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Quasilinear and Hessian Lane-Emden type systems with measure data

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Abstract

We study nonlinear systems of the form $-\Delta_p u = v^{q_1} + \mu$, $-\Delta_p v = u^{q_2} + \eta$ and $F_k[-u] = v^{s_1} + \mu$, $F_k[-v] = u^{s_2} + \eta$ in a bounded domain Ω or in \mathbb{R}^N where μ and η are nonnegative Radon measures, Δ_p and F_k are respectively the p -Laplacian and the k -Hessian operators and q_1, q_2, s_1 and s_2 positive numbers. We give necessary and sufficient conditions for existence expressed in terms of Riesz or Bessel capacities. 2010

Mathematics Subject Classification. 35J70, 35J60, 45G15, 31C15.

Key words: p -Laplacian, k -Hessian, Bessel and Riesz capacities, measures, maximal functions.

1 Introduction and Main results

Let $\Omega \subset \mathbb{R}^N$ be either a bounded domain or the whole \mathbb{R}^N , $p > 1$ and $k \in \{1, 2, \dots, N\}$. We denote by

$$\Delta_p u := \operatorname{div} (|\nabla u|^{p-2} \nabla u)$$

the p -Laplace operator and by

$$F_k[u] = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq N} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_k}$$

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the k -Hessian operator where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of the Hessian matrix D^2u . In the work [20], Phuc and Verbitsky obtained necessary and sufficient conditions for existence of nonnegative solutions to the following equations

$$\begin{aligned} -\Delta_p u &= u^q + \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} F_k[-u] &= u^q + \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (1.2)$$

Their conditions involve the continuity of the measures with respect to Bessel or Riesz capacities and Wolff potentials estimates. For example, if Ω is bounded and μ has compact support in Ω , they proved that it is equivalent to solve (1.1), or to have

$$\mu(E) \leq c_1 \text{Cap}_{\mathbf{G}_{p, \frac{q}{q+1-p}}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.3)$$

for some constant $c_1 > 0$ where $\text{Cap}_{\mathbf{G}_{p, \frac{q}{q+1-p}}}$ is a Bessel capacity, or to have

$$\int_B (\mathbf{W}_{1,p}^R[\mu_B](x))^q dx \leq c_2 \mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \text{supp } \mu \neq \emptyset, \quad (1.4)$$

for some constant $c_2 > 0$, where $R = 2 \text{diam}(\Omega)$ and $\mathbf{W}_{1,p}^R[\mu_B]$ denotes the R -truncated Wolff potential of the measure $\mu_B = \chi_B \mu$. Concerning the k -Hessian operator in a bounded $(k-1)$ -convex domain Ω , they proved that if μ has compact support, the problem (1.2) with $q > k$ admits a nonnegative solution if and only if

$$\mu(E) \leq c_3 \text{Cap}_{\mathbf{G}_{2k, \frac{q}{q-k}}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.5)$$

for some c_3 . In turn this condition is equivalent to

$$\int_B \left[\mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu_B(x)] \right]^q dx \leq c_4 \mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \text{supp } \mu \neq \emptyset, \quad (1.6)$$

for some $c_4 > 0$. The results concerning the linear case $p = 2$ and $k = 1$, can be found in [2, 3, 27].

The natural counterpart of equation (1.1) and (1.2) for systems:

$$\begin{aligned} -\Delta_p u &= v^{q_1} + \mu && \text{in } \Omega \\ -\Delta_p v &= u^{q_2} + \eta && \text{in } \Omega \\ u &= v = 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.7)$$

and

$$\begin{aligned} F_k[-u] &= v^{s_1} + \mu && \text{in } \Omega \\ F_k[-v] &= u^{s_2} + \eta && \text{in } \Omega \\ u &= v = 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.8)$$

where $q_1, q_2 > p-1, s_1, s_2 > k$ and μ, η are Radon measures. If $\Omega = \mathbb{R}^N$, we consider the same equations, except that the boundary conditions are replaced by $\inf_{\mathbb{R}^N} u = \inf_{\mathbb{R}^N} v = 0$ and our statements involve the Riesz potentials and their associated capacities $\text{Cap}_{I_{\alpha, \beta}}$. Our main results are the following.

Theorem A *Let $1 < p < N, q_1, q_2 > 0$ and $q_2 q_1 > (p-1)^2$. Let μ, η be nonnegative Radon measures in \mathbb{R}^N . If the following system*

$$\begin{aligned} -\Delta_p u &= v^{q_1} + \mu && \text{in } \mathbb{R}^N \\ -\Delta_p v &= u^{q_2} + \eta && \text{in } \mathbb{R}^N, \end{aligned} \quad (1.9)$$

admits a nonnegative p -superharmonic solution (u, v) then there exists a positive constant c_5 depending on N, p, q_1, q_2 such that

$$\eta(E) + \int_E (\mathbf{W}_{1,p}[\mu](x))^{q_2} dx \leq c_5 \text{Cap}_{\mathbf{I}_{\frac{p(q_1+p-1)}{q_1}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(E) \quad \text{for all Borel sets } E. \quad (1.10)$$

Conversely, if μ and η are bounded, there exists $c_6 > 0$ depending on N, p, q_1, q_2 such that if $0 < q_1 < \frac{N(p-1)}{N-p}$ and (1.10) holds with c_5 replaced by c_6 , then (1.9) admits a nonnegative p -superharmonic solution (u, v) satisfying

$$v \leq c_8 \mathbf{W}_{1,p}[\omega], \quad u \leq c_9 \mathbf{W}_{1,p}[(\mathbf{W}_{1,p}[\omega])^{q_1}] + c_7 \mathbf{W}_{1,p}[\mu] \quad (1.11)$$

in \mathbb{R}^N for some $c_7, c_8, c_9 > 0$ where $d\omega = (\mathbf{W}_{1,p}[\mu])^{q_2} dx + d\eta$.

We notice that the left-hand side in (1.10) is not symmetric in η and μ and the capacity in the right-hand side is not symmetric in q_1 and q_2 . Hence the following symmetrized inequality holds

$$\mu(E) + \int_E (\mathbf{W}_{1,p}[\eta](x))^{q_1} dx \leq c'_5 \text{Cap}_{\mathbf{I}_{\frac{p(q_2+p-1)}{q_2}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(E) \quad \text{for all Borel sets } E. \quad (1.12)$$

It is known that

$$\text{Cap}_{\mathbf{I}_{\alpha,\beta}}(K) = 0 \quad \forall K \text{ compact},$$

if $\alpha\beta \geq N$, the first part of above implies the following Liouville theorem, obtained by another method in [9, Th 5.3-(i)].

Corollary B Assume that

$$\frac{p(q_1 q_2 + (p-1) \max\{q_1, q_2\})}{q_1 q_2 - (p-1)^2} \geq N.$$

Any nonnegative p -superharmonic solution (u, v) of inequalities

$$\begin{aligned} -\Delta_p u &\geq v^{q_1} && \text{in } \mathbb{R}^N \\ -\Delta_p v &\geq u^{q_2} && \text{in } \mathbb{R}^N, \end{aligned} \quad (1.13)$$

is trivial, i.e. $u = v = 0$.

Classical Liouville results for one equation or inequality, are proved in [4], [5], [11], [22].

When Ω is bounded domain, we have a similar result in which we denote by d the distance function to the boundary $x \mapsto d(x) = \text{dist}(x, \partial\Omega)$.

Theorem C Let $1 < p < N$, $q_1, q_2 > 0$ and $q_2 q_1 > (p-1)^2$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and μ, η nonnegative Radon measures in Ω . If the following problem

$$\begin{aligned} -\Delta_p u &= v^{q_1} + \mu && \text{in } \Omega \\ -\Delta_p v &= u^{q_2} + \eta && \text{in } \Omega \\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.14)$$

admits a nonnegative renormalized solution (u, v) , then then for any compact set $K \subset \Omega$, there exists a positive constant c_{10} depending on N, p, q_1, q_2 and $\text{dist}(K, \partial\Omega)$ such that

$$\eta(E) + \int_E \left(\mathbf{W}_{1,p}^{\frac{d(x)}{4}}[\mu](x) \right)^{q_2} dx \leq c_{10} \text{Cap}_{\mathbf{G}_{\frac{p(q_1+p-1)}{q_1}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(E) \quad \text{for all Borel sets } E \subset K. \quad (1.15)$$

Conversely, let μ and η be bounded with the property that there exists $c_{11} > 0$ depending on N, p, q_1, q_2 and $R = 2\text{diam}(\Omega)$ such that if $0 < q_1 < \frac{N(p-1)}{N-p}$ and

$$\eta(K) + \int_K (\mathbf{W}_{1,p}^{2R}[\mu])^{q_2} dx \leq c_{11} \text{Cap}_{\mathbf{G}_{\frac{p(q_1+p-1)}{q_1}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(K), \quad (1.16)$$

for all compact set $K \subset \Omega$, then (1.14) admits a nonnegative renormalized solution (u, v) satisfying

$$v \leq c_{13} \mathbf{W}_{1,p}^R[\omega], \quad u \leq c_{14} \mathbf{W}_{1,p}^R[(\mathbf{W}_{1,p}^R[\omega])^{q_1}] + c_{12} \mathbf{W}_{1,p}^R[\mu] \quad (1.17)$$

in Ω , where $d\omega = (\mathbf{W}_{1,p}^R[\mu])^{q_2} dx + d\eta$.

It is known that

$$\text{Cap}_{\mathbf{G}_{\alpha,\beta}}(\{x_0\}) > 0$$

if and only if $\alpha\beta > N$. Thus, as an application in a partially subcritical case we have,

Corollary D *Let the assumptions on p, q_1, q_2, Ω and R of Theorem C be satisfied, $x_0 \in \Omega$, $a > 0$ and μ be a nonnegative Radon measures in Ω . If the following problem*

$$\begin{aligned} -\Delta_p u &= v^{q_1} + \mu && \text{in } \Omega \\ -\Delta_p v &= u^{q_2} + a\delta_{x_0} && \text{in } \Omega \\ u &= v = 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.18)$$

admits a nonnegative renormalized solution (u, v) , then there exist positive constants $c_{15} = c_{15}(N, p, q_1, q_2, d(x_0))$ and, for any compact subset K of Ω , $c_{16} = c_{16}(N, p, q_1, q_2, \text{dist}(K, \partial\Omega))$, such that

$$\begin{aligned} (i) \quad & N < \frac{pq_2(q_1 + p - 1)}{q_1 q_2 - (p - 1)^2}, \\ (ii) \quad & a \leq c_{15}, \\ (iii) \quad & \int_K (\mathbf{W}_{1,p}^{2R}[\mu])^{q_2} dx \leq c_{16}. \end{aligned} \quad (1.19)$$

Conversely, assuming that μ is bounded, there exist positive constants $c_{17} = c_{17}(N, p, q_1, q_2, d(x_0))$, $c_{18} = c_{18}(N, p, q_1, q_2)$ such that if $0 < q_1 < \frac{N(p-1)}{N-p}$ and (1.19) holds with c_{15} and c_{16} replaced respectively by c_{17} and c_{18} , then there exists a nonnegative renormalized solution (u, v) of (1.18) satisfying

$$v \leq c_{21} W_{1,p}^R[\omega], \quad u \leq c_{22} \mathbf{W}_{1,p}^R[(W_{1,p}^R[\omega])^{q_1}] + c_{20} \mathbf{W}_{1,p}^R[\mu] \quad (1.20)$$

in Ω , where

$$W_{1,p}^R[\omega] = \mathbf{W}_{1,p}^R \left[(\mathbf{W}_{1,p}^R[\mu])^{q_2} \right] + a^{\frac{1}{p-1}} \left(|x - x_0|^{-\frac{N-p}{p-1}} - R^{-\frac{N-p}{p-1}} \right)_+.$$

Concerning the k -Hessian operator we recall some notions introduced by Trudinger and Wang [23, 24, 25], and we follow their notations. For $k = 1, \dots, N$ and $u \in C^2(\Omega)$ the k -Hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivatives D^2u and S_k is the k -th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}.$$

Since D^2u is symmetric, it is clear that

$$F_k[u] = [D^2u]_k,$$

where we denote by $[A]_k$ the sum of the k -th principal minors of a matrix $A = (a_{ij})$. In order that there exists a smooth k -admissible function which vanishes on $\partial\Omega$, the boundary $\partial\Omega$ must satisfy a uniformly $(k-1)$ -convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega,$$

for some positive constant c_0 , where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal. We also denote by $\Phi^k(\Omega)$ the class of upper-semicontinuous functions $\Omega \rightarrow [-\infty, \infty)$ which are k -convex, or subharmonic in the Perron sense (see Definition 5.1). In this paper we prove the following theorem (in which expression $\mathbb{E}[q]$ is the largest integer less or equal to q)

