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# Quasilinear and Hessian Lane-Emden type systems with measure data

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## Abstract

We study nonlinear systems of the form  $-\Delta_p u = v^{q_1} + \mu$ ,  $-\Delta_p v = u^{q_2} + \eta$  and  $F_k[-u] = v^{s_1} + \mu$ ,  $F_k[-v] = u^{s_2} + \eta$  in a bounded domain  $\Omega$  or in  $\mathbb{R}^N$  where  $\mu$  and  $\eta$  are nonnegative Radon measures,  $\Delta_p$  and  $F_k$  are respectively the  $p$ -Laplacian and the  $k$ -Hessian operators and  $q_1, q_2, s_1$  and  $s_2$  positive numbers. We give necessary and sufficient conditions for existence expressed in terms of Riesz or Bessel capacities. 2010

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*Key words:*  $p$ -Laplacian,  $k$ -Hessian, Bessel and Riesz capacities, measures, maximal functions.

## 1 Introduction and Main results

Let  $\Omega \subset \mathbb{R}^N$  be either a bounded domain or the whole  $\mathbb{R}^N$ ,  $p > 1$  and  $k \in \{1, 2, \dots, N\}$ . We denote by

$$\Delta_p u := \operatorname{div} (|\nabla u|^{p-2} \nabla u)$$

the  $p$ -Laplace operator and by

$$F_k[u] = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq N} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_k}$$

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the  $k$ -Hessian operator where  $\lambda_1, \dots, \lambda_N$  are the eigenvalues of the Hessian matrix  $D^2u$ . In the work [23], Phuc and Verbitsky obtained necessary and sufficient conditions for existence of nonnegative solutions to the following equations

$$\begin{aligned} -\Delta_p u &= u^q + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} F_k[-u] &= u^q + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (1.2)$$

Their conditions involve the continuity of the measures with respect to Bessel or Riesz capacities and Wolff potentials estimates. For example, if  $\Omega$  is bounded and  $\mu$  has compact support in  $\Omega$ , they proved that it is equivalent to solve (1.1), or to have

$$\mu(E) \leq c_1 \text{Cap}_{\mathbf{G}_p, \frac{q}{q+1-p}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.3)$$

for some constant  $c_1 > 0$  where  $\text{Cap}_{\mathbf{G}_p, \frac{q}{q+1-p}}$  is a Bessel capacity, or to have

$$\int_B (\mathbf{W}_{1,p}^R[\mu_B](x))^q dx \leq c_2 \mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \text{supp}\mu \neq \emptyset, \quad (1.4)$$

for some constant  $c_2 > 0$ , where  $R = 2 \text{diam}(\Omega)$  and  $\mathbf{W}_{1,p}^R[\mu_B]$  denotes the  $R$ -truncated Wolff potential of the measure  $\mu_B = \chi_B \mu$ . Concerning the  $k$ -Hessian operator in a bounded  $(k-1)$ -convex domain  $\Omega$ , they proved that if  $\mu$  has compact support, the problem (1.2) with  $q > k$  admits a nonnegative solution if and only if

$$\mu(E) \leq c_3 \text{Cap}_{\mathbf{G}_{2k}, \frac{q}{q-k}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.5)$$

for some  $c_3$ . In turn this condition is equivalent to

$$\int_B \left[ \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu_B(x)] \right]^q dx \leq c_4 \mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \text{supp}\mu \neq \emptyset. \quad (1.6)$$

for some  $c_4 > 0$ . The results concerning the linear case  $p = 2$  and  $k = 1$ , can be found in [2, 3, 31].

The natural counterpart of equation (1.1) and (1.2) for systems:

$$\begin{aligned} -\Delta_p u &= v^{q_1} + \mu && \text{in } \Omega \\ -\Delta_p v &= u^{q_2} + \eta && \text{in } \Omega \\ u &= v = 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.7)$$

and

$$\begin{aligned} F_k[-u] &= v^{s_1} + \mu && \text{in } \Omega \\ F_k[-v] &= u^{s_2} + \eta && \text{in } \Omega \\ u &= v = 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.8)$$

where  $q_1, q_2 > p-1, s_1, s_2 > k$  and  $\mu, \eta$  are Radon measures. If  $\Omega = \mathbb{R}^N$ , we consider the same equations, except that the boundary conditions are replaced by  $\inf_{\mathbb{R}^N} u = \inf_{\mathbb{R}^N} v = 0$  and our statements involve the Riesz potentials and their associated capacities  $\text{Cap}_{I_{\alpha, \beta}}$ . Our main results are the following.

**Theorem A** *Let  $1 < p < N, q_1, q_2 > 0$  and  $q_2 q_1 > (p-1)^2$ . Let  $\mu, \eta$  be nonnegative Radon measures in  $\mathbb{R}^N$ . If the following system*

$$\begin{aligned} -\Delta_p u &= v^{q_1} + \mu && \text{in } \mathbb{R}^N \\ -\Delta_p v &= u^{q_2} + \eta && \text{in } \mathbb{R}^N \end{aligned} \quad (1.9)$$

admits a nonnegative  $p$ -superharmonic solution  $(u, v)$  then there exists a positive constant  $c_5$  depending on  $N, p, q_1, q_2$  such that

$$\eta(E) + \int_E (\mathbf{W}_{1,p}[\mu](x))^{q_2} dx \leq c_5 \text{Cap}_{\mathbf{I}_{\frac{p(q_1+p-1)}{q_1}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(E) \quad \text{for all Borel sets } E. \quad (1.10)$$

Conversely, if  $\mu$  and  $\eta$  are bounded, there exists  $c_6 > 0$  depending on  $N, p, q_1, q_2$  such that if  $0 < q_1 < \frac{N(p-1)}{N-p}$  and (1.10) holds with  $c_5$  replaced by  $c_6$ , then (1.9) admits a nonnegative  $p$ -superharmonic solution  $(u, v)$  satisfying

$$v \leq c_8 \mathbf{W}_{1,p}[\omega], \quad u \leq c_9 \mathbf{W}_{1,p}[(\mathbf{W}_{1,p}[\omega])^{q_1}] + c_7 \mathbf{W}_{1,p}[\mu] \quad (1.11)$$

in  $\mathbb{R}^N$  for some  $c_7, c_8, c_9 > 0$  where  $d\omega = (\mathbf{W}_{1,p}[\mu])^{q_2} dx + d\eta$ .

We notice that the left-hand side in (1.10) is not symmetric in  $\eta$  and  $\mu$  and the capacity in the right-hand side is not symmetric in  $q_1$  and  $q_2$ . Hence the following symmetrized inequality holds

$$\mu(E) + \int_E (\mathbf{W}_{1,p}[\eta](x))^{q_1} dx \leq c'_5 \text{Cap}_{\mathbf{I}_{\frac{p(q_2+p-1)}{q_2}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(E) \quad \text{for all Borel sets } E. \quad (1.12)$$

It is known that

$$\text{Cap}_{\mathbf{I}_{\alpha,\beta}}(K) = 0 \quad \forall K \text{ compact,}$$

if  $\alpha\beta \geq N$ , the first part of above implies the following Liouville theorem, obtained by another method in [9, Th 5.3-(i)]

**Corollary B** Assume that

$$\frac{p(q_1 q_2 + (p-1) \max\{q_1, q_2\})}{q_1 q_2 - (p-1)^2} \geq N.$$

Any nonnegative  $p$ -superharmonic solution  $(u, v)$  of inequalities

$$\begin{aligned} -\Delta_p u &\geq v^{q_1} && \text{in } \mathbb{R}^N \\ -\Delta_p v &\geq u^{q_2} && \text{in } \mathbb{R}^N, \end{aligned} \quad (1.13)$$

is trivial, i.e.  $u = v = 0$ .

When  $\Omega$  is bounded domain, we have a similar result in which we denote by  $d$  the distance function to the boundary  $x \mapsto d(x) = \text{dist}(x, \partial\Omega)$ .

**Theorem C** Let  $1 < p < N$ ,  $q_1, q_2 > 0$  and  $q_2 q_1 > (p-1)^2$ . Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain and  $\mu, \eta$  nonnegative Radon measures in  $\Omega$ . If the following problem

$$\begin{aligned} -\Delta_p u &= v^{q_1} + \mu && \text{in } \Omega \\ -\Delta_p v &= u^{q_2} + \eta && \text{in } \Omega \\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.14)$$

admits a nonnegative renormalized solution  $(u, v)$ , then then for any compact set  $K \subset \Omega$ , there exists a positive constant  $c_{10}$  depending on  $N, p, q_1, q_2$  and  $\text{dist}(K, \partial\Omega)$  such that

$$\eta(E) + \int_E \left( \mathbf{W}_{1,p}^{\frac{d(x)}{4}}[\mu](x) \right)^{q_2} dx \leq c_{10} \text{Cap}_{\mathbf{G}_{\frac{p(q_1+p-1)}{q_1}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(E) \quad \text{for all Borel sets } E \subset K. \quad (1.15)$$

Conversely, if  $\mu$  and  $\eta$  are bounded and there exists  $c_{11} > 0$  depending on  $N, p, q_1, q_2$  and  $R = 2\text{diam}(\Omega)$  such that if  $0 < q_1 < \frac{N(p-1)}{N-p}$  and

$$\eta(K) + \int_K (\mathbf{W}_{1,p}^{2R}[\mu])^{q_2} dx \leq c_{11} \text{Cap}_{\mathbf{G}_{\frac{p(q_1+p-1)}{q_1}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(K) \quad (1.16)$$

for all compact set  $K \subset \Omega$ , then (1.14) admits a nonnegative renormalized solution  $(u, v)$  satisfying

$$v \leq c_{13} \mathbf{W}_{1,p}^R[\omega], \quad u \leq c_{14} \mathbf{W}_{1,p}^R[(\mathbf{W}_{1,p}^R[\omega])^{q_1}] + c_{12} \mathbf{W}_{1,p}^R[\mu] \quad (1.17)$$

in  $\Omega$ , where  $d\omega = (\mathbf{W}_{1,p}^R[\mu])^{q_2} dx + d\eta$ .

It is known that

$$\text{Cap}_{\mathbf{G}_{\alpha,\beta}}(\{x_0\}) > 0$$

if and only if  $\alpha\beta > N$ . Thus, as an application in a partially subcritical case we have,

**Corollary D** *Let the assumptions on  $p, q_1, q_2, \Omega$  and  $R$  of Theorem C be satisfied,  $x_0 \in \Omega$ ,  $a > 0$  and  $\mu$  be a nonnegative Radon measures in  $\Omega$ . If the following problem*

$$\begin{aligned} -\Delta_p u &= v^{q_1} + \mu && \text{in } \Omega \\ -\Delta_p v &= u^{q_2} + a\delta_{x_0} && \text{in } \Omega \\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.18)$$

admits a nonnegative renormalized solution  $(u, v)$ , then there exist positive constants  $c_{15} = c_{15}(N, p, q_1, q_2, d(x_0))$  and for any compact subset  $K$  of  $\Omega$ ,  $c_{16} = c_{16}(N, p, q_1, q_2, \text{dist}(K, \partial\Omega))$ , such that

$$\begin{aligned} (i) \quad & N < \frac{pq_2(q_1 + p - 1)}{q_1q_2 - (p - 1)^2}, \\ (ii) \quad & a \leq c_{15} \\ (iii) \quad & \int_K (\mathbf{W}_{1,p}^{2R}[\mu])^{q_2} dx \leq c_{16}. \end{aligned} \quad (1.19)$$

Conversely, assuming that  $\mu$  is bounded, there exist positive constants  $c_{17} = c_{17}(N, p, q_1, q_2, d(x_0))$ ,  $c_{18} = c_{18}(N, p, q_1, q_2)$  such that if  $0 < q_1 < \frac{N(p-1)}{N-p}$  and (1.19) holds with  $c_{15}$  and  $c_{16}$  replaced respectively by  $c_{17}$  and  $c_{18}$  there exists a nonnegative renormalized solution  $(u, v)$  of (1.18) satisfying

$$v \leq c_{21} W_{1,p}^R[\omega], \quad u \leq c_{22} \mathbf{W}_{1,p}^R[(W_{1,p}^R[\omega])^{q_1}] + c_{20} \mathbf{W}_{1,p}^R[\mu] \quad (1.20)$$

in  $\Omega$ , where

$$W_{1,p}^R[\omega] = \mathbf{W}_{1,p}^R \left[ (\mathbf{W}_{1,p}^R[\mu])^{q_2} \right] + a^{\frac{1}{p-1}} \left( |x - x_0|^{-\frac{N-p}{p-1}} - R^{-\frac{N-p}{p-1}} \right)_+.$$

Concerning the  $k$ -Hessian operator we recall some notions introduced by Trudinger and Wang [27, 28, 29], and we follow their notations. For  $k = 1, \dots, N$  and  $u \in C^2(\Omega)$  the  $k$ -Hessian operator  $F_k$  is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where  $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$  denotes the eigenvalues of the Hessian matrix of second partial derivatives  $D^2u$  and  $S_k$  is the  $k$ -th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}.$$

