Quasilinear and Hessian Lane-Emden type systems with measure data

Marie-Françoise Véron, Quoc-Hung Nguyen, Laurent Véron

Quasilinear and Hessian Lane-Emden type systems with measure data

Marie-Françoise Bidaut-Véron*
Quoc-Hung Nguyen†
Laurent Véron‡

Contents

Abstract

We study nonlinear systems of the form $-\Delta_p u = v^{q_1} + \mu,\ -\Delta_p v = u^{q_2} + \eta$ and $F_k[-u] = v^{s_1} + \mu,\ F_k[-v] = u^{s_2} + \eta$ in a bounded domain Ω or in $\mathbb{R}^N$ where μ and η are nonnegative Radon measures, Δ_p and F_k are respectively the p-Laplacian and the k-Hessian operators and q_1, q_2, s_1 and s_2 positive numbers. We give necessary and sufficient conditions for existence expressed in terms of Riesz or Bessel capacities. *2010*

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1 Introduction and Main results

Let $\Omega \subset \mathbb{R}^N$ be either a bounded domain or the whole $\mathbb{R}^N$, $p > 1$ and $k \in \{1, 2, ..., N\}$. We denote by

$$\Delta_p u := div\left(|\nabla u|^{p-2}\nabla u\right)$$

the p-Laplace operator and by

$$F_k[u] = \sum_{1\le j_1<j_2<...<j_k\le N} \lambda_{j_1}\lambda_{j_2}...\lambda_{j_k}$$

*E-mail address: veronmf@univ-tours.fr, Laboratoire de Mathématiques et Physique Théorique, Université François Rabelais, Tours, France

†E-mail address: quochung.nguyen@sns.it, Scuola Normale Superiore, Centro Ennio de Giorgi, Piazza dei Cavalieri 3, I-56100 Pisa, Italy.

‡E-mail address: Laurent.Veron@lmpt.univ-tours.fr, Laboratoire de Mathématiques et Physique Théorique, Université François Rabelais, Tours, France

the k-Hessian operator where $\lambda_1, ..., \lambda_N$ are the eigenvalues of the Hessian matrix D^2u. In the work [23], Phuc and Verbitsky obtained necessary and sufficient conditions for existence of nonnegative solutions to the following equations

$$\begin{array}{cc} -\Delta_p u = u^q + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{1.1}$$

and

$$\begin{array}{cc} F_k[-u] = u^q + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{array} \tag{1.2}$$

Their conditions involve the continuity of the measures with respect to Bessel or Riesz capacities and Wolff potentials estimates. For example, if Ω is bounded and μ has compact support in Ω, they proved that if it is equivalent to solve (1.1), or to have

$$\mu(E) \le c_1 \mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q+1-p}}(E) \qquad \text{for all compact set } E \subset \Omega, \tag{1.3}$$

for some constant $c_1 > 0$ where $\mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q+1-p}}$ is a Bessel capacity, or to have

$$\int_B \left(\mathbf{W}^R_{1,p}[\mu_B](x)\right)^q dx \le c_2 \mu(B) \qquad \text{for all ball } B \text{ s.t. } B \cap \mathrm{supp}\mu \neq \emptyset, \tag{1.4}$$

for some constant $c_2 > 0$, where $R = 2\,\mathrm{diam}(\Omega)$ and $\mathbf{W}^R_{1,p}[\mu_B]$ denotes the R-truncated Wolff potential of the measure $\mu_B = \chi_B \mu$. Concerning the k-Hessian operator in a bounded $(k-1)$-convex domain Ω, they proved that if μ has compact support, the problem (1.2) with $q > k$ admits a nonnegative solution if and only if

$$\mu(E) \le c_3 \mathrm{Cap}_{\mathbf{G}_{2k}, \frac{q}{q-k}}(E) \qquad \text{for all compact set } E \subset \Omega, \tag{1.5}$$

for some c_3. In turn this condition is equivalent to

$$\int_B \left[\mathbf{W}^R_{\frac{2k}{k+1}, k+1}[\mu_B(x)]\right]^q dx \le c_4 \mu(B) \qquad \text{for all ball } B \text{ s.t. } B \cap \mathrm{supp}\mu \neq \emptyset. \tag{1.6}$$

for some $c_4 > 0$. The results concerning the linear case $p = 2$ and $k = 1$, can be found in [2, 3, 31].

The natural counterpart of equation (1.1) and (1.2) for systems:

$$\begin{array}{cc} -\Delta_p u = v^{q_1} + \mu & \text{in } \Omega \\ -\Delta_p v = u^{q_2} + \eta & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{array} \tag{1.7}$$

and

$$\begin{array}{cc} F_k[-u] = v^{s_1} + \mu & \text{in } \Omega \\ F_k[-v] = u^{s_2} + \eta & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{array} \tag{1.8}$$

where $q_1, q_2 > p-1, s_1, s_2 > k$ and μ, η are Radon measures. If $\Omega = \mathbb{R}^N$, we consider the same equations, except that the boundary conditions are replaced by $\inf_{\mathbb{R}^N} u = \inf_{\mathbb{R}^N} u = 0$ and our statements involve the Riesz potentials and their associated capacities $\mathrm{Cap}_{I_{\alpha,\beta}}$. Our main results are the following.

Theorem A *Let $1 < p < N$, $q_1, q_2 > 0$ and $q_2 q_1 > (p-1)^2$. Let μ, η be nonnegative Radon measures in $\mathbb{R}^N$. If the following system*

$$\begin{array}{cc} -\Delta_p u = v^{q_1} + \mu & \text{in } \mathbb{R}^N \\ -\Delta_p v = u^{q_2} + \eta & \text{in } \mathbb{R}^N \end{array} \tag{1.9}$$

admits a nonnegative p-superharmonic solution (u,v) *then there exists a positive constant* c_5 *depending on* N,p,q_1,q_2 *such that*

$$\eta(E)+\int_E (\mathbf{W}_{1,p}[\mu](x))^{q_2}\,dx \le c_5\,\mathrm{Cap}_{\mathbf{I}_{\frac{p(q_1+p-1)}{q_1},\frac{q_1q_2}{q_1q_2-(p-1)^2}}}(E) \quad \textit{for all Borel sets } E. \tag{1.10}$$

Conversely, if μ *and* η *are bounded, there exists* $c_6>0$ *depending on* N,p,q_1,q_2 *such that if* $0<q_1<\frac{N(p-1)}{N-p}$ *and* (1.10) *holds with* c_5 *replaced by* c_6*, then* (1.9) *admits a nonnegative p-superharmonic solution* (u,v) *satisfying*

$$v\le c_8\mathbf{W}_{1,p}[\omega],\quad u\le c_9\mathbf{W}_{1,p}[(\mathbf{W}_{1,p}[\omega])^{q_1}]+c_7\mathbf{W}_{1,p}[\mu] \tag{1.11}$$

in $\mathbb{R}^N$ *for some* $c_7,c_8,c_9>0$ *where* $d\omega=(\mathbf{W}_{1,p}[\mu])^{q_2}\,dx+d\eta$.

We notice that the left-hand side in (1.10) is not symmetric in η and μ and the capacity in the right-hade side is not symmetric in q_1 and q_2. Hence the following symmetrized inequality holds

$$\mu(E)+\int_E (\mathbf{W}_{1,p}[\eta](x))^{q_1}\,dx \le c_5'\,\mathrm{Cap}_{\mathbf{I}_{\frac{p(q_2+p-1)}{q_2},\frac{q_1q_2}{q_1q_2-(p-1)^2}}}(E) \quad \text{for all Borel sets } E. \tag{1.12}$$

It is known that

$$\mathrm{Cap}_{\mathbf{I}_{\alpha,\beta}}(K)=0\quad \forall K \text{ compact},$$

if $\alpha\beta\ge N$, the first part of above implies the following Liouville theorem, obtained by another method in [9, Th 5.3-(i)]

Corollary B *Assume that*

$$\frac{p(q_1q_2+(p-1)\max\{q_1,q_2\})}{q_1q_2-(p-1)^2}\ge N.$$

Any nonnegative p-superharmonic solution (u,v) *of inequalities*

$$\begin{array}{ll} -\Delta_p u\ge v^{q_1} & \textit{in } \mathbb{R}^N\\ -\Delta_p v\ge u^{q_2} & \textit{in } \mathbb{R}^N, \end{array} \tag{1.13}$$

is trivial, i.e. $u=v=0$.

When Ω is bounded domain, we have a similar result in which we denote by d the distance function to the boundary $x\mapsto d(x)=\mathrm{dist}\,(x,\partial\Omega)$.

Theorem C *Let* $1<p<N$, $q_1,q_2>0$ *and* $q_2q_1>(p-1)^2$. *Let* $\Omega\subset\mathbb{R}^N$ *be a bounded domain and* μ,η *nonnegative Radon measures in* Ω. *If the following problem*

$$\begin{array}{ll} -\Delta_p u=v^{q_1}+\mu & \textit{in } \Omega\\ -\Delta_p v=u^{q_2}+\eta & \textit{in } \Omega\\ u=v=0 & \textit{on } \partial\Omega, \end{array} \tag{1.14}$$

admits a nonnegative renormalized solution (u,v)*, then then for any compact set* $K\subset\Omega$*, there exists a positive constant* c_{10} *depending on* N,p,q_1,q_2 *and dist*$(K,\partial\Omega)$ *such that*

$$\eta(E)+\int_E \left(\mathbf{W}_{1,p}^{\frac{d(x)}{4}}[\mu](x)\right)^{q_2}dx\le c_{10}\,\mathrm{Cap}_{\mathbf{G}_{\frac{p(q_1+p-1)}{q_1},\frac{q_1q_2}{q_1q_2-(p-1)^2}}}(E) \quad \textit{for all Borel sets } E\subset K. \tag{1.15}$$

Conversely, if μ *and* η *are bounded and there exists* $c_{11}>0$ *depending on* N,p,q_1,q_2 *and* $R=2diam\,(\Omega)$ *such that if* $0<q_1<\frac{N(p-1)}{N-p}$ *and*

$$\eta(K)+\int_K \left(\mathbf{W}_{1,p}^{2R}[\mu]\right)^{q_2}dx\le c_{11}\,\mathrm{Cap}_{\mathbf{G}_{\frac{p(q_1+p-1)}{q_1},\frac{q_1q_2}{q_1q_2-(p-1)^2}}}(K) \tag{1.16}$$

for all compact set $K \subset \Omega$, then (1.14) *admits a nonnegative renormalized solution* (u, v) *satisfying*

$$v \le c_{13}\mathbf{W}^R_{1,p}[\omega], \quad u \le c_{14}\mathbf{W}^R_{1,p}[\left(\mathbf{W}^R_{1,p}[\omega]\right)^{q_1}] + c_{12}\mathbf{W}^R_{1,p}[\mu] \tag{1.17}$$

in Ω, *where* $d\omega = \left(\mathbf{W}^R_{1,p}[\mu]\right)^{q_2} dx + d\eta$.

It is known that

$$\mathrm{Cap}_{\mathbf{G}_{\alpha,\beta}}(\{x_0\}) > 0$$

if and only if $\alpha\beta > N$. Thus, as an application in a partially subcritical case we have,

Corollary D *Let the assumptions on* p, q_1, q_2, Ω *and* R *of Theorem C be satisfied,* $x_0 \in \Omega$, $a > 0$ *and* μ *be a nonnegative Radon measures in* Ω. *If the following problem*

$$\begin{array}{ll} -\Delta_p u = v^{q_1} + \mu & \text{in } \Omega \\ -\Delta_p v = u^{q_2} + a\delta_{x_0} & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{array} \tag{1.18}$$

admits a nonnegative renormalized solution (u, v), *then there exist positive constants* $c_{15} = c_{15}(N, p, q_1, q_2, d(x_0))$ *and for any compact subset* K *of* Ω, $c_{16} = c_{16}(N, p, q_1, q_2, \mathrm{dist}\,(K, \partial\Omega))$, *such that*

$$\begin{array}{ll} (i) & N < \dfrac{pq_2(q_1 + p - 1)}{q_1q_2 - (p-1)^2}, \\ (ii) & a \le c_{15} \\ (iii) & \displaystyle\int_K \left(\mathbf{W}^{2R}_{1,p}[\mu]\right)^{q_2} dx \le c_{16}. \end{array} \tag{1.19}$$

Conversely, assuming that μ *is bounded, there exist positive constants* $c_{17} = c_{17}(N, p, q_1, q_2, d(x_0))$, $c_{18} = c_{18}(N, p, q_1, q_2)$ *such that if* $0 < q_1 < \frac{N(p-1)}{N-p}$ *and* (1.19) *holds with* c_{15} *and* c_{16} *replaced respectively by* c_{17} *and* c_{18} *there exists a nonnegative renormalized solution* (u, v) *of* (1.18) *satisfying*

$$v \le c_{21}W^R_{1,p}[\omega], \quad u \le c_{22}\mathbf{W}^R_{1,p}[\left(W^R_{1,p}[\omega]\right)^{q_1}] + c_{20}\mathbf{W}^R_{1,p}[\mu] \tag{1.20}$$

in Ω, *where*

$$W^R_{1,p}[\omega] = \mathbf{W}^R_{1,p}\left[\left(\mathbf{W}^R_{1,p}[\mu]\right)^{q_2}\right] + a^{\frac{1}{p-1}}\left(|x - x_0|^{-\frac{N-p}{p-1}} - R^{-\frac{N-p}{p-1}}\right)_+.$$

Concerning the k-Hessian operator we recall some notions introduced by Trudinger and Wang [27, 28, 29], and we follow their notations. For $k = 1, ..., N$ and $u \in C^2(\Omega)$ the k-Hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivatives D^2u and S_k is the k-th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \le i_1 < ... < i_k \le N} \lambda_{i_1} ... \lambda_{i_k}.$$