Theorem E *Let $2k < N, s_1, s_2 > 0, s_1 s_2 > k^2$. Let Ω be a bounded uniformly $(k-1)$ -convex domain in \mathbb{R}^N with diameter R . Let $\mu = \mu_1 + f$ and $\eta = \eta_1 + g$ be nonnegative Radon measures where μ_1, η_1 has compact support in Ω and $f, g \in L^l(\Omega)$ for some $l > \frac{N}{2k}$. If the following problem*

$$\begin{aligned} F_k[-u] &= v^{s_1} + \mu && \text{in } \Omega \\ F_k[-v] &= u^{s_2} + \eta && \text{in } \Omega \\ u &= v = 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.21)$$

admits a nonnegative solutions (u, v) , continuous near $\partial\Omega$, with $-u$ and $-v$ elements of $\Phi^k(\Omega)$, then for any compact set $K \subset \Omega$, there exists a positive constant c_{23} depending on N, k, s_1, s_2 and $\text{dist}(K, \partial\Omega)$ such that there holds

$$\eta(E) + \int_E \left(\mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x)}{4}}[\mu](x) \right)^{s_2} dx \leq c_{23} \text{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}} (E) \quad \forall E \subset K, E \text{ Borel.} \quad (1.22)$$

Conversely,, if μ and η are bounded, there exist a positive constant c_{24} depending on N, k, s_1, s_2 and $\text{diam}(\Omega)$ such that, if $k \leq s_1 < \frac{Nk}{N-2k}$ and

$$\eta(K) + \int_K \left(\mathbf{W}_{\frac{2k}{k+1}, k+1}^{2R}[\mu] \right)^{s_2} dx \leq c_{24} \text{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}} (K) \quad (1.23)$$

for all Borel set $K \subset \Omega$, then (1.21) admits a nonnegative solution (u, v) , continuous near $\partial\Omega$, with $-u, -v \in \Phi^k(\Omega)$ satisfying

$$v \leq c_{28} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega], \quad u \leq c_{29} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R \left[\left(\mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega] \right)^{s_1} \right] + c_{27} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu] \quad (1.24)$$

in Ω for some constants c_j ($j = 27, 28, 29$) depending on N, k, s_1, s_2 , and $\text{diam}(\Omega)$.

If Ω is replaced by the whole space we prove,

Theorem F *Let $2k < N, s_1, s_2 > 0, s_1 s_2 > k^2$. Let μ, η be a nonnegative Radon measures in \mathbb{R}^N . If the following problem*

$$\begin{aligned} F_k[-u] &= v^{s_1} + \mu && \text{in } \mathbb{R}^N \\ F_k[-v] &= u^{s_2} + \eta && \text{in } \mathbb{R}^N, \end{aligned} \quad (1.25)$$

admits a nonnegative solutions (u, v) with $-u$ and $-v$ belonging to $\Phi^k(\mathbb{R}^N)$, then there exists a positive constant c_{30} depending on N, k, s_1, s_2 such that there holds

$$\eta(E) + \int_E \left(\mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu](x) \right)^{s_2} dx \leq c_{30} \text{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}} (E) \quad \forall E \text{ Borel.} \quad (1.26)$$

Conversely,, if μ and η are bounded, there exists positive constant c_{31} depending on N, k, s_1, s_2 such that, if $0 < s_1 < \frac{Nk}{N-2k}$ and (1.26) holds with c_{31} instead of c_{30} , then (1.25) admits a nonnegative solution (u, v) with $-u$ and $-v$ in $\Phi^k(\mathbb{R}^N)$ satisfying

$$v \leq c_{33} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\omega], \quad u \leq c_{34} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\left(\mathbf{W}_{\frac{2k}{k+1}, k+1}[\omega]\right)^{s_1}] + c_{32} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu] \quad (1.27)$$

in \mathbb{R}^N , where the c_j ($j = 32, 33, 34$) depend on N, k, s_1, s_2 .

As in the p-Laplace case, we have a Liouville property for Hessian systems.

Corollary G Assume that

$$\frac{2k(s_2 s_1 + k \max\{s_1, s_2\})}{s_1 s_2 - k^2} \geq N. \quad (1.28)$$

Any nonnegative solution (u, v) of inequalities

$$\begin{aligned} F_k[-u] &\geq v^{s_1} && \text{in } \mathbb{R}^N \\ F_k[-v] &\geq u^{s_2} && \text{in } \mathbb{R}^N, \end{aligned} \quad (1.29)$$

with $-u$ and $-v$ in $\Phi^k(\mathbb{R}^N)$ is trivial.

2 Estimates on potentials

Throughout this article $c_j, j=1,2,\dots$, denote structural positive constants and c_N is the volume of the unit ball in \mathbb{R}^N . The following inequality will be used several times in the sequel.

Lemma 2.1 Let $\kappa, \gamma, \theta \in \mathbb{R}$, such that $\kappa, \gamma > 0$. Let $h : (0, \infty) \rightarrow (0, \infty)$ be nondecreasing. Then,

$$\int_0^R t^\kappa \left(\int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma \frac{dt}{t} \leq c_{35} \int_0^{2R} t^{\kappa+\theta\gamma} h^\gamma(t) \frac{dt}{t} \quad \forall R \in (0, \infty], \quad (2.1)$$

for some $c_{35} > 0$ depending on κ, γ, θ .

Proof. Case 1: $\gamma \leq 1$. Since there holds

$$\left(\sum_{j=0}^{\infty} a_j \right)^\gamma \leq \sum_{j=0}^{\infty} a_j^\gamma \quad \forall a_j \geq 0,$$

we deduce

$$\begin{aligned} \left(\int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma &\leq c_{\gamma, \theta} \left(\sum_{j=0}^{j_0} h(2^{\frac{j+1}{4}} t) (2^{\frac{j}{4}} t)^\theta \right)^\gamma \\ &\leq c_{\gamma, \theta} \sum_{j=0}^{j_0} \left(h^\gamma(2^{\frac{j+1}{4}} t) \right) (2^{\frac{j}{4}} t)^{\theta\gamma} \\ &\leq c_{\gamma, \theta} \int_t^{2R} h^\gamma(r) r^{\theta\gamma} \frac{dr}{r}, \end{aligned}$$

where $c_{\gamma,\theta} = 2^{\frac{\gamma}{4}} \max\{1, 2^{-\frac{\gamma\theta}{4}}\}$ and $2^{\frac{j_0}{4}}t < R \leq 2^{\frac{j_0+1}{4}}t$ if $R < \infty$ and $j_0 = \infty$ if $R = \infty$. By Fubini's theorem,

$$\begin{aligned} \int_0^R t^\kappa \left(\int_t^R h(r)r^\theta \frac{dr}{r} \right)^\gamma \frac{dt}{t} &\leq c_{\gamma,\theta} \int_0^R t^\kappa \int_t^{2R} h^\gamma(r)r^{\theta\gamma} \frac{dr}{r} \frac{dt}{t} \\ &\leq \frac{c_{\gamma,\theta}}{\kappa} \int_0^{2R} t^{\kappa+\theta\gamma} h^\gamma(t) \frac{dt}{t}, \end{aligned}$$

which is (2.1).

Case 2: $\gamma > 1$. Since

$$\left(\int_t^R h(r)r^\theta \frac{dr}{r} \right)^\gamma \leq \left(\int_t^R r^{-\frac{\gamma}{\gamma-1}} \frac{dr}{r} \right)^{\gamma-1} \int_t^R h^\gamma(r)r^{\gamma(1+\theta)} \frac{dr}{r},$$

we obtain

$$\int_0^R t^\kappa \left(\int_t^R h(r)r^\theta \frac{dr}{r} \right)^\gamma \frac{dt}{t} \leq c_{\gamma,\kappa} \int_0^{2R} t^{\kappa+\theta\gamma} h^\gamma(t) \frac{dt}{t},$$

by Fubini's theorem, which completes the proof. \square

We recall that if $\alpha > 0$, $1 < \beta < \frac{N}{\alpha}$ and μ belongs to the set of positive Radon measures in \mathbb{R}^N that we denote $\mathfrak{M}^+(\mathbb{R}^N)$, the Wolff potential of μ is defined by

$$\mathbf{W}_{\alpha,\beta}[\mu](x) = \int_0^\infty \left(\frac{\mu(B_r(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{p-1}} \frac{dr}{r}, \quad (2.2)$$

and if $R > 0$, the R -truncated Wolff potential of μ is

$$\mathbf{W}_{\alpha,\beta}^R[\mu](x) = \int_0^R \left(\frac{\mu(B_r(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \quad (2.3)$$

If μ is a Radon measure on a Borel set G , its Wolff potential (or truncated Wolff potential) is the potential of its extension by 0 in G^c . We start with the following composition estimate on Wolff potentials.

Lemma 2.2 *Let $1 < \beta < N/\alpha$. Then for any $q > 0$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ we have*

$$\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu] \leq c_{36} \mathbf{W}_{\alpha,\beta} [(\mathbf{W}_{\alpha,\beta}[\mu])^q], \quad (2.4)$$

in \mathbb{R}^N for some $c_{36} > 0$ depending on α, β, N, q . Moreover, if $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$, there holds

$$\mathbf{W}_{\alpha,\beta} [(\mathbf{W}_{\alpha,\beta}[\mu])^q](x) \leq c_{37} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu], \quad (2.5)$$

in \mathbb{R}^N , where $c_{37} > 0$ depends on α, β, N, q .

Proof. For any $x \in \mathbb{R}^N$, using the fact if $y \in B_t(x)$ then $B_t(x) \subset B_{2t}(y)$, we have

$$\begin{aligned} \mathbf{W}_{\alpha,\beta} [(\mathbf{W}_{\alpha,\beta}[\mu])^q](x) &= \int_0^\infty \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\int_0^\infty \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\ &\geq c_{38} \int_0^\infty \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\frac{\mu(B_{2t}(y))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\ &\geq c_{36} \int_0^\infty \left(t^{\frac{\alpha\beta(\beta-1)}{q}} \frac{\mu(B_t(x))}{t^{N-\alpha\beta}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dt}{t} \\ &= c_{36} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu](x), \end{aligned}$$

where $c_{38} = c_{38}(\alpha, \beta, N, q) > 0$, which proves (2.4).

In order to prove (2.4) we recall the following estimate on Wolff potentials [7]

$$\|\mathbf{W}_{\alpha, \beta}[\omega]\|_{L^{\frac{(\beta-1)N}{N-\alpha\beta}, \infty}} \leq c_{39} (\omega(\mathbb{R}^N))^{\frac{1}{\beta-1}} \quad \forall \omega \in \mathfrak{M}_b^+(\mathbb{R}^N), \quad (2.6)$$

where $L^{\frac{(\beta-1)N}{N-\alpha\beta}, \infty}$ denotes the weak- $L^{\frac{(\beta-1)N}{N-\alpha\beta}}$ space. In particular, since $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$,

$$\int_{B_r(x)} (\mathbf{W}_{\alpha, \beta}[\omega])^q dy \leq c_{40} r^N \left(\frac{\omega(\mathbb{R}^N)}{r^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} \quad \forall x \in \mathbb{R}^N, \forall r > 0. \quad (2.7)$$

Applying this inequality to $\omega = \chi_{B_{2r}(x)}\mu$ yields

$$\int_{B_r(x)} (\mathbf{W}_{\alpha, \beta}^r[\mu])^q dy \leq c_{40} r^N \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} \quad \forall x \in \mathbb{R}^N, \forall r > 0. \quad (2.8)$$

We claim that

$$\begin{aligned} I &:= \int_0^\infty \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\ &\leq c_{37} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}[\mu](x). \end{aligned} \quad (2.9)$$

Since $B_r(y) \subset B_{2r}(x)$ for any $y \in B_t(x), r \geq t$, we have

$$\begin{aligned} \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy &\leq \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \\ &\leq c_N t^N \left(\int_t^\infty \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q. \end{aligned}$$

Hence,

$$I \leq c_N^{\frac{1}{\beta-1}} \int_0^\infty t^{\frac{\alpha\beta}{\beta-1}} \left(\int_t^\infty \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^{\frac{q}{\beta-1}} \frac{dt}{t}.$$

Using Lemma 2.1, we infer

$$I \leq c_{37} \int_0^\infty r^{\frac{\alpha\beta}{\beta-1}} \left(\frac{\mu(B_r(x))}{r^{N-\alpha\beta}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} = c_{37} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}[\mu](x),$$

which completes the proof. \square

The following is a version of Lemma 2.2 for truncated Wolff potentials,

Lemma 2.3 *Let $1 < \beta < N/\alpha$ and $q > 0$. If $\delta \in (0, 1)$ there holds for any $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$*

$$\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta d}{2}}[\mu](x) \leq c_{42} \mathbf{W}_{\alpha, \beta}^{\delta d} \left[\left(\mathbf{W}_{\alpha, \beta}^{\delta d(\cdot)}[\mu] \right)^q \right](x) \quad (2.10)$$

in Ω . Moreover, if $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$, there holds for any $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$,

$$\mathbf{W}_{\alpha, \beta}^R \left[\left(\mathbf{W}_{\alpha, \beta}^R[\mu] \right)^q \right](x) \leq c_{43} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{4R}[\mu](x) \quad (2.11)$$

in \mathbb{R}^N .