Since  $D^2u$  is symmetric, it is clear that

$$F_k[u] = [D^2u]_k,$$

where we denote by  $[A]_k$  the sum of the  $k$ -th principal minors of a matrix  $A = (a_{ij})$ . In order that there exists a smooth  $k$ -admissible function which vanishes on  $\partial\Omega$ , the boundary  $\partial\Omega$  must satisfy a uniformly  $(k-1)$ -convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega,$$

for some positive constant  $c_0$ , where  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$  denote the principal curvatures of  $\partial\Omega$  with respect to its inner normal. We also denote by  $\Phi^k(\Omega)$  the class of upper-semicontinuous functions  $\Omega \rightarrow [-\infty, \infty)$  which are  $k$ -convex, or subharmonic in the Perron sense (see Definition 5.1). In this paper we prove the following theorem (in which expression  $\mathbb{E}[q]$  is the largest integer less or equal to  $q$ )

**Theorem E** *Let  $2k < N, s_1, s_2 > 0, s_1 s_2 > k^2$ . Let  $\Omega$  be a bounded uniformly  $(k-1)$ -convex domain in  $\mathbb{R}^N$  with diameter  $R$ . Let  $\mu = \mu_1 + f$  and  $\eta = \eta_1 + g$  be nonnegative Radon measures where  $\mu_1, \eta_1$  has compact support in  $\Omega$  and  $f, g \in L^l(\Omega)$  for some  $l > \frac{N}{2k}$ . If then the following problem*

$$\begin{aligned} F_k[-u] &= v^{s_1} + \mu && \text{in } \Omega \\ F_k[-v] &= u^{s_2} + \eta && \text{in } \Omega \\ u &= v = 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.21)$$

*admits a nonnegative solutions  $(u, v)$ , continuous near  $\partial\Omega$ , with  $-u$  and  $-v$  elements of  $\Phi^k(\Omega)$ , then for any compact set  $K \subset \Omega$ , there exists a positive constant  $c_{23}$  depending on  $N, k, s_1, s_2$  and  $\text{dist}(K, \partial\Omega)$  such that there holds*

$$\eta(E) + \int_E \left( \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x)}{4}}[\mu](x) \right)^{s_2} dx \leq c_{23} \text{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}} (E) \quad \forall E \subset K, E \text{ Borel.} \quad (1.22)$$

*Conversely,, if  $\mu$  and  $\eta$  are bounded, there exist a positive constant  $c_{24}$  depending on  $N, k, s_1, s_2$  and  $\text{diam}(\Omega)$  such that, if  $k \leq s_1 < \frac{Nk}{N-2k}$ , and*

$$\eta(K) + \int_K \left( \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2R}[\mu] \right)^{s_2} dx \leq c_{24} \text{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}} (K) \quad (1.23)$$

*for all Borel set  $K \subset \Omega$ , then (1.21) admits a nonnegative solution  $(u, v)$ , continuous near  $\partial\Omega$ , with  $-u, -v \in \Phi^k(\Omega)$  satisfying*

$$v \leq c_{28} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega], \quad u \leq c_{29} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R \left[ \left( \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega] \right)^{s_1} \right] + c_{27} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu] \quad (1.24)$$

*in  $\Omega$  for some constants  $c_j$  ( $j = 27, 28, 29$ ) depending on  $N, k, s_1, s_2$ , and  $\text{diam}(\Omega)$ .*

If  $\Omega$  is replaced by the whole space we prove,

**Theorem F** *Let  $2k < N, s_1, s_2 > 0, s_1 s_2 > k^2$ . Let  $\mu, \eta$  be a nonnegative Radon measures in  $\mathbb{R}^N$ . If the following problem*

$$\begin{aligned} F_k[-u] &= v^{s_1} + \mu && \text{in } \mathbb{R}^N \\ F_k[-v] &= u^{s_2} + \eta && \text{in } \mathbb{R}^N, \end{aligned} \quad (1.25)$$

*admits a nonnegative solutions  $(u, v)$  with  $-u$  and  $-v$  belonging to  $\Phi^k(\mathbb{R}^N)$ , then there exists a positive constant  $c_{30}$  depending on  $N, k, s_1, s_2$  such that there holds*

$$\eta(E) + \int_E \left( \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu](x) \right)^{s_2} dx \leq c_{30} \text{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}} (E) \quad \forall E \text{ Borel.} \quad (1.26)$$

*Conversely,, if  $\mu$  and  $\eta$  are bounded, there exists positive constant  $c_{31}$  depending on  $N, k, s_1, s_2$  such that, if  $0 < s_1 < \frac{Nk}{N-2k}$  and (1.26) holds with  $c_{31}$  instead of  $c_{30}$  then (1.25) admits a nonnegative solutions  $(u, v)$  with  $-u$  and  $-v$  in  $\Phi^k(\mathbb{R}^N)$  satisfying*

$$v \leq c_{33} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\omega], \quad u \leq c_{34} \mathbf{W}_{\frac{2k}{k+1}, k+1} \left[ \left( \mathbf{W}_{\frac{2k}{k+1}, k+1}[\omega] \right)^{s_1} \right] + c_{32} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu] \quad (1.27)$$

in  $\mathbb{R}^N$ , where the  $c_j$  ( $j = 32, 33, 34$ ) depend on  $N, k, s_1, s_2$ .

As in p-Laplace case, we have a Liouville property for Hessian systems.

**Corollary G** Assume that

$$\frac{2k(s_2s_1 + k \max\{s_1, s_2\})}{s_1s_2 - k^2} \geq N. \quad (1.28)$$

Any nonnegative solution  $(u, v)$  of inequalities

$$\begin{aligned} F_k[-u] &\geq v^{s_1} && \text{in } \mathbb{R}^N, \\ F_k[-v] &\geq u^{s_2} && \text{in } \mathbb{R}^N, \end{aligned} \quad (1.29)$$

with  $-u$  and  $-v$  in  $\Phi^k(\mathbb{R}^N)$  is trivial.

## 2 Estimates on potentials

Throughout this article  $c_j$ ,  $j=1,2,\dots$ , denote structural positive constants and  $c_N$  is the volume of the unit ball in  $\mathbb{R}^N$ . The following inequality will be used several times in the sequel.

**Lemma 2.1** Let  $\kappa, \gamma, \theta \in \mathbb{R}$ , such that  $\kappa, \gamma > 0$ . Let  $h : (0, \infty) \rightarrow (0, \infty)$  be nondecreasing. Then,

$$\int_0^R t^\kappa \left( \int_t^R h(r)r^\theta \frac{dr}{r} \right)^\gamma \frac{dt}{t} \leq c_{35} \int_0^{2R} t^{\kappa+\theta\gamma} h^\gamma(t) \frac{dt}{t} \quad \forall R \in (0, \infty], \quad (2.1)$$

for some  $c_{35} > 0$  depending on  $\kappa, \gamma, \theta$ .

**Proof.** Case 1:  $\gamma \leq 1$ . Since there holds

$$\left( \sum_{j=0}^{\infty} a_j \right)^\gamma \leq \sum_{j=0}^{\infty} a_j^\gamma \quad \forall a_j \geq 0,$$

we derive

$$\begin{aligned} \left( \int_t^R h(r)r^\theta \frac{dr}{r} \right)^\gamma &\leq c_{\gamma,\theta} \left( \sum_{j=0}^{j_0} h(2^{\frac{j+1}{4}}t)(2^{\frac{j}{4}}t)^\theta \right)^\gamma \\ &\leq c_{\gamma,\theta} \sum_{j=0}^{j_0} \left( h^\gamma(2^{\frac{j+1}{4}}t) \right) (2^{\frac{j}{4}}t)^{\theta\gamma} \\ &\leq c_{\gamma,\theta} \int_t^{2R} h^\gamma(r)r^{\theta\gamma} \frac{dr}{r} \end{aligned}$$

where  $c_{\gamma,\theta} = 2^{\frac{\gamma}{4}} \max\{1, 2^{-\frac{\gamma\theta}{4}}\}$  and  $2^{\frac{j_0}{4}}t < R \leq 2^{\frac{j_0+1}{4}}t$  if  $R < \infty$  and  $j_0 = \infty$  if  $R = \infty$ . By Fubini's theorem,

$$\begin{aligned} \int_0^R t^\kappa \left( \int_t^R h(r)r^\theta \frac{dr}{r} \right)^\gamma \frac{dt}{t} &\leq c_{\gamma,\theta} \int_0^R t^\kappa \int_t^{2R} h^\gamma(r)r^{\theta\gamma} \frac{dr}{r} \frac{dt}{t} \\ &\leq \frac{c_{\gamma,\theta}}{\kappa} \int_0^{2R} t^{\kappa+\theta\gamma} h^\gamma(t) \frac{dt}{t}. \end{aligned}$$

which is (2.1).

*Case 2:*  $\gamma > 1$  Since

$$\left( \int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma \leq \left( \int_t^R r^{-\frac{\gamma}{\gamma-1}} \frac{dr}{r} \right)^{\gamma-1} \int_t^R h^\gamma(r) r^{\gamma(1+\theta)} \frac{dr}{r},$$

we derive by Fubini's theorem

$$\int_0^R t^\kappa \left( \int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma \frac{dt}{t} \leq c_{\gamma, \kappa} \int_0^{2R} t^{\kappa+\theta\gamma} h^\gamma(t) \frac{dt}{t},$$

which completes the proof.  $\square$

We recall that if  $\alpha > 0$ ,  $1 < \beta < \frac{N}{\alpha}$  and  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ , the set of positive Radon measures in  $\mathbb{R}^N$ , the Wolff potential of  $\mu$  is defined by

$$\mathbf{W}_{\alpha, \beta}[\mu](x) = \int_0^\infty \left( \frac{\mu(B_r(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r}, \quad (2.2)$$

and if  $R > 0$ , the  $R$ -truncated Wolff potential of  $\mu$  is

$$\mathbf{W}_{\alpha, \beta}^R[\mu](x) = \int_0^R \left( \frac{\mu(B_r(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r}. \quad (2.3)$$

If  $\mu$  is a Radon measure on a Borel set  $G$ , its Wolff potential (or truncated Wolff potential) is the potential of its extension by 0 in  $G^c$ . We start with the following composition estimate on Wolff potentials.

**Lemma 2.2** *Let  $1 < \beta < N/\alpha$ . Then for any  $q > 0$  and  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$  we have*

$$\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu] \leq c_{36} \mathbf{W}_{\alpha, \beta} [(\mathbf{W}_{\alpha, \beta}[\mu])^q], \quad (2.4)$$

in  $\mathbb{R}^N$  for some  $c_{36} > 0$  depending on  $\alpha, \beta, N, q$ . Moreover, if  $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$ , there holds

$$\mathbf{W}_{\alpha, \beta} [(\mathbf{W}_{\alpha, \beta}[\mu])^q](x) \leq c_{37} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu], \quad (2.5)$$

in  $\mathbb{R}^N$ , where  $c_{37} > 0$  depends on  $\alpha, \beta, N, q$ .

**Proof.** For any  $x \in \mathbb{R}^N$ , using the fact that  $y \in B_t(x) \implies B_t(x) \subset B_{2t}(y)$ , we have

$$\begin{aligned} \mathbf{W}_{\alpha, \beta} [(\mathbf{W}_{\alpha, \beta}[\mu])^q](x) &= \int_0^\infty \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left( \int_0^\infty \left( \frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\ &\geq c_{38} \int_0^\infty \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left( \frac{\mu(B_{2t}(y))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\ &\geq c_{36} \int_0^\infty \left( t^{\frac{\alpha\beta(\beta-1)}{q}} \frac{\mu(B_t(x))}{t^{N-\alpha\beta}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dt}{t} \\ &= c_{36} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu](x). \end{aligned}$$

where  $c_{38} = c_{38}(\alpha, \beta, N, q) > 0$ , which proves (2.4).

In order to prove (2.4) we recall the following estimate on Wolff potentials [7]

$$\|\mathbf{W}_{\alpha, \beta}[\omega]\|_{L^{\frac{(\beta-1)N}{N-\alpha\beta}, \infty}} \leq c_{39} (\omega(\mathbb{R}^N))^{\frac{1}{\beta-1}} \quad \forall \omega \in \mathfrak{M}_b^+(\mathbb{R}^N), \quad (2.6)$$



where  $L^{\frac{(p-1)N}{N-\alpha\beta}, \infty}$  denotes the weak- $L^{\frac{(p-1)N}{N-\alpha\beta}}$  space. In particular, since  $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$ ,

$$\int_{B_r(x)} (\mathbf{W}_{\alpha,\beta}[\omega])^q dy \leq c_{40} r^N \left( \frac{\omega(\mathbb{R}^N)}{r^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} \quad \forall x \in \mathbb{R}^N, \forall r > 0. \quad (2.7)$$

Applying this inequality to  $\omega = \chi_{B_{2r}(x)}\mu$  yields

$$\int_{B_r(x)} (\mathbf{W}_{\alpha,\beta}^r[\mu])^q dy \leq c_{40} r^N \left( \frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} \quad \forall x \in \mathbb{R}^N, \forall r > 0. \quad (2.8)$$

We claim that

$$\begin{aligned} I &:= \int_0^\infty \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left( \int_t^\infty \left( \frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\ &\leq c_{37} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu](x). \end{aligned} \quad (2.9)$$

Since  $B_r(y) \subset B_{2r}(x)$  for any  $y \in B_t(x), r \geq t$ , we have

$$\begin{aligned} \int_{B_t(x)} \left( \int_t^\infty \left( \frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy &\leq \int_{B_t(x)} \left( \int_t^\infty \left( \frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \\ &\leq c_N t^N \left( \int_t^\infty \left( \frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q. \end{aligned}$$

Hence,

$$I \leq c_N^{\frac{1}{\beta-1}} \int_0^\infty t^{\frac{\alpha\beta}{\beta-1}} \left( \int_t^\infty \left( \frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^{\frac{q}{\beta-1}} \frac{dt}{t}.$$

Using Lemma 2.1, we infer

$$I \leq c_{37} \int_0^\infty r^{\frac{\alpha\beta}{\beta-1}} \left( \frac{\mu(B_r(x))}{r^{N-\alpha\beta}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} = c_{37} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu](x),$$

which completes the proof.  $\square$

The following is a version of Lemma 2.2 for truncated Wolff potentials,

**Lemma 2.3** *Let  $1 < \beta < N/\alpha$  and  $q > 0$ . If  $\delta \in (0, 1)$  there holds for any  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$*

$$\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}^{\frac{\delta d}{2}}[\mu](x) \leq c_{42} \mathbf{W}_{\alpha,\beta}^{\delta d} \left[ \left( \mathbf{W}_{\alpha,\beta}^{\delta d(\cdot)}[\mu] \right)^q \right](x), \quad (2.10)$$

in  $\Omega$ . Moreover, if  $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$ , there holds for any  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ ,

$$\mathbf{W}_{\alpha,\beta}^R \left[ \left( \mathbf{W}_{\alpha,\beta}^R[\mu] \right)^q \right](x) \leq c_{43} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}^{4R}[\mu](x), \quad (2.11)$$

in  $\mathbb{R}^N$ .

