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On a weighted exponential distribution with a logarithmic weight: Theory and applications

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Abstract

A new weighted exponential distribution is proposed using a logarithmic weight function. Mathematical properties are studied, including moments, residual life function, order statistics and record values. A parametric estimation is performed using the maximum likelihood method, the weighted least-square method and the maximum product of spacing estimators method. Applications are provided using four real data sets. It is shown that our new distribution works better than a number of classical and recent distributions.

Keywords: Statistical distributions, Survival function, Hazard rate function, Record values, Parametric estimation.

2000 MSC: 60E05, 62E15, 62F10

1. Introduction

The general weighted exponential family of distributions has a probability density function (pdf) of the form:

$$f(x) = Cw(x)g(x),$$

where $g(x)$ denotes a pdf of the exponential distribution with parameter $\lambda > 0$, $w(x) \geq 0$ is a weight function and C represents the normalization constant given by $C^{-1} = \int_{-\infty}^{+\infty} w(x)g(x)dx$. Recent weighted exponential distributions with great ability in statistical applications include the exponentiated exponential (EE) distribution introduced by [9], the weighted exponential (WE) distribution introduced by [7], the Gamma exponentiated exponential (GEE) distribution introduced by [14], the weighted generalized exponential (WGE) distribution introduced by [10] and the exponentiated weighted exponential (EWE) distribution introduced by [13]. A brief review on weighted distributions (exponential or non-exponential) can be found in [15]. The statistical literature on proposing new weighted exponential distributions is vast and growing fast.

In this paper, we introduce a new weighted exponential distribution characterized by the pdf:

$$f(x) = \frac{\lambda\beta}{(\beta+1)\log(\beta+1)} \log(\beta + e^{\lambda x})e^{-\lambda x}, \quad x, \beta, \lambda > 0. \quad (1)$$

Thus the associated weight is $w(x) = \log(\beta + e^{\lambda x})$ and $C = \frac{\beta}{(\beta+1)\log(\beta+1)}$. To the best of our knowledge, the consideration of this weight is new in the literature, opening the door of new perspectives of applications. We shall refer to the distribution given by (1) as the logarithmic weighted exponential (LWE). Also, note that $w(x)$ is an increasing and concave up, hence one of the possible shapes of the hazard rate function (hrf) of this distribution is increasing ([8]), as it shall be seen later.

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Moreover, some plots are given to show the graphical behavior of the pdf and hrf of the distribution, illustrating its flexibility in data modeling. Other motivations to the LWE distribution are as follows.

- The pdf (1) can be written as a two-component mixture of two pdfs:

$$f(x) = pf_1(x) + (1-p)f_2(x), \quad x > 0,$$

where $p = \frac{\beta}{(\beta+1)\log(\beta+1)}$ is the mixing proportion, $f_1(x) = \lambda^2 x e^{-\lambda x}$ is the pdf of the Gamma distribution with parameters $(2, \lambda)$, $f_2(x) = \frac{\lambda\beta}{(\beta+1)\log(\beta+1) - \beta} \log(\beta+1 - \beta(1 - e^{-\lambda x})) e^{-\lambda x}$ is an extension of the pdf of gamma-G distribution introduced by [17]. To be more specific, it is associated to a cumulative distribution function (cdf) of the form $H(G(x))$, belonging to the family of composition of two cdfs: $H(y)$ and $G(x)$ (see [2]). Here $G(x)$ depends only on λ ; it is the cdf associated to the exponential distribution of parameter λ and $H(y)$ depends only on β :

$$H(y) = 1 - \frac{1}{(\beta+1)\log(\beta+1) - \beta} [(1 + \beta(1-y)) \log(1 + \beta(1-y)) - \beta(1-y)], \quad y \in [0, 1].$$

One can show that $H(y)$ is a slowly decreasing according to $\beta > 0$ with $H(y) \sim 1 - (1-y)^2$ when $\beta \rightarrow 0$ and $H(y) \sim y$ when $\beta \rightarrow +\infty$.

- Let X follow the LWE distribution and consider the transformation $Y = \beta + e^{\lambda X}$. Then distribution of Y is characterized by the pdf given by

$$f(y) = \frac{\beta}{(\beta+1)\log(\beta+1)} \frac{\log(y)}{(y-\beta)^2}, \quad y \geq \beta+1, \quad \beta > 0,$$

which is a new extension of the Pareto distribution and may be called log-Pareto distribution.

The rest of the paper is organized as follows. Section 2 presents the cdf and hrf related to the LWE distribution, with some plots. Section 3 is devoted to some of its statistical properties. Residual life function is studied in Section 4. Order statistics and record values are investigated in Section 5. Three different estimation procedures are presented in Section 6. Finally, four real data applications are presented in Section 7 to show the performance of the distribution.

2. Functions related to the LWE distribution

2.1. Survival, cumulative density and hazard rate functions

Proposition 1. *Let X be a random variable with pdf $f(x)$ (1). The associated survival function (sf) is given by :*

$$S(x) = P(X > x) = \frac{\log(1 + \beta e^{-\lambda x}) + \beta e^{-\lambda x} \log(\beta + e^{\lambda x})}{(\beta+1)\log(\beta+1)}, \quad x, \beta, \lambda > 0. \quad (2)$$

Proof. Let us set $c = \frac{\lambda\beta}{(\beta+1)\log(\beta+1)}$ and remark that

$$f(x) = c \log(\beta + e^{\lambda x}) e^{-\lambda x} = c \log(e^{\lambda x} (1 + \beta e^{-\lambda x})) e^{-\lambda x} = c (\lambda x e^{-\lambda x} + \log(1 + \beta e^{-\lambda x}) e^{-\lambda x}).$$

Using definition of $S(x)$, we find that

$$S(x) = \int_x^{+\infty} f(t) dt = c \left(\int_x^{+\infty} \lambda t e^{-\lambda t} dt + \int_x^{+\infty} \log(1 + \beta e^{-\lambda t}) e^{-\lambda t} dt \right).$$

Using this decomposition, integration by parts, change of variables and after some algebra, we obtain

$$S(x) = \frac{\log(1 + \beta e^{-\lambda x}) + \beta e^{-\lambda x} \log(\beta + e^{\lambda x})}{(\beta+1)\log(\beta+1)}.$$

This ends the proof. □

The cumulative distribution function follows immediately:

$$F(x) = 1 - S(x) = 1 - \frac{\log(1 + \beta e^{-\lambda x}) + \beta e^{-\lambda x} \log(\beta + e^{\lambda x})}{(\beta + 1) \log(\beta + 1)}, \quad x, \beta, \lambda > 0. \quad (3)$$