Proof. For any $x \in \Omega$,

$$\begin{aligned} \mathbf{W}_{\alpha,\beta}^{\delta d(x)} \left[\left(\mathbf{W}_{\alpha,\beta}^{\delta d(\cdot)} [\mu](\cdot) \right)^q \right] (x) \\ = \int_0^{\delta d(x)} \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\int_0^{\delta d(y)} \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t}. \end{aligned}$$

Since $\delta d(y) \geq \frac{7\delta}{8}d(x)$ for all $y \in B_{\frac{t}{8}}(x)$, provided $0 < t < \delta d(x)$,

$$\begin{aligned} \int_{B_t(x)} \left(\int_0^{\delta d(y)} \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy &\geq \int_{B_{t/8}(x)} \left(\int_0^{\frac{7\delta}{8}d(x)} \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \\ &\geq \int_{B_{t/8}(x)} \left(\int_0^{\frac{7t}{8}} \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \\ &\geq c_{44} \int_{B_{t/8}(x)} \left(\frac{\mu(B_{\frac{3t}{4}}(y))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} dy \\ &\geq c_{44} \int_{B_{t/8}(x)} \left(\frac{\mu(B_{\frac{3t}{4}-\frac{t}{8}}(x))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} dy \\ &\geq c_{45} t^N \left(\frac{\mu(B_{\frac{t}{2}}(x))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}}. \end{aligned}$$

Hence

$$\mathbf{W}_{\alpha,\beta}^{\delta d(x)} \left[\left(\mathbf{W}_{\alpha,\beta}^{\delta d(\cdot)} [\mu](\cdot) \right)^q \right] (x) \geq c_{46} \int_0^{\delta d(x)} \left(t^{\alpha\beta} \left(\frac{\mu(B_{\frac{t}{2}}(x))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} \right)^{\frac{1}{\beta-1}} \frac{dt}{t},$$

which implies (2.10).

Because of (2.8), it is sufficient to prove that there holds

$$\int_0^R \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\int_t^R \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \leq c_{47} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}^{4R} [\mu](x), \quad (2.12)$$

in order to obtain (2.11). Since $B_\rho(y) \subset B_{2\rho}(x)$ for any $y \in B_r(x)$, $\rho \geq r$, we have

$$\begin{aligned} \int_{B_t(x)} \left(\int_t^R \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy &\leq \int_{B_t(x)} \left(\int_t^R \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \\ &\leq c_N t^N \left(\int_t^R \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^R \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\int_t^R \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\ \leq c_N \int_0^R \left(t^{\alpha\beta} \left(\int_t^R \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q \right)^{\frac{1}{\beta-1}} \frac{dt}{t}. \end{aligned}$$

We infer (2.12) by Lemma 2.1, which completes the proof. \square

The next two propositions link Wolff potentials of a measure with Riesz capacities (in the case of whole space) and truncated Wolff potentials with Bessel capacities (in the bounded domain case). Their proof can be found in [20, 21] (and [8] with a different method).

Proposition 2.4 *Let $1 < \beta < N/\alpha$, $q > \beta - 1$, $\nu \in \mathfrak{M}^+(\mathbb{R}^N)$. Then, the following statements are equivalent:*

(a) *The inequality*

$$\nu(K) \leq c_{48} \text{Cap}_{\mathbf{I}_{\alpha\beta}, \frac{q}{q-\beta+1}}(K) \quad (2.13)$$

holds for any compact set $K \subset \mathbb{R}^N$, for some $c_{48} > 0$.

(b) *The inequality*

$$\int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,\beta}[\chi_{B_t(x)}\nu](y))^q dy \leq c_{49}\nu(B_t(x)) \quad (2.14)$$

holds for any ball $B_t(x) \subset \mathbb{R}^N$, for some $c_{49} > 0$.

(c) *The inequality*

$$\mathbf{W}_{\alpha,\beta} [(\mathbf{W}_{\alpha,\beta}[\nu])^q] \leq c_{50} \mathbf{W}_{\alpha,\beta}[\nu] < \infty \text{ a.e in } \mathbb{R}^N \quad (2.15)$$

holds for some $c_{50} > 0$.

Proposition 2.5 *Let $1 < \beta < N/\alpha$, $q > \beta - 1$, $R > 0$ and $\nu \in \mathfrak{M}_b^+(B_R(x_0))$ for some $x_0 \in \mathbb{R}^N$. Then, the following statements are equivalent:*

(a) *The inequality*

$$\nu(K) \leq c_{51} \text{Cap}_{\mathbf{G}_{\alpha\beta}, \frac{q}{q-\beta+1}}(K) \quad (2.16)$$

holds for any compact set $K \subset \mathbb{R}^N$, for some $c_{51} = c_{51}(R) > 0$.

(b) *The inequality*

$$\int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,\beta}^{4R}[\chi_{B_t(x)}\nu](y))^q dy \leq c_{52}\nu(B_t(x)) \quad (2.17)$$

holds for any ball $B_t(x) \subset \mathbb{R}^N$, for some $c_{52} = c_{52}(R) > 0$.

(c) *The inequality*

$$\mathbf{W}_{\alpha,\beta}^{4R} [(\mathbf{W}_{\alpha,\beta}^{4R}[\nu])^q] \leq c_{53} \mathbf{W}_{\alpha,\beta}^{4R}[\nu] \text{ a.e in } B_{2R}(x_0) \quad (2.18)$$

holds for some $c_{53} = c_{53}(R) > 0$.

In the following statement we obtain capacity estimates on combination of measures.

Proposition 2.6 *Let η, μ be in $\mathfrak{M}^+(\mathbb{R}^N)$. Assume that $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$ and $qs > (\beta-1)^2$.*

(i) *If there holds*

$$\eta(K) + \int_K (\mathbf{W}_{\alpha,\beta}[\mu])^s dx \leq \text{Cap}_{\mathbf{I}_{\alpha\beta(q+\beta-1)}, \frac{qs}{qs-(\beta-1)^2}}(K), \quad (2.19)$$

for any compact set $K \subset \mathbb{R}^N$, then

$$\mathbf{W}_{\alpha,\beta} [(\mathbf{W}_{\alpha,\beta} [(\mathbf{W}_{\alpha,\beta}[\omega])^q])^s] \leq c_{54} \mathbf{W}_{\alpha,\beta}[\omega] < \infty \text{ a.e in } \mathbb{R}^N, \quad (2.20)$$

where $\omega = (\mathbf{W}_{\alpha,\beta}[\mu])^s + \eta$.

(ii) If there holds

$$\eta(K) + \int_K (\mathbf{W}_{\alpha,\beta}^{2R}[\mu])^s dx \leq \text{Cap}_{\mathbf{G}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(K), \quad (2.21)$$

for any compact set $K \subset \mathbb{R}^N$, then

$$\mathbf{W}_{\alpha,\beta}^{2R} \left[\left(\mathbf{W}_{\alpha,\beta}^{2R} \left[\left(\mathbf{W}_{\alpha,\beta}^{2R}[\omega]^q \right)^s \right] \right) \right] \leq c_{55} \mathbf{W}_{\alpha,\beta}^{2R}[\omega] < \infty \quad \text{a.e in } B_R(x_0), \quad (2.22)$$

where $\omega = \chi_{B_R(x_0)} \left(\mathbf{W}_{\alpha,\beta}^{2R}[\mu] \right)^s + \chi_{B_R(x_0)} \eta$.

Proof. *Statement (i):* We assume that (2.19) holds. Put $\omega = (\mathbf{W}_{\alpha,\beta}[\mu])^s + \eta$ and apply (2.19) to $K = \overline{B}_{2\rho}(x)$. Since by homogeneity

$$\text{Cap}_{\mathbf{I}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(\overline{B}_{2\rho}(x)) = \rho^{N - \frac{\alpha\beta(q+\beta-1)s}{qs-(\beta-1)^2}} \text{Cap}_{\mathbf{I}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(\overline{B}_2(0)),$$

we deduce from (2.19)

$$\omega(B_\rho(x)) \leq c_{55} \rho^{N - \frac{\alpha\beta(q+\beta-1)s}{qs-(\beta-1)^2}} \quad \forall \rho > 0,$$

which is equivalent to

$$\rho^{\frac{\alpha\beta}{\beta-1}} \left(\frac{\omega(B_\rho(x))}{\rho^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{qs}{(\beta-1)^3}} \leq c_{56} \left(\frac{\omega(B_\rho(x))}{\rho^{N - \alpha\beta}} \right)^{\frac{1}{\beta-1}} \quad \forall \rho > 0. \quad (2.23)$$

We apply Proposition 2.4 to $\nu = \omega$ with $(\alpha, \beta, q) = \left(\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1, s \right)$, (2.19) implies

$$\int_{\mathbb{R}^N} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}[\chi_{B_t(x)}\omega] \right)^s dy \leq c_{57} \omega(B_t(x)). \quad (2.24)$$

By Lemma 2.2, (2.20) is equivalent to

$$\mathbf{W}_{\alpha,\beta} \left[\left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}[\omega] \right)^s \right] \leq c_{58} \mathbf{W}_{\alpha,\beta}[\omega] < \infty \quad \text{a.e } \mathbb{R}^N. \quad (2.25)$$

Therefore, it is enough to show that (2.23) and (2.24) imply (2.25). In fact, since for $t > 0$

$$\int_{B_t(x)} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}[\omega](y) \right)^s dy = \int_{B_t(x)} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}[\chi_{B_{2t}(x)}\omega](y) \right)^s dy,$$

we apply (2.24) and obtain

$$\int_{B_t(x)} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}[\omega](y) \right)^s dy \leq c_{57} \omega(B_{2t}(x)).$$

So, it is enough to show that

$$I := \int_0^\infty \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\omega(B_r(y))}{r^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} \right)^s dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \leq c_{58} \mathbf{W}_{\alpha,\beta}[\omega](x). \quad (2.26)$$

Since $B_r(y) \subset B_{2r}(x)$ for any $y \in B_t(x), r \geq t$, we have

$$\begin{aligned} I &\leq c_N \int_0^\infty \left(t^{\alpha\beta} \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} \right)^s \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\ &= c_N \int_0^\infty t^{\frac{\alpha\beta}{\beta-1}} \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} \right)^{\frac{s}{\beta-1}} \frac{dt}{t}. \end{aligned}$$

It follows from Lemma 2.1 and (2.23) that

$$I \leq c_{59} \int_0^\infty t^{\frac{\alpha\beta}{\beta-1}} \left(\frac{\omega(B_{2t}(x))}{t^{N-\frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{qs}{(\beta-1)^3}} \frac{dt}{t} \leq c_{56} c_{59} \int_0^\infty \left(\frac{\omega(B_{2t}(x))}{t^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dt}{t},$$

which is (2.26).