**Proof.** For any  $x \in \Omega$ ,

$$\begin{aligned} &\mathbf{W}_{\alpha,\beta}^{\delta d(x)} \left[ \left( \mathbf{W}_{\alpha,\beta}^{\delta d(\cdot)}[\mu](\cdot) \right)^q \right](x) \\ &= \int_0^{\delta d(x)} \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left( \int_0^{\delta d(y)} \left( \frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t}. \end{aligned}$$

Since  $\delta d(y) \geq \frac{7\delta}{8}d(x)$  for all  $y \in B_{\frac{t}{8}}(x)$ ,  $0 < t < \delta d(x)$ ,

$$\begin{aligned}
\int_{B_t(x)} \left( \int_0^{\delta d(y)} \left( \frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy &\geq \int_{B_{t/8}(x)} \left( \int_0^{\frac{7\delta}{8}d(x)} \left( \frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \\
&\geq \int_{B_{t/8}(x)} \left( \int_0^{\frac{7t}{8}} \left( \frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \\
&\geq c_{44} \int_{B_{t/8}(x)} \left( \frac{\mu(B_{\frac{3t}{4}}(y))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} dy \\
&\geq c_{44} \int_{B_{t/8}(x)} \left( \frac{\mu(B_{\frac{3t}{4}-\frac{t}{8}}(x))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} dy \\
&\geq c_{45} t^N \left( \frac{\mu(B_{\frac{t}{2}}(x))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}}.
\end{aligned}$$

Hence

$$\mathbf{W}_{\alpha,\beta}^{\delta d(x)} \left[ \left( \mathbf{W}_{\alpha,\beta}^{\delta d(\cdot)} [\mu](\cdot) \right)^q \right] (x) \geq c_{46} \int_0^{\delta d(x)} \left( t^{\alpha\beta} \left( \frac{\mu(B_{\frac{t}{2}}(x))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} \right)^{\frac{1}{\beta-1}} \frac{dt}{t}$$

which implies (2.10).

Because of (2.8), it is enough to show that there holds

$$\int_0^R \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left( \int_t^R \left( \frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \leq c_{47} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}^{4R} [\mu](x), \quad (2.12)$$

in order to prove (2.11). Since  $B_\rho(y) \subset B_{2\rho}(x)$  for any  $y \in B_r(x)$ ,  $\rho \geq r$ , we have

$$\begin{aligned}
\int_{B_t(x)} \left( \int_t^R \left( \frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy &\leq \int_{B_t(x)} \left( \int_t^R \left( \frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \\
&\leq c_N t^N \left( \int_t^R \left( \frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^R \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left( \int_t^R \left( \frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\
\leq c_N \int_0^R \left( t^{\alpha\beta} \left( \int_t^R \left( \frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q \right)^{\frac{1}{\beta-1}} \frac{dt}{t}.
\end{aligned}$$

We infer (2.12) by Lemma 2.1, which completes the proof.  $\square$

The next two propositions link Wolff potentials of a measure with Riesz capacities (in the case of whole space) and truncated Wolff potentials with Bessel capacities (in the bounded domain case). Their proof can be found in [23, 24] (and [8] with a different method).

**Proposition 2.4** *Let  $1 < \beta < N/\alpha$ ,  $q > \beta - 1$ ,  $\nu \in \mathfrak{M}^+(\mathbb{R}^N)$ . Then, the following statements are equivalent:*

(a) The inequality

$$\nu(K) \leq c_{48} \text{Cap}_{\mathbf{I}_{\alpha\beta}, \frac{q}{q-\beta+1}}(K) \quad (2.13)$$

holds for any compact set  $K \subset \mathbb{R}^N$ , for some  $c_{48} > 0$ .

(b) The inequality

$$\int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,\beta}[\chi_{B_t(x)}\nu](y))^q dy \leq c_{49}\nu(B_t(x)) \quad (2.14)$$

holds for any ball  $B_t(x) \subset \mathbb{R}^N$ , for some  $c_{49} > 0$ .

(c) The inequality

$$\mathbf{W}_{\alpha,\beta} [(\mathbf{W}_{\alpha,\beta}[\nu])^q] \leq c_{50} \mathbf{W}_{\alpha,\beta}[\nu] < \infty \text{ a.e in } \mathbb{R}^N \quad (2.15)$$

holds for some  $c_{50} > 0$ .

**Proposition 2.5** Let  $1 < \beta < N/\alpha$ ,  $q > \beta - 1$ ,  $R > 0$  and  $\nu \in \mathfrak{M}_b^+(B_R(x_0))$  for some  $x_0 \in \mathbb{R}^N$ . Then, the following statements are equivalent:

(a) The inequality

$$\nu(K) \leq c_{51} \text{Cap}_{\mathbf{G}_{\alpha\beta}, \frac{q}{q-\beta+1}}(K) \quad (2.16)$$

holds for any compact set  $K \subset \mathbb{R}^N$ , for some  $c_{51} = c_{51}(R) > 0$ .

(b) The inequality

$$\int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,\beta}^{4R}[\chi_{B_t(x)}\nu](y))^q dy \leq c_{52}\nu(B_t(x)) \quad (2.17)$$

holds for any ball  $B_t(x) \subset \mathbb{R}^N$ , for some  $c_{52} = c_{52}(R) > 0$ .

(c) The inequality

$$\mathbf{W}_{\alpha,\beta}^{4R} [(\mathbf{W}_{\alpha,\beta}^{4R}[\nu])^q] \leq c_{53} \mathbf{W}_{\alpha,\beta}^{4R}[\nu] \text{ a.e in } B_{2R}(x_0) \quad (2.18)$$

holds for some  $c_{53} = c_{53}(R) > 0$ .

In the following statement we show how to give capacity estimates on combination of measures.

**Proposition 2.6** Let  $\eta, \mu$  be in  $\mathfrak{M}^+(\mathbb{R}^N)$ . Assume that  $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$  and  $qs > (\beta-1)^2$ .

(i) If there holds

$$\eta(K) + \int_K (\mathbf{W}_{\alpha,\beta}[\mu])^s dx \leq \text{Cap}_{\mathbf{I}_{\alpha\beta(q+\beta-1)}, \frac{qs}{qs-(\beta-1)^2}}(K) \quad (2.19)$$

for any compact set  $K \subset \mathbb{R}^N$ , then

$$\mathbf{W}_{\alpha,\beta} [(\mathbf{W}_{\alpha,\beta} [(\mathbf{W}_{\alpha,\beta}[\omega])^q])^s] \leq c_{54} \mathbf{W}_{\alpha,\beta}[\omega] < \infty \text{ a.e in } \mathbb{R}^N, \quad (2.20)$$

where  $\omega = (\mathbf{W}_{\alpha,\beta}[\mu])^s + \eta$ .

(ii) If there holds

$$\eta(K) + \int_K (\mathbf{W}_{\alpha,\beta}^{2R}[\mu])^s dx \leq \text{Cap}_{\mathbf{G}_{\alpha\beta(q+\beta-1)}, \frac{qs}{qs-(\beta-1)^2}}(K) \quad (2.21)$$

for any compact set  $K \subset \mathbb{R}^N$ , then

$$\mathbf{W}_{\alpha,\beta}^{2R} \left[ \left( \mathbf{W}_{\alpha,\beta}^{2R} \left[ (\mathbf{W}_{\alpha,\beta}^{2R}[\omega])^q \right] \right)^s \right] \leq c_{55} \mathbf{W}_{\alpha,\beta}^{2R}[\omega] < \infty \text{ a.e in } B_R(x_0), \quad (2.22)$$

where  $\omega = \chi_{B_R(x_0)} \left( \mathbf{W}_{\alpha,\beta}^{2R}[\mu] \right)^s + \chi_{B_R(x_0)} \eta$ .

**Proof.** *Statement (i):* We assume that (2.19) holds. Put  $\omega = (\mathbf{W}_{\alpha,\beta}[\mu])^s + \eta$  and apply (2.19) to  $K = \overline{B}_{2\rho}(x)$ . Since by homogeneity

$$\text{Cap}_{\mathbf{I}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(\overline{B}_{2\rho}(x)) = \rho^{N - \frac{\alpha\beta(q+\beta-1)s}{qs-(\beta-1)^2}} \text{Cap}_{\mathbf{I}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(\overline{B}_2(0)),$$

we derive from (2.19)

$$\omega(B_\rho(x)) \leq c_{55} \rho^{N - \frac{\alpha\beta(q+\beta-1)s}{qs-(\beta-1)^2}} \quad \forall \rho > 0,$$

which is equivalent to

$$\rho^{\frac{\alpha\beta}{\beta-1}} \left( \frac{\omega(B_\rho(x))}{\rho^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{qs}{(\beta-1)^3}} \leq c_{56} \left( \frac{\omega(B_\rho(x))}{\rho^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \quad \forall \rho > 0. \quad (2.23)$$

We apply Proposition 2.4 to  $\nu = \omega$  with  $(\alpha, \beta, q) = \left( \frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1, s \right)$ , (2.19) implies

$$\int_{\mathbb{R}^N} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1} [\chi_{B_t(x)} \omega] \right)^s dy \leq c_{57} \omega(B_t(x)). \quad (2.24)$$

By Lemma 2.2, (2.20) is equivalent to

$$\mathbf{W}_{\alpha,\beta} \left[ \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1} [\omega] \right)^s \right] \leq c_{58} \mathbf{W}_{\alpha,\beta}[\omega] < \infty \text{ a.e. } \mathbb{R}^N \quad (2.25)$$

Therefore, it is enough to show that (2.23) and (2.24) imply (2.25). In fact, since for  $t > 0$

$$\int_{B_t(x)} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1} [\omega](y) \right)^s dy = \int_{B_t(x)} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1} [\chi_{B_{2t}(x)} \omega](y) \right)^s dy,$$

we apply (2.24) and obtain

$$\int_{B_t(x)} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1} [\omega](y) \right)^s dy \leq c_{57} \omega(B_{2t}(x)).$$

So, it is enough to show that

$$I := \int_0^\infty \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left( \int_t^\infty \left( \frac{\omega(B_r(y))}{r^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} \right)^s dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \leq c_{58} \mathbf{W}_{\alpha,\beta}[\omega](x). \quad (2.26)$$

Since  $B_r(y) \subset B_{2r}(x)$  for any  $y \in B_t(x), r \geq t$ , we have

$$\begin{aligned} I &\leq c_N \int_0^\infty \left( t^{\alpha\beta} \left( \int_t^\infty \left( \frac{\omega(B_{2r}(x))}{r^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} \right)^s \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\ &= c_N \int_0^\infty t^{\frac{\alpha\beta}{\beta-1}} \left( \int_t^\infty \left( \frac{\omega(B_{2r}(x))}{r^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} \right)^{\frac{s}{\beta-1}} \frac{dt}{t} \end{aligned}$$

It follows from Lemma 2.1 and (2.23) that

$$I \leq c_{59} \int_0^\infty t^{\frac{\alpha\beta}{\beta-1}} \left( \frac{\omega(B_{2t}(x))}{t^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{qs}{(\beta-1)^3}} \frac{dt}{t} \leq c_{56} c_{59} \int_0^\infty \left( \frac{\omega(B_{2t}(x))}{t^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dt}{t},$$

which is (2.26).