Since D^2u is symmetric, it is clear that

$$F_k[u] = \left[D^2u\right]_k,$$

where we denote by $[A]_k$ the sum of the k-th principal minors of a matrix $A = (a_{ij})$. In order that there exists a smooth k-admissible function which vanishes on $\partial\Omega$, the boundary $\partial\Omega$ must satisfy a uniformly (k-1)-convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega,$$

for some positive constant c_0, where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal. We also denote by $\Phi^k(\Omega)$ the class of upper-semicontinuous functions $\Omega \to [-\infty, \infty)$ which are k-convex, or subharmonic in the Perron sense (see Definition 5.1). In this paper we prove the following theorem (in which expression $\mathbb{E}[q]$ is the largest integer less or equal to q)

Theorem E *Let $2k < N, s_1, s_2 > 0$, $s_1 s_2 > k^2$. Let Ω be a bounded uniformly (k-1)-convex domain in $\mathbb{R}^N$ with diameter R. Let $\mu = \mu_1 + f, \eta = \eta_1 + g$ be a nonnegative Radon measures where μ_1, η_1 has compact support in Ω and $f, g \in L^l(\Omega)$ for some $l > \frac{N}{2k}$.—— for simplicity I want to state the theorem $u = v = 0$ If then the following problem*

$$\begin{array}{ll} F_k[-u] = v^{s_1} + \mu & \text{in } \Omega \\ F_k[-v] = u^{s_2} + \eta & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{array} \tag{1.21}$$

admits a nonnegative solutions (u, v), continuous near $\partial\Omega$, with $-u$ and $-v$ elements of $\Phi^k(\Omega)$, then for any compact set $K \subset \Omega$, there exists a positive constant c_{23} depending on N, k, s_1, s_2 and $dist(K, \partial\Omega)$ such that there holds

$$\eta(E) + \int_E \left(\mathbf{W}^{\frac{d(x)}{4}}_{\frac{2k}{k+1}, k+1}[\mu](x) \right)^{s_2} dx \leq c_{23} \operatorname{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}}(E) \qquad \forall E \subset K, E \text{ Borel.} \tag{1.22}$$

Conversely,, if μ and η are bounded, there exist a positive constant c_{24} depending on N, k, s_1, s_2 and diam (Ω) such that, if $k \leq s_1 < \frac{Nk}{N-2k}$, and

$$\eta(K) + \int_K \left(\mathbf{W}^{2R}_{\frac{2k}{k+1}, k+1}[\mu] \right)^{s_2} dx \leq c_{24} \operatorname{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}}(K) \tag{1.23}$$

for all Borel set $K \subset \Omega$, then (1.21) admits a nonnegative solution (u, v), continuous near $\partial\Omega$, with $-u, -v \in \Phi^k(\Omega)$ satisfying

$$v \leq c_{28} \mathbf{W}^R_{\frac{2k}{k+1}, k+1}[\omega], \quad u \leq c_{29} \mathbf{W}^R_{\frac{2k}{k+1}, k+1}[\left(\mathbf{W}^R_{\frac{2k}{k+1}, k+1}[\omega] \right)^{s_1}] + c_{27} \mathbf{W}^R_{\frac{2k}{k+1}, k+1}[\mu] \tag{1.24}$$

in Ω for some constants c_j $(j = 27, 28, 29)$ depending on N, k, s_1, s_2, and diam (Ω).

If Ω is replaced by the whole space we prove,

Theorem F *Let $2k < N, s_1, s_2 > 0$, $s_1 s_2 > k^2$. Let μ, η be a nonnegative Radon measures in $\mathbb{R}^N$. If the following problem*

$$\begin{array}{ll} F_k[-u] = v^{s_1} + \mu & \text{in } \mathbb{R}^N \\ F_k[-v] = u^{s_2} + \eta & \text{in } \mathbb{R}^N, \end{array} \tag{1.25}$$

admits a nonnegative solutions (u, v) with $-u$ and $-v$ belonging to $\Phi^k(\mathbb{R}^N)$, then there exists a positive constant c_{30} depending on N, k, s_1, s_2 such that there holds

$$\eta(E) + \int_E \left(\mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu](x) \right)^{s_2} dx \leq c_{30} \operatorname{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}}(E) \qquad \forall E \text{ Borel.} \tag{1.26}$$

Conversely,, if μ and η are bounded, there exists positive constant c_{31} depending on N, k, s_1, s_2 such that, if $0 < s_1 < \frac{Nk}{N-2k}$ and (1.26) holds with c_{31} instead of c_{30} then (1.25) admits a nonnegative solutions (u, v) with $-u$ and $-v$ in $\Phi^k(\mathbb{R}^N)$ satisfying

$$v \leq c_{33} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\omega], \quad u \leq c_{34} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\left(\mathbf{W}_{\frac{2k}{k+1}, k+1}[\omega] \right)^{s_1}] + c_{32} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu] \tag{1.27}$$

in $\mathbb{R}^N$*, where the* c_j *(*$j = 32, 33, 34$*) depend on* N, k, s_1, s_2*.*

As in p-Laplace case, we have a Liouville property for Hessian systems.

Corollary G *Assume that*

$$\frac{2k(s_2 s_1 + k\max\{s_1, s_2\})}{s_1 s_2 - k^2} \geq N. \tag{1.28}$$

Any nonnegative solution (u,v) of inequalities

$$\begin{array}{ll} F_k[-u] \geq v^{s_1} & in\ \ \mathbb{R}^N, \\ F_k[-v] \geq u^{s_2} & in\ \ \mathbb{R}^N, \end{array} \tag{1.29}$$

with $-u$ *and* $-v$ *in* $\Phi^k(\mathbb{R}^N)$ *is trivial.*

2 Estimates on potentials

Throughout this article c_j, j=1,2,..., denote structural positive constants and c_N is the volume of the unit ball in $\mathbb{R}^N$. The following inequality will be used several times in the sequel.

Lemma 2.1 *Let* $\kappa, \gamma, \theta \in \mathbb{R}$*, such that* $\kappa, \gamma > 0$*. Let* $h : (0, \infty) \to (0, \infty)$ *be nondecreasing. Then,*

$$\int_0^R t^\kappa \left(\int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma \frac{dt}{t} \leq c_{35} \int_0^{2R} t^{\kappa+\theta\gamma} h^\gamma(t) \frac{dt}{t} \qquad \forall R \in (0, \infty], \tag{2.1}$$

for some $c_{35} > 0$ *depending on* κ*,* γ*,* θ*.*

Proof. *Case 1:* $\gamma \leq 1$. Since there holds

$$\left(\sum_{j=0}^\infty a_j \right)^\gamma \leq \sum_{j=0}^\infty a_j^\gamma \qquad \forall a_j \geq 0,$$

we derive

$$\begin{aligned} \left(\int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma &\leq c_{\gamma,\theta} \left(\sum_{j=0}^{j_0} h(2^{\frac{j+1}{4}} t)(2^{\frac{j}{4}} t)^\theta \right)^\gamma \\ &\leq c_{\gamma,\theta} \sum_{j=0}^{j_0} \left(h^\gamma(2^{\frac{j+1}{4}} t) \right) (2^{\frac{j}{4}} t)^{\theta\gamma} \\ &\leq c_{\gamma,\theta} \int_t^{2R} h^\gamma(r) r^{\theta\gamma} \frac{dr}{r} \end{aligned}$$

where $c_{\gamma,\theta} = 2^{\frac{\gamma}{4}} \max\{1, 2^{-\frac{\gamma\theta}{4}}\}$ and $2^{\frac{j_0}{4}} t < R \leq 2^{\frac{j_0+1}{4}} t$ if $R < \infty$ and $j_0 = \infty$ if $R = \infty$. By Fubini's theorem,

$$\begin{aligned} \int_0^R t^\kappa \left(\int_t^R h(r) r^\theta \frac{dr}{r} \right)^\gamma \frac{dt}{t} &\leq c_{\gamma,\theta} \int_0^R t^\kappa \int_t^{2R} h^\gamma(r) r^{\theta\gamma} \frac{dr}{r} \frac{dt}{t} \\ &\leq \frac{c_{\gamma,\theta}}{\kappa} \int_0^{2R} t^{\kappa+\theta\gamma} h^\gamma(t) \frac{dt}{t}. \end{aligned}$$

which is (2.1).

Case 2: $\gamma > 1$ Since

$$\left(\int_t^R h(r) r^\theta \frac{dr}{r}\right)^\gamma \le \left(\int_t^R r^{-\frac{\gamma}{\gamma-1}} \frac{dr}{r}\right)^{\gamma-1} \int_t^R h^\gamma(r) r^{\gamma(1+\theta)} \frac{dr}{r},$$

we derive by Fubini's theorem

$$\int_0^R t^\kappa \left(\int_t^R h(r) r^\theta \frac{dr}{r}\right)^\gamma \frac{dt}{t} \le c_{_{\gamma,\kappa}} \int_0^{2R} t^{\kappa+\theta\gamma} h^\gamma(t) \frac{dt}{t},$$

which completes the proof. □

We recall that if $\alpha > 0$, $1 < \beta < \frac{N}{\alpha}$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$, the set of positive Radon measures in $\mathbb{R}^N$, the Wolff potential of μ is defined by

$$\mathbf{W}_{\alpha,\beta}[\mu](x) = \int_0^\infty \left(\frac{\mu(B_r(x))}{r^{N-\alpha\beta}}\right)^{\frac{1}{p-1}} \frac{dr}{r}, \tag{2.2}$$

and if $R > 0$, the R-truncated Wolff potential of μ is

$$\mathbf{W}^R_{\alpha,\beta}[\mu](x) = \int_0^R \left(\frac{\mu(B_r(x))}{r^{N-\alpha\beta}}\right)^{\frac{1}{p-1}} \frac{dr}{r}. \tag{2.3}$$

If μ is a Radon measure on a Borel set G, it's Wolff potential (or truncated Wolff potential) is the potential of its extension by 0 in G^c. We start with the following composition estimate on Wolff potentials.

Lemma 2.2 *Let $1 < \beta < N/\alpha$. Then for any $q > 0$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ we have*

$$\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu] \le c_{_{36}} \mathbf{W}_{\alpha,\beta}\left[(\mathbf{W}_{\alpha,\beta}[\mu])^q\right], \tag{2.4}$$

in $\mathbb{R}^N$ for some $c_{_{36}} > 0$ depending on α, β, N, q. Moreover, if $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$, there holds

$$\mathbf{W}_{\alpha,\beta}\left[(\mathbf{W}_{\alpha,\beta}[\mu])^q\right](x) \le c_{_{37}} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu], \tag{2.5}$$

in $\mathbb{R}^N$, where $c_{_{37}} > 0$ depends on α, β, N, q.

Proof. For any $x \in \mathbb{R}^N$, using the fact that $y \in B_t(x) \Longrightarrow B_t(x) \subset B_{2t}(y)$, we have

$$\begin{aligned}
\mathbf{W}_{\alpha,\beta}\left[(\mathbf{W}_{\alpha,\beta}[\mu])^q\right](x) &= \int_0^\infty \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\int_0^\infty \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}}\right)^{\frac{1}{\beta-1}} \frac{dr}{r}\right)^q dy\right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\
&\ge c_{_{38}} \int_0^\infty \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\frac{\mu(B_{2t}(y))}{t^{N-\alpha\beta}}\right)^{\frac{q}{\beta-1}} dy\right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\
&\ge c_{_{36}} \int_0^\infty \left(t^{\frac{\alpha\beta(\beta-1)}{q}} \frac{\mu(B_t(x))}{t^{N-\alpha\beta}}\right)^{\frac{q}{(\beta-1)^2}} \frac{dt}{t} \\
&= c_{_{36}} \mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu](x).
\end{aligned}$$

where $c_{_{38}} = c_{_{38}}(\alpha, \beta, N, q) > 0$, which proves (2.4).

