mathbb{E}(\mathcal{E}_k)$. This proves Claim 2.7.

Since $2 \leq i < j \leq d$, we have $\frac{i+j-d}{2} \leq \frac{d}{2} - 1$, and in particular it follows that $G_{d/2}$ is a (possibly empty) complete graph, whose number of vertices is denoted by $\omega \geq 0$. Note that the energy $\mathcal{E}$ of a complete graph on $\omega$ vertices is equal to $\omega(q-1)^{\omega-1}$, while the energy $\mathcal{E}_0$ of $G_0$ is equal to $|A_0 \cap 2N \cap (d/2, d]| \leq \frac{d}{4}$. For a vertex $u \in L(r, \frac{d}{2})$, let $\omega_u = |A_u \cap 2N \cap (d/2, d]|$ (this is the number of distinct even depths at a vertex colored $c$ appears in the subtree of height $\frac{d}{2}$ rooted in $u$). It follows from Claim 2.7 that the average of $\omega_u(q-1)^{\omega_u-1}$, over all vertices $u \in L(r, \frac{d}{2})$, is at most $\frac{d}{4}$. Let $a$ be the average of $\omega_u$, over all vertices $u \in L(r, \frac{d}{2})$. By Jensen’s inequality and the convexity of the function $x \mapsto x(q-1)^{x-1}$ for $x \geq 0$, we have that $a(q-1)^{a-1} \leq \frac{d}{4}$, and thus $a \leq \frac{\log(d/2)}{\log(q-1)} + 1.$
Note that \( a \) depends on the color \( c \) under consideration (to make this more explicit, let us now write \( a_c \) instead of \( a \)). Since there are \( \frac{d}{2} \) even depths between depth \( \frac{d}{2} \) and depth \( d \), there is a color \( c \in \{1, \ldots, C\} \) such that \( a_c \cdot C \geq \frac{d}{2} \) and thus, \( C \geq \frac{d}{a_c} \geq \frac{d \log(q-1)}{4 \log(d/2) + 4 \log(q-1)} \), as desired.

We now explain how the results proved above give a negative answer to Problem 1.1. Let \( U_3^d \) (resp. \( Q_3^d \)) be obtained from \( T_3^d \) (resp. \( P_3^d \)) by adding an edge \( uv \) for any pair of vertices \( u, v \) having the same parent. Note that for any \( d \), \( U_3^d \) and \( Q_3^d \) are planar (and thus, planar) and chordal, and \( Q_3^d \) has pathwidth 2 (\( U_3^d \) and \( Q_3^d \) are depicted in Figure 2) and the original copies of \( T_3^d \) and \( P_3^d \) are spanning trees of \( U_3^d \) and \( Q_3^d \), respectively. In the remainder of this section, whenever we write \( T_3^d \), we mean the original copy of \( T_3^d \) in \( U_3^d \).

![Figure 2. The graphs \( U_3^3 \) (left) and \( Q_3^5 \) (right). The bold edges represent the original copies of \( T_3^3 \) and \( P_3^5 \), respectively.](image)

Observe that for any two vertices \( u \) and \( v \) distinct from the root of \( T_3^d \), \( u \) and \( v \) are at distance \( d \) in \( T_3^d \) if and only if they are at distance \( d - 1 \) in \( U_3^d \) (since the depth of \( T_3^d \) is \( d \), the fact that \( u \) and \( v \) differ from the root and are at distance \( d \) apart implies that none of the two vertices is an ancestor of the other). The same property holds for \( Q_3^d \) and \( P_3^d \). As a consequence, for any odd integer \( d \), \( \chi(U_3^{d+1}, d) \) and \( \chi(T_3^{d+1}, d + 1) \) differ by at most one, and \( \chi(Q_3^{d+1}, d) \) and \( \chi(P_3^{d+1}, d + 1) \) also differ by at most one. Using this observation, we immediately obtain the following corollary of Theorem 2.5 and Corollary 2.4, which gives a negative answer to Problem 1.1.

**Corollary 2.8.** For any odd integer \( d \),

\[
\chi(U_3^{d+1}, d) \geq \frac{(d+1) \log(2)}{4 \log((d+1)/2) + 4 \log(2)} - 1 \quad \text{and} \quad \chi(Q_3^{d+1}, d) \geq \log_2(d + 8) - 3.
\]

The graphs \( U_3^{d+1} \) and its exact \( d \)-th power have \( n = 2^{d+2} \) vertices, and thus the chromatic number of the exact \( d \)-th power of \( U_3^{d+1} \) grows as \( \Omega \left( \frac{n}{\log \log n} \right) \). The graphs \( Q_3^{d+1} \) and its exact \( d \)-th power have \( n = \binom{d+2}{2} \) vertices, and thus the chromatic number of the exact \( d \)-th power of \( Q_3^{d+1} \) grows as \( \Omega(\log n) \). It is not difficult (using Theorem 2.1 for \( U_3^{d+1} \)) to show that these bounds are asymptotically tight.