Statement (ii): We assume that (2.21) holds. Put $d\omega = \chi_\Omega (\mathbf{W}_{\alpha,\beta}[\mu])^s + \chi_\Omega \eta$, then

$$\omega(B_\rho(x)) \leq c_{60} \rho^{N-\frac{\alpha\beta(q+\beta-1)s}{qs-(\beta-1)^2}} \quad \forall 0 < \rho < 2R.$$

As in the proof of statement (i), the above inequality is equivalent to

$$\rho^{\frac{\alpha\beta}{\beta-1}} \left(\frac{\omega(B_\rho(x))}{\rho^{N-\frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{qs}{(\beta-1)^3}} \leq c_{61} \left(\frac{\omega(B_\rho(x))}{\rho^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \quad \forall 0 < \rho < 2R. \quad (2.27)$$

Applying Proposition 2.5 with $\nu = \omega$ and $(\alpha, \beta, q) = \left(\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1, s \right)$,

$$\int_{\mathbb{R}^N} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\alpha\beta} [\chi_{B_t(x)} \omega] \right)^s dy \leq c_{62} \omega(B_t(x)). \quad (2.28)$$

By Lemma 2.3, (2.22) is equivalent to

$$\mathbf{W}_{\alpha,\beta}^{4R} \left[\left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\alpha\beta} [\omega] \right)^s \right] \leq c_{63} \mathbf{W}_{\alpha,\beta}^{4R} [\omega] \quad \text{a.e in } B_R(x_0). \quad (2.29)$$

Therefore, it is sufficient to prove that (2.27) and (2.28) imply (2.29). Actually, since

$$\int_{B_t(x)} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\alpha\beta} [\omega](y) \right)^s dy = \int_{B_t(x)} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\alpha\beta} [\chi_{B_{2t}(x)} \omega](y) \right)^s dy$$

for all $0 < t < 4R$, thus applying (2.28), we obtain

$$\int_{B_t(x)} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\alpha\beta} [\omega](y) \right)^s dy \leq c_{64} \omega(B_{2t}(x)).$$

So, it is sufficient to show that for any $x \in B_R(x_0)$

$$II := \int_0^{4R} \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\int_t^{4R} \left(\frac{\omega(B_r(y))}{r^{N-\frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} \right)^s dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \leq c_{65} \mathbf{W}_{\alpha,\beta}^{4R} [\omega](x). \quad (2.30)$$

Since $B_r(y) \subset B_{2r}(x)$ for any $y \in B_t(x)$ with $r \geq t$, we have

$$II \leq c_N \int_0^{4R} t^{\frac{\alpha\beta}{\beta-1}} \left(\int_t^{4R} \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} \right)^{\frac{s}{\beta-1}} \frac{dt}{t}.$$

Combining this with Lemma 2.1 and (2.27) yields

$$II \leq c_{66} \mathbf{W}_{\alpha,\beta}^{16R}[\omega](x).$$

Therefore, (2.29) follows since $\mathbf{W}_{\alpha,\beta}^{16R}[\omega] \leq c_{67} \mathbf{W}_{\alpha,\beta}^{4R}[\omega]$ in $B_R(x_0)$. \square

Proposition 2.7 *Let η, μ be in $\mathfrak{M}^+(\mathbb{R}^N)$. Assume that $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$ and $qs > (\beta-1)^2$. Let (u_m, v_m) be nonnegative measurable functions in \mathbb{R}^N verifying, for all $m \geq 0$,*

$$u_{m+1} \leq c^* \mathbf{W}_{\alpha,\beta}[v_m^q + \mu], \quad v_{m+1} \leq c^* \mathbf{W}_{\alpha,\beta}[u_m^s + \eta] \quad \text{a.e. in } \mathbb{R}^N,$$

for some $c^* > 0$ and $(u_0, v_0) = 0$. Then, there exists a constant $M^* > 0$ depending only on $N, \alpha, \beta, q, s, c^*$ such that if the measure $d\omega = (\mathbf{W}_{\alpha,\beta}[\mu])^s dx + d\eta$ satisfies

$$\omega(K) \leq M^* \text{Cap}_{\mathbf{I}, \frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}(K), \quad (2.31)$$

for any compact set $K \subset \mathbb{R}^N$, then

$$v_m \leq c_{69} \mathbf{W}_{\alpha,\beta}[\omega], \quad u_m \leq c_{70} \mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q] + c_{68} \mathbf{W}_{\alpha,\beta}[\mu] \quad \forall m \geq 0, \quad (2.32)$$

for some constants c_{68}, c_{69}, c_{70} depending only on N, α, β, q, s and c^* .

Proof. By Proposition 2.6, (2.31) implies

$$\mathbf{W}_{\alpha,\beta} [(\mathbf{W}_{\alpha,\beta} [(\mathbf{W}_{\alpha,\beta}[\omega])^q])^s] \leq c_{71} M^{\frac{qs}{(\beta-1)^3}} \mathbf{W}_{\alpha,\beta}[\omega] < \infty \quad \text{a.e. in } \mathbb{R}^N. \quad (2.33)$$

We set

$$\begin{aligned} c_{68} &= c^* 2^{\frac{1}{\beta-1}}, \\ c_{69} &= c^* 2^{1+\frac{1}{\beta-1}} (c_{68}^s 2^{s-1} + 1)^{\frac{1}{\beta-1}}, \\ c_{70} &= c^* 2^{\frac{1}{\beta-1}} c_{69}^{\frac{q}{\beta-1}}, \end{aligned}$$

and choose $M^* > 0$ such that

$$c^* 2^{\frac{1}{\beta-1}} (c_{70}^s 2^{s-1})^{\frac{1}{\beta-1}} c_{71} M^* \frac{qs}{(\beta-1)^3} = \frac{c_{69}}{2}.$$

We claim that

$$v_m \leq c_{69} \mathbf{W}_{\alpha,\beta}[\omega], \quad u_m \leq c_{70} \mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q] + c_{68} \mathbf{W}_{\alpha,\beta}[\mu] \quad \forall m \geq 0. \quad (2.34)$$

Clearly, by definition of c_{68}, c_{69} and c_{70} , we have (2.34) for $m = 0, 1$. Next we assume that (2.34) holds for all integer $m \leq l$ for some $l \in \mathbb{N}_+$, then

$$\begin{aligned} u_{l+1} &\leq c^* \mathbf{W}_{\alpha,\beta}[v_l^q + \mu] \\ &\leq c^* 2^{\frac{1}{\beta-1}} c_{69}^{\frac{q}{\beta-1}} \mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q] + c^* 2^{\frac{1}{\beta-1}} \mathbf{W}_{\alpha,\beta}[\mu] \\ &= c_{70} \mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q] + c_{68} \mathbf{W}_{\alpha,\beta}[\mu], \end{aligned}$$

and

$$\begin{aligned} v_{l+1} &\leq c^* \mathbf{W}_{\alpha,\beta}[(c_{70} \mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q] + c_{68} \mathbf{W}_{\alpha,\beta}[\mu])^s + \eta] \\ &\leq c^* \mathbf{W}_{\alpha,\beta}[c_{70}^s 2^{s-1} (\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q])^s + c_{68}^s 2^{s-1} (\mathbf{W}_{\alpha,\beta}[\mu])^s + \eta] \\ &\leq c^* 2^{\frac{1}{\beta-1}} (c_{70}^s 2^{s-1})^{\frac{1}{\beta-1}} \mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q])^s] \\ &\quad + c^* 2^{\frac{1}{\beta-1}} (c_{68}^s 2^{s-1} + 1)^{\frac{1}{\beta-1}} \mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\mu])^s + \eta] \\ &\leq c^* 2^{\frac{1}{\beta-1}} (c_{70}^s 2^{s-1})^{\frac{1}{\beta-1}} c_{71} M^* \frac{qs}{(\beta-1)^3} \mathbf{W}_{\alpha,\beta}[\omega] + c^* 2^{\frac{1}{\beta-1}} (c_{68}^s 2^{s-1} + 1)^{\frac{1}{\beta-1}} \mathbf{W}_{\alpha,\beta}[\omega] \\ &= \frac{c_{69}}{2} \mathbf{W}_{\alpha,\beta}[\omega] + \frac{c_{69}}{2} \mathbf{W}_{\alpha,\beta}[\omega] \\ &= c_{69} \mathbf{W}_{\alpha,\beta}[\omega]. \end{aligned}$$

Thus, (2.34) holds true for $m = l + 1$. Hence, (2.34) is valid for all $l \geq 0$. \square

The next result is an adaptation of Proposition 2.7 to truncated Wolff potentials.

Proposition 2.8 *Let η, μ be in $\mathfrak{M}_b^+(B_R(x_0))$. Assume that $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$ and $qs > (\beta-1)^2$. Let (u_m, v_m) be nonnegative measurable functions in \mathbb{R}^N such that for all $m \geq 0$*

$$u_{m+1} \leq c_* \mathbf{W}_{\alpha,\beta}^R[\chi_{B_R(x_0)} v_m^q + \mu], \quad v_{m+1} \leq c_* \mathbf{W}_{\alpha,\beta}^R[\chi_{B_R(x_0)} u_m^s + \eta] \quad \text{a.e. in } B_R(x_0),$$

and $(u_0, v_0) = 0$. If we set $d\omega = \left(\mathbf{W}_{\alpha,\beta}^{2R}[\mu] \right)^s dx + d\eta$, there exists a constant $M_* > 0$ depending only on $N, \alpha, \beta, q, s, R$ and c_* such that if

$$\omega(K) \leq M_* \text{Cap}_{\mathbf{G}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(K), \quad (2.35)$$

for any compact set $K \subset \mathbb{R}^N$, then

$$v_m \leq c_{73} \mathbf{W}_{\alpha,\beta}^{2R}[\omega], \quad u_m \leq c_{74} \mathbf{W}_{\alpha,\beta}^{2R}[(\mathbf{W}_{\alpha,\beta}^{2R}[\omega])^q] + c_{72} \mathbf{W}_{\alpha,\beta}^{2R}[\mu] \quad \forall k \geq 0 \quad (2.36)$$

in $B_R(x_0)$ for some constants c_{72}, c_{73}, c_{74} depending only on $N, \alpha, \beta, q, s, R$ and c_* .

Proof. The proof is similar to the one of Proposition 2.7 and we omit the details. \square

Proposition 2.9 *Let $1 < \beta < N/\alpha$ and $q, s > 0$ such that $qs > (\beta-1)^2$.*

(i) *Assume that η and μ belong to $\mathfrak{M}_b^+(\mathbb{R}^N)$ and (u, v) are nonnegative measurable functions satisfying*

$$\begin{aligned} (i) \quad & \mathbf{W}_{\alpha,\beta}[v^q] + \mathbf{W}_{\alpha,\beta}[\mu] \leq c_{75} u, \\ (ii) \quad & \mathbf{W}_{\alpha,\beta}[u^s] + \mathbf{W}_{\alpha,\beta}[\eta] \leq c_{75} v \quad \text{a.e. in } \mathbb{R}^N, \end{aligned} \quad (2.37)$$

for some $c_{75} > 0$. Then there exists a constant $c_{76} > 0$ depending only on N, α, β, q, s and c_{75} such that

$$\eta(K) + \int_K (\mathbf{W}_{\alpha,\beta}[\mu](x))^s dx \leq c_{76} \text{Cap}_{\mathbf{I}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(K), \quad (2.38)$$

for any compact set $K \subset \mathbb{R}^N$.

(ii) *Assume that η and μ belong to $\mathfrak{M}_b^+(\Omega)$ and (u, v) are nonnegative functions satisfying*

$$\begin{aligned} (i) \quad & \mathbf{W}_{\alpha,\beta}^{\delta d(\cdot)}[v^q] + \mathbf{W}_{\alpha,\beta}^{\delta d}[\mu] \leq c_{77} u, \\ (ii) \quad & \mathbf{W}_{\alpha,\beta}^{\delta d(\cdot)}[u^s] + \mathbf{W}_{\alpha,\beta}^{\delta d}[\eta] \leq c_{77} v \quad \text{a.e. in } \Omega, \end{aligned} \quad (2.39)$$

for some $c_{77} > 0$. Then for any $\Omega' \subset\subset \Omega$, there exists a constant $c_{78} > 0$ depending only on $n, \alpha, \beta, q, s, c_{77}$ and $\text{dist}(\Omega', \partial\Omega)$ such that

$$\eta(K) + \int_K \left(\mathbf{W}_{\alpha,\beta}^{\delta d(x)}[\mu](x) \right)^s dx \leq c_{78} \text{Cap}_{\mathbf{G}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(K), \quad (2.40)$$

for any compact set $K \subset \Omega'$.

Proof. (i): Set $\omega = u^s + \eta$, then

$$\omega \geq u^s \geq (\mathbf{W}_{\alpha,\beta}[v^q])^s \geq c_{79} (\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q])^s.$$

By (2.4) in Lemma 2.2, we get

$$\omega \geq c_{80} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}[\omega] \right)^s,$$

which implies

$$\int_{\mathbb{R}^N} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1} [\chi_{B_t(x)} \omega] \right)^s dy \leq c_{81} \omega(B_t(x)) \quad \forall x \in \mathbb{R}^N, \forall t > 0.$$

Applying Proposition 2.4 to $\mu = \omega$ with $(\alpha, \beta, q) = \left(\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1, s \right)$, we get (2.38).