In order to prove (2.4) we recall the following estimate on Wolff potentials [7]

$$||\mathbf{W}_{\alpha,\beta}[\omega]||_{L^{\frac{(\beta-1)N}{N-\alpha\beta},\infty}} \le c_{_{39}} \left(\omega(\mathbb{R}^N)\right)^{\frac{1}{\beta-1}} \qquad \forall\, \omega \in \mathfrak{M}^+_b(\mathbb{R}^N), \tag{2.6}$$

where $L^{\frac{(p-1)N}{N-\alpha\beta},\infty}$ denotes the weak-$L^{\frac{(p-1)N}{N-\alpha\beta}}$ space. In particular, since $0<q<\frac{N(\beta-1)}{N-\alpha\beta}$,

$$\int_{B_r(x)}(\mathbf{W}_{\alpha,\beta}[\omega])^q\,dy\le c_{40}r^N\left(\frac{\omega(\mathbb{R}^N)}{r^{N-\alpha\beta}}\right)^{\frac{q}{\beta-1}}\qquad\forall x\in\mathbb{R}^N,\ \forall r>0. \tag{2.7}$$

Applying this inequality to $\omega=\chi_{B_{2r}(x)}\mu$ yields

$$\int_{B_r(x)}\left(\mathbf{W}^r_{\alpha,\beta}[\mu]\right)^q dy\le c_{40}r^N\left(\frac{\mu(B_{2r}(x))}{r^{n-\alpha\beta}}\right)^{\frac{q}{\beta-1}}\qquad\forall x\in\mathbb{R}^N,\ \forall r>0. \tag{2.8}$$

We claim that

$$\begin{aligned}I&:=\int_0^\infty\left(\frac{1}{t^{N-\alpha\beta}}\int_{B_t(x)}\left(\int_t^\infty\left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}}\right)^{\frac{1}{\beta-1}}\frac{dr}{r}\right)^q dy\right)^{\frac{1}{\beta-1}}\frac{dt}{t}\\&\le c_{37}\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\mu](x).\end{aligned} \tag{2.9}$$

Since $B_r(y)\subset B_{2r}(x)$ for any $y\in B_t(x), r\ge t$, we have

$$\begin{aligned}\int_{B_t(x)}\left(\int_t^\infty\left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}}\right)^{\frac{1}{\beta-1}}\frac{dr}{r}\right)^q dy&\le\int_{B_t(x)}\left(\int_t^\infty\left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}}\right)^{\frac{1}{\beta-1}}\frac{dr}{r}\right)^q dy\\&\le c_N t^N\left(\int_t^\infty\left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}}\right)^{\frac{1}{\beta-1}}\frac{dr}{r}\right)^q.\end{aligned}$$

Hence,

$$I\le c_N^{\frac{1}{\beta-1}}\int_0^\infty t^{\frac{\alpha\beta}{\beta-1}}\left(\int_t^\infty\left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}}\right)^{\frac{1}{\beta-1}}\frac{dr}{r}\right)^{\frac{q}{\beta-1}}\frac{dt}{t}.$$

Using Lemma 2.1, we infer

$$I\le c_{37}\int_0^\infty r^{\frac{\alpha\beta}{\beta-1}}\left(\frac{\mu(B_r(x))}{r^{N-\alpha\beta}}\right)^{\frac{q}{(\beta-1)^2}}\frac{dr}{r}=c_{37}\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\mu](x),$$

which completes the proof. □

The following is a version of Lemma 2.2 for truncated Wolff potentials,

Lemma 2.3 *Let $1<\beta<N/\alpha$ and $q>0$. If $\delta\in(0,1)$ there holds for any $\mu\in\mathfrak{M}^+(\mathbb{R}^N)$*

$$\mathbf{W}^{\frac{\delta d}{2}}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\mu](x)\le c_{42}\mathbf{W}^{\delta d}_{\alpha,\beta}\left[\left(\mathbf{W}^{\delta d(.)}_{\alpha,\beta}[\mu]\right)^q\right](x), \tag{2.10}$$

in Ω. Moreover, if $0<q<\frac{N(\beta-1)}{N-\alpha\beta}$, there holds for any $\mu\in\mathfrak{M}^+(\mathbb{R}^N)$,

$$\mathbf{W}^R_{\alpha,\beta}\left[\left(\mathbf{W}^R_{\alpha,\beta}[\mu]\right)^q\right](x)\le c_{43}\mathbf{W}^{4R}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\mu](x), \tag{2.11}$$

in $\mathbb{R}^N$.

Proof. For any $x\in\Omega$,

$$\begin{aligned}&\mathbf{W}^{\delta d(x)}_{\alpha,\beta}\left[\left(\mathbf{W}^{\delta d(.)}_{\alpha,\beta}[\mu](.)\right)^q\right](x)\\&\qquad=\int_0^{\delta d(x)}\left(\frac{1}{t^{N-\alpha\beta}}\int_{B_t(x)}\left(\int_0^{\delta d(y)}\left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}}\right)^{\frac{1}{\beta-1}}\frac{dr}{r}\right)^q dy\right)^{\frac{1}{\beta-1}}\frac{dt}{t}.\end{aligned}$$

Since $\delta d(y) \geq \frac{7\delta}{8} d(x)$ for all $y \in B_{\frac{t}{8}}(x), 0 < t < \delta d(x)$,

$$\begin{aligned}\int_{B_t(x)} \left(\int_0^{\delta d(y)} \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy &\geq \int_{B_{t/8}(x)} \left(\int_0^{\frac{7\delta}{8} d(x)} \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \\ &\geq c_{44} (\delta d(x))^{-\frac{q(N-\alpha\beta)}{\beta-1}} \int_{B_{t/8}(x)} \left(\frac{\mu(B_{\frac{3t}{4}}(y))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} dy \\ &\geq c_{44} (\delta d(x))^{-\frac{q(N-\alpha\beta)}{\beta-1}} \int_{B_{t/8}(x)} \left(\frac{\mu(B_{\frac{3t}{4}-\frac{t}{8}}(y))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} dy \\ &\geq c_{45} (\delta d(x))^{-\frac{q(N-\alpha\beta)}{\beta-1}} t^N \left(\frac{\mu(B_{\frac{t}{2}}(x))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}}.\end{aligned}$$

Hence

$$\mathbf{W}_{\alpha,\beta}^{\delta d(x)} \left[\left(\mathbf{W}_{\alpha,\beta}^{\delta d(.)}[\mu](.) \right)^q \right](x) \geq c_{46} (\delta d(x))^{-\frac{q(N-\alpha\beta)}{(\beta-1)^2}} \left(\int_0^{\delta d(x)} t^{\alpha\beta} \left(\frac{\mu(B_{\frac{t}{2}}(x))}{t^{N-\alpha\beta}} \right)^{\frac{q}{\beta-1}} \frac{dt}{t} \right)^{\frac{1}{\beta-1}},$$

which implies (2.10).

Because of (2.8), it is enough to show that there holds

$$\int_0^R \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\int_t^R \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \leq c_{47} \mathbf{W}^{4R}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\mu](x), \tag{2.12}$$

in order to prove (2.11). Since $B_\rho(y) \subset B_{2\rho}(x)$ for any $y \in B_r(x), \rho \geq r$, we have

$$\begin{aligned}\int_{B_t(x)} \left(\int_t^R \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy &\leq \int_{B_t(x)} \left(\int_t^R \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \\ &\leq c_N t^N \left(\int_t^R \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^R \left(\frac{1}{t^{N-\alpha\beta}} \int_{B_t(x)} \left(\int_t^R \left(\frac{\mu(B_r(y))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q dy \right)^{\frac{1}{\beta-1}} \frac{dt}{t} \\ &\qquad \leq c_N \int_0^R \left(t^{\alpha\beta} \left(\int_t^R \left(\frac{\mu(B_{2r}(x))}{r^{N-\alpha\beta}} \right)^{\frac{1}{\beta-1}} \frac{dr}{r} \right)^q \right)^{\frac{1}{\beta-1}} \frac{dt}{t}.\end{aligned}$$

We infer (2.12) by Lemma 2.1, which completes the proof. □

The next two propositions link Wolff potentials of a measure with Riesz capaciticies (in the case of whole space) and truncated Wolff potentials with Bessel capaciticies (in the bounded domain case). Their proof can be found in [23, 24] (and [8] with a different method).

Proposition 2.4 *Let $1 < \beta < N/\alpha$, $q > \beta - 1$, $\nu \in \mathfrak{M}^+(\mathbb{R}^N)$. Then, the following statements are equivalent:*

(a) The inequality

$$\nu(K) \leq c_{48} \, Cap_{\mathbf{I}_{\alpha\beta}, \frac{q}{q-\beta+1}}(K) \tag{2.13}$$

holds for any compact set $K \subset \mathbb{R}^N$, for some $c_{48} > 0$.
(b) The inequality

$$\int_{\mathbb{R}^N} \left(\mathbf{W}_{\alpha,\beta}[\chi_{B_t(x)}\nu](y)\right)^q dy \le c_{49}\nu(B_t(x)) \tag{2.14}$$

holds for any ball $B_t(x) \subset \mathbb{R}^N$, for some $c_{49} > 0$.
(c) The inequality

$$\mathbf{W}_{\alpha,\beta}\left[(\mathbf{W}_{\alpha,\beta}[\nu])^q\right] \le c_{50}\mathbf{W}_{\alpha,\beta}[\nu] < \infty \ \ a.e \ in \ \ \mathbb{R}^N \tag{2.15}$$

holds for some $c_{50} > 0$.

Proposition 2.5 *Let $1 < \beta < N/\alpha$, $q > \beta - 1$, $R > 0$ and $\nu \in \mathfrak{M}_b^+(B_R(x_0))$ for some $x_0 \in \mathbb{R}^N$. Then, the following statements are equivalent:*
(a) The inequality

$$\nu(K) \le c_{51}\, Cap_{\mathbf{G}_{\alpha\beta},\frac{q}{q-\beta+1}}(K) \tag{2.16}$$

holds for any compact set $K \subset \mathbb{R}^N$, for some $c_{51} = c_{51}(R) > 0$.
(b) The inequality

$$\int_{\mathbb{R}^N} \left(\mathbf{W}^{4R}_{\alpha,\beta}[\chi_{B_t(x)}\nu](y)\right)^q dy \le c_{52}\nu(B_t(x)) \tag{2.17}$$

holds for any ball $B_t(x) \subset \mathbb{R}^N$, for some $c_{52} = c_{52}(R) > 0$.
(c) The inequality

$$\mathbf{W}^{4R}_{\alpha,\beta}\left[\left(\mathbf{W}^{4R}_{\alpha,\beta}[\nu]\right)^q\right] \le c_{53}\mathbf{W}^{4R}_{\alpha,\beta}[\nu] \quad a.e \ in \ \ B_{2R}(x_0) \tag{2.18}$$

holds for some $c_{53} = c_{53}(R) > 0$.

In the following statement we show how to give capacitary estimates on combination of measures.

Proposition 2.6 *Let η, μ be in $\mathfrak{M}^+(\mathbb{R}^N)$. Assume that $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$ and $qs > (\beta-1)^2$.*

(i) If there holds

$$\eta(K) + \int_K (\mathbf{W}_{\alpha,\beta}[\mu])^s\, dx \le \mathrm{Cap}_{\mathbf{I}_{\frac{\alpha\beta(q+\beta-1)}{q}},\frac{qs}{qs-(\beta-1)^2}}(K) \tag{2.19}$$

for any compact set $K \subset \mathbb{R}^N$, then

$$\mathbf{W}_{\alpha,\beta}\left[(\mathbf{W}_{\alpha,\beta}\left[(\mathbf{W}_{\alpha,\beta}[\omega])^q\right])^s\right] \le c_{54}\mathbf{W}_{\alpha,\beta}[\omega] < \infty \quad a.e \ in \quad \mathbb{R}^N, \tag{2.20}$$

where $\omega = (\mathbf{W}_{\alpha,\beta}[\mu])^s + \eta$.
(ii) If there holds

$$\eta(K) + \int_K \left(\mathbf{W}^{2R}_{\alpha,\beta}[\mu]\right)^s dx \le \mathrm{Cap}_{\mathbf{G}_{\frac{\alpha\beta(q+\beta-1)}{q}},\frac{qs}{qs-(\beta-1)^2}}(K) \tag{2.21}$$

for any compact set $K \subset \mathbb{R}^N$, then

$$\mathbf{W}^{2R}_{\alpha,\beta}\left[\left(\mathbf{W}^{2R}_{\alpha,\beta}\left[\left(\mathbf{W}^{2R}_{\alpha,\beta}[\omega]\right)^q\right]\right)^s\right] \le c_{55}\mathbf{W}^{2R}_{\alpha,\beta}[\omega] < \infty \quad a.e \ in \quad B_R(x_0), \tag{2.22}$$

where $\omega = \chi_{B_R(x_0)}\left(\mathbf{W}^{2R}_{\alpha,\beta}[\mu]\right)^s + \chi_{B_R(x_0)}\eta$.

Proof. *Statement (i)*: We assume that (2.19) holds. Put $\omega = (\mathbf{W}_{\alpha,\beta}[\mu])^s + \eta$ and apply (2.19) to $K = \overline{B}_{2\rho}(x)$. Since by homogeneity

$$\mathrm{Cap}_{\mathbf{I}_{\frac{\alpha\beta(q+\beta-1)}{q},\frac{qs}{qs-(\beta-1)^2}}}(\overline{B}_{2\rho}(x)) = \rho^{N-\frac{\alpha\beta(q+\beta-1)s}{qs-(\beta-1)^2}}\,\mathrm{Cap}_{\mathbf{I}_{\frac{\alpha\beta(q+\beta-1)}{q},\frac{qs}{qs-(\beta-1)^2}}}(\overline{B}_2(0)),$$

we derive from (2.19)

$$\omega(B_\rho(x)) \le c_{55}\rho^{N-\frac{\alpha\beta(q+\beta-1)s}{qs-(\beta-1)^2}} \qquad \forall\ \ \rho>0,$$

which is equivalent to

$$\rho^{\frac{\alpha\beta}{\beta-1}}\left(\frac{\omega(B_\rho(x))}{\rho^{N-\frac{\alpha\beta(q+\beta-1)}{q}}}\right)^{\frac{qs}{(\beta-1)^3}} \le c_{56}\left(\frac{\omega(B_\rho(x))}{\rho^{N-\alpha\beta}}\right)^{\frac{1}{\beta-1}} \qquad \forall\ \ \rho>0. \tag{2.23}$$

We apply Proposition 2.4 to $\nu=\omega$ with $(\alpha,\beta,q)=\left(\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1,s\right)$, (2.19) implies

$$\int_{\mathbb{R}^N}\left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\chi_{B_t(x)}\omega]\right)^s dy \le c_{57}\omega(B_t(x)). \tag{2.24}$$

By Lemma 2.2, (2.20) is equivalent to

$$\mathbf{W}_{\alpha,\beta}\left[\left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega]\right)^s\right] \le c_{58}\mathbf{W}_{\alpha,\beta}[\omega]<\infty \ \ \text{a.e}\ \ \mathbb{R}^N \tag{2.25}$$

Therefore, it is enough to show that (2.23) and (2.24) imply (2.25). In fact, since for $t>0$

$$\int_{B_t(x)}\left(\mathbf{W}^t_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega](y)\right)^s dy = \int_{B_t(x)}\left(\mathbf{W}^t_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\chi_{B_{2t}(x)}\omega](y)\right)^s dy,$$

we apply (2.24) and obtain

$$\int_{B_t(x)}\left(\mathbf{W}^t_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega](y)\right)^s dy \le c_{57}\omega(B_{2t}(x)).$$