*Statement (ii):* We assume that (2.21) holds. Put  $d\omega = \chi_\Omega (\mathbf{W}_{\alpha,\beta}[\mu])^s + \chi_\Omega \eta$ , then

$$\omega(B_\rho(x)) \leq c_{60} \rho^{N - \frac{\alpha\beta(q+\beta-1)s}{qs - (\beta-1)^2}} \quad \forall 0 < \rho < 2R.$$

As in the proof of statement (i), the above inequality is equivalent to

$$\rho^{\frac{\alpha\beta}{\beta-1}} \left( \frac{\omega(B_\rho(x))}{\rho^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{qs}{(\beta-1)^3}} \leq c_{61} \left( \frac{\omega(B_\rho(x))}{\rho^{N - \alpha\beta}} \right)^{\frac{1}{\beta-1}} \quad \forall 0 < \rho < 2R. \quad (2.27)$$

Applying Proposition 2.5 with  $\nu = \omega$  and  $(\alpha, \beta, q) = \left( \frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1, s \right)$ ,

$$\int_{\mathbb{R}^N} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{4R} [\chi_{B_t(x)} \omega] \right)^s dy \leq c_{62} \omega(B_t(x)). \quad (2.28)$$

By Lemma 2.3, (2.22) is equivalent to

$$\mathbf{W}_{\alpha,\beta}^{4R} \left[ \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{4R} [\omega] \right)^s \right] \leq c_{63} \mathbf{W}_{\alpha,\beta}^{4R} [\omega] \quad \text{a.e in } B_R(x_0). \quad (2.29)$$

Therefore, it is enough to show that (2.27) and (2.28) imply (2.29). In fact, since

$$\int_{B_t(x)} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^t [\omega](y) \right)^s dy = \int_{B_t(x)} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{2t} [\chi_{B_{2t}(x)} \omega](y) \right)^s dy$$

for all  $0 < t < 4R$  thus applying (2.28), we obtain

$$\int_{B_t(x)} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^t [\omega](y) \right)^s dy \leq c_{64} \omega(B_{2t}(x)).$$

So, it is enough to show that for any  $x \in B_R(x_0)$

$$II := \int_0^{4R} \left( \frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left( \int_t^{4R} \left( \frac{\omega(B_r(y))}{r^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} \right)^s dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \leq c_{65} \mathbf{W}_{\alpha,\beta}^{4R} [\omega](x). \quad (2.30)$$

Since  $B_r(y) \subset B_{2r}(x)$  for any  $y \in B_t(x), r \geq t$ , we have

$$II \leq c_N \int_0^{4R} t^{\frac{\alpha\beta}{\beta-1}} \left( \int_t^{4R} \left( \frac{\omega(B_{2r}(x))}{r^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{q}{(\beta-1)^2}} \frac{dr}{r} \right)^{\frac{s}{\beta-1}} \frac{dt}{t}.$$

Combining this with Lemma 2.1 and (2.27) yields

$$II \leq c_{66} \mathbf{W}_{\alpha,\beta}^{16R} [\omega](x).$$

Therefore, (2.29) follows since  $\mathbf{W}_{\alpha,\beta}^{16R} [\omega] \leq c_{67} \mathbf{W}_{\alpha,\beta}^{4R} [\omega]$  in  $B_R(x_0)$ .  $\square$

**Proposition 2.7** *Let  $\eta, \mu$  be in  $\mathfrak{M}^+(\mathbb{R}^N)$ . Assume that  $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$  and  $qs > (\beta-1)^2$ . Let  $(u_m, v_m)$  be nonnegative measurable functions in  $\mathbb{R}^N$  such that for all  $m \geq 0$*

$$u_{m+1} \leq c^* \mathbf{W}_{\alpha,\beta} [v_m^q + \mu], \quad v_{m+1} \leq c^* \mathbf{W}_{\alpha,\beta} [u_m^s + \eta] \quad \text{a.e. in } \mathbb{R}^N$$

for some  $c^* > 0$  and  $(u_0, v_0) = 0$ . Then, there exists a constant  $M^* > 0$  depending only on  $N, \alpha, \beta, q, s, c^*$  such that if  $d\omega = (\mathbf{W}_{\alpha, \beta}[\mu])^s dx + d\eta$  satisfies

$$\omega(K) \leq M^* \text{Cap}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}(K), \quad (2.31)$$

for any compact set  $K \subset \mathbb{R}^N$ , then

$$v_m \leq c_{69} \mathbf{W}_{\alpha, \beta}[\omega], \quad u_m \leq c_{70} \mathbf{W}_{\alpha, \beta}[(\mathbf{W}_{\alpha, \beta}[\omega])^q] + c_{68} \mathbf{W}_{\alpha, \beta}[\mu] \quad \forall m \geq 0 \quad (2.32)$$

for some constants  $c_{68}, c_{69}, c_{70}$  depending only on  $N, \alpha, \beta, q, s$  and  $c^*$ .

**Proof.** By Proposition 2.6, (2.31) implies

$$\mathbf{W}_{\alpha, \beta} [(\mathbf{W}_{\alpha, \beta} [(\mathbf{W}_{\alpha, \beta}[\omega])^q])^s] \leq c_{71} M^{\frac{qs}{(\beta-1)^3}} \mathbf{W}_{\alpha, \beta}[\omega] < \infty \quad \text{a.e in } \mathbb{R}^N. \quad (2.33)$$

We set

$$\begin{aligned} c_{68} &= c^* 2^{\frac{1}{\beta-1}}, \\ c_{69} &= c^* 2^{1+\frac{1}{\beta-1}} (c_{68}^s 2^{s-1} + 1)^{\frac{1}{\beta-1}}, \\ c_{70} &= c^* 2^{\frac{1}{\beta-1}} c_{69}^{\frac{q}{\beta-1}}, \end{aligned}$$

and choose  $M^* > 0$  such that

$$c^* 2^{\frac{1}{\beta-1}} (c_{70}^s 2^{s-1})^{\frac{1}{\beta-1}} c_{71} M^{\frac{qs}{(\beta-1)^3}} = \frac{c_{69}}{2}.$$

We claim that

$$v_m \leq c_{69} \mathbf{W}_{\alpha, \beta}[\omega], \quad u_m \leq c_{70} \mathbf{W}_{\alpha, \beta}[(\mathbf{W}_{\alpha, \beta}[\omega])^q] + c_{68} \mathbf{W}_{\alpha, \beta}[\mu] \quad \forall m \geq 0. \quad (2.34)$$

Clearly, by definition of  $c_{68}, c_{69}, c_{70}$ , we have (2.34) for  $m = 0, 1$ . Next we assume that (2.34) holds for all  $m \leq l$  for some  $l \in \mathbb{N}_+^*$ , then

$$\begin{aligned} u_{l+1} &\leq c^* \mathbf{W}_{\alpha, \beta}[v_l^q + \mu] \\ &\leq c^* 2^{\frac{1}{\beta-1}} c_{69}^{\frac{q}{\beta-1}} \mathbf{W}_{\alpha, \beta}[(\mathbf{W}_{\alpha, \beta}[\omega])^q] + c^* 2^{\frac{1}{\beta-1}} \mathbf{W}_{\alpha, \beta}[\mu] \\ &= c_{70} \mathbf{W}_{\alpha, \beta}[(\mathbf{W}_{\alpha, \beta}[\omega])^q] + c_{68} \mathbf{W}_{\alpha, \beta}[\mu] \end{aligned}$$

and

$$\begin{aligned} v_{l+1} &\leq c^* \mathbf{W}_{\alpha, \beta}[(c_{70} \mathbf{W}_{\alpha, \beta}[(\mathbf{W}_{\alpha, \beta}[\omega])^q] + c_{68} \mathbf{W}_{\alpha, \beta}[\mu])^s + \eta] \\ &\leq c^* \mathbf{W}_{\alpha, \beta}[c_{70}^s 2^{s-1} (\mathbf{W}_{\alpha, \beta}[(\mathbf{W}_{\alpha, \beta}[\omega])^q])^s + c_{68}^s 2^{s-1} (\mathbf{W}_{\alpha, \beta}[\mu])^s + \eta] \\ &\leq c^* 2^{\frac{1}{\beta-1}} (c_{70}^s 2^{s-1})^{\frac{1}{\beta-1}} \mathbf{W}_{\alpha, \beta}[(\mathbf{W}_{\alpha, \beta}[(\mathbf{W}_{\alpha, \beta}[\omega])^q])^s] \\ &\quad + c^* 2^{\frac{1}{\beta-1}} (c_{68}^s 2^{s-1} + 1)^{\frac{1}{\beta-1}} \mathbf{W}_{\alpha, \beta}[(\mathbf{W}_{\alpha, \beta}[\mu])^s + \eta] \\ &\leq c^* 2^{\frac{1}{\beta-1}} (c_{70}^s 2^{s-1})^{\frac{1}{\beta-1}} c_{71} M^{\frac{qs}{(\beta-1)^3}} \mathbf{W}_{\alpha, \beta}[\omega] + c^* 2^{\frac{1}{\beta-1}} (c_{68}^s 2^{s-1} + 1)^{\frac{1}{\beta-1}} \mathbf{W}_{\alpha, \beta}[\omega] \\ &= \frac{c_{69}}{2} \mathbf{W}_{\alpha, \beta}[\omega] + \frac{c_{69}}{2} \mathbf{W}_{\alpha, \beta}[\omega] \\ &= c_{69} \mathbf{W}_{\alpha, \beta}[\omega]. \end{aligned}$$

Thus, (2.34) is true for  $m = l + 1$ . Hence, (2.34) is true for all  $l \geq 0$ .  $\square$

The next result is an adaptation of Proposition 2.7 to truncated Wolff potentials.

**Proposition 2.8** Let  $\eta, \mu$  be in  $\mathfrak{M}_b^+(B_R(x_0))$ . Assume that  $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$  and  $qs > (\beta-1)^2$ . Let  $(u_m, v_m)$  be nonnegative measurable functions in  $\mathbb{R}^N$  such that for all  $m \geq 0$

$$u_{m+1} \leq c_* \mathbf{W}_{\alpha,\beta}^R[\chi_{B_R(x_0)} v_m^q + \mu], \quad v_{m+1} \leq c_* \mathbf{W}_{\alpha,\beta}^R[\chi_{B_R(x_0)} u_m^s + \eta] \quad \text{a.e. in } B_R(x_0)$$

and  $(u_0, v_0) = 0$ . If we set  $d\omega = \left(\mathbf{W}_{\alpha,\beta}^{2R}[\mu]\right)^s dx + d\eta$ , there exists a constant  $M_* > 0$  depending only on  $N, \alpha, \beta, q, s, R$  and  $c_*$  such that if

$$\omega(K) \leq M_* \text{Cap}_{\mathbf{G}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(K) \quad (2.35)$$

for any compact set  $K \subset \mathbb{R}^N$ , then

$$v_m \leq c_{73} \mathbf{W}_{\alpha,\beta}^{2R}[\omega], \quad u_m \leq c_{74} \mathbf{W}_{\alpha,\beta}^{2R}[(\mathbf{W}_{\alpha,\beta}^{2R}[\omega])^q] + c_{72} \mathbf{W}_{\alpha,\beta}^{2R}[\mu] \quad \forall k \geq 0 \quad (2.36)$$

in  $B_R(x_0)$  for some constants  $c_{72}, c_{73}, c_{74}$  depending only on  $N, \alpha, \beta, q, s, R$  and  $c_*$ .

**Proof.** The proof is similar to the one of Proposition 2.7 and we omit the details.  $\square$

**Proposition 2.9** Let  $1 < \beta < N/\alpha$  and  $q, s > 0$  such that  $qs > (\beta-1)^2$ .

(i) Assume that  $\eta$  and  $\mu$  belong to  $\mathfrak{M}_b^+(\mathbb{R}^N)$  and  $(u, v)$  are nonnegative measurable functions satisfying

$$\begin{cases} \mathbf{W}_{\alpha,\beta}[v^q] + \mathbf{W}_{\alpha,\beta}[\mu] \leq c_{75} u, \\ \mathbf{W}_{\alpha,\beta}[u^s] + \mathbf{W}_{\alpha,\beta}[\eta] \leq c_{75} v \end{cases} \quad \text{a.e. in } \mathbb{R}^N, \quad (2.37)$$

for some  $c_{75} > 0$ . Then there exists a constant  $c_{76} > 0$  depending only on  $N, \alpha, \beta, q, s$  and  $c_{75}$  such that

$$\eta(K) + \int_K (\mathbf{W}_{\alpha,\beta}[\mu](x))^s dx \leq c_{76} \text{Cap}_{\mathbf{I}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(K), \quad (2.38)$$

for any compact set  $K \subset \mathbb{R}^N$ .

(ii) Assume that  $\eta$  and  $\mu$  belong to  $\mathfrak{M}_b^+(\Omega)$  and  $(u, v)$  are nonnegative functions satisfying

$$\begin{cases} \mathbf{W}_{\alpha,\beta}^{\delta d(\cdot)}[v^q] + \mathbf{W}_{\alpha,\beta}^{\delta d}[\mu] \leq c_{77} u, \\ \mathbf{W}_{\alpha,\beta}^{\delta d(\cdot)}[u^s] + \mathbf{W}_{\alpha,\beta}^{\delta d}[\eta] \leq c_{77} v \end{cases} \quad \text{a.e. in } \Omega, \quad (2.39)$$

for some  $c_{77} > 0$ . Then for any  $\Omega' \subset\subset \Omega$ , there exists a constant  $c_{78} > 0$  depending only on  $n, \alpha, \beta, q, s, c_{77}$  and  $\text{dist}(\Omega', \partial\Omega)$  such that

$$\eta(K) + \int_K \left(\mathbf{W}_{\alpha,\beta}^{\delta d(x)}[\mu](x)\right)^s dx \leq c_{78} \text{Cap}_{\mathbf{G}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(K), \quad (2.40)$$

for any compact set  $K \subset \Omega'$ .