The hazard rate function is given by

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\lambda \beta \log(\beta + e^{\lambda x}) e^{-\lambda x}}{\log(1 + \beta e^{-\lambda x}) + \beta e^{-\lambda x} \log(\beta + e^{\lambda x})}, \quad x, \beta, \lambda > 0. \quad (4)$$

2.2. Behaviour of the functions

First of all, let us investigate some features of the functions inherent to the LWE distribution. For the pdf (1), we have

$$f(x) \sim \frac{\lambda \beta}{\beta + 1} e^{-\lambda x} \rightarrow \frac{\lambda \beta}{\beta + 1}, \quad x \rightarrow 0, \quad f(x) \sim \frac{\lambda^2 \beta}{(\beta + 1) \log(\beta + 1)} x e^{-\lambda x} \rightarrow 0, \quad x \rightarrow +\infty.$$

For the sf (2), we have

$$S(x) \sim 1 - \frac{\lambda}{\log(\beta + 1)} x \rightarrow 1, \quad x \rightarrow 0, \quad S(x) \sim \frac{\beta}{(\beta + 1) \log(\beta + 1)} (1 + \lambda x) e^{-\lambda x} \rightarrow 0, \quad x \rightarrow +\infty.$$

And for the cdf (3), we have

$$F(x) \sim \frac{\lambda}{\log(\beta + 1)} x \rightarrow 0, \quad x \rightarrow 0, \quad F(x) \sim 1 - \frac{\beta}{(\beta + 1) \log(\beta + 1)} (1 + \lambda x) e^{-\lambda x} \rightarrow 1, \quad x \rightarrow +\infty.$$

For the hrf (4), we have

$$h(x) \sim \frac{\lambda \beta}{\beta + 1} e^{-\lambda x} \rightarrow \frac{\lambda \beta}{\beta + 1}, \quad x \rightarrow 0, \quad h(x) \sim \frac{\lambda^2 x}{1 + \lambda x} \rightarrow \lambda, \quad x \rightarrow +\infty.$$

Let us now study the change points of $h(x)$. By denoting $y = \log(\beta + e^{\lambda x})$, we have

$$h'(x) = \frac{\beta \lambda^2 e^{-\lambda x} [-\lambda x e^{\lambda x} - (\beta + e^{\lambda x} y (-1 - \lambda x + (1 + 2\beta e^{\lambda x}) y))]}{(\beta + e^{\lambda x}) (x \lambda - (1 + \beta e^{\lambda x}) y)^2}.$$

The change points of the hrf function are given by b^* by solving the two following equations:

$$b^* \lambda - (1 + \beta e^{\lambda b^*}) y_1 = 0 \text{ and } -\lambda b^* e^{\lambda b^*} - (\beta + e^{\lambda b^*} y_1 (-1 - \lambda b^* + (1 + 2\beta e^{\lambda b^*}) y_1)) = 0.$$

Figures 1 and 2 show the graphical features of the pdf and hrf of the LWE distribution for several choices of parameters (λ, β) . In particular, we see that the pdf is very flexible and highly right skewed. Interesting comportment of the hrf, with possible change-point(s), can be observed.

3. Statistical properties

This section is devoted to some statistical properties of the LWE distribution.

3.1. Expansion of the pdf

Proposition 2. *Let $f(x)$ be (1). Then we have the following expansion:*

$$f(x) = \frac{\lambda^2 \beta}{(\beta + 1) \log(\beta + 1)} x e^{-\lambda x} + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} e^{-\lambda(\ell+1)x}, \quad (5)$$

with

$$A_{0,0} = \frac{\lambda \beta}{(\beta + 1) \log(\beta + 1)} \log(\beta + 1), \quad A_{k,\ell} = \frac{\lambda \beta}{(\beta + 1) \log(\beta + 1)} \binom{k}{\ell} \frac{1}{k} \left(\frac{\beta}{\beta + 1} \right)^k (-1)^{\ell+1},$$

for $k > 0$ or $\ell > 0$.

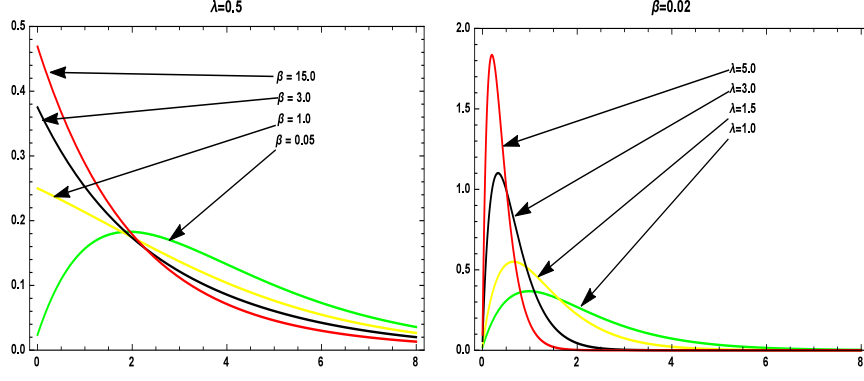


Figure 1: Plots of the LWE pdf.

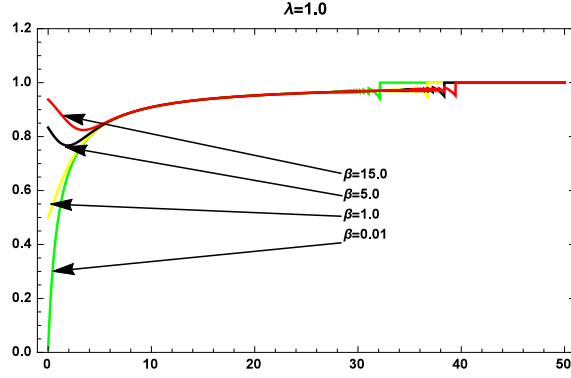


Figure 2: Plots of the LWE hrf.