(ii) We define ω as above and we have

$$\omega \geq u^s \geq (\mathbf{W}_{\alpha, \beta}^{\delta d} [v^q])^s \geq c_{82} \left(\mathbf{W}_{\alpha, \beta}^{\delta d} \left[\left(\mathbf{W}_{\alpha, \beta}^{\delta d(\cdot)} [\omega] \right)^q \right] \right)^s \quad \text{a.e. in } \Omega,$$

which leads to

$$\omega \geq c_{83} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{2} d} [\omega] \right)^s \quad \text{a.e. in } \Omega,$$

by inequality (2.10) in Lemma 2.3. Let M_ω denote the centered Hardy-Littlewood maximal function which is defined for any $f \in L^1_{loc}(\mathbb{R}^N, d\omega)$ by

$$M_\omega f(x) = \sup_{t>0} \frac{1}{\omega(B_t(x))} \int_{B_t(x)} |f| d\omega.$$

Let $K \subset \Omega$ be compact. Set $r_K = \text{dist}(K, \partial\Omega)$ and $\Omega_K = \{x \in \Omega : d(x, K) < r_K/2\}$. Then, for any Borel set $E \subset K$,

$$c_{84} \int_{\Omega} (M_\omega \chi_E)^{\frac{sq}{(\beta-1)^2}} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{2} d(x)} [\omega] \right)^s dx \leq \int_{\Omega} (M_\omega \chi_E)^{\frac{sq}{(\beta-1)^2}} d\omega.$$

Since M_ω is a bounded linear map on $L^p(\mathbb{R}^N, d\omega)$ for any $p > 1$ and

$$(M_\omega \chi_E)^{\frac{sq}{(\beta-1)^2}} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{2} d(x)} [\omega] \right)^s \geq \int_0^{\frac{\delta}{2} d(x)} \left(\frac{\omega(B_t(x) \cap E)}{\omega(B_t(x))} \frac{\omega(B_t(x))}{t^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{sq}{(\beta-1)^2}} \frac{dt}{t},$$

we obtain

$$\int_{\Omega} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{2} d(x)} [\omega_E] \right)^s dx \leq c_{85} \omega(E),$$

where $\omega_E = \chi_E \omega$. Note that if $x \in \Omega$ and $d(x) \leq r_K/8$, then $B_t(x) \subset \Omega \setminus \Omega_K$ for all $t \in (0, \frac{\delta d(x)}{2})$; indeed, for all $y \in B_t(x)$

$$d(y, \partial\Omega) \leq d(x, \partial\Omega) + |x - y| < (1 + \delta)d(x, \partial\Omega) < \frac{1}{4}r_K,$$

thus

$$d(y, K) \geq d(K, \partial\Omega) - d(y, \partial\Omega) > \frac{3}{4}r_K > \frac{1}{2}r_K,$$

which implies $y \notin \Omega_K$. We deduce that

$$\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{2} d(x, \partial\Omega)} [\omega_E](x) \geq \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{16} r_K} [\omega_E](x) \quad \forall x \in \Omega,$$

and

$$\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{16} r_K} [\omega_E](x) = 0 \quad \forall x \in \Omega^c.$$

Hence we obtain

$$\int_{\mathbb{R}^N} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{16} r_K} [\omega_E] \right)^s dx \leq c_{85} \omega(E) \quad \forall E \subset K, E \text{ Borel.} \quad (2.41)$$

Applying Proposition 2.5 with $\mu = \chi_{K \cap B_{2^{-6}\delta r_K}(x)} \omega$ we get (2.40), which completes the proof. \square

3 Quasilinear Dirichlet problems

Let Ω be a bounded domain in \mathbb{R}^N . If $\mu \in \mathfrak{M}_b(\Omega)$, we denote by μ^+ and μ^- respectively its positive and negative parts in the Jordan decomposition. We denote by $\mathfrak{M}_0(\Omega)$ the space of measures in Ω which are absolutely continuous with respect to the $c_{1,p}^\Omega$ -capacity defined on a compact set $K \subset \Omega$ by

$$c_{1,p}^\Omega(K) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p dx : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega) \right\}.$$

We also denote $\mathfrak{M}_s(\Omega)$ the space of measures in Ω with support on a set of zero $c_{1,p}^\Omega$ -capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega) \cap \mathfrak{M}_b(\Omega)$. It is well known that any $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ can be written under the form $\mu_0 = f - \operatorname{div} g$ where $f \in L^1(\Omega)$ and $g \in L^{p'}(\Omega, \mathbb{R}^N)$.

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s, k\}, -k\}$. If u is a measurable function defined in Ω , finite a.e. and such that $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, there exists a measurable function $v : \Omega \rightarrow \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{\{|u| \leq k\}} v$ a.e. in Ω and for all $k > 0$. We define the gradient a.e. ∇u of u by $v = \nabla u$. We recall the definition of a renormalized solution given in [12].

Definition 3.1 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of*

$$\begin{aligned} -\Delta_p u &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (3.1)$$

if $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$, and u has the property that for any $k > 0$ there exist λ_k^+ and λ_k^- belonging to $\mathfrak{M}_b^+ \cap \mathfrak{M}_0(\Omega)$, respectively concentrated on the sets $u = k$ and $u = -k$, with the property that $\mu_k^+ \rightharpoonup \mu_s^+$, $\mu_k^- \rightharpoonup \mu_s^-$ in the narrow topology of measures and such that

$$\int_{\{|u| < k\}} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_{\{|u| < k\}} \varphi d\mu_0 + \int_{\Omega} \varphi d\lambda_k^+ - \int_{\Omega} \varphi d\lambda_k^-,$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.2 *We recall that if u is a renormalized solution to problem (3.1), then $\frac{|\nabla u|^p}{(|u|+1)^r} \in L^1(\Omega)$ for all $r > 1$. Furthermore, $u \geq 0$ a.e. in Ω if $\mu \in \mathfrak{M}_b^+(\Omega)$.*

The following general stability result has been proved in [12, Th 4.1].

Theorem 3.3 *Let $\mu = \mu_0 + \mu_s^+ - \mu_s^-$, with $\mu_0 = F - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$ and μ_s^+, μ_s^- belonging to $\mathfrak{M}_s^+(\Omega)$. Let $\mu_n = F_n - \operatorname{div} g_n + \rho_n - \eta_n$ with $F_n \in L^1(\Omega)$, $g_n \in (L^{p'}(\Omega))^N$ and ρ_n, η_n belonging to $\mathfrak{M}_b^+(\Omega)$. Assume that $\{F_n\}$ converges to F weakly in $L^1(\Omega)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(\Omega))^N$ and $(\operatorname{div} g_n)$ is bounded in $\mathfrak{M}_b(\Omega)$; assume also that $\{\rho_n\}$ converges to μ_s^+ and $\{\eta_n\}$ to μ_s^- in the narrow topology. If $\{u_n\}$ is a sequence of renormalized solutions of (3.1) with data μ_n , then, up to a subsequence, it converges a.e. in Ω to a renormalized solution u of problem (3.1). Furthermore, $T_k(u_n)$ converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$ for any $k > 0$.*

We also recall the following estimate [20, Th 2.1].

Proposition 3.4 *Let Ω be a bounded domain of \mathbb{R}^N . Then there exists a constant $C > 0$, depending on p and N such that if $\mu \in \mathfrak{M}_b^+(\Omega)$ and u is a nonnegative renormalized solution of problem (3.1) with data μ , there holds*

$$\frac{1}{c_{s6}} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\mu](x) \leq u(x) \leq c_{s6} \mathbf{W}_{1,p}^{2 \operatorname{diam}(\Omega)}[\mu](x) \quad \text{a.e. in } \Omega. \quad (3.2)$$

Proof of Theorem C. *The condition is necessary.* Assume that (1.14) admits a nonnegative renormalized solutions (u, v) . By Proposition 3.4 there holds

$$\begin{aligned} u(x) &\geq c_{87} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}} [v^{q_1} + \mu](x) \\ v(x) &\geq c_{87} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}} [u^{q_2} + \mu](x) \quad \text{a.e. in } \Omega. \end{aligned}$$

Hence, we infer (1.15) from Proposition 2.9-(ii).

Sufficient conditions. Let $\{(u_m, v_m)\}_{m \in \mathbb{N}}$ be a sequence of nonnegative renormalized solutions of the following problems for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= v_m^{q_1} + \mu && \text{in } \Omega \\ -\Delta_p v_{m+1} &= u_m^{q_2} + \eta && \text{in } \Omega \\ u_{m+1} &= v_{m+1} = 0 && \text{on } \partial\Omega, \end{aligned} \quad (3.3)$$

with initial condition $(u_0, v_0) = 0$. The sequences $\{u_m\}$ and $\{v_m\}$ can be constructed in such a way that they are nondecreasing (see e.g. [21]). By Proposition 3.4 we have

$$\begin{aligned} u_{m+1} &\leq c_{86} \mathbf{W}_{1,p}^R [v_m^{q_1} + \mu](x) \\ v_{m+1} &\leq c_{86} \mathbf{W}_{1,p}^R [u_m^{q_2} + \eta](x) \quad \text{a.e. in } \Omega, \end{aligned}$$

where $R = 2 \operatorname{diam}(\Omega)$. Thus, by Proposition 2.8 there exists a constant $M_* > 0$ depending only on N, p, q_1, q_2, R such that if

$$\omega(K) \leq M_* \operatorname{Cap}_{\mathbf{G}_{\frac{p(q_1+p-1)}{q_1}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(K) \quad (3.4)$$

for any compact set $K \subset \mathbb{R}^N$ with $d\omega = (\mathbf{W}_{1,p}^R[\mu])^{q_2} dx + d\eta$, then

$$v_m \leq c_{73} \mathbf{W}_{1,p}^R[\omega], \quad u_m \leq c_{74} \mathbf{W}_{1,p}^R[(\mathbf{W}_{1,p}^R[\omega])^{q_1}] + c_{72} \mathbf{W}_{1,p}^R[\mu] \quad \forall k \geq 0 \quad (3.5)$$

in Ω , and

$$\mathbf{W}_{1,p}^R[\omega] \in L^{q_2}(\Omega), \quad \mathbf{W}_{1,p}^R[(\mathbf{W}_{1,p}^R[\omega])^{q_1}] + \mathbf{W}_{1,p}^R[\mu] \in L^{q_1}(\Omega). \quad (3.6)$$

This implies that $\{u_m\}, \{v_m\}_{m \in \mathbb{N}}$ are well defined and nondecreasing. Thus $\{(u_m, v_m)\}$ converges a.e in Ω to some functions (u, v) which satisfies (1.17) in Ω . Furthermore, we deduce from (3.6) and the monotone convergence theorem that $u_m^{q_1} \rightarrow u^{q_1}$ and $v_m^{q_2} \rightarrow v^{q_2}$ in $L^1(\Omega)$. Finally we infer that u is a renormalized solution of (1.14) by Theorem 3.3. \square

4 p-superharmonic functions and quasilinear equations in \mathbb{R}^N

We recall some definitions and properties of p -superharmonic functions (see e.g. [13], [14], [15] for general properties and [28] for a simple presentation).

Definition 4.1 *A function u is said to be p -harmonic in \mathbb{R}^N if $u \in W_{loc}^{1,p}(\mathbb{R}^N)$ and $-\Delta_p u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$; it is always C^1 . A function u is called a p -supersolution in \mathbb{R}^N if $u \in W_{loc}^{1,p}(\mathbb{R}^N)$ and $-\Delta_p u \geq 0$ in $\mathcal{D}'(\mathbb{R}^N)$.*

Definition 4.2 *A lower semicontinuous (l.s.c) function $u : \mathbb{R}^N \rightarrow (-\infty, \infty]$ is called p -super-harmonic if u is not identically infinite and if, for all open $D \subset \subset \mathbb{R}^N$ and all $v \in C(\overline{D})$, p -harmonic in D , $v \leq u$ on ∂D implies $v \leq u$ in D .*

Let u be a p -superharmonic in \mathbb{R}^N . It is well known that $u \wedge k := \min\{u, k\} \in W_{loc}^{1,p}(\mathbb{R}^N)$ is a p -supersolution for all $k > 0$ and $u < \infty$ a.e in \mathbb{R}^N , thus, u has a gradient (see the previous section). We also have $|\nabla u|^{p-1} \in L_{loc}^q(\mathbb{R}^N)$, $\frac{|\nabla u|^p}{(|u|+1)^r} \in L_{loc}^1(\mathbb{R}^N)$ and $u \in L_{loc}^s(\mathbb{R}^N)$ for $1 \leq q < \frac{N}{N-1}$ and $r > 1$, $1 \leq s < \frac{N(p-1)}{N-p}$ (see [13, Theorem 7.46]). Thus for any $0 \leq \varphi \in C_c^1(\Omega)$, by the dominated convergence theorem,

$$\langle -\Delta_p u, \varphi \rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla(u \wedge k)|^{p-2} \nabla(u \wedge k) \cdot \nabla \varphi \geq 0.$$

Hence, by the Riesz Representation Theorem, there is a nonnegative Radon measure denoted by $\mu[u]$, called the Riesz measure, such that $-\Delta_p u = \mu[u]$ in $\mathcal{D}'(\mathbb{R}^N)$.

The following weak convergence result for Riesz measures proved in [26] will be used to obtain the existence of p -superharmonic solutions to quasilinear equations.

Proposition 4.3 *Suppose that $\{u_n\}$ is a sequence of nonnegative p -superharmonic functions in \mathbb{R}^N that converges a.e to a p -superharmonic function u . Then the sequence of measures $\{\mu[u_n]\}$ converges to $\mu[u]$ in the weak sense of measures.*

The proof of the next result can be found in [20].

Proposition 4.4 *Let μ be a measure in $\mathfrak{M}^+(\mathbb{R}^N)$. Suppose that $\mathbf{W}_{1,p}[\mu] < \infty$ a.e. Then there exists a nonnegative p -superharmonic function u in \mathbb{R}^N such that $-\Delta_p u = \mu$ in $\mathcal{D}'(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} u = 0$ and*

$$\frac{1}{c_{86}} \mathbf{W}_{1,p}[\mu](x) \leq u(x) \leq c_{86} \mathbf{W}_{1,p}[\mu](x), \quad (4.1)$$

for almost all x in \mathbb{R}^N , where the constant c_{86} is the one of Proposition 3.4. Furthermore any p -superharmonic function u in \mathbb{R}^N , such that $\inf_{\mathbb{R}^N} u = 0$ satisfies (4.1) with $\mu = -\Delta_p u$.