**Proof.** (i): Set  $\omega = u^s + \eta$ , then

$$\omega \geq u^s \geq (\mathbf{W}_{\alpha,\beta}[v^q])^s \geq c_{79} (\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q])^s.$$

By (2.4) in Lemma 2.2, we get

$$\omega \geq c_{80} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}[\omega] \right)^s$$

which implies

$$\int_{\mathbb{R}^N} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1} [\chi_{B_t(x)} \omega] \right)^s dy \leq c_{81} \omega(B_t(x)) \quad \forall x \in \mathbb{R}^N, \forall t > 0.$$

Applying Proposition 2.4 to  $\mu = \omega$  with  $(\alpha, \beta, q) = \left( \frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1, s \right)$ , we get (2.38).

(ii) We define  $\omega$  as above and we have

$$\omega \geq u^s \geq (\mathbf{W}_{\alpha, \beta}^{\delta d} [v^q])^s \geq c_{82} \left( \mathbf{W}_{\alpha, \beta}^{\delta d} \left[ \left( \mathbf{W}_{\alpha, \beta}^{\delta d(\cdot)} [\omega] \right)^q \right] \right)^s \quad \text{a.e. in } \Omega,$$

which leads to

$$\omega \geq c_{83} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{2} d} [\omega] \right)^s \quad \text{a.e. in } \Omega,$$

by inequality (2.10) in Lemma 2.3. Let  $M_\omega$  denote the centered Hardy-Littlewood maximal function which is defined for any  $f \in L^1_{loc}(\mathbb{R}^N, d\omega)$  by

$$M_\omega f(x) = \sup_{t>0} \frac{1}{\omega(B_t(x))} \int_{B_t(x)} |f| d\omega.$$

Let  $K \subset \Omega$  be compact. Set  $r_K = \text{dist}(K, \partial\Omega)$  and  $\Omega_K = \{x \in \Omega : d(x, K) < r_K/2\}$ . Then, for any Borel set  $E \subset K$ ,

$$c_{84} \int_{\Omega} (M_\omega \chi_E)^{\frac{sq}{(\beta-1)^2}} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{2} d(x)} [\omega] \right)^s dx \leq \int_{\Omega} (M_\omega \chi_E)^{\frac{sq}{(\beta-1)^2}} d\omega.$$

Since  $M_\omega$  is a bounded linear map on  $L^p(\mathbb{R}^N, d\omega)$  for any  $p > 1$  and

$$(M_\omega \chi_E)^{\frac{sq}{(\beta-1)^2}} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{2} d(x)} [\omega] \right)^s \geq \int_0^{\frac{\delta}{2} d(x)} \left( \frac{\omega(B_t(x) \cap E)}{\omega(B_t(x))} \frac{\omega(B_t(x))}{t^{N - \frac{\alpha\beta(q+\beta-1)}{q}}} \right)^{\frac{sq}{(\beta-1)^2}} \frac{dt}{t}$$

we derive

$$\int_{\Omega} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{2} d(x)} [\omega_E] \right)^s dx \leq c_{85} \omega(E),$$

where  $\omega_E = \chi_E \omega$ . Note that if  $x \in \Omega$  and  $d(x) \leq r_K/8$ , then  $B_t(x) \subset \Omega \setminus \Omega_K$  for all  $t \in (0, \frac{\delta d(x)}{2})$ ; indeed, for all  $y \in B_t(x)$

$$d(y, \partial\Omega) \leq d(x, \partial\Omega) + |x - y| < (1 + \delta)d(x, \partial\Omega) < \frac{1}{4}r_K,$$

thus

$$d(y, K) \geq d(K, \partial\Omega) - d(y, \partial\Omega) > \frac{3}{4}r_K > \frac{1}{2}r_K,$$

which implies  $y \notin \Omega_K$ . We deduce that

$$\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{2} d(x, \partial\Omega)} [\omega_E](x) \geq \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{16} r_K} [\omega_E](x) \quad \forall x \in \Omega,$$

and

$$\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{16} r_K} [\omega_E](x) = 0 \quad \forall x \in \Omega^c.$$

Hence we obtain

$$\int_{\mathbb{R}^N} \left( \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q} + 1}^{\frac{\delta}{16} r_K} [\omega_E] \right)^s dx \leq c_{85} \omega(E) \quad \forall E \subset K, E \text{ Borel.} \quad (2.41)$$

Applying Proposition 2.5 with  $\mu = \chi_{K \cap B_{2^{-6}\delta} r_K}(x) \omega$  we get (2.40), which completes the proof.  $\square$



### 3 Quasilinear Dirichlet problems

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ . If  $\mu \in \mathfrak{M}_b(\Omega)$ , we denote by  $\mu^+$  and  $\mu^-$  respectively its positive and negative parts in the Jordan decomposition. We denote by  $\mathfrak{M}_0(\Omega)$  the space of measures in  $\Omega$  which are absolutely continuous with respect to the  $c_{1,p}^\Omega$ -capacity defined on a compact set  $K \subset \Omega$  by

$$c_{1,p}^\Omega(K) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p dx : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega) \right\}.$$

We also denote  $\mathfrak{M}_s(\Omega)$  the space of measures in  $\Omega$  with support on a set of zero  $c_{1,p}^\Omega$ -capacity. Classically, any  $\mu \in \mathfrak{M}_b(\Omega)$  can be written in a unique way under the form  $\mu = \mu_0 + \mu_s$  where  $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$  and  $\mu_s \in \mathfrak{M}_s(\Omega) \cap \mathfrak{M}_b(\Omega)$ . It is well known that any  $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$  can be written under the form  $\mu_0 = f - \operatorname{div} g$  where  $f \in L^1(\Omega)$  and  $g \in L^{p'}(\Omega, \mathbb{R}^N)$ .

For  $k > 0$  and  $s \in \mathbb{R}$  we set  $T_k(s) = \max\{\min\{s, k\}, -k\}$ . If  $u$  is a measurable function defined in  $\Omega$ , finite a.e. and such that  $T_k(u) \in W_{loc}^{1,p}(\Omega)$  for any  $k > 0$ , there exists a measurable function  $v : \Omega \rightarrow \mathbb{R}^N$  such that  $\nabla T_k(u) = \chi_{\{|u| \leq k\}} v$  a.e. in  $\Omega$  and for all  $k > 0$ . We define the gradient a.e.  $\nabla u$  of  $u$  by  $v = \nabla u$ . We recall the definition of a renormalized solution given in [12].

**Definition 3.1** *Let  $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$ . A measurable function  $u$  defined in  $\Omega$  and finite a.e. is called a renormalized solution of*

$$\begin{aligned} -\Delta_p u &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (3.1)$$

if  $T_k(u) \in W_0^{1,p}(\Omega)$  for any  $k > 0$ ,  $|\nabla u|^{p-1} \in L^r(\Omega)$  for any  $0 < r < \frac{N}{N-1}$ , and  $u$  has the property that for any  $k > 0$  there exist  $\lambda_k^+$  and  $\lambda_k^-$  belonging to  $\mathfrak{M}_b^+ \cap \mathfrak{M}_0(\Omega)$ , respectively concentrated on the sets  $u = k$  and  $u = -k$ , with the property that  $\mu_k^+ \rightharpoonup \mu_s^+$ ,  $\mu_k^- \rightharpoonup \mu_s^-$  in the narrow topology of measures and such that

$$\int_{\{|u| < k\}} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_{\{|u| < k\}} \varphi d\mu_0 + \int_{\Omega} \varphi d\lambda_k^+ - \int_{\Omega} \varphi d\lambda_k^-,$$

for every  $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ .

**Remark 3.2** *We recall that if  $u$  is a renormalized solution to problem (3.1), then  $\frac{|\nabla u|^p}{(|u|+1)^r} \in L^1(\Omega)$  for all  $r > 1$ . Furthermore,  $u \geq 0$  a.e. in  $\Omega$  if  $\mu \in \mathfrak{M}_b^+(\Omega)$ .*

The following general stability result has been proved in [12, Th 4.1].

**Theorem 3.3** *Let  $\mu = \mu_0 + \mu_s^+ - \mu_s^-$ , with  $\mu_0 = F - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$  and  $\mu_s^+, \mu_s^-$  belonging to  $\mathfrak{M}_s^+(\Omega)$ . Let  $\mu_n = F_n - \operatorname{div} g_n + \rho_n - \eta_n$  with  $F_n \in L^1(\Omega)$ ,  $g_n \in (L^{p'}(\Omega))^N$  and  $\rho_n, \eta_n$  belonging to  $\mathfrak{M}_b^+(\Omega)$ . Assume that  $\{F_n\}$  converges to  $F$  weakly in  $L^1(\Omega)$ ,  $\{g_n\}$  converges to  $g$  strongly in  $(L^{p'}(\Omega))^N$  and  $(\operatorname{div} g_n)$  is bounded in  $\mathfrak{M}_b(\Omega)$ ; assume also that  $\{\rho_n\}$  converges to  $\mu_s^+$  and  $\{\eta_n\}$  to  $\mu_s^-$  in the narrow topology. If  $\{u_n\}$  is a sequence of renormalized solutions of (3.1) with data  $\mu_n$ , then, up to a subsequence, it converges a.e. in  $\Omega$  to a renormalized solution  $u$  of problem (3.1). Furthermore,  $T_k(u_n)$  converges to  $T_k(u)$  in  $W_0^{1,p}(\Omega)$  for any  $k > 0$ .*

We also recall the following estimate [23, Th 2.1].

**Proposition 3.4** *Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$ . Then there exists a constant  $C > 0$ , depending on  $p$  and  $N$  such that if  $\mu \in \mathfrak{M}_b^+(\Omega)$  and  $u$  is a nonnegative renormalized solution of problem (3.1) with data  $\mu$ , there holds*

$$\frac{1}{c_{s6}} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\mu](x) \leq u(x) \leq c_{s6} \mathbf{W}_{1,p}^{2 \operatorname{diam}(\Omega)}[\mu](x) \quad \text{a.e. in } \Omega. \quad (3.2)$$

**Proof of Theorem C.** *The condition is necessary.* Assume that (1.14) admits a nonnegative renormalized solutions  $(u, v)$ . By Proposition 3.4 there holds

$$\begin{aligned} u(x) &\geq c_{87} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}} [v^{q_1} + \mu](x), \\ v(x) &\geq c_{87} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}} [u^{q_2} + \mu](x) \quad \text{a.e. in } \Omega. \end{aligned}$$

Hence, we infer (1.15) from Proposition 2.9-(ii).

*Sufficient conditions.* Let  $\{(u_m, v_m)\}_{m \in \mathbb{N}}$  be a sequence of nonnegative renormalized solutions of the following problems for  $m \in \mathbb{N}$ ,

$$\begin{aligned} -\Delta_p u_{m+1} &= v_m^{q_1} + \mu && \text{in } \Omega, \\ -\Delta_p v_{m+1} &= u_m^{q_2} + \eta && \text{in } \Omega, \\ u_{m+1} &= v_{m+1} = 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.3}$$

with initial condition  $(u_0, v_0) = 0$ . The sequences  $\{u_m\}$  and  $\{v_m\}$  can be constructed in such a way that they are nondecreasing (see e.g. [24]). By Proposition 3.4 we have

$$\begin{aligned} u_{m+1} &\leq c_{86} \mathbf{W}_{1,p}^R [v_m^{q_1} + \mu](x) \\ v_{m+1} &\leq c_{86} \mathbf{W}_{1,p}^R [u_m^{q_2} + \eta](x) \quad \text{a.e. in } \Omega, \end{aligned}$$

where  $R = 2 \operatorname{diam}(\Omega)$ . Thus, by Proposition 2.8 there exists a constant  $M_* > 0$  depending only on  $N, p, q_1, q_2, R$  such that if

$$\omega(K) \leq M_* \operatorname{Cap}_{\mathbf{G}_{\frac{p(q_1+p-1)}{q_1}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(K) \tag{3.4}$$

for any compact set  $K \subset \mathbb{R}^N$  with  $d\omega = (\mathbf{W}_{1,p}^R[\mu])^{q_2} dx + d\eta$  then

$$v_m \leq c_{73} \mathbf{W}_{1,p}^R[\omega], \quad u_m \leq c_{74} \mathbf{W}_{1,p}^R[(\mathbf{W}_{1,p}^R[\omega])^{q_1}] + c_{72} \mathbf{W}_{1,p}^R[\mu] \quad \forall k \geq 0 \tag{3.5}$$

in  $\Omega$  and

$$\mathbf{W}_{1,p}^R[\omega] \in L^{q_2}(\Omega), \quad \mathbf{W}_{1,p}^R[(\mathbf{W}_{1,p}^R[\omega])^{q_1}] + \mathbf{W}_{1,p}^R[\mu] \in L^{q_1}(\Omega). \tag{3.6}$$

This implies that  $\{u_m\}, \{v_m\}_{m \in \mathbb{N}}$  are well defined and nondecreasing. Thus  $\{(u_m, v_m)\}$  converges a.e in  $\Omega$  to some functions  $(u, v)$  which satisfies (1.17) in  $\Omega$ . Furthermore, we deduce from (3.6) and the monotone convergence theorem that  $u_m^{q_1} \rightarrow u^{q_1}, v_m^{q_2} \rightarrow v^{q_2}$  in  $L^1(\Omega)$ . Finally, by Theorem 3.3 we infer that  $u$  is a renormalized solution of (1.14).  $\square$

## 4 p-superharmonic functions and quasilinear equations in $\mathbb{R}^N$

We recall some definitions and properties of  $p$ -superharmonic functions.