Proof. Let us set $c = \frac{\lambda\beta}{(\beta+1)\log(\beta+1)}$ and remark that

$$\begin{aligned} f(x) &= c \log(\beta + e^{\lambda x}) e^{-\lambda x} = c \log \left((\beta + 1) e^{\lambda x} \left(1 - \frac{\beta}{\beta + 1} (1 - e^{-\lambda x}) \right) \right) e^{-\lambda x} \\ &= c \left(\log(\beta + 1) + \lambda x + \log \left(1 - \frac{\beta}{\beta + 1} (1 - e^{-\lambda x}) \right) \right) e^{-\lambda x}. \end{aligned}$$

Using the expansion: $\log(1 - y) = -\sum_{k=1}^{\infty} \frac{y^k}{k}$, $y \in [-1, 1)$, noticing that $0 \leq \frac{\beta}{\beta+1}(1 - e^{-\lambda x}) < 1$, and the binomial series expansion, we have

$$\begin{aligned} \log \left(1 - \frac{\beta}{\beta + 1} (1 - e^{-\lambda x}) \right) &= -\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\beta}{\beta + 1} \right)^k (1 - e^{-\lambda x})^k \\ &= \sum_{k=1}^{\infty} \sum_{\ell=0}^k \binom{k}{\ell} \frac{1}{k} \left(\frac{\beta}{\beta + 1} \right)^k (-1)^{\ell+1} e^{-\lambda \ell x}. \end{aligned}$$

We end the proof by putting these equalities together. \square

3.2. Quantile function

We can obtain the quantile function using the cdf (3); it is the function $Q(x)$ satisfying the nonlinear equation:

$$F(Q(x)) = x \Leftrightarrow \log(1 + \beta e^{-\lambda Q(x)}) + \beta e^{-\lambda Q(x)} \log(\beta + e^{\lambda Q(x)}) = (1 - x)(\beta + 1) \log(\beta + 1). \quad (6)$$

Variates of the LWE distribution can be simulated by using $X = Q(U)$ where U is a random variable following the uniform distribution on $[0, 1]$.

3.3. Moments

Proposition 3. Let X be a random variable with pdf $f(x)$ given by (1). Consider the Gamma function: $\Gamma(\nu) = \int_0^{+\infty} x^{\nu-1} e^{-x} dx$ and the coefficient $A_{k,\ell}$ of expansion (5). Then the r -th moment of X is given by

$$E(X^r) = \frac{\beta\Gamma(r+2)}{(\beta+1)\log(\beta+1)\lambda^r} + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} \frac{\Gamma(r+1)}{\lambda^{r+1}(\ell+1)^{r+1}}. \quad (7)$$

Proof. Let us set $c = \frac{\lambda\beta}{(\beta+1)\log(\beta+1)}$ and consider the expansion (5). Hence

$$\begin{aligned} E(X^r) &= \int_{-\infty}^{+\infty} x^r f(x) dx = \int_0^{+\infty} x^r \left(c\lambda x e^{-\lambda x} + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} e^{-\lambda(\ell+1)x} \right) dx \\ &= c\lambda \int_0^{+\infty} x^{r+1} e^{-\lambda x} dx + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} \int_0^{+\infty} x^r e^{-\lambda(\ell+1)x} dx \\ &= c \frac{\Gamma(r+2)}{\lambda^{r+1}} + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} \frac{\Gamma(r+1)}{\lambda^{r+1}(\ell+1)^{r+1}}. \end{aligned}$$

The proof is completed. □

3.4. Moment generating function

Proposition 4. Let X be a random variable with pdf $f(x)$ given by (1). Consider the coefficient $A_{k,\ell}$ of expansion (5). Then the moment generating function of X is given by

$$M(t) = \frac{\lambda^2\beta}{(\beta+1)\log(\beta+1)(\lambda-t)} + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} \frac{1}{\lambda(\ell+1)-t}, \quad t < \lambda.$$

Proof. Let us set $c = \frac{\lambda\beta}{(\beta+1)\log(\beta+1)}$ and consider the expansion (5). Hence, for $t < \lambda$,

$$\begin{aligned} M(t) &= E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f(x) dx = \int_0^{+\infty} e^{tx} \left(c\lambda x e^{-\lambda x} + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} e^{-\lambda(\ell+1)x} \right) dx \\ &= c\lambda \int_0^{+\infty} e^{-(\lambda-t)x} dx + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} \int_0^{+\infty} e^{-(\lambda(\ell+1)-t)x} dx \\ &= c \frac{\lambda}{\lambda-t} + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} \frac{1}{\lambda(\ell+1)-t}. \end{aligned}$$

This ends the proof. □

3.5. Conditional moments and mean deviations

Proposition 5. Let X be a random variable with pdf $f(x)$ given by (1). Consider the lower incomplete Gamma function $\Gamma(t, \nu) = \int_0^t x^{\nu-1} e^{-x} dx$ and the coefficient $A_{k,\ell}$ of expansion (5). Then we have

$$\int_0^t x^r f(x) dx = \frac{\beta\Gamma(\lambda t, r+2)}{(\beta+1)\log(\beta+1)\lambda^r} + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} \frac{\Gamma(\lambda(\ell+1)t, r+1)}{\lambda^{r+1}(\ell+1)^{r+1}}, \quad t > 0. \quad (8)$$

Proof. Let us set $c = \frac{\lambda\beta}{(\beta+1)\log(\beta+1)}$ and consider the expansion (5). Therefore

$$\begin{aligned}\int_0^t x^r f(x) dx &= \int_0^t x^r \left(c\lambda x e^{-\lambda x} + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} e^{-\lambda(\ell+1)x} \right) dx \\ &= c\lambda \int_0^t x^{r+1} e^{-\lambda x} dx + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} \int_0^t x^r e^{-\lambda(\ell+1)x} dx \\ &= c \frac{\Gamma(\lambda t, r+2)}{\lambda^{r+1}} + \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{k,\ell} \frac{\Gamma(\lambda(\ell+1)t, r+1)}{\lambda^{r+1}(\ell+1)^{r+1}}.\end{aligned}$$

The proof is completed. \square

The r -th conditional moments of the LWE distribution is given by

$$E(X^r | X > t) = \frac{1}{1-F(t)} \int_t^{+\infty} x^r f(x) dx = \frac{1}{1-F(t)} \left(E(X^r) - \int_0^t x^r f(x) dx \right).$$

The r -th reversed moments of the LWE distribution is given by

$$E(X^r | X \leq t) = \frac{1}{F(t)} \int_0^t x^r f(x) dx.$$

Both moments can be expressed using (8). The mean deviations of X about the mean $\mu = E(X)$ can be expressed as $\delta = 2\mu F(\mu) - 2 \int_0^{\mu} x f(x) dx$ and the mean deviations of X about the median M has the form $\eta = \mu - 2 \int_0^M x f(x) dx$.