So, it is enough to show that

$$I := \int_0^\infty\left(\frac{1}{t^{N-\alpha\beta}}\int_{B_t(x)}\left(\int_t^\infty\left(\frac{\omega(B_r(y))}{r^{N-\frac{\alpha\beta(q+\beta-1)}{q}}}\right)^{\frac{q}{(\beta-1)^2}}\frac{dr}{r}\right)^s dy\right)^{\frac{1}{\beta-1}}\frac{dt}{t} \le c_{58}\mathbf{W}_{\alpha,\beta}[\omega](x). \tag{2.26}$$

Since $B_r(y)\subset B_{2r}(x)$ for any $y\in B_t(x), r\ge t$, we have

$$\begin{aligned} I &\le c_{_N}\int_0^\infty\left(t^{\alpha\beta}\left(\int_t^\infty\left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{\alpha\beta(q+\beta-1)}{q}}}\right)^{\frac{q}{(\beta-1)^2}}\frac{dr}{r}\right)^s\right)^{\frac{1}{\beta-1}}\frac{dt}{t}\\ &= c_{_N}\int_0^\infty t^{\frac{\alpha\beta}{\beta-1}}\left(\int_t^\infty\left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{\alpha\beta(q+\beta-1)}{q}}}\right)^{\frac{q}{(\beta-1)^2}}\frac{dr}{r}\right)^{\frac{s}{\beta-1}}\frac{dt}{t}\end{aligned}$$

It follows from Lemma 2.1 and (2.23) that

$$I \le c_{59}\int_0^\infty t^{\frac{\alpha\beta}{\beta-1}}\left(\frac{\omega(B_{2t}(x))}{t^{N-\frac{\alpha\beta(q+\beta-1)}{q}}}\right)^{\frac{qs}{(\beta-1)^3}}\frac{dt}{t} \le c_{56}c_{59}\int_0^\infty\left(\frac{\omega(B_{2t}(x))}{t^{N-\alpha\beta}}\right)^{\frac{1}{\beta-1}}\frac{dt}{t},$$

which is (2.26).
Statement (ii): We assume that (2.21) holds. Put $d\omega = \chi_\Omega \left(\mathbf{W}_{\alpha,\beta}[\mu]\right)^s + \chi_\Omega \eta$, then

$$\omega(B_\rho(x)) \le c_{60}\rho^{N-\frac{\alpha\beta(q+\beta-1)s}{qs-(\beta-1)^2}} \qquad \forall\ 0<\rho<2R.$$

As in the proof of statement (i), the above inequality is equivalent to

$$\rho^{\frac{\alpha\beta}{\beta-1}}\left(\frac{\omega(B_\rho(x))}{\rho^{N-\frac{\alpha\beta(q+\beta-1)}{q}}}\right)^{\frac{qs}{(\beta-1)^3}} \le c_{61}\left(\frac{\omega(B_\rho(x))}{\rho^{N-\alpha\beta}}\right)^{\frac{1}{\beta-1}} \qquad \forall\ 0<\rho<2R. \tag{2.27}$$

Applying Proposition 2.5 with $\nu=\omega$ and $(\alpha,\beta,q)=\left(\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1,s\right)$,

$$\int_{\mathbb{R}^N}\left(\mathbf{W}^{4R}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\chi_{B_t(x)}\omega]\right)^s dy \le c_{62}\omega(B_t(x)). \tag{2.28}$$

By Lemma 2.3, (2.22) is equivalent to

$$\mathbf{W}^{4R}_{\alpha,\beta}\left[\left(\mathbf{W}^{4R}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega]\right)^s\right] \le c_{63}\mathbf{W}^{4R}_{\alpha,\beta}[\omega] \qquad \text{a.e in } B_R(x_0). \tag{2.29}$$

Therefore, it is enough to show that (2.27) and (2.28) imply (2.29). In fact, since

$$\int_{B_t(x)}\left(\mathbf{W}^{t}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega](y)\right)^s dy = \int_{B_t(x)}\left(\mathbf{W}^{t}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\chi_{B_{2t}(x)}\omega](y)\right)^s dy$$

for all $0<t<4R$ thus applying (2.28), we obtain

$$\int_{B_t(x)}\left(\mathbf{W}^{t}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega](y)\right)^s dy \le c_{64}\omega(B_{2t}(x)).$$

So, it is enough to show that for any $x\in B_R(x_0)$

$$II := \int_0^{4R}\left(\frac{1}{t^{N-\alpha\beta}}\int_{B_t(x)}\left(\int_t^{4R}\left(\frac{\omega(B_r(y))}{r^{N-\frac{\alpha\beta(q+\beta-1)}{q}}}\right)^{\frac{q}{(\beta-1)^2}}\frac{dr}{r}\right)^s dy\right)^{\frac{1}{\beta-1}}\frac{dt}{t} \le c_{65}\mathbf{W}^{4R}_{\alpha,\beta}[\omega](x). \tag{2.30}$$

Since $B_r(y)\subset B_{2r}(x)$ for any $y\in B_t(x), r\ge t$, we have

$$II \le c_N\int_0^{4R} t^{\frac{\alpha\beta}{\beta-1}}\left(\int_t^{4R}\left(\frac{\omega(B_{2r}(x))}{r^{n-\frac{\alpha\beta(q+\beta-1)}{q}}}\right)^{\frac{q}{(\beta-1)^2}}\frac{dr}{r}\right)^{\frac{s}{\beta-1}}\frac{dt}{t}.$$

Combining this with Lemma 2.1 and (2.27) yields

$$II \le c_{66}\mathbf{W}^{16R}_{\alpha,\beta}[\omega](x).$$

Therefore, (2.29) follows since $\mathbf{W}^{16R}_{\alpha,\beta}[\omega]\le c_{67}\mathbf{W}^{4R}_{\alpha,\beta}[\omega]$ in $B_R(x_0)$. □

Proposition 2.7 *Let η,μ be in $\mathfrak{M}^+(\mathbb{R}^N)$. Assume that $0<q<\frac{N(\beta-1)}{N-\alpha\beta}$ and $qs>(\beta-1)^2$. Let (u_m,v_m) be nonnegative measurable funtions in $\mathbb{R}^N$ such that for all $m\ge 0$*

$$u_{m+1}\le c^*\mathbf{W}_{\alpha,\beta}[v_m^q+\mu], \qquad v_{m+1}\le c^*\mathbf{W}_{\alpha,\beta}[u_m^s+\eta] \qquad \text{a.e. in} \quad \mathbb{R}^N$$

for some $c^*>0$ and $(u_0,v_0)=0$. Then, there exists a constant $M^*>0$ depending only on N,α,β,q,s,c^* such that if $d\omega=(\mathbf{W}_{\alpha,\beta}[\mu])^s\,dx+d\eta$ satisfies

$$\omega(K)\le M^*\,\mathrm{Cap}_{\mathbf{I}_{\frac{\alpha\beta(q+\beta-1)}{q},\frac{qs}{qs-(\beta-1)^2}}}(K), \tag{2.31}$$

for any compact set $K\subset\mathbb{R}^N$, then

$$v_m\le c_{69}\mathbf{W}_{\alpha,\beta}[\omega],\quad u_m\le c_{70}\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q]+c_{68}\mathbf{W}_{\alpha,\beta}[\mu]\ \ \forall\ \ m\ge 0 \tag{2.32}$$

for some constants c_{68},c_{69},c_{70} depending only on N,α,β,q,s and c^*.

Proof. By Proposition 2.6, (2.31) implies

$$\mathbf{W}_{\alpha,\beta}\left[(\mathbf{W}_{\alpha,\beta}\left[(\mathbf{W}_{\alpha,\beta}[\omega])^q\right])^s\right]\le c_{71}M^{\frac{qs}{(\beta-1)^3}}\mathbf{W}_{\alpha,\beta}[\omega]<\infty\qquad\text{a.e in }\ \mathbb{R}^N. \tag{2.33}$$

We set

$$\begin{aligned}
c_{68}&=c^*2^{\frac{1}{\beta-1}},\\
c_{69}&=c^*2^{1+\frac{1}{\beta-1}}(c_{68}^s2^{s-1}+1)^{\frac{1}{\beta-1}},\\
c_{70}&=c^*2^{\frac{1}{\beta-1}}c_{69}^{\frac{q}{\beta-1}},
\end{aligned}$$

and choose $M^*>0$ such that

$$c^*2^{\frac{1}{\beta-1}}\left(c_{70}^s2^{s-1}\right)^{\frac{1}{\beta-1}}c_{71}M^{*\,\frac{qs}{(\beta-1)^3}}=\frac{c_{69}}{2}.$$

We claim that

$$v_m\le c_{69}\mathbf{W}_{\alpha,\beta}[\omega],\quad u_m\le c_{70}\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q]+c_{68}\mathbf{W}_{\alpha,\beta}[\mu]\qquad\forall\ \ m\ge 0. \tag{2.34}$$

Clearly, by definition of c_{68},c_{69},c_{70}, we have (2.34) for $m=0,1$. Next we assume that (2.34) holds for all $m\le l$ for some $l\in\mathbb{N}_+^*$, then

$$\begin{aligned}
u_{l+1}&\le c^*\mathbf{W}_{\alpha,\beta}[v_l^q+\mu]\\
&\le c^*2^{\frac{1}{\beta-1}}c_{69}^{\frac{q}{\beta-1}}\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q]+c^*2^{\frac{1}{\beta-1}}\mathbf{W}_{\alpha,\beta}[\mu]\\
&=c_{70}\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q]+c_{68}\mathbf{W}_{\alpha,\beta}[\mu]
\end{aligned}$$

and

$$\begin{aligned}
v_{l+1}&\le c^*\mathbf{W}_{\alpha,\beta}[(c_{70}\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q]+c_{68}\mathbf{W}_{\alpha,\beta}[\mu])^s+\eta]\\
&\le c^*\mathbf{W}_{\alpha,\beta}[c_{70}^s2^{s-1}\left(\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q]\right)^s+c_{68}^s2^{s-1}(\mathbf{W}_{\alpha,\beta}[\mu])^s+\eta]\\
&\le c^*2^{\frac{1}{\beta-1}}\left(c_{70}^s2^{s-1}\right)^{\frac{1}{\beta-1}}\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\omega])^q])^s]\\
&\quad+c^*2^{\frac{1}{\beta-1}}(c_{68}^s2^{s-1}+1)^{\frac{1}{\beta-1}}\mathbf{W}_{\alpha,\beta}[(\mathbf{W}_{\alpha,\beta}[\mu])^s+\eta]\\
&\le c^*2^{\frac{1}{\beta-1}}\left(c_{70}^s2^{s-1}\right)^{\frac{1}{\beta-1}}c_{71}M^{*\,\frac{qs}{(\beta-1)^3}}\mathbf{W}_{\alpha,\beta}[\omega]+c^*2^{\frac{1}{\beta-1}}(c_{68}^s2^{s-1}+1)^{\frac{1}{\beta-1}}\mathbf{W}_{\alpha,\beta}[\omega]\\
&=\frac{c_{69}}{2}\mathbf{W}_{\alpha,\beta}[\omega]+\frac{c_{69}}{2}\mathbf{W}_{\alpha,\beta}[\omega]\\
&=c_{69}\mathbf{W}_{\alpha,\beta}[\omega].
\end{aligned}$$

Thus, (2.34) is true for $m=l+1$. Hence, (2.34) is true for all $l\ge 0$. □

The next result is an adaptation of Proposition 2.7 to truncated Wolff potentials.

Proposition 2.8 *Let η, μ be in $\mathfrak{M}_b^+(B_R(x_0))$. Assume that $0 < q < \frac{N(\beta-1)}{N-\alpha\beta}$ and $qs > (\beta-1)^2$. Let (u_m, v_m) be nonnegative measurable funtions in $\mathbb{R}^N$ such that for all $m \geq 0$*

$$u_{m+1} \leq c_* \mathbf{W}^R_{\alpha,\beta}[\chi_{B_R(x_0)} v_m^q + \mu], \quad v_{m+1} \leq c_* \mathbf{W}^R_{\alpha,\beta}[\chi_{B_R(x_0)} u_m^s + \eta] \qquad a.e. \ in \quad B_R(x_0)$$

and $(u_0, v_0) = 0$. If we set $d\omega = \left(\mathbf{W}^{2R}_{\alpha,\beta}[\mu]\right)^s dx + d\eta$, there exists a constant $M_* > 0$ depending only on $N, \alpha, \beta, q, s, R$ and c_* such that if

$$\omega(K) \leq M_* \operatorname{Cap}_{\mathbf{G}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(K) \tag{2.35}$$

for any compact set $K \subset \mathbb{R}^N$, then

$$v_m \leq c_{73} \mathbf{W}^{2R}_{\alpha,\beta}[\omega], \quad u_m \leq c_{74} \mathbf{W}^{2R}_{\alpha,\beta}[\left(\mathbf{W}^{2R}_{\alpha,\beta}[\omega]\right)^q] + c_{72} \mathbf{W}^{2R}_{\alpha,\beta}[\mu] \qquad \forall\ k \geq 0 \tag{2.36}$$

in $B_R(x_0)$ for some constants c_{72}, c_{73}, c_{74} depending only on $N, \alpha, \beta, q, s, R$ and c_*.