Proof of Theorem A. *The condition is necessary.* Assume that (1.14) admits a nonnegative p -superharmonic functions (u, v) . By Proposition 4.4 there holds

$$\begin{aligned} u(x) &\geq c_{87} \mathbf{W}_{1,p}[v^{q_1} + \mu](x), \\ v(x) &\geq c_{87} \mathbf{W}_{1,p}[u^{q_2} + \eta](x) \quad \text{for almost all } x \in \Omega. \end{aligned}$$

Hence, we obtain (1.10) from Proposition 2.9-(i).

The condition is sufficient. Let $\{(u_m, v_m)\}_{m \in \mathbb{N}}$ be a sequence of nonnegative p -superharmonic solutions of the following problems for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= v_m^{q_1} + \mu && \text{in } \mathbb{R}^N \\ -\Delta_p v_{m+1} &= u_m^{q_2} + \eta && \text{in } \mathbb{R}^N \\ \inf_{\mathbb{R}^N} u_{m+1} &= \inf_{\mathbb{R}^N} v_{m+1} = 0, \end{aligned} \quad (4.2)$$

with $(u_0, v_0) = (0, 0)$. As in the proof of Theorem C we can assume that $\{u_m\}$ and $\{v_m\}$ are nondecreasing. By Proposition 4.4 we have

$$\begin{aligned} u_{m+1} &\leq c_{86} \mathbf{W}_{1,p}[v_m^{q_1} + \mu](x) \\ v_{m+1} &\leq c_{86} \mathbf{W}_{1,p}[u_m^{q_2} + \eta](x) \quad \text{for all } x \in \Omega. \end{aligned}$$

Thus, by Proposition 2.7 there exists a constant $c > 0$ depending only on N, p, q_1, q_2 such that, if

$$\omega(K) \leq M^* \text{Cap}_{\mathbf{I}_{\frac{p(q_1+p-1)}{q_1}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(K) \quad (4.3)$$

for any compact set $K \subset \mathbb{R}^N$ with $d\omega = (\mathbf{W}_{1,p}[\mu])^{q_2} dx + d\eta$, then there holds in Ω ,

$$v_m \leq c_{69} \mathbf{W}_{1,p}[\omega], \quad u_m \leq c_{70} \mathbf{W}_{1,p}[(\mathbf{W}_{1,p}[\omega])^{q_1}] + c_{68} \mathbf{W}_{1,p}[\mu] \quad \text{for all } m \geq 0, \quad (4.4)$$

and

$$\mathbf{W}_{1,p}[\omega] \in L_{loc}^{q_2}(\mathbb{R}^N), \quad \mathbf{W}_{1,p}[(\mathbf{W}_{1,p}[\omega])^{q_1}] + \mathbf{W}_{1,p}[\mu] \in L_{loc}^{q_1}(\mathbb{R}^N). \quad (4.5)$$

This implies that $\{u_m\}, \{v_m\}$ are well defined and nondecreasing. Thus $\{(u_m, v_m)\}$ converges a.e in \mathbb{R}^N to some functions (u, v) which satisfies (1.17) in \mathbb{R}^N . Furthermore, we infer from (3.6) and the monotone convergence theorem that $u_m^{q_1} \rightarrow u^{q_1}, v_m^{q_2} \rightarrow u^{q_2}$ in $L_{loc}^1(\mathbb{R}^N)$. By Proposition 4.3 we deduce that (u, v) are nonnegative p -superharmonic solutions of (1.9). \square

5 Hessian equations

In this section $\Omega \subset \mathbb{R}^N$ is either a bounded domain with a C^2 boundary or the whole \mathbb{R}^N . For $k = 1, \dots, N$ and $u \in C^2(\Omega)$ the k -hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivative D^2u and S_k is the k -th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}.$$

We can see that

$$F_k[u] = [D^2u]_k,$$

where for a matrix $A = (a_{ij})$, $[A]_k$ denotes the sum of the k -th principal minors. We assume that $\partial\Omega$ is uniformly $(k-1)$ -convex, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \quad \text{on } \partial\Omega,$$

for some positive constant c_0 , where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal.

Definition 5.1 *An upper-semicontinuous function $u : \Omega \rightarrow [-\infty, \infty)$ is k -convex (k -subharmonic) if, for every open set $\Omega' \subset \bar{\Omega}' \subset \Omega$ and for every function $v \in C^2(\Omega') \cap C(\bar{\Omega}')$ satisfying $F_k[v] \leq 0$ in Ω' , the following implication is true*

$$u \leq v \quad \text{on } \partial\Omega' \quad \implies \quad u \leq v \quad \text{in } \Omega'.$$

We denote by $\Phi^k(\Omega)$ the class of all k -subharmonic functions in Ω which are not identically equal to $-\infty$.

The following weak convergence result for k -Hessian operators proved in [24] is fundamental in our study.

Proposition 5.2 *Let Ω be either a bounded uniformly $(k-1)$ -convex in \mathbb{R}^N or the whole \mathbb{R}^N . For each $u \in \Phi^k(\Omega)$, there exists a nonnegative Radon measure $\mu_k[u]$ in Ω such that*

1 $\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$.

2 If $\{u_n\}$ is a sequence of k -convex functions which converges a.e to u , then $\mu_k[u_n] \rightharpoonup \mu_k[u]$ in the weak sense of measures.

As in the case of quasilinear equations with measure data, precise estimates of solutions of k -Hessian equations with measures data are expressed in terms of Wolff potentials. The next results are proved in [24, 17, 20].

Theorem 5.3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 , uniformly $(k-1)$ -convex domain. Let μ be a nonnegative Radon measure in Ω which can be decomposed under the form*

$$\mu = \mu_1 + f,$$

where μ_1 is a measure with compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$ if $k \leq \frac{N}{2}$, or $p = 1$ if $k > \frac{N}{2}$. Then there exists a nonnegative function u in Ω , continuous near $\partial\Omega$, such that $-u \in \Phi^k(\Omega)$ and u is a solution of the problem

$$\begin{aligned} F_k[-u] &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Furthermore, any nonnegative function u such that $-u \in \Phi^k(\Omega)$ which is continuous near $\partial\Omega$ and is a solution of above equation, satisfies

$$\frac{1}{c_{\text{ss}}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{8}}[\mu] \leq u(x) \leq c_{\text{ss}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2 \text{diam } \Omega}[\mu](x), \quad (5.1)$$

where c_{ss} is a positive constant independent of x, u and Ω .

Theorem 5.4 *Let μ be a measure in $\mathfrak{M}^+(\mathbb{R}^N)$ and $2k < N$. Suppose that $\mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu] < \infty$ a.e. Then there exists u , $-u \in \Phi^k(\mathbb{R}^N)$ such that $\inf_{\mathbb{R}^N} u = 0$ and $F_k[-u] = \mu$ in \mathbb{R}^N and*

$$\frac{1}{c_{\text{ss}}} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu](x) \leq u(x) \leq c_{\text{ss}} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu](x), \quad (5.2)$$

for all x in \mathbb{R}^N . Furthermore, if u is a nonnegative function such that $\inf_{\mathbb{R}^N} u = 0$ and $-u \in \Phi^k(\mathbb{R}^N)$, then (5.2) holds with $\mu = F_k[-u]$.

Proof of Theorem E. *The condition is necessary.* Assume that (1.21) admits a nonnegative solution (u, v) , continuous near $\partial\Omega$, such that $-u, -v \in \Phi^k(\Omega)$ and $u^{s_2}, v^{s_1} \in L^1(\Omega)$. Then by Theorem 5.3 we have

$$\begin{aligned} u(x) &\geq \frac{1}{c_{\text{ss}}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{8}}[v^{s_1} + \mu](x) \\ v(x) &\geq \frac{1}{c_{\text{ss}}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{8}}[u^{s_2} + \eta](x) \quad \text{for almost all } x \in \Omega. \end{aligned}$$

Using the part 2 of Proposition 2.9, we conclude that (1.22) holds.

The condition is sufficient. We define a sequence of nonnegative functions u_m, v_m , continuous near $\partial\Omega$ and such that $-u_m, -v_m \in \Phi^k(\Omega)$, by the following iterative scheme for $m \geq 0$,

$$\begin{aligned} F_k[-u_{m+1}] &= v_m^{s_1} + \mu && \text{in } \Omega, \\ F_k[-v_{m+1}] &= u_m^{s_2} + \eta && \text{in } \Omega, \\ u_{m+1} &= v_{m+1} = 0 && \text{on } \partial\Omega. \end{aligned} \quad (5.3)$$

Clearly, we can assume that $\{u_m\}$ is nondecreasing as in [21]. By Theorem 5.3 we have

$$u_{m+1} \leq c_{\text{ss}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[v_m^{s_1} + \mu], \quad v_{m+1} \leq c_{\text{ss}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[u_m^{s_2} + \mu] \quad \text{in } \Omega, \quad (5.4)$$

where $R = 2 \text{diam}(\Omega)$.

Then, by Proposition 2.8, there exists a constant $M_* > 0$ depending only on N, p, q_1, q_2, R such that if

$$\omega(K) \leq M_* \text{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}}(K)$$

for any compact set $K \subset \mathbb{R}^N$ with $d\omega = \left(\mathbf{W}_{\frac{2k}{k+1}, k+1}^R \mu\right)^{s_2} dx + d\eta$, then there holds,

$$v_m \leq c_{73} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega], \quad u_m \leq c_{74} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R\left[\left(\mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega]\right)^{s_1}\right] + c_{72} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu]$$

in Ω , for all $m \in \mathbb{N}$, for some positive constants c_{72}, c_{73} and c_{74} depending only on N, k, s_1, s_2, R . Note that we can write

$$v_m^{s_1} + \mu = \left(\mu_1 + \chi_{\Omega_\delta} v_m^{s_1}\right) + \left((1 - \chi_{\Omega_\delta})v_m^{s_1} + f\right),$$

and

$$u_m^{s_2} + \eta = \left(\eta_1 + \chi_{\Omega_\delta} u_m^{s_2}\right) + \left((1 - \chi_{\Omega_\delta})u_m^{s_2} + g\right),$$

where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$ and $\delta > 0$ is small enough and since u_m is continuous near $\partial\Omega$, then $v_m^{s_1} + \mu, u_m^{s_2} + \eta$ satisfy the assumptions of the data in Theorem 5.3. Therefore the sequence $\{u_m\}$ is well defined and nondecreasing. Thus, $\{u_m\}$ converges a.e in Ω to some function u which satisfies (1.24) in Ω . Furthermore, by the monotone convergence theorem there holds $v_m^{s_1} \rightarrow v, u_m^{s_2} \rightarrow u$ in $L^1(\Omega)$. Finally, by Proposition 5.2, we infer that (1.21) admits a nonnegative solutions u, v , continuous near $\partial\Omega$, with $-u, -v \in \Phi^k(\Omega)$ satisfying (1.24). \square

Proof of Theorem F *The condition is necessary.* Assume that (1.21) admits nonnegative solution (u, v) , such that $-u, -v \in \Phi^k(\mathbb{R}^N)$ and $u^{s_2}, v^{s_1} \in L_{loc}^1(\mathbb{R}^N)$. Then by Theorem 5.3 we have

$$\begin{aligned} u(x) &\geq \frac{1}{c_{88}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[v^{s_1} + \mu](x) \\ v(x) &\geq \frac{1}{c_{88}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[u^{s_2} + \eta](x) \quad \text{for almost all } x \in \mathbb{R}^N. \end{aligned}$$

Using Proposition 2.9-(ii), we conclude that (1.22) holds.