4. Residuals life function

Let X be a random variable having the pdf $f(x)$ given by (1). The residual life is described by the conditional random variable $R_{(t)} = X - t | X > t$, $t \geq 0$. It naturally appears in life length studies (see [6] and [12]). Using the sf $S(x)$ (2), the sf of the residual lifetime $R_{(t)}$ is given by

$$S_{R_{(t)}}(x) = \frac{S(x+t)}{S(t)} = \frac{\log(1 + \beta e^{-\lambda(x+t)}) + \beta e^{-\lambda(x+t)} \log(\beta + e^{\lambda(x+t)})}{\log(1 + \beta e^{-\lambda t}) + \beta e^{-\lambda t} \log(\beta + e^{\lambda t})}, \quad x > 0.$$

The associated cdf is given by

$$F_{R_{(t)}}(x) = \frac{\log\left(\frac{1 + \beta e^{-\lambda t}}{1 + \beta e^{-\lambda(x+t)}}\right) + \beta e^{-\lambda t} (\log(\beta + e^{\lambda t}) - e^{-\lambda x} \log(\beta + e^{\lambda(x+t)}))}{\log(1 + \beta e^{-\lambda t}) + \beta e^{-\lambda t} \log(\beta + e^{\lambda t})}, \quad x > 0.$$

Then, the corresponding pdf is given by

$$f_{R_{(t)}}(x) = \frac{\lambda\beta \log(\beta + e^{\lambda(x+t)}) e^{-\lambda(x+t)}}{\log(1 + \beta e^{-\lambda t}) + \beta e^{-\lambda t} \log(\beta + e^{\lambda t})}, \quad x > 0.$$

The associated hrf is given by

$$h_{R_{(t)}}(x) = \frac{\lambda\beta \log(\beta + e^{\lambda(x+t)}) e^{-\lambda(x+t)}}{\log(1 + \beta e^{-\lambda(x+t)}) + \beta e^{-\lambda(x+t)} \log(\beta + e^{\lambda(x+t)})}, \quad x > 0.$$

The mean residual life is defined as

$$K(t) = E(R_{(t)}) = E(X - t | X > t) = \frac{1}{S(t)} \int_t^{+\infty} x f(x) dx - t = \frac{1}{S(t)} \left(E(X) - \int_0^t x f(x) dx \right) - t,$$

where $f(x)$ is given by (1), $S(t)$ is given by (2), $E(X)$ is given by (7) with $r = 1$ and $\int_0^t xf(x)dx$ is given by (8) with $r = 1$.

On the other hand, the variance residual life is given by

$$\begin{aligned} V(t) &= \text{Var}(R_{(t)}) = \text{Var}(X - t \mid X > t) = \frac{2}{S(t)} \int_t^{+\infty} xS(x)dx - 2tK(t) - [K(t)]^2 \\ &= \frac{1}{S(t)} \left(E(X^2) - \int_0^t x^2 f(x)dx \right) - t^2 - 2tK(t) - [K(t)]^2, \end{aligned}$$

where $E(X^2)$ is given by (7) with $r = 2$ and $\int_0^t x^2 f(x)dx$ is given by (8) with $r = 2$.

The reverse residual life is described by the conditional random variable $\bar{R}_{(t)} = t - X \mid X \leq t$, $t \geq 0$. Using the cdf (3), the sf of the reversed residual lifetime $\bar{R}_{(t)}$ is given by

$$S_{\bar{R}_{(t)}}(x) = \frac{F(t-x)}{F(t)} = \frac{(\beta+1)\log(\beta+1) - \log(1+\beta e^{-\lambda(t-x)}) + \beta e^{-\lambda(t-x)} \log(\beta + e^{\lambda(t-x)})}{(\beta+1)\log(\beta+1) - \log(1+\beta e^{-\lambda t}) + \beta e^{-\lambda t} \log(\beta + e^{\lambda t})}, \quad 0 < x \leq t.$$

The associated cdf is given by

$$F_{\bar{R}_{(t)}}(x) = \frac{\log(1+\beta e^{-\lambda(t-x)}) + \beta e^{-\lambda(t-x)} \log(\beta + e^{\lambda(t-x)}) - \log(1+\beta e^{-\lambda t}) + \beta e^{-\lambda t} \log(\beta + e^{\lambda t})}{(\beta+1)\log(\beta+1) - \log(1+\beta e^{-\lambda t}) + \beta e^{-\lambda t} \log(\beta + e^{\lambda t})}, \quad x > 0.$$

Therefore, the corresponding pdf is given by

$$f_{\bar{R}_{(t)}}(x) = \frac{\lambda \beta \log(\beta + e^{\lambda(t-x)}) e^{-\lambda(t-x)}}{(\beta+1)\log(\beta+1) - \log(1+\beta e^{-\lambda t}) + \beta e^{-\lambda t} \log(\beta + e^{\lambda t})}, \quad x > 0.$$

The associated hrf is given by

$$h_{\bar{R}_{(t)}}(x) = \frac{\lambda \beta \log(\beta + e^{\lambda(t-x)}) e^{-\lambda(t-x)}}{(\beta+1)\log(\beta+1) - \log(1+\beta e^{-\lambda(t-x)}) + \beta e^{-\lambda(t-x)} \log(\beta + e^{\lambda(t-x)})}, \quad x > 0.$$

The mean reversed residual life is defined as

$$L(t) = E(\bar{R}_{(t)}) = E(t - X \mid X \leq t) = t - \frac{1}{F(t)} \int_0^t xf(x)dx,$$

where $f(x)$ is given by (1), $F(t)$ is given by (3) and $\int_0^t xf(x)dx$ is given by (8) with $r = 1$.

The variance reversed residual life is given by

$$\begin{aligned} W(t) &= \text{Var}(\bar{R}_{(t)}) = \text{Var}(t - X \mid X \leq t) = 2tL(t) - [L(t)]^2 - \frac{2}{F(t)} \int_0^t xF(x)dx \\ &= 2tL(t) - [L(t)]^2 - t^2 + \frac{1}{F(t)} \int_0^t x^2 f(x)dx, \end{aligned}$$

where $\int_0^t x^2 f(x)dx$ is given by (8) with $r = 2$.

5. Order statistics and record values

Order statistics naturally appear in many areas of statistics, as reliability and quality control testing. Let X_1, X_2, \dots, X_n be n i.i.d. random variables having the pdf (1). Let us consider its order statistics is $X_{1:n}, X_{2:n}, \dots, X_{n:n}$. Using the expressions of $f(x)$ given by (1) and $F(x)$ (3), the pdf of the i -th order statistic $X_{i:n}$ is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} [F(x)]^{i-1} [1-F(x)]^{n-i} f(x) \\ &= \frac{n!}{(i-1)!(n-i)!} \left[1 - \frac{\log(1+\beta e^{-\lambda x}) + \beta e^{-\lambda x} \log(\beta + e^{\lambda x})}{(\beta+1)\log(\beta+1)} \right]^{i-1} \\ &\times \left[\frac{\log(1+\beta e^{-\lambda x}) + \beta e^{-\lambda x} \log(\beta + e^{\lambda x})}{(\beta+1)\log(\beta+1)} \right]^{n-i} \frac{\lambda \beta}{(\beta+1)\log(\beta+1)} \log(\beta + e^{\lambda x}) e^{-\lambda x}, \quad x > 0. \end{aligned}$$