Proof. The proof is similar to the one of Proposition 2.7 and we omit the details. □

Proposition 2.9 *Let $1 < \beta < N/\alpha$ and $q, s > 0$ such that $qs > (\beta-1)^2$.*
(i) Assume that η and μ belong to $\mathfrak{M}_b^+(\mathbb{R}^N)$ and (u, v) are nonnegative measurable functions satisfying

$$\begin{cases} \mathbf{W}_{\alpha,\beta}[v^q] + \mathbf{W}_{\alpha,\beta}[\mu] \leq c_{75} u, \\ \mathbf{W}_{\alpha,\beta}[u^s] + \mathbf{W}_{\alpha,\beta}[\eta] \leq c_{75} v \qquad a.e. \ in \ \mathbb{R}^N, \end{cases} \tag{2.37}$$

for some $c_{75} > 0$. Then there exists a constant $c_{76} > 0$ depending only on N, α, β, q, s and c_{75} such that

$$\eta(K) + \int_K (\mathbf{W}_{\alpha,\beta}[\mu](x))^s \, dx \leq c_{76} \operatorname{Cap}_{\mathbf{I}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(K), \tag{2.38}$$

for any compact set $K \subset \mathbb{R}^N$.
(ii) Assume that η and μ belong to $\mathfrak{M}_b^+(\Omega)$ and (u, v) are nonnegative functions satisfying

$$\begin{cases} \mathbf{W}^{\delta d(.)}_{\alpha,\beta}[v^q] + \mathbf{W}^{\delta d}_{\alpha,\beta}[\mu] \leq c_{77} u, \\ \mathbf{W}^{\delta d(.)}_{\alpha,\beta}[u^s] + \mathbf{W}^{\delta d}_{\alpha,\beta}[\eta] \leq c_{77} v \qquad a.e. \ in \ \Omega, \end{cases} \tag{2.39}$$

for some $c_{77} > 0$. Then for any $\Omega' \subset\subset \Omega$, there exists a constant $c_{78} > 0$ depending only on $n, \alpha, \beta, q, s, c_{77}$ and $dist(\Omega', \partial\Omega)$ such that

$$\eta(K) + \int_K \left(\mathbf{W}^{\delta d(x)}_{\alpha,\beta}[\mu](x)\right)^s dx \leq c_{78} \operatorname{Cap}_{\mathbf{G}_{\frac{\alpha\beta(q+\beta-1)}{q}, \frac{qs}{qs-(\beta-1)^2}}}(K), \tag{2.40}$$

for any compact set $K \subset \Omega'$.

Proof. (i): Set $\omega = u^s + \eta$, then

$$\omega \geq u^s \geq (\mathbf{W}_{\alpha,\beta}[v^q])^s \geq c_{79} \left(\mathbf{W}_{\alpha,\beta}\left[(\mathbf{W}_{\alpha,\beta}[\omega])^q\right]\right)^s.$$

By (2.4) in Lemma 2.2, we get

$$\omega \geq c_{80} \left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2}, \frac{(\beta-1)^2}{q}+1}[\omega]\right)^s$$

which implies

$$\int_{\mathbb{R}^N}\left(\mathbf{W}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\chi_{B_t(x)}\omega]\right)^s dy \le c_{81}\omega(B_t(x)) \qquad \forall\, x\in\mathbb{R}^N,\ \forall\, t>0.$$

Applying Proposition 2.4 to $\mu=\omega$ with $(\alpha,\beta,q)=\left(\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1,s\right)$, we get (2.38).

(ii) We define ω as above and we have

$$\omega\ge u^s\ge\left(\mathbf{W}^{\delta d}_{\alpha,\beta}[v^q]\right)^s\ge c_{82}\left(\mathbf{W}^{\delta d}_{\alpha,\beta}\left[\left(\mathbf{W}^{\delta d(.)}_{\alpha,\beta}[\omega]\right)^q\right]\right)^s \qquad \text{a.e. in } \Omega,$$

which leads to

$$\omega\ge c_{83}\left(\mathbf{W}^{\frac{\delta}{2}d}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega]\right)^s \qquad \text{a.e. in } \Omega,$$

by inequality (2.10) in Lemma 2.3. Let M_ω denote the centered Hardy-Littlewood maximal function which is defined for any $f\in L^1_{loc}(\mathbb{R}^N,d\omega)$ by

$$M_\omega f(x)=\sup_{t>0}\frac{1}{\omega(B_t(x))}\int_{B_t(x)}|f|d\omega.$$

Let $K\subset\Omega$ be compact. Set $r_K=dist(K,\partial\Omega)$ and $\Omega_K=\{x\in\Omega: d(x,K)<r_K/2\}$. Then, for any Borel set $E\subset K$,

$$c_{84}\int_\Omega\left(M_\omega\chi_E\right)^{\frac{sq}{(\beta-1)^2}}\left(\mathbf{W}^{\frac{\delta}{2}d(x)}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega]\right)^s dx\le\int_\Omega\left(M_\omega\chi_E\right)^{\frac{sq}{(\beta-1)^2}}d\omega.$$

Since M_ω is a bounded linear map on $L^p(\mathbb{R}^N,d\omega)$ for any $p>1$ and

$$\left(M_\omega\chi_E\right)^{\frac{sq}{(\beta-1)^2}}\left(\mathbf{W}^{\frac{\delta}{2}d(x)}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega]\right)^s\ge\int_0^{\frac{\delta}{2}d(x)}\left(\frac{\omega(B_t(x)\cap E)}{\omega(B_t(x))}\frac{\omega(B_t(x)}{t^{N-\frac{\alpha\beta(q+\beta-1)}{q}}}\right)^{\frac{sq}{(\beta-1)^2}}\frac{dt}{t}$$

we derive

$$\int_\Omega\left(\mathbf{W}^{\frac{\delta}{2}d(x)}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega_E]\right)^s dx\le c_{85}\omega(E),$$

where $\omega_E=\chi_E\omega$. Note that if $x\in\Omega$ and $d(x)\le r_K/8$, then $B_t(x)\subset\Omega\backslash\Omega_K$ for all $t\in(0,\frac{\delta d(x)}{2})$; indeed, for all $y\in B_t(x)$

$$d(y,\partial\Omega)\le d(x,\partial\Omega)+|x-y|<(1+\delta)d(x,\partial\Omega)<\frac{1}{4}r_K,$$

thus

$$d(y,K)\ge d(K,\partial\Omega)-d(y,\partial\Omega)>\frac{3}{4}r_K>\frac{1}{2}r_K,$$

which implies $y\notin\Omega_K$. We deduce that

$$\mathbf{W}^{\frac{\delta}{2}d(x,\partial\Omega)}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega_E](x)\ge\mathbf{W}^{\frac{\delta}{16}r_K}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega_E](x) \qquad \forall x\in\Omega,$$

and

$$\mathbf{W}^{\frac{\delta}{16}r_K}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega_E](x)=0 \qquad \forall x\in\Omega^c.$$

Hence we obtain

$$\int_{\mathbb{R}^N}\left(\mathbf{W}^{\frac{\delta}{16}r_K}_{\frac{\alpha\beta(q+\beta-1)}{q+(\beta-1)^2},\frac{(\beta-1)^2}{q}+1}[\omega_E]\right)^s dx\le c_{85}\omega(E) \qquad \forall E\subset K,\ E \text{ Borel}. \tag{2.41}$$

Applying Proposition 2.5 with $\mu=\chi_{K\cap B_{2^{-6}\delta r_K^{(x)}}}\omega$ we get (2.40), which completes the proof. □

3 Quasilinear Dirichlet problems

Let Ω be a bounded domain in $\mathbb{R}^N$. If $\mu \in \mathfrak{M}_b(\Omega)$, we denote by μ^+ and μ^- respectively its positive and negative parts in the Jordan decomposition. We denote by $\mathfrak{M}_0(\Omega)$ the space of measures in Ω which are absolutely continuous with respect to the $c^\Omega_{1,p}$-capacity defined on a compact set $K \subset \Omega$ by

$$c^\Omega_{1,p}(K) = \inf\left\{\int_\Omega |\nabla\varphi|^p dx : \varphi \geq \chi_K, \varphi \in C^\infty_c(\Omega)\right\}.$$

We also denote $\mathfrak{M}_s(\Omega)$ the space of measures in Ω with support on a set of zero $c^\Omega_{1,p}$-capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega) \cap \mathfrak{M}_b(\Omega)$. It is well known that any $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ can be written under the form $\mu_0 = f - div\ g$ where $f \in L^1(\Omega)$ and $g \in L^{p'}(\Omega, \mathbb{R}^N)$.

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s, k\}, -k\}$. If u is a measurable function defined in Ω, finite a.e. and such that $T_k(u) \in W^{1,p}_{loc}(\Omega)$ for any $k > 0$, there exists a measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{\{|u|\leq k\}} v$ a.e. in Ω and for all $k > 0$. We define the gradient a.e. ∇u of u by $v = \nabla u$. We recall the definition of a renormalized solution given in [12].

Definition 3.1 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of*

$$\begin{array}{rl} -\Delta_p u = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{3.1}$$

if $T_k(u) \in W^{1,p}_0(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$, and u has the property that for any $k > 0$ there exist λ^+_k and λ^-_k belonging to $\mathfrak{M}^+_b \cap \mathfrak{M}_0(\Omega)$, respectively concentrated on the sets $u = k$ and $u = -k$, with the property that $\mu^+_k \rightharpoonup \mu^+_s$, $\mu^-_k \rightharpoonup \lambda^-_s$ in the narrow topology of measures and such that

$$\int_{\{|u|<k\}} |\nabla u|^{p-2}\nabla u.\nabla\varphi dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_\Omega \varphi d\lambda^+_k - \int_\Omega \varphi d\lambda^-_k,$$

for every $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$.

Remark 3.2 *We recall that if u is a renormalized solution to problem (3.1), then $\frac{|\nabla u|^p}{(|u|+1)^r} \in L^1(\Omega)$ for all $r > 1$. Furthermore, $u \geq 0$ a.e. in Ω if $\mu \in \mathfrak{M}^+_b(\Omega)$.*

The following general stability result has been proved in [12, Th 4.1].

Theorem 3.3 *Let $\mu = \mu_0 + \mu^+_s - \mu^-_s$, with $\mu_0 = F - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$ and μ^+_s, μ^-_s belonging to $\mathfrak{M}^+_s(\Omega)$. Let $\mu_n = F_n - \operatorname{div} g_n + \rho_n - \eta_n$ with $F_n \in L^1(\Omega)$, $g_n \in (L^{p'}(\Omega))^N$ and ρ_n, η_n belonging to $\mathfrak{M}^+_b(\Omega)$. Assume that $\{F_n\}$ converges to F weakly in $L^1(\Omega)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(\Omega))^N$ and $(\operatorname{div} g_n)$ is bounded in $\mathfrak{M}_b(\Omega)$; assume also that $\{\rho_n\}$ converges to μ^+_s and $\{\eta_n\}$ to μ^-_s in the narrow topology. If $\{u_n\}$ is a sequence of renormalized solutions of (3.1) with data μ_n, then, up to a subsequence, it converges a.e. in Ω to a renormalized solution u of problem (3.1). Furthermore, $T_k(u_n)$ converges to $T_k(u)$ in $W^{1,p}_0(\Omega)$ for any $k > 0$.*

We also recall the following estimate [23, Th 2.1].

Proposition 3.4 *Let Ω be a bounded domain of $\mathbb{R}^N$. Then there exists a constant $C > 0$, depending on p and N such that if $\mu \in \mathfrak{M}^+_b(\Omega)$ and u is a nonnegative renormalized solution of problem (3.1) with data μ, there holds*

$$\frac{1}{c_{86}}\mathbf{W}^{\frac{d(x,\partial\Omega)}{3}}_{1,p}[\mu](x) \leq u(x) \leq c_{86}\mathbf{W}^{2\,diam\,(\Omega)}_{1,p}[\mu](x) \qquad a.e.\ in\ \Omega. \tag{3.2}$$

Proof of Theorem C. *The condition is necessary.* Assume that (1.14) admits a nonnegative renormalized solutions (u, v). By Proposition 3.4 there holds

$$\begin{aligned} u(x) &\geq c_{87}\mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[v^{q_1}+\mu](x),\\ v(x) &\geq c_{87}\mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[u^{q_2}+\mu](x) \qquad \text{a.e. in } \Omega. \end{aligned}$$

Hence, we infer (1.15) from Proposition 2.9-(ii).

Sufficient conditions. Let $\{(u_m, v_m)\}_{m\in\mathbb{N}}$ be a sequence of nonnegative renormalized solutions of the following problems for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= v_m^{q_1}+\mu && \text{in } \Omega,\\ -\Delta_p v_{m+1} &= u_m^{q_2}+\eta && \text{in } \Omega,\\ u_{m+1} &= v_{m+1} = 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.3}$$

with initial condition $(u_0, v_0) = 0$. The sequences $\{u_m\}$ and $\{v_m\}$ can be constructed in such a way that they are nondecreasing (see e.g. [24]). By Proposition 3.4 we have

$$\begin{aligned} u_{m+1} &\leq c_{86}\mathbf{W}_{1,p}^{R}[v_m^{q_1}+\mu](x)\\ v_{m+1} &\leq c_{86}\mathbf{W}_{1,p}^{R}[u_m^{q_2}+\eta](x) \qquad \text{a.e. in } \Omega, \end{aligned}$$

where $R = 2\,diam\,(\Omega)$. Thus, by Proposition 2.8 there exists a constant $M_* > 0$ depending only on N, p, q_1, q_2, R such that if

$$\omega(K) \leq M_*\,\mathrm{Cap}_{\mathbf{G}_{\frac{p(q_1+p-1)}{q_1},\frac{q_1q_2}{q_1q_2-(p-1)^2}}}(K) \tag{3.4}$$

for any compact set $K \subset \mathbb{R}^N$ with $d\omega = \left(\mathbf{W}_{1,p}^{R}[\mu]\right)^{q_2}dx + d\eta$ then

$$v_m \leq c_{73}\mathbf{W}_{1,p}^{R}[\omega], \quad u_m \leq c_{74}\mathbf{W}_{1,p}^{R}[\left(\mathbf{W}_{1,p}^{R}[\omega]\right)^{q_1}] + c_{72}\mathbf{W}_{1,p}^{R}[\mu] \ \ \forall\ \ k \geq 0 \tag{3.5}$$

in Ω and

$$\mathbf{W}_{1,p}^{R}[\omega] \in L^{q_2}(\Omega), \qquad \mathbf{W}_{1,p}^{R}[\left(\mathbf{W}_{1,p}^{R}[\omega]\right)^{q_1}] + \mathbf{W}_{1,p}^{R}[\mu] \in L^{q_1}(\Omega). \tag{3.6}$$

This implies that $\{u_m\}, \{v_m\}_{m\in\mathbb{N}}$ are well defined and nondecreasing. Thus $\{(u_m, v_m)\}$ converges a.e in Ω to some functions (u, v) which satisfies (1.17) in Ω. Furthermore, we deduce from (3.6) and the monotone convergence theorem that $u_m^{q_1} \to u^{q_1}, v_m^{q_2} \to u^{q_2}$ in $L^1(\Omega)$. Finally, by Theorem 3.3 we infer that u is a renormalized solution of (1.14). □

4 p-superharmonic functions and quasilinear equations in $\mathbb{R}^N$

We recall some definitions and properties of p-superharmonic functions.