**Definition 4.1** *A function  $u$  is said to be  $p$ -harmonic in  $\mathbb{R}^N$  if  $u \in W_{loc}^{1,p}(\mathbb{R}^N)$  and  $-\Delta_p u = 0$  in  $\mathcal{D}'(\mathbb{R}^N)$ ; it is always  $C^1$ . A function  $u$  is called a  $p$ -supersolution in  $\mathbb{R}^N$  if  $u \in W_{loc}^{1,p}(\mathbb{R}^N)$  and  $-\Delta_p u \geq 0$  in  $\mathcal{D}'(\mathbb{R}^N)$ .*

**Definition 4.2** *A lower semicontinuous (l.s.c) function  $u : \mathbb{R}^N \rightarrow (-\infty, \infty]$  is called  $p$ -super-harmonic if  $u$  is not identically infinite and if, for all open  $D \subset \subset \mathbb{R}^N$  and all  $v \in C(\bar{D})$ ,  $p$ -harmonic in  $D$ ,  $v \leq u$  on  $\partial D$  implies  $v \leq u$  in  $D$ .*

Let  $u$  be a  $p$ -superharmonic in  $\mathbb{R}^N$ . It is well known that  $u \wedge k := \min\{u, k\} \in W_{loc}^{1,p}(\mathbb{R}^N)$  is a  $p$ -supersolution for all  $k > 0$  and  $u < \infty$  a.e in  $\mathbb{R}^N$ , thus,  $u$  has a gradient (see the previous section). We also have  $|\nabla u|^{p-1} \in L_{loc}^q(\mathbb{R}^N)$ ,  $\frac{|\nabla u|^p}{(|u|+1)^r} \in L_{loc}^1(\mathbb{R}^N)$  and  $u \in L_{loc}^s(\mathbb{R}^N)$  for  $1 \leq q < \frac{N}{N-1}$  and  $r > 1$ ,  $1 \leq s < \frac{N(p-1)}{N-p}$  (see [16, Theorem 7.46]). Thus for any  $0 \leq \varphi \in C_c^1(\Omega)$ , by the dominated convergence theorem,

$$\langle -\Delta_p u, \varphi \rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla(u \wedge k)|^{p-2} \nabla(u \wedge k) \cdot \nabla \varphi \geq 0.$$

Hence, by the Riesz Representation Theorem, there is a nonnegative Radon measure denoted by  $\mu[u]$ , called Riesz measure, such that  $-\Delta_p u = \mu[u]$  in  $\mathcal{D}'(\mathbb{R}^N)$ .

The following weak convergence result for Riesz measures proved in [30] will be used to obtain the existence of  $p$ -superharmonic solutions to quasilinear equations.

**Proposition 4.3** *Suppose that  $\{u_n\}$  is a sequence of nonnegative  $p$ -superharmonic functions in  $\mathbb{R}^N$  that converges a.e to a  $p$ -superharmonic function  $u$ . Then the sequence of measures  $\{\mu[u_n]\}$  converges to  $\mu[u]$  in the weak sense of measures.*

The proof of the next result can be found in in [23]

**Proposition 4.4** *Let  $\mu$  be a measure in  $\mathfrak{M}^+(\mathbb{R}^N)$ . Suppose that  $\mathbf{W}_{1,p}[\mu] < \infty$  a.e. Then there exists a nonnegative  $p$ -superharmonic function  $u$  in  $\mathbb{R}^N$  such that  $-\Delta_p u = \mu$  in  $\mathcal{D}'(\mathbb{R}^N)$ ,  $\inf_{\mathbb{R}^N} u = 0$  and*

$$\frac{1}{c_{86}} \mathbf{W}_{1,p}[\mu](x) \leq u(x) \leq c_{86} \mathbf{W}_{1,p}[\mu](x), \quad (4.1)$$

for almost all  $x$  in  $\mathbb{R}^N$ , where the constant  $c_{86}$  is the one of Proposition 3.4. Furthermore any  $p$ -superharmonic function  $u$  in  $\mathbb{R}^N$ , such that  $\inf_{\mathbb{R}^N} u = 0$  satisfies (4.1) with  $\mu = -\Delta_p u$ .

**Proof of Theorem A.** *The condition is necessary.* Assume that (1.14) admits a nonnegative  $p$ -superharmonic functions  $(u, v)$ . By Proposition 4.4 there holds

$$\begin{aligned} u(x) &\geq c_{87} \mathbf{W}_{1,p}[v^{q_1} + \mu](x), \\ v(x) &\geq c_{87} \mathbf{W}_{1,p}[u^{q_2} + \eta](x) \quad \text{for almost all } x \in \Omega. \end{aligned}$$

Hence, we derive (1.10) from Proposition 2.9-(i).

*The condition is sufficient.* Let  $\{(u_m, v_m)\}_{m \in \mathbb{N}}$  be a sequence of nonnegative  $p$ -superharmonic solutions of the following problems for  $m \in \mathbb{N}$ ,

$$\begin{aligned} -\Delta_p u_{m+1} &= v_m^{q_1} + \mu && \text{in } \mathbb{R}^N \\ -\Delta_p v_{m+1} &= u_m^{q_2} + \eta && \text{in } \mathbb{R}^N \\ \inf_{\mathbb{R}^N} u_{m+1} &= \inf_{\mathbb{R}^N} v_{m+1} = 0, \end{aligned} \quad (4.2)$$

with  $(u_0, v_0) = 0$ . As in the proof of Theorem C we can assume that  $\{u_m\}$  and  $\{v_m\}$  are nondecreasing. By Proposition 4.4 we have

$$\begin{aligned} u_{m+1} &\leq c_{86} \mathbf{W}_{1,p}[v_m^{q_1} + \mu](x) \\ v_{m+1} &\leq c_{86} \mathbf{W}_{1,p}[u_m^{q_2} + \eta](x) \quad \forall x \in \Omega \end{aligned}$$

Thus, by Proposition 2.7 there exists a constant  $c > 0$  depending only on  $N, p, q_1, q_2$  such that if

$$\omega(K) \leq M^* \text{Cap}_{\mathbf{I}_{\frac{p(q_1+p-1)}{q_1}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}}(K) \quad (4.3)$$

for any compact set  $K \subset \mathbb{R}^N$  with  $d\omega = (\mathbf{W}_{1,p}[\mu])^{q_2} dx + d\eta$  then

$$v_m \leq c_{69} \mathbf{W}_{1,p}[\omega], \quad u_m \leq c_{70} \mathbf{W}_{1,p}[(\mathbf{W}_{1,p}[\omega])^{q_1}] + c_{68} \mathbf{W}_{1,p}[\mu] \quad \forall k \geq 0 \quad (4.4)$$

in  $\Omega$  and

$$\mathbf{W}_{1,p}[\omega] \in L_{loc}^{q_2}(\mathbb{R}^N), \quad \mathbf{W}_{1,p}[(\mathbf{W}_{1,p}[\omega])^{q_1}] + \mathbf{W}_{1,p}[\mu] \in L_{loc}^{q_1}(\mathbb{R}^N). \quad (4.5)$$

This implies that  $\{u_m\}, \{v_m\}$  are well defined and nondecreasing. Thus  $\{(u_m, v_m)\}$  converges a.e in  $\mathbb{R}^N$  to some functions  $(u, v)$  which satisfies (1.17) in  $\mathbb{R}^N$ . Furthermore, we infer from (3.6) and the monotone convergence theorem that  $u_m^{q_1} \rightarrow u^{q_1}, v_m^{q_2} \rightarrow v^{q_2}$  in  $L_{loc}^1(\mathbb{R}^N)$ . By Proposition 4.3 we derive that  $(u, v)$  are nonnegative  $p$ -superharmonic solutions of (1.9).  $\square$

## 5 Hessian equations

In this section  $\Omega \subset \mathbb{R}^N$  is either a bounded domain with a  $C^2$  boundary or the whole  $\mathbb{R}^N$ . For  $k = 1, \dots, N$  and  $u \in C^2(\Omega)$  the  $k$ -hessian operator  $F_k$  is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where  $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$  denotes the eigenvalues of the Hessian matrix of second partial derivative  $D^2u$  and  $S_k$  is the  $k$ -th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}.$$

We can see that

$$F_k[u] = [D^2u]_k,$$

where for a matrix  $A = (a_{ij})$ ,  $[A]_k$  denotes the sum of the  $k$ -th principal minors. We assume that  $\partial\Omega$  is uniformly  $(k-1)$ -convex, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega,$$

for some positive constant  $c_0$ , where  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$  denote the principal curvatures of  $\partial\Omega$  with respect to its inner normal.

**Definition 5.1** *An upper-semicontinuous function  $u : \Omega \rightarrow [-\infty, \infty)$  is  $k$ -convex ( $k$ -subharmonic) if, for every open set  $\Omega' \subset \bar{\Omega}' \subset \Omega$  and for every function  $v \in C^2(\Omega') \cap C(\bar{\Omega}')$  satisfying  $F_k[v] \leq 0$  in  $\Omega'$ , the following implication is true*

$$u \leq v \text{ on } \partial\Omega' \implies u \leq v \text{ in } \Omega'.$$

We denote by  $\Phi^k(\Omega)$  the class of all  $k$ -subharmonic functions in  $\Omega$  which are not identically equal to  $-\infty$ .

The following weak convergence result for  $k$ -Hessian operators proved in [28] is fundamental in our study.

**Proposition 5.2** *Let  $\Omega$  be either a bounded uniformly  $(k-1)$ -convex in  $\mathbb{R}^N$  or the whole  $\mathbb{R}^N$ . For each  $u \in \Phi^k(\Omega)$ , there exists a nonnegative Radon measure  $\mu_k[u]$  in  $\Omega$  such that*

**1**  $\mu_k[u] = F_k[u]$  for  $u \in C^2(\Omega)$ .

**2** If  $\{u_n\}$  is a sequence of  $k$ -convex functions which converges a.e to  $u$ , then  $\mu_k[u_n] \rightharpoonup \mu_k[u]$  in the weak sense of measures.

As in the case of quasilinear equations with measure data, precise estimates of solutions of  $k$ -Hessian equations with measures data are expressed in terms of Wolff potentials. The next results are proved in [28, 20, 23].

**Theorem 5.3** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded  $C^2$ , uniformly  $(k-1)$ -convex domain. Let  $\mu$  be a nonnegative Radon measure in  $\Omega$  which can be decomposed under the form*

$$\mu = \mu_1 + f,$$

where  $\mu_1$  is a measure with compact support in  $\Omega$  and  $f \in L^q(\Omega)$  for some  $q > \frac{N}{2k}$  if  $k \leq \frac{N}{2}$ , or  $p = 1$  if  $k > \frac{N}{2}$ . Then there exists a nonnegative function  $u$  in  $\Omega$  such that  $-u \in \Phi^k(\Omega)$ , continuous near  $\partial\Omega$  and  $u$  is a solution of the problem

$$\begin{aligned} F_k[-u] &= \mu & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned}$$

Furthermore, any nonnegative function  $u$  such that  $-u \in \Phi^k(\Omega)$  which is continuous near  $\partial\Omega$  and is a solution of above equation, satisfies

$$\frac{1}{c_{ss}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{s}}[\mu] \leq u(x) \leq c_{ss} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2diam\Omega}[\mu](x), \quad (5.1)$$

where  $c_{ss}$  is a positive constant independent of  $x, u$  and  $\Omega$ .

**Theorem 5.4** *Let  $\mu$  be a measure in  $\mathfrak{M}^+(\mathbb{R}^N)$  and  $2k < N$ . Suppose that  $\mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu] < \infty$  a.e. Then there exists  $u$ ,  $-u \in \Phi^k(\mathbb{R}^N)$  such that  $\inf_{\mathbb{R}^N} u = 0$  and  $F_k[-u] = \mu$  in  $\mathbb{R}^N$  and*

$$\frac{1}{c_{ss}} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu](x) \leq u(x) \leq c_{ss} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu](x), \quad (5.2)$$

for all  $x$  in  $\mathbb{R}^N$ . Furthermore, if  $u$  is a nonnegative function such that  $\inf_{\mathbb{R}^N} u = 0$  and  $-u \in \Phi^k(\mathbb{R}^N)$ , then (5.2) holds with  $\mu = F_k[-u]$ .

**Proof of Theorem E.** *The condition is necessary.* Assume that (1.21) admits a nonnegative solution  $(u, v)$ , continuous near  $\partial\Omega$ , such that  $-u, -v \in \Phi^k(\Omega)$  and  $u^{s_2}, v^{s_1} \in L^1(\Omega)$ . Then by Theorem 5.3 we have

$$\begin{aligned} u(x) &\geq \frac{1}{c_{ss}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{s}}[v^{s_1} + \mu](x) \\ v(x) &\geq \frac{1}{c_{ss}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{s}}[u^{s_2} + \eta](x) \quad \text{for almost all } x \in \Omega. \end{aligned}$$

Using the part 2 of Proposition 2.9, we conclude that (1.22) holds.

*The condition is sufficient.* We define a sequence of nonnegative functions  $u_m, v_m$ , continuous near  $\partial\Omega$  and such that  $-u_m, -v_m \in \Phi^k(\Omega)$ , by the following iterative scheme for  $m \geq 0$ ,

$$\begin{aligned} F_k[-u_{m+1}] &= v_m^{s_1} + \mu & \text{in } \Omega, \\ F_k[-v_{m+1}] &= u_m^{s_2} + \eta & \text{in } \Omega, \\ u_{m+1} &= v_{m+1} = 0 & \text{on } \partial\Omega. \end{aligned} \quad (5.3)$$

Clearly, we can assume that  $\{u_m\}$  is nondecreasing as in [24]. By Theorem 5.3 we have

$$\begin{aligned} u_{m+1} &\leq c_{ss} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[v_m^{s_1} + \mu], \\ v_{m+1} &\leq c_{ss} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[u_m^{s_2} + \mu] \quad \text{in } \Omega \end{aligned} \quad (5.4)$$

where  $R = 2 \text{diam}(\Omega)$ .