In particular, let us mention that the pdf of $X_{1:n} = \inf(X_1, \dots, X_n)$ is given by

$$\begin{aligned} f_{1:n}(x) &= n[1 - F(x)]^{n-1} f(x) \\ &= \frac{\lambda\beta}{(\beta + 1)\log(\beta + 1)} n \left[\frac{\log(1 + \beta e^{-\lambda x}) + \beta e^{-\lambda x} \log(\beta + e^{\lambda x})}{(\beta + 1)\log(\beta + 1)} \right]^{n-1} \log(\beta + e^{\lambda x}) e^{-\lambda x} \end{aligned}$$

and the pdf of $X_{n:n} = \sup(X_1, \dots, X_n)$ is given by

$$\begin{aligned} f_{n:n}(x) &= n[F(x)]^{n-1} f(x) \\ &= \frac{\lambda\beta}{(\beta + 1)\log(\beta + 1)} n \left[1 - \frac{\log(1 + \beta e^{-\lambda x}) + \beta e^{-\lambda x} \log(\beta + e^{\lambda x})}{(\beta + 1)\log(\beta + 1)} \right]^{n-1} \log(\beta + e^{\lambda x}) e^{-\lambda x}. \end{aligned}$$

The cdf of the i -th order statistic $X_{i:n}$ is given by

$$\begin{aligned} F_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{i+k} [F(x)]^{i+k} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{i+k} \left[1 - \frac{\log(1 + \beta e^{-\lambda x}) + \beta e^{-\lambda x} \log(\beta + e^{\lambda x})}{(\beta + 1)\log(\beta + 1)} \right]^{i+k}, \quad x > 0. \end{aligned}$$

For $i < j$, the joint pdf of $(X_{i:n}, X_{j:n})$ is given by

$$\begin{aligned} f_{(i:n, j:n)}(x_i, x_j) &= \frac{n!}{(i-1)!(n-j)!(j-i-1)!} [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-i-1} [1 - F(x_j)]^{n-j} f(x_i) f(x_j) \\ &= \frac{n!}{(i-1)!(n-j)!(j-i-1)!} \left[1 - \frac{\log(1 + \beta e^{-\lambda x_i}) + \beta e^{-\lambda x_i} \log(\beta + e^{\lambda x_i})}{(\beta + 1)\log(\beta + 1)} \right]^{i-1} \\ &\quad \times \left[\frac{\log\left(\frac{1 + \beta e^{-\lambda x_i}}{1 + \beta e^{-\lambda x_j}}\right) + \beta(e^{-\lambda x_i} \log(\beta + e^{\lambda x_i}) - e^{-\lambda x_j} \log(\beta + e^{\lambda x_j}))}{(\beta + 1)\log(\beta + 1)} \right]^{j-i-1} \\ &\quad \times \left[\frac{\log(1 + \beta e^{-\lambda x_j}) + \beta e^{-\lambda x_j} \log(\beta + e^{\lambda x_j})}{(\beta + 1)\log(\beta + 1)} \right]^{n-j} \left(\frac{\lambda\beta}{(\beta + 1)\log(\beta + 1)} \right)^2 \\ &\quad \times \log(\beta + e^{\lambda x_i}) \log(\beta + e^{\lambda x_j}) e^{-\lambda(x_i + x_j)}. \end{aligned}$$

As order statistics, record values arise in many situations in applied statistics. Theoretical and practical aspects of record values can be found in [1] and [3]. Let X_1, X_2, \dots , be a sequence of i.i.d. random variables having the pdf (1). We define a sequence of record times $U(n)$ as follows: $U(1) = 1$, $U(n) = \min\{j; j > U(n-1), X_j > X_{U(n-1)}\}$ for $n \geq 2$. We define the i -th upper record value by $R_i = X_{U(i)}$, with $R_1 = X_1$. Using the pdf $f(x)$ (1) and the cdf $F(x)$ (3), the pdf of R_i is given by

$$\begin{aligned} f_{R_i}(x) &= \frac{[-\log(1 - F(x))]^{i-1}}{(i-1)!} f(x) \\ &= \frac{1}{(i-1)!} \left[-\log\left(\frac{\log(1 + \beta e^{-\lambda x}) + \beta e^{-\lambda x} \log(\beta + e^{\lambda x})}{(\beta + 1)\log(\beta + 1)}\right) \right]^{i-1} \\ &\quad \times \frac{\lambda\beta}{(\beta + 1)\log(\beta + 1)} \log(\beta + e^{\lambda x}) e^{-\lambda x}, \quad x > 0. \end{aligned}$$

Using pdf (1) and the hrf (4), the joint pdf of (R_1, \dots, R_n) is given by

$$\begin{aligned} f_{(R_1, \dots, R_n)}(x_1, \dots, x_n) &= f(x_n) \prod_{k=1}^{n-1} h(x_k) = \frac{\lambda\beta}{(\beta + 1)\log(\beta + 1)} \log(\beta + e^{\lambda x_n}) e^{-\lambda x_n} \\ &\quad \times \prod_{k=1}^{n-1} \frac{\lambda\beta \log(\beta + e^{\lambda x_k}) e^{-\lambda x_k}}{\log(1 + \beta e^{-\lambda x_k}) + \beta e^{-\lambda x_k} \log(\beta + e^{\lambda x_k})}, \end{aligned}$$

for $0 < x_1 < \dots < x_n$.

For $i < j$, the joint pdf of (R_i, R_j) is given by

$$\begin{aligned}
f_{(R_i, R_j)}(x_i, x_j) &= \frac{[-\log(1 - F(x_i))]^{i-1}}{(i-1)!} \frac{\left[\log\left(\frac{1-F(x_i)}{1-F(x_j)}\right)\right]^{j-i-1}}{(j-i-1)!} h(x_i) f(x_j) \\
&= \frac{1}{(i-1)!} \left[-\log\left(\frac{\log(1 + \beta e^{-\lambda x_i}) + \beta e^{-\lambda x_i} \log(\beta + e^{\lambda x_i})}{(\beta + 1) \log(\beta + 1)}\right) \right]^{i-1} \\
&\times \frac{1}{(j-i-1)!} \left[\log\left(\frac{\log(1 + \beta e^{-\lambda x_i}) + \beta e^{-\lambda x_i} \log(\beta + e^{\lambda x_i})}{\log(1 + \beta e^{-\lambda x_j}) + \beta e^{-\lambda x_j} \log(\beta + e^{\lambda x_j})}\right) \right]^{j-i-1} \\
&\times \frac{\lambda \beta \log(\beta + e^{\lambda x_i}) e^{-\lambda x_i}}{\log(1 + \beta e^{-\lambda x_i}) + \beta e^{-\lambda x_i} \log(\beta + e^{\lambda x_i})} \times \frac{\lambda \beta}{(\beta + 1) \log(\beta + 1)} \log(\beta + e^{\lambda x_j}) e^{-\lambda x_j},
\end{aligned}$$

for $0 < x_i < x_j$.