Definition 4.1 *A function u is said to be p-harmonic in $\mathbb{R}^N$ if $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ and $-\Delta_p u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$; it is always C^1. A function u is called a p-supersolution in $\mathbb{R}^N$ if $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ and $-\Delta_p u \geq 0$ in $\mathcal{D}'(\mathbb{R}^N)$.*

Definition 4.2 *A lower semicontinuous (l.s.c) function $u : \mathbb{R}^N \to (-\infty, \infty]$ is called p-super-harmonic if u is not identically infinite and if, for all open $D \subset\subset \mathbb{R}^N$ and all $v \in C(\overline{D})$, p-harmonic in D, $v \leq u$ on ∂D implies $v \leq u$ in D.*

Let u be a p-superharmonic in $\mathbb{R}^N$. It is well known that $u \wedge k := \min\{u, k\} \in W^{1,p}_{loc}(\mathbb{R}^N)$ is a p-supersolution for all $k > 0$ and $u < \infty$ a.e in $\mathbb{R}^N$, thus, u has a gradient (see the previous section). We also have $|\nabla u|^{p-1} \in L^q_{loc}(\mathbb{R}^N)$, $\frac{|\nabla u|^p}{(|u|+1)^r} \in L^1_{loc}(\mathbb{R}^N)$ and $u \in L^s_{loc}(\mathbb{R}^N)$ for $1 \le q < \frac{N}{N-1}$ and $r > 1$, $1 \le s < \frac{N(p-1)}{N-p}$ (see [16, Theorem 7.46]). Thus for any $0 \le \varphi \in C^1_c(\Omega)$, by the dominated convergence theorem,

$$\langle -\Delta_p u, \varphi\rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u.\nabla\varphi dx = \lim_{k\to\infty} \int_{\mathbb{R}^N} |\nabla(u\wedge k)|^{p-2}\nabla(u\wedge k).\nabla\varphi \ge 0.$$

Hence, by the Riesz Representation Theorem, there is a nonnegative Radon measure denoted by $\mu[u]$, called Riesz measure, such that $-\Delta_p u = \mu[u]$ in $\mathcal{D}'(\mathbb{R}^N)$.

The following weak convergence result for Riesz measures proved in [30] will be used to obtain the existence of p-superharmonic solutions to quasilinear equations.

Proposition 4.3 *Suppose that $\{u_n\}$ is a sequence of nonnegative p-superharmonic functions in $\mathbb{R}^N$ that converges a.e to a p-superharmonic function u. Then the sequence of measures $\{\mu[u_n]\}$ converges to $\mu[u]$ in the weak sense of measures.*

The proof of the next result can be found in in [23]

Proposition 4.4 *Let μ be a measure in $\mathfrak{M}^+(\mathbb{R}^N)$. Suppose that $\mathbf{W}_{1,p}[\mu] < \infty$ a.e. Then there exists a nonnegative p-superharmonic function u in $\mathbb{R}^N$ such that $-\Delta_p u = \mu$ in $\mathcal{D}'(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} u = 0$ and*

$$\frac{1}{c_{86}}\mathbf{W}_{1,p}[\mu](x) \le u(x) \le c_{86}\mathbf{W}_{1,p}[\mu](x), \tag{4.1}$$

for almost all x in $\mathbb{R}^N$, where the constant c_{86} is the one of Proposition 3.4. Furthermore any p-superharmonic function u in $\mathbb{R}^N$, such that $\inf_{\mathbb{R}^N} u = 0$ satisfies (4.1) with $\mu = -\Delta_p u$.

Proof of Theorem A. *The condition is necessary.* Assume that (1.14) admits a nonnegative p-superharmonic functions (u, v). By Proposition 4.4 there holds

$$\begin{aligned} u(x) &\ge c_{87}\mathbf{W}_{1,p}[v^{q_1} + \mu](x),\\ v(x) &\ge c_{87}\mathbf{W}_{1,p}[u^{q_2} + \eta](x) \qquad \text{for almost all } x \in \Omega. \end{aligned}$$

Hence, we derive (1.10) from Proposition 2.9-(i).

The condition is sufficient. Let $\{(u_m, v_m)\}_{m\in\mathbb{N}}$ be a sequence of nonnegative p-superharmonic solutions of the following problems for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= v_m^{q_1} + \mu && \text{in } \mathbb{R}^N\\ -\Delta_p v_{m+1} &= u_m^{q_2} + \eta && \text{in } \mathbb{R}^N\\ \inf_{\mathbb{R}^N} u_{m+1} &= \inf_{\mathbb{R}^N} v_{m+1} = 0, \end{aligned} \tag{4.2}$$

with $(u_0, v_0) = 0$. As in the proof of Theorem C we can assume that $\{u_m\}$ and $\{v_m\}$ are nondecreasing. By Proposition 4.4 we have

$$\begin{aligned} u_{m+1} &\le c_{86}\mathbf{W}_{1,p}[v_m^{q_1} + \mu](x)\\ v_{m+1} &\le c_{86}\mathbf{W}_{1,p}[u_m^{q_2} + \eta](x) \quad \forall \ \ x \in \Omega \end{aligned}$$

Thus, by Proposition 2.7 there exists a constant $c > 0$ depending only on N, p, q_1, q_2 such that if

$$\omega(K) \le M^* \operatorname{Cap}_{\mathbf{I}_{\frac{p(q_1+p-1)}{q_1}}, \frac{q_1 q_2}{q_1 q_2 - (p-1)^2}}(K) \tag{4.3}$$

for any compact set $K \subset \mathbb{R}^N$ with $d\omega = (\mathbf{W}_{1,p}[\mu])^{q_2}\,dx + d\eta$ then

$$v_m \le c_{69}\mathbf{W}_{1,p}[\omega], \quad u_m \le c_{70}\mathbf{W}_{1,p}[(\mathbf{W}_{1,p}[\omega])^{q_1}] + c_{68}\mathbf{W}_{1,p}[\mu] \ \ \forall \ \ k \ge 0 \tag{4.4}$$

in Ω and

$$\mathbf{W}_{1,p}[\omega] \in L^{q_2}_{loc}(\mathbb{R}^N), \quad \mathbf{W}_{1,p}[(\mathbf{W}_{1,p}[\omega])^{q_1}] + \mathbf{W}_{1,p}[\mu] \in L^{q_1}_{loc}(\mathbb{R}^N). \tag{4.5}$$

This implies that $\{u_m\}, \{v_m\}$ are well defined and nondecreasing. Thus $\{(u_m, v_m)\}$ converges a.e in $\mathbb{R}^N$ to some functions (u, v) which satisfies (1.17) in $\mathbb{R}^N$. Furthermore, we infer from (3.6) and the monotone convergence theorem that $u_m^{q_1} \to u^{q_1}, v_m^{q_2} \to u^{q_2}$ in $L^1_{loc}(\mathbb{R}^N)$. By Proposition 4.3 we derive that (u, v) are nonnegative p-superharmonic solutions of (1.9). □

5 Hessian equations

In this section $\Omega \subset \mathbb{R}^N$ is either a bounded domain with a C^2 boundary or the whole $\mathbb{R}^N$. For $k = 1, ..., N$ and $u \in C^2(\Omega)$ the k-hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivative D^2u and S_k is the k-th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \le i_1 < ... < i_k \le N} \lambda_{i_1} ... \lambda_{i_k}.$$

We can see that

$$F_k[u] = \left[D^2u\right]_k,$$

where for a matrix $A = (a_{ij})$, $[A]_k$ denotes the sum of the k-th principal minors. We assume that $\partial\Omega$ is uniformly (k-1)-convex, that is

$$S_{k-1}(\kappa) \ge c_0 > 0 \ on \ \ \partial\Omega,$$

for some positive constant c_0, where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal.

Definition 5.1 *An upper-semicontinuous function $u : \Omega \to [-\infty, \infty)$ is k-convex (k-subharmonic) if, for every open set $\Omega' \subset \overline{\Omega}' \subset \Omega$ and for every function $v \in C^2(\Omega') \cap C(\overline{\Omega'})$ satisfying $F_k[v] \le 0$ in Ω', the following implication is true*

$$u \le v \ on \ \partial\Omega' \quad \Longrightarrow \quad u \le v \ \ in \ \ \Omega'.$$

We denote by $\Phi^k(\Omega)$ the class of all k-subharmonic functions in Ω which are not identically equal to $-\infty$.

The following weak convergence result for k-Hessian operators proved in [28] is fundamental in our study.

Proposition 5.2 *Let Ω be either a bounded uniformly (k-1)-convex in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$. For each $u \in \Phi^k(\Omega)$, there exists a nonnegative Radon measure $\mu_k[u]$ in Ω such that*

1 *$\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$.*

2 *If $\{u_n\}$ is a sequence of k-convex functions which converges a.e to u, then $\mu_k[u_n] \rightharpoonup \mu_k[u]$ in the weak sense of measures.*

As in the case of quasilinear equations with measure data, precise estimates of solutions of k-Hessian equations with measures data are expressed in terms of Wolff potentials. The next results are proved in [28, 20, 23].

Theorem 5.3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2, uniformly (k-1)-convex domain. Let μ be a nonnegative Radon measure in Ω which can be decomposed under the form*

$$\mu = \mu_1 + f,$$

where μ_1 is a measure with compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$ if $k \leq \frac{N}{2}$, or $p = 1$ if $k > \frac{N}{2}$. Then there exists a nonnegative function u in Ω such that $-u \in \Phi^k(\Omega)$, continuous near $\partial\Omega$ and u is a solution of the problem

$$\begin{aligned} F_k[-u] &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Furthermore, any nonnegative function u such that $-u \in \Phi^k(\Omega)$ which is continuous near $\partial\Omega$ and is a solution of above equation, satisfies

$$\frac{1}{c_{88}} \mathbf{W}^{\frac{d(x,\partial\Omega)}{8}}_{\frac{2k}{k+1},k+1}[\mu] \leq u(x) \leq c_{88} \mathbf{W}^{2diam\,\Omega}_{\frac{2k}{k+1},k+1}[\mu](x), \tag{5.1}$$

where c_{88} is a positive constant independent of x, u and Ω.

Theorem 5.4 *Let μ be a measure in $\mathfrak{M}^+(\mathbb{R}^N)$ and $2k < N$. Suppose that $\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu] < \infty$ a.e. Then there exists u, $-u \in \Phi^k(\mathbb{R}^N)$ such that $\inf_{\mathbb{R}^N} u = 0$ and $F_k[-u] = \mu$ in $\mathbb{R}^N$ and*

$$\frac{1}{c_{88}} \mathbf{W}_{\frac{2k}{k+1},k+1}[\mu](x) \leq u(x) \leq c_{88} \mathbf{W}_{\frac{2k}{k+1},k+1}[\mu](x), \tag{5.2}$$

for all x in $\mathbb{R}^N$. Furthermore, if u is a nonnegative function such that $\inf_{\mathbb{R}^N} u = 0$ and $-u \in \Phi^k(\mathbb{R}^N)$, then (5.2) holds with $\mu = F_k[-u]$.

Proof of Theorem E. *The condition is necessary.* Assume that (1.21) admits a nonnegative solution (u, v), continuous near $\partial\Omega$, such that $-u, -v \in \Phi^k(\Omega)$ and $u^{s_2}, v^{s_1} \in L^1(\Omega)$. Then by Theorem 5.3 we have

$$\begin{aligned} u(x) &\geq \frac{1}{c_{88}} \mathbf{W}^{\frac{d(x,\partial\Omega)}{8}}_{\frac{2k}{k+1},k+1}[v^{s_1} + \mu](x) \\ v(x) &\geq \frac{1}{c_{88}} \mathbf{W}^{\frac{d(x,\partial\Omega)}{8}}_{\frac{2k}{k+1},k+1}[u^{s_2} + \eta](x) \qquad \text{for almost all } x \in \Omega. \end{aligned}$$

Using the part 2 of Proposition 2.9, we conclude that (1.22) holds.