Then, by Proposition 2.8, there exists a constant  $M_* > 0$  depending only on  $N, p, q_1, q_2, R$  such that if

$$\omega(K) \leq M_* \text{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}}(K)$$

for any compact set  $K \subset \mathbb{R}^N$  with  $d\omega = \left( \mathbf{W}_{\frac{2k}{k+1}, k+1}^R \mu \right)^{s_2} dx + d\eta$  then

$$v_m \leq c_{73} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega], \quad u_m \leq c_{74} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R \left[ \left( \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega] \right)^{s_1} \right] + c_{72} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu] \quad \forall m \geq 0$$

in  $\Omega$  for some positive constants  $c_{72}, c_{73}$  and  $c_{74}$  depending only on  $N, k, s_1, s_2, R$ . Note that we can write

$$v_m^{s_1} + \mu = \left( \mu_1 + \chi_{\Omega_\delta} v_m^{s_1} \right) + \left( (1 - \chi_{\Omega_\delta}) v_m^{s_1} + f \right),$$

and

$$u_m^{s_2} + \eta = \left( \eta_1 + \chi_{\Omega_\delta} u_m^{s_2} \right) + \left( (1 - \chi_{\Omega_\delta}) u_m^{s_2} + g \right),$$

where  $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$  and  $\delta > 0$  is small enough and since  $u_m$  is continuous near  $\partial\Omega$ , then  $v_m^{s_1} + \mu, u_m^{s_2} + \eta$  satisfy the assumptions of the data in Theorem 5.3. Therefore the sequence  $\{u_m\}$  is well defined and nondecreasing. Thus,  $\{u_m\}$  converges a.e in  $\Omega$  to some function  $u$  which satisfies (1.24) in  $\Omega$ . Furthermore, by the monotone convergence theorem there holds  $v_m^{s_1} \rightarrow v, u_m^{s_2} \rightarrow u$  in  $L^1(\Omega)$ . Finally, by Proposition 5.2, we infer that (1.21) admits a nonnegative solutions  $u, v$ , continuous near  $\partial\Omega$ , with  $-u, -v \in \Phi^k(\Omega)$  satisfying (1.24).  $\square$

**Proof of Theorem F.** *The condition is necessary.* Assume that (1.21) admits nonnegative solution  $(u, v)$ , such that  $-u, -v \in \Phi^k(\mathbb{R}^N)$  and  $u^{s_2}, v^{s_1} \in L_{loc}^1(\mathbb{R}^N)$ . Then by Theorem 5.3 we have

$$\begin{aligned} u(x) &\geq \frac{1}{c_{88}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [v^{s_1} + \mu](x) \\ v(x) &\geq \frac{1}{c_{88}} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [u^{s_2} + \eta](x) \quad \text{for almost all } x \in \mathbb{R}^N. \end{aligned}$$

Using Proposition 2.9-(ii), we conclude that (1.22) holds.

*The condition is sufficient.* We defined a sequence of nonnegative functions  $u_m, v_m$ , continuous near  $\partial\Omega$  and such that  $-u_m, -v_m \in \Phi^k(\Omega)$ , by the following iterative scheme for  $m \geq 0$ ,

$$\begin{aligned} F_k[-u_{m+1}] &= v_m^{s_1} + \mu && \text{in } \mathbb{R}^N, \\ F_k[-v_{m+1}] &= u_m^{s_2} + \eta && \text{in } \mathbb{R}^N, \\ \inf_{\mathbb{R}^N} u_{m+1} &= \inf_{\mathbb{R}^N} v_{m+1} = 0. \end{aligned}$$

As in the previous proofs  $\{u_m\}$  is nondecreasing. By Theorem 5.3 we have

$$\begin{aligned} u_{m+1} &\leq c_{88} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [v_m^{s_1} + \mu], \\ v_{m+1} &\leq c_{88} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [u_m^{s_2} + \mu] \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Then, by Proposition 2.7, there exists a constant  $M^* > 0$  depending only on  $N, p, q_1, q_2, R$  such that if

$$\omega(K) \leq M^* \text{Cap}_{\mathbf{I}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}}(K)$$

for any compact set  $K \subset \mathbb{R}^N$  with  $d\omega = \left( \mathbf{W}_{\frac{2k}{k+1}, k+1}^R \mu \right)^{s_2} dx + d\eta$ , then

$$v_m \leq c_{69} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega], \quad u_m \leq c_{70} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R \left[ \left( \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega] \right)^{s_1} \right] + c_{68} \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu] \quad \forall m \geq 0$$

in  $\Omega$  where  $c_{68}, c_{69}$  and  $c_{70}$  depend on  $N, k, s_1, s_2, R$ . Therefore the sequence  $\{u_m\}$  is well defined and nondecreasing. Thus,  $\{u_m\}$  converges a.e in  $\Omega$  to some function  $u$  for which (1.27) is satisfied in  $\mathbb{R}^N$ . Furthermore, by the monotone convergence theorem we have  $v_m^{s_1} \rightarrow v, u_m^{s_2} \rightarrow u$  in  $L_{loc}^1(\mathbb{R}^N)$ . Finally, by Proposition 5.2, we obtain that (1.21) admits a nonnegative solutions  $u, v$  with  $-u, -v \in \Phi^k(\mathbb{R}^N)$  satisfying (1.27).  $\square$

## 6 Further results

The method exposed in the previous sections, can be applied to types of problems. We give below an example for a semilinear system in  $\mathbb{R}_+^N = \{x = (x', x_N), x' \in \mathbb{R}^{N-1}, x_N > 0\}$ .

$$\begin{aligned} -\Delta u &= v^{q_1} && \text{in } \mathbb{R}_+^N \\ -\Delta v &= u^{q_2} && \text{in } \mathbb{R}_+^N \\ u &= \sigma_1, v = \sigma_2 && \text{in } \mathbb{R}^{N-1}, \end{aligned} \quad (6.1)$$

where we have identified  $\partial\mathbb{R}_+^N$  and  $\mathbb{R}^{N-1}$ . We denote by  $\mathbf{P}$  (resp.  $\mathbf{G}$ ) the Poisson kernel in  $\mathbb{R}_+^N$  (resp the Green kernel in  $\mathbb{R}^N$ ). The Poisson potential and the Green potential,  $\mathbf{P}[\cdot]$  and  $\mathbf{G}[\cdot]$ , associated to  $-\Delta$  are defined respectively by

$$\mathbf{P}[\sigma](y) = \int_{\partial\mathbb{R}_+^N} P(y, z) d\sigma(z), \quad G[f](y) = \int_{\mathbb{R}_+^N} \mathbf{G}(y, x) f(x) dx,$$

see [21]. We set  $\rho(x) = x_N$ , and define the capacity  $\text{Cap}_{\alpha, s}^\rho$  by

$$\text{Cap}_{\alpha, s}^\rho(K) = \inf \left\{ \int_{\mathbb{R}_+^N} f^s \rho dx : f \geq 0, \mathbf{I}_\alpha[f \rho \chi_{\mathbb{R}_+^N}] \geq \chi_K \right\},$$

for all Borel set  $K \subset \mathbb{R}^N$ , where  $\mathbf{I}_\alpha$  is the Riesz kernel of order  $\alpha$  in  $\mathbb{R}^N$ .

**Theorem 6.1** *Let  $1 \leq q_1 < \frac{N}{N-1}$ ,  $q_1 q_2 > 1$ . If there exists a constant  $\tilde{c} > 0$  such that if*

$$\begin{aligned} \int_K \rho(x) (\mathbf{P}[\sigma_1](x))^{q_2} dx &\leq \tilde{c} \text{Cap}_{\frac{q_1+2}{q_1}, \frac{q_1 q_2}{q_1 q_2 - 1}}^\rho(K) \\ \sigma_2(G) &\leq \tilde{c} \text{Cap}_{I_{\frac{2(q_2+1)}{q_1 q_2}, \frac{q_1 q_2}{q_1 q_2 - 1}}}^\rho(G), \end{aligned} \quad (6.2)$$

for all Borel sets  $K \subset \mathbb{R}_+^N$  and  $G \subset \mathbb{R}^{N-1}$ , then the problem (6.1) admits a solution.

All solutions in above theorem are understood in the usual very weak sense:  $u \in L_{loc}^1(\mathbb{R}_+^N \cap B)$ ,  $u^{q_2}, v^{q_1} \in L_\rho^1(\mathbb{R}_+^N \cap B)$  for any ball  $B$  and

$$\begin{aligned} \int_{\mathbb{R}_+^N} u(-\Delta \xi) dx &= \int_{\mathbb{R}_+^N} v^{q_1} \xi dx - \int_{\partial\mathbb{R}_+^N} \frac{\partial \xi}{\partial n} d\sigma_1, \\ \int_{\mathbb{R}_+^N} v(-\Delta \xi) dx &= \int_{\mathbb{R}_+^N} u^{q_2} \xi dx - \int_{\partial\mathbb{R}_+^N} \frac{\partial \xi}{\partial n} d\sigma_2, \end{aligned}$$

for any  $\xi \in C^2(\overline{\mathbb{R}_+^N}) \cap C_c(\mathbb{R}^N)$  with  $\xi = 0$  on  $\partial\mathbb{R}_+^N$ . It is well-known that such a solution  $u$  satisfies

$$u = \mathbf{G}[v^{q_1}] + \mathbf{P}[\sigma_1], \quad v = \mathbf{G}[u^{q_2}] + \mathbf{P}[\sigma_2] \quad \text{a.e. in } \mathbb{R}_+^N.$$

To prove theorem we need the following a basic estimate

**Lemma 6.2** *Assume that  $0 < q_1 < \frac{N}{N-1}$ . Then for any  $\omega \in \mathfrak{M}_b^+(\mathbb{R}^N)$ ,*

$$\mathbf{I}_2[(\mathbf{I}_1[\omega])^{q_1}] \leq c_{89} \mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}[\omega] \quad \text{a.e. in } \mathbb{R}^N, \quad (6.3)$$

where  $c_{89} > 0$  depends on  $q_1, q_2$  and  $N$ .

**Proof.** The proof of Lemma 6.2 is similar to the one of Lemma 2.2 and details are omitted. Note that if  $\omega \in \mathfrak{M}_b(\overline{\mathbb{R}_+^N})$  it is extended by 0 in  $\mathbb{R}_-^N$ .  $\square$

**Remark 6.3** The condition  $0 < q_1 < \frac{N}{N-1}$  is a necessary and sufficient condition for  $(\mathbf{I}_1[\omega])^{q_1} \in L^1_{loc}(\mathbb{R}^N)$  for any  $\omega \in \mathfrak{M}_b^+(\mathbb{R}^N)$ .

**Theorem 6.4** Let  $q_1 \geq 1$ ,  $q_1 q_2 > 1$  and  $\omega \in \mathfrak{M}_b(\overline{\mathbb{R}^N_+})$ . If

$$\omega(K) \leq c_{90} \text{Cap}_{\frac{q_1+2}{q_1}, \frac{q_1 q_2}{q_1 q_2 - 1}}^\rho(K) \quad \forall K \subset \overline{\mathbb{R}^N_+}, K \text{ Borel},$$

for some  $c_{90} > 0$ , then

$$\mathbf{I}_1 \left[ \left( \mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}[\omega] \right)^{q_2} \rho \chi_{\mathbb{R}^N_+} \right] \leq c_{91} \mathbf{I}_1[\omega] \quad \text{a.e. in } \mathbb{R}^N_+. \quad (6.4)$$

**Proof.** *Step 1.* For any compact  $K \subset \left\{ x \in \mathbb{R}^N_+ : \mathbf{I}_{\frac{q_1+2}{q_1}}[f \rho \chi_{\mathbb{R}^N_+}](x) > \lambda \right\}$ , we have

$$\omega(K) \leq c_{90} \text{Cap}_{\frac{q_1+2}{q_1}, \frac{q_1 q_2}{q_1 q_2 - 1}}^\rho(K) \leq c_{90} \lambda^{-\frac{q_1 q_2}{q_1 q_2 - 1}} \int_{\mathbb{R}^N_+} f^{\frac{q_1 q_2}{q_1 q_2 - 1}} \rho dx$$

by assumption and the definition of the capacity. Hence,

$$\lambda^{\frac{q_1 q_2}{q_1 q_2 - 1}} \omega \left( \left\{ \mathbf{I}_{\frac{q_1+2}{q_1}}[f \rho \chi_{\mathbb{R}^N_+}] > \lambda \right\} \right) \leq c_{90} \int_{\mathbb{R}^N_+} f^{\frac{q_1 q_2}{q_1 q_2 - 1}} \rho dx \quad \forall \lambda > 0.$$

This implies an estimate in Lorentz space,

$$\| \mathbf{I}_{\frac{q_1+2}{q_1}}[f \rho \chi_{\mathbb{R}^N_+}] \|_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}, \infty}(\mathbb{R}^N, d\omega)} \leq \| f \|_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}}(\mathbb{R}^N, \chi_{\mathbb{R}^N_+} \rho dx)} \quad \forall f \geq 0. \quad (6.5)$$