The condition is sufficient. We defined a sequence of nonnegative functions u_m, v_m , continuous near $\partial\Omega$ and such that $-u_m, -v_m \in \Phi^k(\Omega)$, by the following iterative scheme for $m \geq 0$,

$$\begin{aligned} F_k[-u_{m+1}] &= v_m^{s_1} + \mu && \text{in } \mathbb{R}^N, \\ F_k[-v_{m+1}] &= u_m^{s_2} + \eta && \text{in } \mathbb{R}^N, \\ \inf_{\mathbb{R}^N} u_{m+1} &= \inf_{\mathbb{R}^N} v_{m+1} = 0. \end{aligned}$$

As in the previous proofs $\{u_m\}$ is nondecreasing. By Theorem 5.3 we have

$$\begin{aligned} u_{m+1} &\leq c_{88} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[v_m^{s_1} + \mu] \\ v_{m+1} &\leq c_{88} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[u_m^{s_2} + \eta] \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Then, by Proposition 2.7, there exists a constant $M^* > 0$ depending only on N, p, q_1, q_2, R such that if

$$\omega(K) \leq M^* \text{Cap}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}(K)$$

for any compact set $K \subset \mathbb{R}^N$ with $d\omega = \left(\mathbf{W}_{\frac{2k}{k+1}, k+1}^R \mu\right)^{s_2} dx + d\eta$, then

$$v_m \leq c_{69} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega], \quad u_m \leq c_{70} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R\left[\left(\mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega]\right)^{s_1}\right] + c_{68} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu]$$

in Ω , for all $m \in \mathbb{N}$, where c_{68}, c_{69} and c_{70} depend on N, k, s_1, s_2, R . Therefore the sequence $\{u_m\}$ is well defined and nondecreasing. Thus, $\{u_m\}$ converges a.e in Ω to some function u for which (1.27) is satisfied in \mathbb{R}^N . Furthermore, by the monotone convergence theorem we have $v_m^{s_1} \rightarrow v, u_m^{s_2} \rightarrow u$ in $L_{loc}^1(\mathbb{R}^N)$. Finally, by Proposition 5.2, we obtain that (1.21) admits a nonnegative solutions u, v with $-u, -v \in \Phi^k(\mathbb{R}^N)$ satisfying (1.27). \square

6 Further results

The method exposed in the previous sections, can be applied to types of problems. We give below an example for a semilinear system in $\mathbb{R}_+^N = \{x = (x', x_N), x' \in \mathbb{R}^{N-1}, x_N > 0\}$.

$$\begin{aligned} -\Delta u &= v^{q_1} && \text{in } \mathbb{R}_+^N \\ -\Delta v &= u^{q_2} && \text{in } \mathbb{R}_+^N \\ u &= \sigma_1, v = \sigma_2 && \text{on } \partial\mathbb{R}_+^N \approx \mathbb{R}^{N-1}, \end{aligned} \tag{6.1}$$

where we have identified $\partial\mathbb{R}_+^N$ and \mathbb{R}^{N-1} . We denote by \mathbf{P} (resp. \mathbf{G}) the Poisson kernel in \mathbb{R}_+^N (resp the Green kernel in \mathbb{R}^N). The Poisson potential and the Green potential, $\mathbf{P}[\cdot]$ and $\mathbf{G}[\cdot]$, associated to $-\Delta$ are defined respectively by

$$\mathbf{P}[\sigma](y) = \int_{\partial\mathbb{R}_+^N} P(y, z) d\sigma(z), \quad G[f](y) = \int_{\mathbb{R}_+^N} \mathbf{G}(y, x) f(x) dx,$$

see [18]. We set $\rho(x) = x_N$ and define the capacity $\text{Cap}_{\alpha, s}^\rho$ by

$$\text{Cap}_{\alpha, s}^\rho(K) = \inf \left\{ \int_{\mathbb{R}_+^N} f^s \rho dx : f \geq 0, \mathbf{I}_\alpha[f\rho\chi_{\mathbb{R}_+^N}] \geq \chi_K \right\},$$

for all Borel set $K \subset \mathbb{R}^N$, where \mathbf{I}_α is the Riesz kernel of order α in \mathbb{R}^N .

Theorem 6.1 *Let $1 \leq q_1 < \frac{N}{N-1}$, $q_1 q_2 > 1$. If there exists a constant $\tilde{c} > 0$ such that if*

$$\begin{aligned} (i) \quad & \int_K \rho(x) (\mathbf{P}[\sigma_1](x))^{q_2} dx \leq \tilde{c} \text{Cap}_{\frac{q_1+2}{q_1}, \frac{q_1 q_2}{q_1 q_2 - 1}}^\rho(K), \\ (ii) \quad & \sigma_2(G) \leq \tilde{c} \text{Cap}_{\frac{2(q_2+1)}{q_1 q_2}, \frac{q_1 q_2}{q_1 q_2 - 1}}^\rho(G), \end{aligned} \tag{6.2}$$

for all Borel sets $K \subset \mathbb{R}_+^N$ and $G \subset \mathbb{R}^{N-1}$, then the problem (6.1) admits a solution.

All solutions in above theorem are understood in the usual very weak sense: $u \in L_{loc}^1(\mathbb{R}_+^N \cap B)$, $u^{q_2}, v^{q_1} \in L_\rho^1(\mathbb{R}_+^N \cap B)$ for any ball B and

$$\begin{aligned} \int_{\mathbb{R}_+^N} u(-\Delta\xi) dx &= \int_{\mathbb{R}_+^N} v^{q_1} \xi dx - \int_{\partial\mathbb{R}_+^N} \frac{\partial\xi}{\partial n} d\sigma_1, \\ \int_{\mathbb{R}_+^N} v(-\Delta\xi) dx &= \int_{\mathbb{R}_+^N} u^{q_2} \xi dx - \int_{\partial\mathbb{R}_+^N} \frac{\partial\xi}{\partial n} d\sigma_2, \end{aligned}$$

for any $\xi \in C^2(\overline{\mathbb{R}_+^N}) \cap C_c(\mathbb{R}^N)$ with $\xi = 0$ on $\partial\mathbb{R}_+^N$. It is well-known that such a solution u satisfies

$$u = \mathbf{G}[v^{q_1}] + \mathbf{P}[\sigma_1], \quad v = \mathbf{G}[u^{q_2}] + \mathbf{P}[\sigma_2] \quad \text{a.e. in } \mathbb{R}_+^N.$$

To prove Theorem 6.1 we need the following basic estimate,

Lemma 6.2 *Assume that $0 < q_1 < \frac{N}{N-1}$. Then for any $\omega \in \mathfrak{M}_b^+(\mathbb{R}^N)$,*

$$\mathbf{I}_2[(\mathbf{I}_1[\omega])^{q_1}] \leq c_{s_9} \mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}[\omega] \quad \text{a.e. in } \mathbb{R}^N, \tag{6.3}$$

where $c_{s_9} > 0$ depends on q_1, q_2 and N .

Proof. The proof of Lemma 6.2 is similar to the one of Lemma 2.2 and details are omitted. Note that if $\omega \in \mathfrak{M}_b(\overline{\mathbb{R}_+^N})$ it is extended by 0 in \mathbb{R}_-^N . \square

Remark 6.3 The condition $0 < q_1 < \frac{N}{N-1}$ is a necessary and sufficient condition in order $(\mathbf{I}_1[\omega])^{q_1}$ be locally integrable in \mathbb{R}^N for any $\omega \in \mathfrak{M}_b^+(\mathbb{R}^N)$.

Theorem 6.4 Let $q_1 \geq 1$, $q_1 q_2 > 1$ and $\omega \in \mathfrak{M}_b(\overline{\mathbb{R}_+^N})$. If

$$\omega(K) \leq c_{90} \text{Cap}_{\frac{q_1+2}{q_1}, \frac{q_1 q_2}{q_1 q_2 - 1}}^\rho(K) \quad \forall K \subset \overline{\mathbb{R}_+^N}, K \text{ Borel},$$

for some $c_{90} > 0$, then

$$\mathbf{I}_1 \left[\left(\mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}[\omega] \right)^{q_2} \rho \chi_{\mathbb{R}_+^N} \right] \leq c_{91} \mathbf{I}_1[\omega] \quad \text{a.e. in } \mathbb{R}_+^N. \quad (6.4)$$

Proof. *Step 1.* For any compact $K \subset \left\{ x \in \mathbb{R}_+^N : \mathbf{I}_{\frac{q_1+2}{q_1}}[f \rho \chi_{\mathbb{R}_+^N}](x) > \lambda \right\}$, we have

$$\omega(K) \leq c_{90} \text{Cap}_{\frac{q_1+2}{q_1}, \frac{q_1 q_2}{q_1 q_2 - 1}}^\rho(K) \leq c_{90} \lambda^{-\frac{q_1 q_2}{q_1 q_2 - 1}} \int_{\mathbb{R}_+^N} f^{\frac{q_1 q_2}{q_1 q_2 - 1}} \rho dx$$

by assumption and the definition of the capacity. Hence,

$$\lambda^{\frac{q_1 q_2}{q_1 q_2 - 1}} \omega \left(\left\{ \mathbf{I}_{\frac{q_1+2}{q_1}}[f \rho \chi_{\mathbb{R}_+^N}] > \lambda \right\} \right) \leq c_{90} \int_{\mathbb{R}_+^N} f^{\frac{q_1 q_2}{q_1 q_2 - 1}} \rho dx \quad \forall \lambda > 0.$$

This implies an estimate in Lorentz space,

$$\| \mathbf{I}_{\frac{q_1+2}{q_1}}[f \rho \chi_{\mathbb{R}_+^N}] \|_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}, \infty}(\mathbb{R}^N, d\omega)} \leq \| f \|_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}}(\mathbb{R}^N, \chi_{\mathbb{R}_+^N} \rho dx)} \quad \forall f \geq 0. \quad (6.5)$$

Step 2. Since, for any $g \in C_c(\mathbb{R}_+^N)$,

$$\int_{\mathbb{R}_+^N} \mathbf{I}_{\frac{q_1+2}{q_1}}[g\omega] f \rho dx = \int_{\mathbb{R}^N} \mathbf{I}_{\frac{q_1+2}{q_1}}[f \rho \chi_{\mathbb{R}_+^N}] g d\omega,$$

we infer, using duality between $L^{p,1}$ and $L^{p',\infty}$, Holder's inequality therein and (6.5), that

$$\begin{aligned} \int_{\mathbb{R}_+^N} \mathbf{I}_{\frac{q_1+2}{q_1}}[g\omega] f \rho dx &\leq \| \mathbf{I}_{\frac{q_1+2}{q_1}}[f \rho \chi_{\mathbb{R}_+^N}] \|_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}, \infty}(\mathbb{R}^N, d\omega)} \| g \|_{L^{q_1 q_2, 1}(\mathbb{R}^N, d\omega)} \\ &\leq \| f \|_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}}(\mathbb{R}^N, \chi_{\mathbb{R}_+^N} \rho dx)} \| g \|_{L^{q_1 q_2, 1}(\mathbb{R}^N, d\omega)} \quad \forall f, g \geq 0. \end{aligned}$$

Therefore,

$$\| \mathbf{I}_{\frac{q_1+2}{q_1}}[g\omega] \|_{L^{q_1 q_2}(\mathbb{R}^N, \chi_{\mathbb{R}_+^N} \rho dx)} \leq \| g \|_{L^{q_1 q_2, 1}(\mathbb{R}^N, d\omega)}. \quad (6.6)$$

Step 3. Taking $g = \chi_{B_t(x)}$ and since for $q_1 \geq 1$

$$\begin{aligned} \mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}[\nu](x) &= \int_0^\infty \left(\frac{\nu(B_\rho(x))}{\rho^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} dx \\ &\leq c_{89} \left(\int_0^\infty \frac{\nu(B_\rho(x))}{\rho^{N-\frac{q_1+2}{q_1}}} dx \right)^{q_1} \\ &= c_{89} \left(\mathbf{I}_{\frac{q_1+2}{q_1}}[\nu](x) \right)^{q_1} \quad \forall \nu \in \mathfrak{M}_b^+(\mathbb{R}^N), \quad \forall x \in \mathbb{R}^N, \end{aligned}$$

we deduce that for almost all $x \in \mathbb{R}_+^N$,

$$\int_{\mathbb{R}_+^N} \left(\mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}} [\chi_{B_t(x)} \omega] \right)^{q_2} \rho dy \leq c_{90} \omega(B_t(x)),$$

from (6.6), which implies

$$\omega(B_t(x)) \leq c_{91} \frac{t^{(N-\frac{q_1+2}{q_1})-\frac{q_1 q_2}{q_1 q_2-1}}}{\left(\int_{B_{2t}(x)} \chi_{\mathbb{R}_+^N} \rho dy \right)^{\frac{1}{q_1 q_2-1}}} \leq c_{92} \frac{t^{n-\frac{q_1+2}{q_1}-\frac{q_1 q_2}{q_1 q_2-1}}}{(\max\{x_n, t\})^{\frac{1}{q_1 q_2-1}}}, \quad (6.7)$$

since $\int_{B_r(x)} \chi_{\mathbb{R}_+^N} \rho dy \asymp r^N \max\{x_N, r\}$ for any $x \in \mathbb{R}_+^N, r > 0$ where the symbol \asymp is defined by

$$A \asymp B \iff \frac{1}{c} B \leq A \leq cB \quad \text{for some constant } c > 0.$$

It implies also

$$\int_{B_t(x)} \left(\mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}^t [\omega] \right)^{q_2} \chi_{\mathbb{R}_+^N} \rho dy \leq c_{90} \omega(B_{2t}(x)), \quad (6.8)$$

from which follows

$$\int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left(\mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}^t [\omega] \right)^{q_2} \chi_{\mathbb{R}_+^N} \rho dy dt \leq c_{93} \mathbf{I}_1[\omega](x).$$

Therefore, if the following inequality holds

$$\int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\omega(B_r(y))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} \chi_{\mathbb{R}_+^N} \rho dy dt \leq c_{93} \mathbf{I}_1[\omega](x), \quad (6.9)$$

it will imply (6.4).