The condition is sufficient. We define a sequence of nonnegative functions u_m, v_m, continuous near $\partial\Omega$ and such that $-u_m, -v_m \in \Phi^k(\Omega)$, by the following iterative scheme for $m \geq 0$,

$$\begin{aligned} F_k[-u_{m+1}] &= v_m^{s_1} + \mu && \text{in } \Omega, \\ F_k[-v_{m+1}] &= u_m^{s_2} + \eta && \text{in } \Omega, \\ u_{m+1} &= v_{m+1} = 0 && \text{on } \partial\Omega. \end{aligned} \tag{5.3}$$

Clearly, we can assume that $\{u_m\}$ is nondecreasing as in [24]. By Theorem 5.3 we have

$$\begin{aligned} u_{m+1} &\leq c_{88} \mathbf{W}^R_{\frac{2k}{k+1},k+1}[v_m^{s_1} + \mu], \\ v_{m+1} &\leq c_{88} \mathbf{W}^R_{\frac{2k}{k+1},k+1}[u_m^{s_2} + \mu] \qquad \text{in } \Omega \end{aligned} \tag{5.4}$$

where $R = 2\,diam\,(\Omega)$.
Then, by Proposition 2.8, there exists a constant $M_* > 0$ depending only on N, p, q_1, q_2, R such that if

$$\omega(K) \leq M_* \operatorname{Cap}_{\mathbf{G}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}}(K)$$

for any compact set $K \subset \mathbb{R}^N$ with $d\omega = \left(\mathbf{W}^R_{\frac{2k}{k+1},k+1}\mu]\right)^{s_2} dx + d\eta$ then

$$v_m \le c_{73}\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega], \quad u_m \le c_{74}\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\left(\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega]\right)^{s_1}] + c_{72}\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu] \ \ \forall \ \ m \ge 0$$

in Ω for some positive constants c_{72}, c_{73} and c_{74} depending only on N, k, s_1, s_2, R. Note that we can write

$$v_m^{s_1} + \mu = \left(\mu_1 + \chi_{\Omega_\delta} v_m^{s_1}\right) + \left((1-\chi_{\Omega_\delta})v_m^{s_1} + f\right),$$

and

$$u_m^{s_2} + \eta = \left(\eta_1 + \chi_{\Omega_\delta} u_m^{s_2}\right) + \left((1-\chi_{\Omega_\delta})u_m^{s_2} + g\right),$$

where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$ and $\delta > 0$ is small enough and since u_m is continuous near $\partial\Omega$, then $v_m^{s_1} + \mu, u_m^{s_2} + \eta$ satisfy the assumptions of the data in Theorem 5.3. Therefore the sequence $\{u_m\}$ is well defined and nondecreasing. Thus, $\{u_m\}$ converges a.e in Ω to some function u which satisfies (1.24) in Ω. Furthermore, by the monotone convergence theorem there holds $v_m^{s_1} \to v, u_m^{s_2} \to u$ in $L^1(\Omega)$. Finally, by Proposition 5.2, we infer that (1.21) admits a nonnegative solutions u, v, continuous near $\partial\Omega$, with $-u, -v \in \Phi^k(\Omega)$ satisfying (1.24). □

Proof of Theorem F. *The condition is necessary.* Assume that (1.21) admits nonnegative solution (u, v), such that $-u, -v \in \Phi^k(\mathbb{R}^N)$ and $u^{s_2}, v^{s_1} \in L^1_{loc}(\mathbb{R}^N)$. Then by Theorem 5.3 we have

$$\begin{aligned} u(x) &\ge \frac{1}{c_{88}}\mathbf{W}_{\frac{2k}{k+1},k+1}[v^{s_1} + \mu](x) \\ v(x) &\ge \frac{1}{c_{88}}\mathbf{W}_{\frac{2k}{k+1},k+1}[u^{s_2} + \eta](x) \qquad \text{for almost all } x \in \mathbb{R}^N. \end{aligned}$$

Using Proposition 2.9-(ii), we conclude that (1.22) holds.

The condition is sufficient. We defined a sequence of nonnegative functions u_m, v_m, continuous near $\partial\Omega$ and such that $-u_m, -v_m \in \Phi^k(\Omega)$, by the following iterative scheme for $m \ge 0$,

$$\begin{aligned} &F_k[-u_{m+1}] = v_m^{s_1} + \mu \qquad \text{in } \mathbb{R}^N, \\ &F_k[-v_{m+1}] = u_m^{s_2} + \eta \qquad \text{in } \mathbb{R}^N, \\ &\inf_{\mathbb{R}^N} u_{m+1} = \inf_{\mathbb{R}^N} v_{m+1} = 0. \end{aligned}$$

As in the previous proofs $\{u_m\}$ is nondecreasing. By Theorem 5.3 we have

$$\begin{aligned} &u_{m+1} \le c_{88}\mathbf{W}_{\frac{2k}{k+1},k+1}[v_m^{s_1} + \mu], \\ &v_{m+1} \le c_{88}\mathbf{W}_{\frac{2k}{k+1},k+1}[u_m^{s_2} + \mu] \qquad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Then, by Proposition 2.7, there exists a constant $M^* > 0$ depending only on N, p, q_1, q_2, R such that if

$$\omega(K) \le M^* \operatorname{Cap}_{\mathbf{I}_{\frac{2k(s_1+k)}{s_1}, \frac{s_1 s_2}{s_1 s_2 - k^2}}}(K)$$

for any compact set $K \subset \mathbb{R}^N$ with $d\omega = \left(\mathbf{W}_{\frac{2k}{k+1},k+1}\mu]\right)^{s_2} dx + d\eta$, then

$$v_m \le c_{69}\mathbf{W}_{\frac{2k}{k+1},k+1}[\omega], \quad u_m \le c_{70}\mathbf{W}_{\frac{2k}{k+1},k+1}[\left(\mathbf{W}_{\frac{2k}{k+1},k+1}[\omega]\right)^{s_1}] + c_{68}\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu] \ \ \forall \ \ m \ge 0$$

in Ω where c_{68}, c_{69} and c_{70} depend on N, k, s_1, s_2, R. Therefore the sequence $\{u_m\}$ is well defined and nondecreasing. Thus, $\{u_m\}$ converges a.e in Ω to some function u for which (1.27) is satisfied in $\mathbb{R}^N$. Furthermore, by the monotone convergence theorem we have $v_m^{s_1} \to v, u_m^{s_2} \to u$ in $L^1_{loc}(\mathbb{R}^N)$. Finally, by Proposition 5.2, we obtain that (1.21) admits a nonnegative solutions u, v with $-u, -v \in \Phi^k(\mathbb{R}^N)$ satisfying (1.27). □

6 Further results

The method exposed in the previous sections, can be applied to types of problems. We give below an example for a semilinear system in $\mathbb{R}^N_+ = \{x = (x', x_{_N}), x' \in \mathbb{R}^{N-1}, x_{_N} > 0\}$.

$$\begin{aligned} -\Delta u &= v^{q_1} && \text{in } \mathbb{R}^N_+ \\ -\Delta v &= u^{q_2} && \text{in } \mathbb{R}^N_+ \\ u &= \sigma_1, v = \sigma_2 && \text{in } \mathbb{R}^{N-1}, \end{aligned} \tag{6.1}$$

where we have identified $\partial\mathbb{R}^N_+$ and $\mathbb{R}^{N-1}$. We denote by $\mathbf{P}$ (resp. $\mathbf{G}$) the Poisson kernel in $\mathbb{R}^N_+$ (resp the Green kernel in $\mathbb{R}^N$). The Poisson potential and the Green potential, $\mathbf{P}[.]$ and $\mathbf{G}[.]$, associated to $-\Delta$ are defined resepctively by

$$\mathbf{P}[\sigma](y) = \int_{\partial\mathbb{R}^N_+} P(y,z)d\sigma(z), \quad G[f](y) = \int_{\mathbb{R}^N_+} \mathbf{G}(y,x)f(x)dx,$$

see [21]. We set $\rho(x) = x_{_N}$, and define the capacity $\mathrm{Cap}^{\rho}_{\alpha,s}$ by

$$\mathrm{Cap}^{\rho}_{\alpha,s}(K) = \inf\left\{\int_{\mathbb{R}^N_+} f^s \rho\, dx : f \geq 0, \mathbf{I}_\alpha[f\rho\,\chi_{_{\mathbb{R}^N_+}}] \geq \chi_{_K}\right\},$$

for all Borel set $K \subset \mathbb{R}^N$, where $\mathbf{I}_\alpha$ is the Riesz kernel of order α in $\mathbb{R}^N$.

Theorem 6.1 *Let $1 \leq q_1 < \frac{N}{N-1}$, $q_1q_2 > 1$. If there exists a constant $\tilde{c} > 0$ such that if*

$$\begin{aligned} \int_K \rho(x)(\mathbf{P}[\sigma_1](x))^{q_2}dx &\leq \tilde{c}\,\mathrm{Cap}^{\rho}_{\frac{q_1+2}{q_1},\frac{q_1q_2}{q_1q_2-1}}(K) \\ \sigma_2(G) &\leq \tilde{c}\,\mathrm{Cap}_{I_{\frac{2(q_2+1)}{q_1q_2}},\frac{q_1q_2}{q_1q_2-1}}(G), \end{aligned} \tag{6.2}$$

for all Borel sets $K \subset \mathbb{R}^N_+$ and $G \subset \mathbb{R}^{N-1}$, then the problem (6.1) *admits a solution.*

All solutions in above theorem are understood in the usual very weak sense: $u \in L^1_{loc}(\mathbb{R}^N_+ \cap B)$, $u^{q_2}, v^{q_1} \in L^1_\rho(\mathbb{R}^N_+ \cap B)$ for any ball B and

$$\begin{aligned} \int_{\mathbb{R}^N_+} u(-\Delta\xi)dx &= \int_{\mathbb{R}^N_+} v^{q_1}\xi dx - \int_{\partial\mathbb{R}^N_+} \frac{\partial\xi}{\partial n}d\sigma_1, \\ \int_{\mathbb{R}^N_+} v(-\Delta\xi)dx &= \int_{\mathbb{R}^N_+} u^{q_2}\xi dx - \int_{\partial\mathbb{R}^N_+} \frac{\partial\xi}{\partial n}d\sigma_2, \end{aligned}$$

for any $\xi \in C^2(\overline{\mathbb{R}^N_+}) \cap C_c(\mathbb{R}^N)$ with $\xi = 0$ on $\partial\mathbb{R}^N_+$. It is well-known that such a solution u satisfies

$$u = \mathbf{G}[v^{q_1}] + \mathbf{P}[\sigma_1], \quad v = \mathbf{G}[u^{q_2}] + \mathbf{P}[\sigma_2] \qquad \text{a.e. in } \mathbb{R}^N_+.$$

To prove theorem we need the following a basic estimate

Lemma 6.2 *Assume that $0 < q_1 < \frac{N}{N-1}$. Then for any $\omega \in \mathfrak{M}^+_b(\mathbb{R}^N)$,*

$$\mathbf{I}_2\left[(\mathbf{I}_1[\omega])^{q_1}\right] \leq c_{89}\mathbf{W}_{\frac{q_1+2}{q_1+1},\frac{q_1+1}{q_1}}[\omega] \qquad \textit{a.e. in } \mathbb{R}^N, \tag{6.3}$$

where $c_{89} > 0$ depends on q_1, q_2 and N.

Proof. The proof of Lemma 6.2 is similar to the one of Lemma 2.2 and details are omitted. Note that if $\omega \in \mathfrak{M}_b(\overline{\mathbb{R}^N_+})$ it is extended by 0 in $\mathbb{R}^N_-$. □

Remark 6.3 *The condition* $0 < q_1 < \frac{N}{N-1}$ *is necessary and sufficient condition for* $(\mathbf{I}_1[\omega])^{q_1} \in L^1_{loc}(\mathbb{R}^N)$ *for any* $\omega \in \mathfrak{M}^+_b(\mathbb{R}^N)$.

Theorem 6.4 *Let* $q_1 \geq 1$, $q_1 q_2 > 1$ *and* $\omega \in \mathfrak{M}_b(\overline{\mathbb{R}^N_+})$. *If*

$$\omega(K) \leq c_{90}\, \mathrm{Cap}^{\rho}_{\frac{q_1+2}{q_1}, \frac{q_1 q_2}{q_1 q_2 - 1}}(K) \qquad \forall\ K \subset \overline{\mathbb{R}^N_+},\ K \text{ Borel},$$

for some $c_{90} > 0$, *then*

$$\mathbf{I}_1 \left[\left(\mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}[\omega] \right)^{q_2} \rho\, \chi_{\mathbb{R}^N_+} \right] \leq c_{91} \mathbf{I}_1[\omega] \qquad \text{a.e. in } \mathbb{R}^N_+. \tag{6.4}$$

Proof. *Step 1.* For any compact $K \subset \left\{ x \in \mathbb{R}^N_+ : \mathbf{I}_{\frac{q_1+2}{q_1}}[f\rho\, \chi_{\mathbb{R}^N_+}](x) > \lambda \right\}$, we have

$$\omega(K) \leq c_{90}\, \mathrm{Cap}^{\rho}_{\frac{q_1+2}{q_1}, \frac{q_1 q_2}{q_1 q_2 - 1}}(K) \leq c_{90} \lambda^{-\frac{q_1 q_2}{q_1 q_2 - 1}} \int_{\mathbb{R}^N_+} f^{\frac{q_1 q_2}{q_1 q_2 - 1}} \rho\, dx$$

by assumption and the definition of the capacity. Hence,

$$\lambda^{\frac{q_1 q_2}{q_1 q_2 - 1}} \omega \left(\left\{ \mathbf{I}_{\frac{q_1+2}{q_1}}[f\rho\, \chi_{\mathbb{R}^N_+}] > \lambda \right\} \right) \leq c_{90} \int_{\mathbb{R}^N_+} f^{\frac{q_1 q_2}{q_1 q_2 - 1}} \rho\, dx \qquad \forall\ \lambda > 0.$$

This implies an estimate in Lorentz space,

$$||\mathbf{I}_{\frac{q_1+2}{q_1}}[f\rho\, \chi_{\mathbb{R}^N_+}]||_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}, \infty}(\mathbb{R}^N, d\omega)} \leq ||f||_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}}(\mathbf{R}^N, \chi_{\mathbb{R}^N_+}\rho\, dx)} \qquad \forall\ f \geq 0. \tag{6.5}$$