6. Estimation procedures

This section is devoted to three estimation procedures for the LWE model parameters: the maximum likelihood estimation, weighted least square estimation and maximum product of spacings.

6.1. Maximum likelihood estimators

Let X_1, X_2, \dots, X_n be a random sample from the LWE distribution with parameter vector $\Theta = (\beta, \lambda)$ and x_1, x_2, \dots, x_n are the corresponding observed values. By considering $f(x)$ (1), the likelihood function is given by

$$L(\Theta) = \prod_{i=1}^n f(x_i) = \frac{\lambda^n \beta^n}{(\beta + 1)^n (\log(\beta + 1))^n} \left[\prod_{i=1}^n \log(\beta + e^{\lambda x_i}) \right] \left[\prod_{i=1}^n e^{-\lambda x_i} \right].$$

The log-likelihood function can be expressed as

$$\ell(\Theta) = n \log(\lambda) + n \log(\beta) - n \log(\beta + 1) - n \log(\log(\beta + 1)) + \sum_{i=1}^n \log(\log(\beta + e^{\lambda x_i})) - \lambda \sum_{i=1}^n x_i.$$

The nonlinear log-likelihood equations given by $\frac{\partial \ell(\Theta)}{\partial \Theta} = 0$ are listed below

$$\frac{\partial \ell(\Theta)}{\partial \beta} = \frac{n}{\beta} - \frac{n}{\beta + 1} - \frac{n}{(\beta + 1) \log(\beta + 1)} + \sum_{i=1}^n \frac{1}{(\beta + e^{\lambda x_i}) \log(\beta + e^{\lambda x_i})} = 0$$

and

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \frac{\lambda e^{\lambda x_i}}{(\beta + e^{\lambda x_i}) \log(\beta + e^{\lambda x_i})} - \sum_{i=1}^n x_i = 0.$$

Solving the equations above simultaneously, we obtain the maximum likelihood estimators (MLEs) of the model parameters. Some numerical iterative methods may be used to estimate the model parameters and the global maxima of the log-likelihood can be justified by considering different starting values for the parameters. To show the likelihood equations have a unique solution in the parameters, we plot the profile log-likelihood function of the model parameters in Figure 3, for one of the data sets (D1) used in the application section.

Using the asymptotic distribution of the MLEs, the information matrix is used to establish the confidence intervals for the model parameters. The elements of the 2×2 observed information matrix $J(\Theta) = Jrs(\Theta)$ for $r, s \in \{\beta, \lambda\}$ can be obtained from the authors upon request.

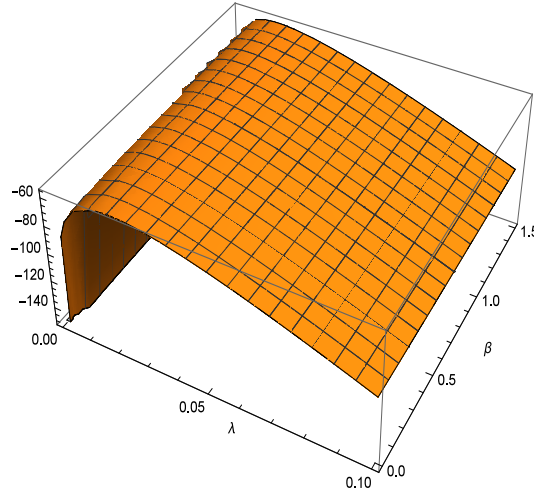


Figure 3: The profile of log-likelihood function for (λ, β) .

6.2. Weighted least-square estimators

Let x_1, x_2, \dots, x_n be an ordered sample of the random sample of size n from the LWE distribution. Let $F(x)$ be the cdf (3). Then the weighted least square estimators (WLSE) can be obtained by minimizing

$$\begin{aligned} S(\Theta) &= \sum_{i=1}^n w_i \{F(x_i) - E(F(X_{i:n}))\}^2 \\ &= \sum_{i=1}^n w_i \left\{ 1 - \frac{\log(1 + \beta e^{-\lambda x_i}) + \beta e^{-\lambda x_i} \log(\beta + e^{\lambda x_i})}{(\beta + 1) \log(\beta + 1)} - \frac{i}{n+1} \right\}^2, \end{aligned}$$

with respect to the unknown parameters of the LWE distribution, where

$$w_i = \frac{1}{\text{Var}(F(X_{i:n}))} = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

The associated nonlinear equations $\frac{\partial S(\Theta)}{\partial \Theta} = 0$ are given by

$$\frac{\partial S(\Theta)}{\partial \beta} = 2 \sum_{i=1}^n w_i \left\{ 1 - \frac{\log(1 + \beta e^{-\lambda x_i}) + \beta e^{-\lambda x_i} \log(\beta + e^{\lambda x_i})}{(\beta + 1) \log(\beta + 1)} - \frac{i}{n+1} \right\} \eta_1(x_i, \Theta) = 0$$

and

$$\frac{\partial S(\Theta)}{\partial \lambda} = 2 \sum_{i=1}^n w_i \left\{ 1 - \frac{\log(1 + \beta e^{-\lambda x_i}) + \beta e^{-\lambda x_i} \log(\beta + e^{\lambda x_i})}{(\beta + 1) \log(\beta + 1)} - \frac{i}{n+1} \right\} \eta_2(x_i, \Theta) = 0,$$

where

$$\begin{aligned} \eta_1(x_i, \Theta) &= -\frac{1}{(\beta + 1)^2 (\log(\beta + 1))^2} \left(\log(\beta + 1) + \beta \log(\beta + 1) - \beta \log(\beta + e^{\lambda x_i}) \right. \\ &\quad \left. + \log(\beta + 1) \log(\beta + e^{\lambda x_i}) - e^{\lambda x_i} (1 + \log(\beta + 1)) \log(1 + \beta e^{-\lambda x_i}) \right) e^{-\lambda x_i} \end{aligned} \quad (9)$$

and

$$\eta_2(x_i, \Theta) = \frac{\beta}{(\beta + 1) \log(\beta + 1)} x e^{-\lambda x_i} \log(\beta + e^{\lambda x_i}). \quad (10)$$

Solving this system of nonlinear equations simultaneously will yield the WLSE's of the distribution parameters.