It was recently proved by Quiroz [8] that if \( G \) is a chordal graph of clique number at most \( t \geq 2 \), and \( d \) is an odd number, then \( \chi(G, d) \leq \binom{t}{2}(d + 1) \). By Corollary 2.8, the
3. INTERVAL COLORING

For an integer $d$ and a real $c > 1$, recall that $\chi(T_q, [d, cd])$ denotes the smallest number of colors in a coloring of the vertices of $T_q$ such that any two vertices of $T_q$ at distance at least $d$ and at most $cd$ apart have distinct colors. Parlier and Petit [6] proved that

$$q(q - 1)^{\lceil cd/2 \rceil - \lfloor d/2 \rfloor} \leq \chi(T_q, [d, cd]) \leq (q - 1)^{\lceil cd/2 \rceil + 1} (\lfloor cd \rfloor + 1).$$

In this final section, we prove that their lower bound (which is proved by finding a set of $\ell$ colors in which each pair of vertices at distance at least $d$ and at most $cd$ apart have distinct colors, as desired. □

**Theorem 3.1.** For any integers $q \geq 3$ and $d$ and any real $c > 1$, $\chi(T_q, [d, cd]) \leq \frac{q}{q-2}(q - 1)^{\left\lceil \frac{cd}{2} \right\rceil - \frac{d}{2} + 1} + cd + 1$.

**Proof.** The proof is similar to the proof of Theorem 2.1. We consider any ordering $e_1, e_2, \ldots$ of the edges of $T_q$ obtained from a breadth-first search starting at $r$. Then, for any $i = 1, 2, \ldots$ in order, we assign a color $c(e_i)$ to the edge $e_i$ as follows. Let $e_i = uv$, with $u$ being the parent of $v$, and let $\ell = \lfloor cd/2 \rfloor - d/2$. We assign to $uv$ a color $c(uv)$ distinct from the colors of all the edges $xy$ (with $x$ being the parent of $y$) such that $x$ is at distance at most $\ell$ from $u^k$ (where $k$ is the minimum of $\ell$ and the depth of $u$), or $x$ is an ancestor of $u$ at distance at most $cd$ from $u$ (and $y$ lies on the path from $u$ to $x$). There are at most $cd + \sum_{j=0}^{\ell} q(q - 1)^j \leq \frac{q}{q-2}(q - 1)^{\ell+1} + d - 1$ such edges, so we can color all the edges following this procedure by using a total of at most $\frac{q}{q-2}(q - 1)^{\ell+1} + cd$ colors.

As in the proof of Theorem 2.1, we now define our coloring of the vertices of $T_q$ as follows: first color all the vertices at distance at most $\frac{d}{2} - 1$ from $r$ with a new color that does not appear on any edge of $T_q$, then for each vertex $v$ with parent $u$, we color all the vertices of $L(v, \frac{d}{2} - 1)$ with color $c(uv)$. In this vertex-coloring, at most $\frac{q}{q-2}(q - 1)^{\ell+1} + cd + 1$ colors are used.

Assume that two vertices $s$ and $t$, at distance at least $d$ and at most $cd$ apart, were assigned the same color. This implies that $c(s^{d/2-1}s^{d/2}) = c(t^{d/2-1}t^{d/2})$. Assume without loss of generality that the depth of $s$ is at least the depth of $t$, and consider first the case where $t^{d/2-1}$ is an ancestor of $s$. Then $t^{d/2}$ is an ancestor of $s^{d/2}$ at distance at most $cd$ from $s^{d/2}$ (and $t^{d/2-1}$ lies on the path from $s^{d/2}$ to $t^{d/2}$), which contradicts the definition of our edge-coloring $c$. Thus, we can assume that $t^{d/2-1}$ is not an ancestor of $s$. This implies that $t^{d/2-1}t^{d/2}$ lies on the path between $s$ and $t$, and therefore $t^{d/2}$ is at distance at most $\ell = \lfloor cd/2 \rfloor - d/2$ from the ancestor of $s^{d/2}$ at distance $\ell$ from $s^{d/2}$ (or simply from $r$, if the depth of $s^{d/2}$ is at most $\ell$). Again, this contradicts the definition of our coloring $c$. We obtained a coloring of the vertices of $T_q$ with at most $\frac{q}{q-2}(q - 1)^{\ell+1} + cd + 1$ colors in which each pair of vertices at distance at least $d$ and at most $cd$ apart have distinct colors, as desired. □
EXACT DISTANCE COLORING IN TREES

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