Step 2. Since, for any $g \in C_c(\mathbb{R}^N_+)$,

$$\int_{\mathbb{R}^N_+} \mathbf{I}_{\frac{q_1+2}{q_1}}[g\omega] f\rho\, dx = \int_{\mathbb{R}^N} \mathbf{I}_{\frac{q_1+2}{q_1}}[f\rho\, \chi_{\mathbb{R}^N_+}] g d\omega,$$

we derive, using duality between $L^{p,1}$ and $L^{p',\infty}$, Holder's inequality therein and (6.5), that

$$\begin{aligned} \int_{\mathbb{R}^N_+} \mathbf{I}_{\frac{q_1+2}{q_1}}[g\omega] f\rho\, dx &\leq ||\mathbf{I}_{\frac{q_1+2}{q_1}}[f\rho\, \chi_{\mathbb{R}^N_+}]||_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}, \infty}(\mathbb{R}^N, d\omega)} ||g||_{L^{q_1 q_2, 1}(\mathbb{R}^N, d\omega)} \\ &\leq ||f||_{L^{\frac{q_1 q_2}{q_1 q_2 - 1}}(\mathbb{R}^N, \chi_{\mathbb{R}^N_+}\rho\, dx)} ||g||_{L^{q_1 q_2, 1}(\mathbb{R}^N, d\omega)} \qquad \forall\ f, g \geq 0. \end{aligned}$$

Therefore,

$$||\mathbf{I}_{\frac{q_1+2}{q_1}}[g\omega]||_{L^{q_1 q_2}(\mathbb{R}^N, \chi_{\mathbb{R}^N_+}\rho\, dx)} \leq ||g||_{L^{q_1 q_2, 1}(\mathbb{R}^N, d\omega)}. \tag{6.6}$$

Step 3. Taking $g = \chi_{B_t(x)}$ and since for $q_1 \geq 1$

$$\begin{aligned} \mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}[\nu](x) &= \int_0^\infty \left(\frac{\nu(B_\rho(x))}{\rho^{N - \frac{q_1+2}{q_1}}} \right)^{q_1} dx \\ &\leq c_{89} \left(\int_0^\infty \frac{\nu(B_\rho(x))}{\rho^{N - \frac{q_1+2}{q_1}}} dx \right)^{q_1} \\ &= c_{89} \left(\mathbf{I}_{\frac{q_1+2}{q_1}}[\nu](x) \right)^{q_1} \qquad \forall \nu \in \mathfrak{M}^+_b(\mathbb{R}^N),\ \ \forall x \in \mathbb{R}^N, \end{aligned}$$

we derive that for almost all $x \in \mathbb{R}^N_+$,

$$\int_{\mathbb{R}^N_+} \left(\mathbf{W}_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}[\chi_{B_t(x)}\omega] \right)^{q_2} \rho \, dy \le c_{90}\omega(B_t(x)),$$

from (6.6), which implies

$$\omega(B_t(x)) \le c_{91} \frac{t^{(N-\frac{q_1+2}{q_1})\frac{q_1q_2}{q_1q_2-1}}}{\left(\int_{B_{2t}(x)} \chi_{\mathbb{R}^N_+} \rho \, dy \right)^{\frac{1}{q_1q_2-1}}} \le c_{92} \frac{t^{n-\frac{q_1+2}{q_1}\frac{q_1q_2}{q_1q_2-1}}}{(\max\{x_n, t\})^{\frac{1}{q_1q_2-1}}}, \tag{6.7}$$

since $\int_{B_r(x)} \chi_{\mathbb{R}^N_+} \rho \, dy \asymp r^N \max\{x_N, r\}$ for any $x \in \mathbb{R}^N_+, r > 0$ where the symbol $\asymp$ is defined by

$$A \asymp B \Longleftrightarrow \frac{1}{c}B \le A \le cB \quad \text{for some constant } c > 0.$$

It implies also

$$\int_{B_t(x)} \left(\mathbf{W}^t_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}[\omega] \right)^{q_2} \chi_{\mathbb{R}^N_+} \rho \, dy \le c_{90}\omega(B_{2t}(x)), \tag{6.8}$$

from which follows

$$\int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left(\mathbf{W}^t_{\frac{q_1+2}{q_1+1}, \frac{q_1+1}{q_1}}[\omega] \right)^{q_2} \chi_{\mathbb{R}^N_+} \rho \, dydt \le c_{93}\mathbf{I}_1[\omega](x).$$

Therefore, if the following inequality holds

$$\int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\omega(B_r(y))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} \chi_{\mathbb{R}^N_+} \rho \, dydt \le c_{93}\mathbf{I}_1[\omega](x), \tag{6.9}$$

it will imply (6.4).

Step 4. We claim that (6.9) holds. Since $B_r(y) \in B_{2r}(x)$, $y \in B_t(x), r \ge t$,

$$\begin{aligned}
&\int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\omega(B_r(y))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} \chi_{\mathbb{R}^N_+} \rho \, dydt \\
&\qquad \le \int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \chi_{\mathbb{R}^N_+} \rho \, dy \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} dt \\
&\qquad \asymp \int_0^\infty \max\{x_n, t\} \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} dt.
\end{aligned}$$

By integration by part,

$$\begin{aligned}
&\int_0^\infty \frac{1}{t^N} \int_{B_t(x)} \left(\int_t^\infty \left(\frac{\omega(B_r(y))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2} \chi_{\mathbb{R}^N_+} \rho \, dydt \\
&= q_2 \int_0^\infty \int_0^t \max\{x_N, s\} ds \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} \left(\frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dt}{t} \\
&= q_2 \int_0^\infty \int_0^t \max\{x_N, s\} ds \left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}} \right)^{q_1} \frac{dr}{r} \right)^{q_2-1} \left(\frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}} \right)^{q_1-1} t^{\frac{2}{q_1}} \frac{\omega(B_{2t}(x))}{t^{N-1}} \frac{dt}{t}
\end{aligned}$$

We have

$$\int_0^t \max\{x_{_N}, s\}ds \asymeq t\max\{x_{_N}, t\},$$

$$\begin{aligned}\left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}}\right)^{q_1}\frac{dr}{r}\right)^{q_2-1} &\leq c_{94}\left(\int_t^\infty \left(\frac{r^{-\frac{q_1+2}{q_1(q_1q_2-1)}}}{(\max\{x_{_N}, r\})^{\frac{1}{q_1q_2-1}}}\right)^{q_1}\frac{dr}{r}\right)^{q-1}\\ &\asymeq t^{-\frac{(q_1+2)(q_2-1)}{q_1q_2-1}}(\max\{x_{_N}, t\})^{-\frac{q_1(q_1-1)}{q_1q_2-1}},\end{aligned}$$

by (6.7) and

$$\begin{aligned}\left(\frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}}\right)^{q_1-1} t^{\frac{2}{q_1}} &\leq c_{95}\left(\frac{t^{-\frac{q_1+2}{q_1(q_1q_2-1)}}}{(\max\{x_{_N}, t\})^{\frac{1}{q_1q_2-1}}}\right)^{q_2-1} t^{\frac{2}{q_1}}\\ &= c_{95}t^{-\frac{(q_1+2)(q_1-1)}{q_1(q_1q_2-1)}+\frac{2}{q_1}}(\max\{x_{_N}, t\})^{-\frac{q_1-1}{q_1q_2-1}}\end{aligned}$$

Thus,

$$\int_0^t \max\{x_{_N}, s\}ds\left(\int_t^\infty \left(\frac{\omega(B_{2r}(x))}{r^{N-\frac{q_1+2}{q_1}}}\right)^{q_1}\frac{dr}{r}\right)^{q_2-1}\left(\frac{\omega(B_{2t}(x))}{t^{N-\frac{q_1+2}{q_1}}}\right)^{q_1-1} t^{2/q_1} \leq c_{96}$$

and we obtain (6.9). □

Lemma 6.5 *Let $\alpha > 0$, $s > 1$ such that $\alpha + \frac{2}{s'} < N-1$ where $s' = \frac{s}{s-1}$. For all $\eta \in \mathfrak{M}^+(\mathbb{R}^{N-1})$, there holds*

$$\int_{\mathbb{R}^N}(\mathbf{I}_\alpha[\eta\otimes\delta_{\{x_{_N}=0\}}])^{s'}x_{_N}dx \asymeq \int_{\mathbb{R}^{N-1}}\left(\int_0^\infty \frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}}\frac{dt}{t}\right)^{s'}dx', \tag{6.10}$$

where I_β is the Riesz potential of order β in $\mathbb{R}^{N-1}$. As a consequence, we have

$$\mathrm{Cap}^\rho_{\alpha,s}(E\times\{x_{_N}=0\}) \asymeq \mathrm{Cap}_{I_{\alpha+2/s'-1},s}(E) \qquad \forall E\subset\mathbb{R}^{N-1},\ E \textit{Borel}. \tag{6.11}$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}^N}(\mathbf{I}_\alpha[\eta\otimes\delta_{\{x_{_N}=0\}}])^{s'}x_n dx &\geq \int_{\mathbb{R}^N}\left(\int_{2x_{_N}}^{4x_{_N}}\frac{(\eta\otimes\delta_{\{x_{_N}=0\}})(B_r(x))}{r^{N-\alpha}}\frac{dr}{r}\right)^{s'}x_n dx\\ &\geq c_{97}\int_{\mathbb{R}^N}\left(\frac{\eta(B'_{x_{_N}}(x'))}{x_{_N}^{N-\alpha}}\right)^{s'}x_{_N}dx\\ &\geq c_{98}\int_{\mathbb{R}^{N-1}}\left(\sup_{t>0}\frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}}\right)dx.'\end{aligned} \tag{6.12}$$

By using Lemma 2.1 we derive

$$\begin{aligned}\int_{\mathbb{R}^N}(\mathbf{I}_\alpha[\eta\otimes\delta_{\{x_{_N}=0\}}])^{s'}x_n dx &\leq \int_{\mathbb{R}^N}\left(\int_{x_{_N}}^\infty\frac{\eta(B'_r(x'))}{r^{N-\alpha}}\frac{dr}{r}\right)^{s'}dx_{_N}dx'\\ &\leq c_{99}\int_{\mathbb{R}^{N-1}}\int_0^\infty\left(\frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}}\right)^{s'}\frac{dt}{t}dx'.\end{aligned} \tag{6.13}$$

On the other hand, by [23, Proposition 5.1], there holds

$$\begin{aligned}\int_{\mathbb{R}^{N-1}}\left(\sup_{t>0}\frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}}\right)dx' &\asymp \int_{\mathbb{R}^{N-1}}\int_0^\infty\left(\frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}}\right)^{s'}\frac{dt}{t}dx'\\ &\asymp \int_{\mathbb{R}^{N-1}}\left(\int_0^\infty\frac{\eta(B'_t(x'))}{t^{N-1-\alpha-\frac{2}{s'}}}\frac{dt}{t}\right)^{s'}dx'.\end{aligned}\tag{6.14}$$

Combining (6.12), (6.13) and (6.14) we obtain (6.10). Moreover, we derive (6.11) from (6.10) and [1, Theorem 2.5.1], which ends the proof. □

Proof of Theorem 6.1 The following estimates are cclassical

$$\mathbf{G}(x,y)\asymp\frac{x_{_N}y_{_N}}{|x-y|^{N-2}\max\{|x-y|,x_{_N},y_{_N}\}^2}\le c_{100}\frac{y_{_N}}{|x-y|^{N-1}}\tag{6.15}$$

$$\mathbf{P}(x,z)=c_{101}\frac{x_{_N}}{|x-z|^N}\le c_{101}\frac{1}{|x-z|^{N-1}}\tag{6.16}$$

Thus,

$$\mathbf{G}\left[(\mathbf{P}[\sigma_1])^{q_2}\right]+\mathbf{P}[\sigma_2]\le c_{102}\mathbf{I}_1[\omega]\tag{6.17}$$

where $\omega(x)=\rho(\mathbf{P}[\sigma_1])^{q_2}+\sigma_2$ in $\mathbb{R}^N$. Therefore, we infer that if

$$\mathbf{I}_1\left[\left(\mathbf{I}_2\left[(\mathbf{I}_1[\omega])^{q_1}\right]\right)^{q_2}\chi_{_{\mathbb{R}^N_+}}\rho\right]\le c_{103}\mathbf{I}_1[\omega]\quad\text{in }\ \mathbb{R}^N_+\tag{6.18}$$

for some $c_{103}>0$ small enough, then (6.1) admits a positive solution (u,v). On the other hand, we deduce (6.18) from Lemma 6.2 and Theorem 6.4. The proof is complete. □

Remark 6.6 *The system*

$$\begin{aligned}-\Delta u&=v^{q_1}+\epsilon_1\mu && \text{in }\Omega\\ -\Delta v&=u^{q_2}+\epsilon_2\eta && \text{in }\Omega\\ u&=\epsilon_3\sigma_1, v=\epsilon_4\sigma_2 && \text{in }\partial\Omega,\end{aligned}\tag{6.19}$$

where $d(.)\mu, d(.)\lambda$ belong to $\mathfrak{M}_b^+(\Omega)$, σ_1,σ_2 to $\mathfrak{M}^+(\partial\Omega)$ and the ϵ_j are positive numbers, is analyzed in [10, Th 4.6]. Therein it is proved that if

$$\int_\Omega\left(\mathbf{G}[\mu]+\mathbf{P}[\lambda]\right)^{\max\{q_1,q_2\}}d(x)dx<\infty,\tag{6.20}$$

which is equivalent to a capacitary estimate, and

$$\min\left\{q_2\frac{q_1+1}{q_2+1},q_1\frac{q_2+1}{q_1+1}\right\}<\frac{N+1}{N-1},\tag{6.21}$$

and if the ϵ_j are small enough, then (6.19) admits a positive solution. Now condition (6.21) is a subcriticality assumption (for at least one of the two exponents q_j) since there is no condition on the boundary measures.

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