6.3. Maximum product of spacings estimators

The maximum product of spacings (MPS) method is a powerful alternative to MLE for the estimation of the unknown parameters of continuous univariate distributions. Let x_1, x_2, \dots, x_n be an ordered sample of the random sample of size n from the LWE distribution. Let $F(x)$ be the cdf (3) and $D_i(\Theta) = F(x_i) - F(x_{i-1})$ for $i = 1, 2, \dots, n+1$, be the uniform spacings of a random sample from the LWE distribution, where $F(x_0) = 0$ and $F(x_{n+1}) = 1$. The maximum product spacings estimators $\hat{\Theta}_{MPS}$ are obtained by maximizing the geometric mean of the spacings given by

$$G(\Theta) = \left[\prod_{i=1}^{n+1} D_i(\Theta) \right]^{\frac{1}{n+1}},$$

with respect to Θ . It is equivalent to maximize the logarithm of the geometric mean of sample spacings given by

$$\ell(\Theta) = \log(G(\Theta)) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log(D_i(\Theta)).$$

Using $\eta_1(x_i, \Theta)$ (9) and $\eta_2(x_i, \Theta)$ (10), the estimates $\hat{\Theta}_{MPS}$ can be obtained by solving the non-linear equations

$$\frac{\partial \ell(\Theta)}{\partial \beta} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\Theta)} [\eta_1(x_i, \Theta) - \eta_1(x_{i-1}, \Theta)] = 0$$

and

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\Theta)} [\eta_2(x_i, \Theta) - \eta_2(x_{i-1}, \Theta)] = 0.$$

7. Applications to real data

In this section, the potentiality of the LWE distribution is highlighted. We show that our distribution has a better fit than the compared distributions under all estimation procedures: MLE, WLSE and MPSE, for four real data sets.

We fit the two-parameter gamma (gamma), exponential logarithmic (EL) [16], exponentiated exponential (EE) [9], weighted exponential (WE) [7] and LWE distributions to four real data sets. The pdf of these models are as follows.

- The pdf of the gamma distribution is given by

$$f(x) = \frac{x^{k-1} e^{-\frac{x}{\theta}}}{\theta^k \Gamma(k)}, \quad x, k, \theta > 0.$$

- The pdf of the exponential logarithmic distribution is given by

$$f(x) = -\frac{\beta(1-p)e^{-\beta x}}{\log(p)(1-(1-p)e^{-\beta x})}, \quad x, \beta > 0, p \in (0, 1).$$

- The pdf of the exponentiated exponential distribution is given by

$$f(x) = \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1}, \quad x, \alpha, \lambda > 0.$$

- The pdf of the weighted exponential distribution is given by

$$f(x) = \frac{(\alpha+1)\lambda e^{-\lambda x} (1 - e^{-\alpha \lambda x})}{\alpha}, \quad x, \alpha, \lambda > 0.$$

Table 1: Descriptive statistics of the data sets

D1							
n	Mean	Median	SD	Skewness	Kurtosis	M1	M2
10	100.1	71.6	90.5638	0.943922	2.86235	70.4	47.8
D2							
20	8.42945	8.662	5.32206	0.176969	2.43092	4.2029	3.663
D3							
53	80.883	76	52.2441	0.494003	2.51167	43.2915	41.1
D4							
76	1.95924	1.73615	1.57398	1.97956	8.16079	1.07026	0.7202

SD = Standard Deviation, M1 = Mean deviation about the mean,
M2 = Mean deviation about the median

Table 2: Some moments of the LWE

D1					D2				
Mean	Median	SD	M1	M2	Mean	Median	SD	M1	M2
99.9137	74.4296	91.026	69.3594	66.3962	9.13596	7.51929	7.02205	5.38507	5.22879
D3					D4				
Mean	Median	SD	M1	M2	Mean	Median	SD	M1	M2
84.2705	69.9799	62.7895	48.1234	46.7521	1.89683	0.593826	1.35356	1.03643	1.00689

We check the adequacy of the fitted models via the statistics of Anderson-Darling and the Cramér-von Mises. They allow to determine how closely a specific distribution fits the associated empirical distribution for a given data set. The smaller statistics give the better fit.

The description of the considered real data sets is as follows.

Time to failure in hours for non-repairable Item data Set (D1). The first data set we consider represents the time to failure in hours for non-repairable items. The data set consists of 10 sample points and is reported at [11, Page 279].

Failure times data Set (D2). The second data set we consider represents the failure times of 20 components. The data set can be found in [11, Page 154].

Total monthly rainfall data Set (D3). The third data set is obtained from the department of water resources and power agency manager of water resources of the State of Sao Paulo. The data represents total monthly rainfall during April at Sao Carlos from 1960 to 2014. The data set can be found at [4].

Fatigue fracture data Set (D4). The fourth data represents the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed. For previous studies with the data see [5].

Table 1 provides descriptive statistics of the data sets. From this table, it can be seen that all the used data sets are right skewed. Some moments of the LWE distribution are given in Table 2. We computed the results using the definitions, where the parameters of the LWE are replaced by MLEs for D1 and WLSEs for the other three data sets respectively from Tables 3-6. From Tables 1 and 2, it can be noted that the considered moments of the LWE distribution are approximately close to the sample moments for all the four data sets.

Tables 3-6 list the parameter estimates for the three estimation procedures discussed in Section 6, for the four data sets with goodness-of-fit statistics. From these tables, it can be noted that LWE distribution is a powerful competitor to all the compared distributions. It provides a rather flexible mechanism for fitting a wide spectrum of positive real data sets with shape property of being right skewed. It is also observed that for small sample size MLE performs better than the WLSE and MPSE but for larger sample size WLSE performs better than MLE and MPSE.

The conclusions drawn above are confirmed by Figure 4; it can be seen that the estimated LWE density superimposed on the histogram of the data sets based on MLEs for D1 and WLSEs for D2-D4 provides a good fit to these data.