*Step 2.* Since, for any  $g \in C_c(\mathbb{R}^N_+)$ ,

$$\int_{\mathbb{R}^N_+} \mathbf{I}_{\frac{q_1+2}{q_1}}[g\omega] f \rho dx = \int_{\mathbb{R}^N} \mathbf{I}_{\frac{q_1+2}{q_1}}[f \rho \chi_{\mathbb{R}^N_+}] g d\omega,$$

we derive, using duality between  $L^{p,1}$  and  $L^{p',\infty}$ , Holder's inequality therein and (6.5), that

$$\begin{aligned} \int_{\mathbb{R}^N_+} \mathbf{I}_{\frac{q_1+2}{q_1}}[g\omega] f \rho dx &\leq \| \mathbf{I}_{\frac{q_1+2}{q_1}}[f \rho \chi_{\mathbb{R}^N_+}] \|_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}, \infty}(\mathbb{R}^N, d\omega)} \| g \|_{L^{q_1 q_2, 1}(\mathbb{R}^N, d\omega)} \\ &\leq \| f \|_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}}(\mathbb{R}^N, \chi_{\mathbb{R}^N_+} \rho dx)} \| g \|_{L^{q_1 q_2, 1}(\mathbb{R}^N, d\omega)} \quad \forall f, g \geq 0. \end{aligned}$$

Therefore,

$$\| \mathbf{I}_{\frac{q_1+2}{q_1}}[g\omega] \|_{L^{q_1 q_2}(\mathbb{R}^N, \chi_{\mathbb{R}^N_+} \rho dx)} \leq \| g \|_{L^{q_1 q_2, 1}(\mathbb{R}^N, d\omega)}. \quad (6.6)$$

*Step 3.* Taking  $g = \chi_{B_t(x)}$  and since for  $q_1 \geq 1$

$$\begin{aligned} \mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}[\nu](x) &= \int_0^\infty \left( \frac{\nu(B_\rho(x))}{\rho^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} dx \\ &\leq c_{89} \left( \int_0^\infty \frac{\nu(B_\rho(x))}{\rho^{N-\frac{q_1+2}{q_1}}} dx \right)^{q_1} \\ &= c_{89} \left( \mathbf{I}_{\frac{q_1+2}{q_1}}[\nu](x) \right)^{q_1} \quad \forall \nu \in \mathfrak{M}_b^+(\mathbb{R}^N), \quad \forall x \in \mathbb{R}^N, \end{aligned}$$



we derive that for almost all  $x \in \mathbb{R}_+^N$ ,

$$\int_{\mathbb{R}_+^N} \left( \mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}} [\chi_{B_t(x)} \omega] \right)^{q_2} \rho dy \leq c_{90} \omega(B_t(x)),$$

from (6.6), which implies

$$\omega(B_t(x)) \leq c_{91} \frac{t^{(N-\frac{q_1+2}{q_1})-\frac{q_1 q_2}{q_1 q_2-1}}}{\left( \int_{B_{2t}(x)} \chi_{\mathbb{R}_+^N} \rho dy \right)^{\frac{1}{q_1 q_2-1}}} \leq c_{92} \frac{t^{n-\frac{q_1+2}{q_1}-\frac{q_1 q_2}{q_1 q_2-1}}}{(\max\{x_n, t\})^{\frac{1}{q_1 q_2-1}}}, \quad (6.7)$$

since  $\int_{B_r(x)} \chi_{\mathbb{R}_+^N} \rho dy \asymp r^N \max\{x_N, r\}$  for any  $x \in \mathbb{R}_+^N, r > 0$  where the symbol  $\asymp$  is defined by

$$A \asymp B \iff \frac{1}{c} B \leq A \leq cB \quad \text{for some constant } c > 0.$$

It implies also

$$\int_{B_t(x)} \left( \mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}^t [\omega] \right)^{q_2} \chi_{\mathbb{R}_+^N} \rho dy \leq c_{90} \omega(B_{2t}(x)), \quad (6.8)$$

from which follows

$$\int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left( \mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}^t [\omega] \right)^{q_2} \chi_{\mathbb{R}_+^N} \rho dy dt \leq c_{93} \mathbf{I}_1[\omega](x).$$

Therefore, if the following inequality holds

$$\int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left( \int_t^\infty \left( \frac{\omega(B_r(y))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} \chi_{\mathbb{R}_+^N} \rho dy dt \leq c_{93} \mathbf{I}_1[\omega](x), \quad (6.9)$$

it will imply (6.4).

*Step 4.* We claim that (6.9) holds. Since  $B_r(y) \in B_{2r}(x), y \in B_t(x), r \geq t$ ,

$$\begin{aligned} & \int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left( \int_t^\infty \left( \frac{\omega(B_r(y))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} \chi_{\mathbb{R}_+^N} \rho dy dt \\ & \leq \int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \chi_{\mathbb{R}_+^N} \rho dy \left( \int_t^\infty \left( \frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} dt \\ & \asymp \int_0^\infty \max\{x_n, t\} \left( \int_t^\infty \left( \frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} dt. \end{aligned}$$

By integration by part,

$$\begin{aligned} & \int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left( \int_t^\infty \left( \frac{\omega(B_r(y))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} \chi_{\mathbb{R}_+^N} \rho dy dt \\ & = q_2 \int_0^\infty \int_0^t \max\{x_N, s\} ds \left( \int_t^\infty \left( \frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} \left( \frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dt}{t} \\ & = q_2 \int_0^\infty \int_0^t \max\{x_N, s\} ds \left( \int_t^\infty \left( \frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} \left( \frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}} \right)^{q_1-1} \frac{2}{t^{\frac{2}{q_1}}} \frac{\omega(B_{2t}(x))}{t^{N-1}} \frac{dt}{t} \end{aligned}$$

We have

$$\int_0^t \max\{x_N, s\} ds \asymp t \max\{x_N, t\},$$

$$\begin{aligned} \left( \int_t^\infty \left( \frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} &\leq c_{94} \left( \int_t^\infty \left( \frac{r^{-\frac{q_1+2}{q_1(q_1q_2-1)}}}{(\max\{x_N, r\})^{\frac{1}{q_1q_2-1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} \\ &\asymp t^{-\frac{(q_1+2)(q_2-1)}{q_1q_2-1}} (\max\{x_N, t\})^{-\frac{q_1(q_1-1)}{q_1q_2-1}}, \end{aligned}$$

by (6.7) and

$$\begin{aligned} \left( \frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}} \right)^{q_1-1} t^{\frac{2}{q_1}} &\leq c_{95} \left( \frac{t^{-\frac{q_1+2}{q_1(q_1q_2-1)}}}{(\max\{x_N, t\})^{\frac{1}{q_1q_2-1}}} \right)^{q_2-1} t^{\frac{2}{q_1}} \\ &= c_{95} t^{-\frac{(q_1+2)(q_1-1)}{q_1(q_1q_2-1)} + \frac{2}{q_1}} (\max\{x_N, t\})^{-\frac{q_1-1}{q_1q_2-1}} \end{aligned}$$

Thus,

$$\int_0^t \max\{x_N, s\} ds \left( \int_t^\infty \left( \frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} \left( \frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}} \right)^{q_1-1} t^{2/q_1} \leq c_{96}$$

and we obtain (6.9).  $\square$

**Lemma 6.5** *Let  $\alpha > 0$ ,  $s > 1$  such that  $\alpha + \frac{2}{s'} < N - 1$  where  $s' = \frac{s}{s-1}$ . For all  $\eta \in \mathfrak{M}^+(\mathbb{R}^{N-1})$ , there holds*

$$\int_{\mathbb{R}^N} (\mathbf{I}_\alpha[\eta \otimes \delta_{\{x_N=0\}}])^{s'} x_N dx \asymp \int_{\mathbb{R}^{N-1}} \left( \int_0^\infty \frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} \frac{dt}{t} \right)^{s'} dx', \quad (6.10)$$

where  $I_\beta$  is the Riesz potential of order  $\beta$  in  $\mathbb{R}^{N-1}$ . As a consequence, we have

$$\text{Cap}_{\alpha,s}^p(E \times \{x_N = 0\}) \asymp \text{Cap}_{I_{\alpha+2/s'-1},s}(E) \quad \forall E \subset \mathbb{R}^{N-1}, E \text{ Borel}. \quad (6.11)$$

**Proof.** We have

$$\begin{aligned} \int_{\mathbb{R}^N} (\mathbf{I}_\alpha[\eta \otimes \delta_{\{x_N=0\}}])^{s'} x_N dx &\geq \int_{\mathbb{R}^N} \left( \int_{2x_N}^{4x_N} \frac{(\eta \otimes \delta_{\{x_N=0\}})(B_r(x))}{r^{N-\alpha}} \frac{dr}{r} \right)^{s'} x_N dx \\ &\geq c_{97} \int_{\mathbb{R}^N} \left( \frac{\eta(B'_{x_N}(x'))}{x_N^{N-\alpha}} \right)^{s'} x_N dx \\ &\geq c_{98} \int_{\mathbb{R}^{N-1}} \left( \sup_{t>0} \frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} \right)^{s'} dx'. \end{aligned} \quad (6.12)$$

By using Lemma 2.1 we derive

$$\begin{aligned} \int_{\mathbb{R}^N} (\mathbf{I}_\alpha[\eta \otimes \delta_{\{x_N=0\}}])^{s'} x_N dx &\leq \int_{\mathbb{R}^N} \left( \int_{x_N}^\infty \frac{\eta(B'_r(x'))}{r^{N-\alpha}} \frac{dr}{r} \right)^{s'} dx_N dx' \\ &\leq c_{99} \int_{\mathbb{R}^{N-1}} \int_0^\infty \left( \frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} \right)^{s'} \frac{dt}{t} dx'. \end{aligned} \quad (6.13)$$

On the other hand, by [23, Proposition 5.1], there holds

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \left( \sup_{t>0} \frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} \right) dx' &\asymp \int_{\mathbb{R}^{N-1}} \int_0^\infty \left( \frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} \right)^{s'} \frac{dt}{t} dx' \\ &\asymp \int_{\mathbb{R}^{N-1}} \left( \int_0^\infty \frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}} dt \right)^{s'} dx'. \end{aligned} \quad (6.14)$$

Combining (6.12), (6.13) and (6.14) we obtain (6.10). Moreover, we derive (6.11) from (6.10) and [1, Theorem 2.5.1], which ends the proof.  $\square$

**Proof of Theorem 6.1** The following estimates are classical

$$\mathbf{G}(x, y) \asymp \frac{x_N y_N}{|x - y|^{N-2} \max\{|x - y|, x_N, y_N\}^2} \leq c_{100} \frac{y_N}{|x - y|^{N-1}} \quad (6.15)$$

$$\mathbf{P}(x, z) = c_{101} \frac{x_N}{|x - z|^N} \leq c_{101} \frac{1}{|x - z|^{N-1}} \quad (6.16)$$

Thus,

$$\mathbf{G}[(\mathbf{P}[\sigma_1])^{q_2}] + \mathbf{P}[\sigma_2] \leq c_{102} \mathbf{I}_1[\omega] \quad (6.17)$$

where  $\omega(x) = \rho(\mathbf{P}[\sigma_1])^{q_2} + \sigma_2$  in  $\mathbb{R}^N$ . Therefore, we infer that if

$$\mathbf{I}_1 \left[ (\mathbf{I}_2[(\mathbf{I}_1[\omega])^{q_1}]^{q_2} \chi_{\mathbb{R}_+^N} \rho) \right] \leq c_{103} \mathbf{I}_1[\omega] \quad \text{in } \mathbb{R}_+^N \quad (6.18)$$

for some  $c_{103} > 0$  small enough, then (6.1) admits a positive solution  $(u, v)$ . On the other hand, we deduce (6.18) from Lemma 6.2 and Theorem 6.4. The proof is complete.  $\square$

**Remark 6.6** *The system*

$$\begin{aligned} -\Delta u &= v^{q_1} + \epsilon_1 \mu && \text{in } \Omega \\ -\Delta v &= u^{q_2} + \epsilon_2 \eta && \text{in } \Omega \\ u &= \epsilon_3 \sigma_1, v = \epsilon_4 \sigma_2 && \text{in } \partial\Omega, \end{aligned} \quad (6.19)$$

where  $d(\cdot)\mu, d(\cdot)\lambda$  belong to  $\mathfrak{M}_b^+(\Omega)$ ,  $\sigma_1, \sigma_2$  to  $\mathfrak{M}^+(\partial\Omega)$  and the  $\epsilon_j$  are positive numbers, is analyzed in [10, Th 4.6]. Therein it is proved that if

$$\int_{\Omega} (\mathbf{G}[\mu] + \mathbf{P}[\lambda])^{\max\{q_1, q_2\}} d(x) dx < \infty, \quad (6.20)$$

which is equivalent to a capacity estimate, and

$$\min \left\{ q_2 \frac{q_1 + 1}{q_2 + 1}, q_1 \frac{q_2 + 1}{q_1 + 1} \right\} < \frac{N + 1}{N - 1}, \quad (6.21)$$

and if the  $\epsilon_j$  are small enough, then (6.19) admits a positive solution. Now condition (6.21) is a subcriticality assumption (for at least one of the two exponents  $q_j$ ) since there is no condition on the boundary measures.

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