Step 4. We claim that (6.9) holds. Since $B_r(y) \in B_{2r}(x), y \in B_t(x), r \geq t$,

$$\begin{aligned} & \int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\omega(B_r(y))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} \chi_{\mathbb{R}_+^N} \rho dy dt \\ & \leq \int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \chi_{\mathbb{R}_+^N} \rho dy \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} dt \\ & \asymp \int_0^\infty \max\{x_n, t\} \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} dt. \end{aligned}$$

By integration by part,

$$\begin{aligned} & \int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\omega(B_r(y))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} \chi_{\mathbb{R}_+^N} \rho dy dt \\ & = q_2 \int_0^\infty \int_0^t \max\{x_N, s\} ds \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} \left(\frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dt}{t} \\ & = q_2 \int_0^\infty \int_0^t \max\{x_N, s\} ds \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} \left(\frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}} \right)^{q_1-1} \frac{2}{t^{\frac{2}{q_1}}} \frac{\omega(B_{2t}(x))}{t^{N-1}} \frac{dt}{t} \end{aligned}$$

We have

$$\int_0^t \max\{x_N, s\} ds \asymp t \max\{x_N, t\},$$

$$\begin{aligned} \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} &\leq c_{94} \left(\int_t^\infty \left(\frac{r^{-\frac{q_1+2}{q_1(q_1q_2-1)}}}{(\max\{x_N, r\})^{\frac{1}{q_1q_2-1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} \\ &\asymp t^{-\frac{(q_1+2)(q_2-1)}{q_1q_2-1}} (\max\{x_N, t\})^{-\frac{q_1(q_1-1)}{q_1q_2-1}}, \end{aligned}$$

by (6.7) and

$$\begin{aligned} \left(\frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}} \right)^{q_1-1} t^{\frac{2}{q_1}} &\leq c_{95} \left(\frac{t^{-\frac{q_1+2}{q_1(q_1q_2-1)}}}{(\max\{x_N, t\})^{\frac{1}{q_1q_2-1}}} \right)^{q_2-1} t^{\frac{2}{q_1}} \\ &= c_{95} t^{-\frac{(q_1+2)(q_1-1)}{q_1(q_1q_2-1)} + \frac{2}{q_1}} (\max\{x_N, t\})^{-\frac{q_1-1}{q_1q_2-1}}. \end{aligned}$$

Thus,

$$\int_0^t \max\{x_N, s\} ds \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} \left(\frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}} \right)^{q_1-1} t^{2/q_1} \leq c_{96},$$

and we obtain (6.9). \square

Lemma 6.5 *Let $\alpha > 0$, $s > 1$ such that $\alpha + \frac{2}{s'} < N - 1$ where $s' = \frac{s}{s-1}$. For all $\eta \in \mathfrak{M}^+(\mathbb{R}^{N-1})$, there holds*

$$\int_{\mathbb{R}^N} (\mathbf{I}_\alpha[\eta \otimes \delta_{\{x_N=0\}}])^{s'} x_N dx \asymp \int_{\mathbb{R}^{N-1}} \left(\int_0^\infty \frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} \frac{dt}{t} \right)^{s'} dx', \quad (6.10)$$

where I_β is the Riesz potential of order β in \mathbb{R}^{N-1} . As a consequence, we have

$$\text{Cap}_{\alpha, s}^\rho(E \times \{x_N = 0\}) \asymp \text{Cap}_{I_{\alpha+2/s', s}}(E) \quad \forall E \subset \mathbb{R}^{N-1}, E \text{ Borel}. \quad (6.11)$$

Proof. We have

$$\begin{aligned} \int_{\mathbb{R}^N} (\mathbf{I}_\alpha[\eta \otimes \delta_{\{x_N=0\}}])^{s'} x_N dx &\geq \int_{\mathbb{R}^N} \left(\int_{2x_N}^{4x_N} \frac{(\eta \otimes \delta_{\{x_N=0\}})(B_r(x))}{r^{N-\alpha}} \frac{dr}{r} \right)^{s'} x_N dx \\ &\geq c_{97} \int_{\mathbb{R}^N} \left(\frac{\eta(B'_{x_N}(x'))}{x_N^{N-\alpha}} \right)^{s'} x_N dx \\ &\geq c_{98} \int_{\mathbb{R}^{N-1}} \left(\sup_{t>0} \frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} \right)^{s'} dx'. \end{aligned} \quad (6.12)$$

By using Lemma 2.1 we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (\mathbf{I}_\alpha[\eta \otimes \delta_{\{x_N=0\}}])^{s'} x_N dx &\leq \int_{\mathbb{R}^N} \left(\int_{x_N}^\infty \frac{\eta(B'_r(x'))}{r^{N-\alpha}} \frac{dr}{r} \right)^{s'} dx_N dx' \\ &\leq c_{99} \int_{\mathbb{R}^{N-1}} \int_0^\infty \left(\frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} \right)^{s'} \frac{dt}{t} dx'. \end{aligned} \quad (6.13)$$

On the other hand, by [20, Proposition 5.1], there holds

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \left(\sup_{t>0} \frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} \right) dx' &\asymp \int_{\mathbb{R}^{N-1}} \int_0^\infty \left(\frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} \right)^{s'} \frac{dt}{t} dx' \\ &\asymp \int_{\mathbb{R}^{N-1}} \left(\int_0^\infty \frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} dt \right)^{s'} dx'. \end{aligned} \quad (6.14)$$

Combining (6.12), (6.13) and (6.14) we obtain (6.10). Moreover, we deduce (6.11) from (6.10) and [1, Theorem 2.5.1], which ends the proof. \square

Proof of Theorem 6.1 The following estimates are classical

$$\mathbf{G}(x, y) \asymp \frac{x_N y_N}{|x - y|^{N-2} \max\{|x - y|, x_N, y_N\}^2} \leq c_{100} \frac{y_N}{|x - y|^{N-1}}, \quad (6.15)$$

$$\mathbf{P}(x, z) = c_{101} \frac{x_N}{|x - z|^N} \leq c_{101} \frac{1}{|x - z|^{N-1}}. \quad (6.16)$$

Thus,

$$\mathbf{G}[(\mathbf{P}[\sigma_1])^{q_2}] + \mathbf{P}[\sigma_2] \leq c_{102} \mathbf{I}_1[\omega], \quad (6.17)$$

where $\omega(x) = \rho(\mathbf{P}[\sigma_1])^{q_2} + \sigma_2$ in \mathbb{R}^N . Therefore, we infer that if

$$\mathbf{I}_1 \left[(\mathbf{I}_2[(\mathbf{I}_1[\omega])^{q_1}])^{q_2} \chi_{\mathbb{R}_+^N} \rho \right] \leq c_{103} \mathbf{I}_1[\omega] \quad \text{in } \mathbb{R}_+^N \quad (6.18)$$

for some $c_{103} > 0$ small enough, then (6.1) admits a positive solution (u, v) . On the other hand, we deduce (6.18) from Lemma 6.2 and Theorem 6.4. The proof is complete. \square

Remark 6.6 *The system*

$$\begin{aligned} -\Delta u &= v^{q_1} + \epsilon_1 \mu && \text{in } \Omega \\ -\Delta v &= u^{q_2} + \epsilon_2 \eta && \text{in } \Omega \\ u &= \epsilon_3 \sigma_1, v = \epsilon_4 \sigma_2 && \text{in } \partial\Omega, \end{aligned} \quad (6.19)$$

where $d(\cdot)\mu, d(\cdot)\lambda$ belong to $\mathfrak{M}_b^+(\Omega)$, σ_1, σ_2 to $\mathfrak{M}^+(\partial\Omega)$ and the ϵ_j are positive numbers, is analyzed in [10, Th 4.6]. Therein it is proved that if

$$\int_{\Omega} (\mathbf{G}[\mu] + \mathbf{P}[\lambda])^{\max\{q_1, q_2\}} d(x) dx < \infty, \quad (6.20)$$

which is equivalent to a capacity estimate, and

$$\min \left\{ q_2 \frac{q_1 + 1}{q_2 + 1}, q_1 \frac{q_2 + 1}{q_1 + 1} \right\} < \frac{N + 1}{N - 1}, \quad (6.21)$$

and if the ϵ_j are small enough, then (6.19) admits a positive solution. Now condition (6.21) is a subcriticality assumption (for at least one of the two exponents q_j) since there is no condition on the boundary measures.

References

- [1] D. R. Adams, L.I. Heberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wissenschaften **31**, Springer-Verlag (1999).
- [2] D.R. Adams, M. Pierre, *Capacity strong type estimates in semilinear problems*, Ann. Inst. Fourier (Grenoble) **41** (1991), 117-135.

- [3] P. Baras, M. Pierre, *Critère d'existence des solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré, Anal. Non Lin. **3** (1985), 185-212.
- [4] M. F. Bidaut-Véron, *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type*, Arch. Ration. Mech. Anal. **107** (1989), 293-324.
- [5] M. F. Bidaut-Véron, *Necessary conditions of existence for an elliptic equation with source term and measure data involving p -Laplacian*, in: Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, Electron. J. Differ. Equ. Conf. **8** (2002), 23-34.
- [6] M. F. Bidaut-Véron, *Removable singularities and existence for a quasilinear equation with absorption or source term and measure data*, Adv. Nonlinear Stud. **3** (2003), 25-63.
- [7] M. F. Bidaut-Véron, Q.-H. Nguyen, L. Véron, *Quasilinear Lane-Emden equations with absorption and measure data*, J. Math. Pures Appl. **102** (2014), 315337.
- [8] M.F. Bidaut-Véron, Q.-H. Nguyen, L. Véron; *Quasilinear elliptic equations with source mixed term and measure data*, preprint.
- [9] M. F. Bidaut-Véron, S. Pohozaev, *Nonexistence results and estimates for some nonlinear elliptic problems*, J. Anal. Math. **84** (2001), 1-49.
- [10] M. F. Bidaut-Véron, C. Yarur, *Semilinear elliptic equations and systems with measure data: existence and a priori estimates*, Adv. Diff. Equ. **7** (2002), 257-296.
- [11] I. Birindelli, F. Demengel, *Some Liouville theorems for the p -Laplacian*, in: Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, Electron. J. Differ. Equ. Conf. **8** (2002), 35-46.
- [12] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Sc. Norm. Sup. Pisa **28** (1999), 741-808.
- [13] J. Heinonen, T. Kilpelainen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations*. Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY, 2006. xii+404 pp.
- [14] T. Kilpelainen, J. Malý, *Degenerate elliptic equation with measure data and nonlinear potentials*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. **19** (1992), 591-613.
- [15] T. Kilpelainen, J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137-161.
- [16] P. Honzik, B. Jaye, *On the good- λ inequality for nonlinear potentials*, Proc. Amer. Math. Soc. **140** (2012), 4167-4180.
- [17] D. Labutin, *Potential estimates for a class of fully nonlinear elliptic equations*, Duke Math. J. **111**, 1-49, (2002).
- [18] M. Marcus, L. Véron, *Nonlinear second order elliptic equations involving measures*. De Gruyter Series in Nonlinear Analysis and Applications **21**. De Gruyter, Berlin (2014).
- [19] M. M. Rao, Z. D. Ren, *Theory of Orlicz Spaces*, Textbooks in Pure and Applied Mathematics (1991).
- [20] N. C. Phuc, I. E. Verbitsky, *Quasilinear and Hessian equations of Lane-Emden type*, Ann. Math. **168**, 859-914 (2008).
- [21] N. C. Phuc, I. E. Verbitsky, *Singular quasilinear and Hessian equation and inequalities*, J. Funct. Anal. **256** (2009), 1875-1906.

- [22] J. Serrin, H. Zou, *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. **189** (2002) 79-142.
- [23] N. S. Trudinger and X.J. Wang, *Hessian measures*, Topological Methods in Nonlinear Analysis **10** (1997), 225-239.
- [24] N. S. Trudinger, X.J. Wang, *Hessian measures II*, Annals of Mathematics **150** (1999), 579-604.
- [25] N. S. Trudinger, X. J. Wang, *Hessian measures III*, Journal of Functional Analysis **193** (2002), 1-23.
- [26] N. S. Trudinger, X. J. Wang, *On the weak continuity of elliptic operators and applications to potential theory*, Amer. J. Math. **124** (2002), 369-410.
- [27] L. Véron, *Elliptic equations involving measures*, in Stationary Partial Differential Equations, Handbook of Equations **vol. I**, Elsevier B.V., pp. 593-712 (2004).
- [28] L. Véron, *Local and global aspects of quasilinear degenerate elliptic equations. Quasilinear elliptic singular problems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, xv+457 (2017).