Table 3: Comparison of fit of LWE using different methods of estimation for Time to Failure in Hours for Nonrepairable Item data

MLEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	1.139028	87.881928	0.144832	0.016461
EL(p, β)	0.999999	0.009990	0.146562	0.017179
WE(λ, α)	0.010233	39.953472	0.149407	0.017899
EE(λ, α)	0.010826	1.136636	0.143582	0.016449
LWE(λ, β)	0.015096	1.451571	0.128042	0.014967
WLSEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	0.796078	147.040746	0.187528	0.021962
EL(p, β)	0.346263	0.006459	0.176709	0.020845
WE(λ, α)	0.009125	1549.999999	0.129782	0.0148178
EE(λ, α)	0.007323	0.789169	0.187466	0.021972
LWE(λ, β)	0.010721	7.699068	0.134756	0.015240
MPSEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	0.788220	146.918918	0.189559	0.022436
EL(p, β)	0.353425	0.006614	0.172185	0.020647
WE(λ, α)	0.008841	499.999279	0.146562	0.017179
EE(λ, α)	0.007347	0.781233	0.189547	0.022428
LWE(λ, β)	0.010536	7.490030	0.143201	0.016290

Table 4: Comparison of fit of LWE using different methods of estimation for failure times of 20 components data

MLEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	1.276065	6.605812	0.964807	0.173748
EL(p, β)	0.999999	0.118626	1.20985	0.240738
WE(λ, α)	0.237238	0.000210	1.06544	0.116016
EE(λ, α)	0.134624	1.235388	0.997362	0.182967
LWE(λ, β)	0.211521	0.411468	0.707688	0.126298
WLSEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	1.872033	4.914790	0.925161	0.076585
EL(p, β)	0.999999	0.104051	0.948973	0.161832
WE(λ, α)	0.217647	0.000038	1.002540	0.074213
EE(λ, α)	0.158061	1.918960	0.963278	0.081900
LWE(λ, β)	0.202654	0.238868	0.605758	0.076161
MPSEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	0.994138	8.962964	1.06508	0.200422
EL(p, β)	0.999999	0.112255	1.057910	0.198708
WE(λ, α)	0.228869	0.000494	1.004650	0.092105
EE(λ, α)	0.109947	0.968710	1.096200	0.207921
LWE(λ, β)	0.195684	0.587401	0.662525	0.111726

Table 5: Comparison of fit of LWE using different methods of estimation for Rainfall data

MLEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	1.539744	52.530164	0.877195	0.130219
EL(p, β)	0.999999	0.012363	2.36462	0.425471
WE(λ, α)	0.024726	0.000042	0.820019	0.092466
EE(λ, α)	0.015789	1.514078	0.973941	0.147181
LWE(λ, β)	0.023096	0.213791	0.596077	0.090642
WLSEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	1.823373	46.318640	0.665771	0.060652
EL(p, β)	0.999999	0.011138	1.83096	0.2781
WE(λ, α)	0.023778	0.000020	0.758456	0.064999
EE(λ, α)	0.017183	1.901809	0.727332	0.067376
LWE(λ, β)	0.022642	0.137041	0.483161	0.053447
MPSEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	1.379133	59.777856	1.00204	0.149678
EL(p, β)	0.999999	0.012082	2.19852	0.382795
WE(λ, α)	0.024345	0.000001	0.777771	0.0783983
EE(λ, α)	0.014531	1.353717	1.10728	0.168091
LWE(λ, β)	0.022328	0.277365	0.5744	0.080558

Table 6: Comparison of fit of LWE using different methods of estimation for fatigue data

MLEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	1.641076	1.193875	0.680647	0.113341
EL(p, β)	0.999999	0.510401	3.01876	0.574565
WE(λ, α)	0.711243	1.541313	0.580135	0.0836306
EE(λ, α)	0.702793	1.709493	0.677809	0.113052
LWE(λ, β)	0.970107	0.150408	0.59646	0.100525
WLSEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	2.003304	0.944853	0.583048	0.063036
EL(p, β)	0.999999	0.456246	2.60599	0.456466
WE(λ, α)	0.847727	0.613244	0.557909	0.067250
EE(λ, α)	0.815028	2.150388	0.569838	0.062684
LWE(λ, β)	1.046110	0.021336	0.542988	0.066131
MPSEs				
Distributions	Estimates		A^*	W^*
Gamma(k, θ)	1.476364	1.358301	0.906648	0.160206
EL(p, β)	0.999998	0.496530	2.83668	0.527345
WE(λ, α)	0.650868	2.265658	0.659052	0.101915
EE(λ, α)	0.635134	1.658918	0.844588	0.15135
LWE(λ, β)	0.930942	0.217076	0.727137	0.1306

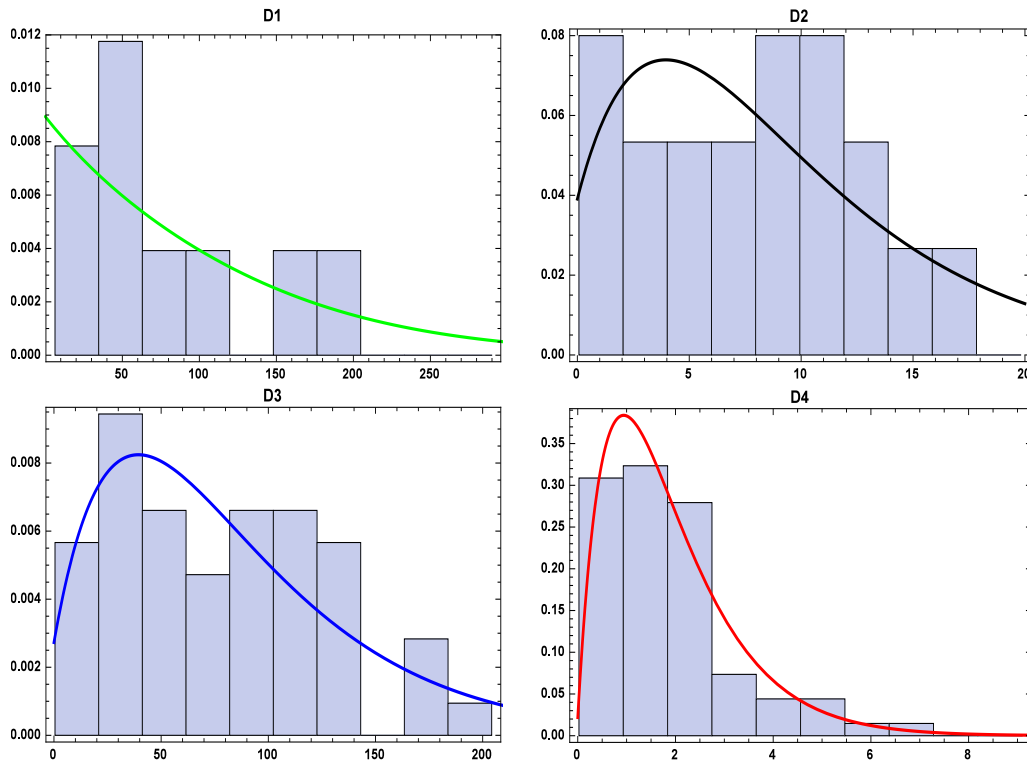


Figure 4: The fitted LWE pdfs superimposed on the histogram of data.

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