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# Two-level preconditioners for the Helmholtz equation

Marcella Bonazzoli<sup>1</sup>, Victorita Dolean<sup>1,2</sup>, Ivan G. Graham<sup>3</sup>, Euan A. Spence<sup>3</sup>, and Pierre-Henri Tournier<sup>4</sup>

## 1 Introduction

Solving the Helmholtz equation  $-\Delta u - k^2 u = f$  is a challenging task because of its indefinite nature and its highly oscillatory solution when the wavenumber  $k$  is high. Although there have been different attempts to solve it efficiently, we believe that there is no established and robust preconditioner, whose behavior is independent of  $k$ , for general decompositions into subdomains. In Conen et al. [2014] a two-level preconditioner was introduced, where the coarse correction involves local eigenproblems of Dirichlet-to-Neumann (DtN) maps. This method proved to be very robust with respect to heterogeneous coefficients compared to the reference preconditioner based on plane waves, and its construction is completely automatic without the need for parameter tuning. Another method was developed in Graham et al. [2017b,a], where two-level domain decomposition approximations of the Helmholtz equation with absorption  $-\Delta u - (k^2 + i\varepsilon)u = f$  were used as preconditioners for the pure Helmholtz equation without absorption; there the coarse correction is based on a coarse mesh with diameter constrained by  $k$ . Our purpose is to compare numerically the performance of the latter with the two-level method based on DtN maps, both in two and three dimensions.

Consider the interior Helmholtz problem of the following form: let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , be a polyhedral, bounded domain; find  $u: \Omega \rightarrow \mathbb{C}$  such that

$$-\Delta u - (k^2 + i\varepsilon)u = f \quad \text{in } \Omega, \quad (1a)$$

$$\frac{\partial u}{\partial n} - i\eta u = 0 \quad \text{on } \Gamma = \partial\Omega. \quad (1b)$$

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<sup>1</sup> Université Côte d'Azur, CNRS, LJAD, France, e-mail: [marcella.bonazzoli@unice.fr](mailto:marcella.bonazzoli@unice.fr)

<sup>2</sup> University of Strathclyde, Glasgow, UK, e-mail: [Victorita.Dolean@strath.ac.uk](mailto:Victorita.Dolean@strath.ac.uk)

<sup>3</sup> University of Bath, UK, e-mail: [I.G.Graham@bath.ac.uk](mailto:I.G.Graham@bath.ac.uk), [E.A.Spence@bath.ac.uk](mailto:E.A.Spence@bath.ac.uk)

<sup>4</sup> UPMC Univ Paris 06, LJLL, Paris, France, e-mail: [tournier@ljl11.upmc.fr](mailto:tournier@ljl11.upmc.fr)

The wavenumber  $k$  is given by  $k(\mathbf{x}) = \omega/c(\mathbf{x})$ , where  $\omega$  is the angular frequency and  $c$  is the speed of propagation that might depend on  $\mathbf{x} \in \Omega$ ; we take  $\eta = \text{sign}(\varepsilon)k$  as Robin boundary condition parameter. We are interested in solving the problem in the case  $\varepsilon = 0$ , using  $\varepsilon$  as a parameter when building the preconditioner. The variational formulation of Problem (1) is: find  $u \in V = H^1(\Omega)$  such that  $a_\varepsilon(u, v) = F(v)$ ,  $\forall v \in V$ , where  $a_\varepsilon(\cdot, \cdot): V \times V \rightarrow \mathbb{C}$  and  $F: V \rightarrow \mathbb{C}$  are defined by

$$a_\varepsilon(u, v) = \int_{\Omega} (\nabla u \cdot \overline{\nabla v} - (k^2 + i\varepsilon)u\overline{v}) - \int_{\Gamma} i\eta u\overline{v}, \quad F(v) = \int_{\Omega} f\overline{v}.$$

Note that if  $\varepsilon \neq 0$  and  $\eta = \text{sign}(\varepsilon)k$ ,  $a_\varepsilon$  is coercive (see §2 in Graham et al. [2017b]). We consider a discretization of the variational problem using piecewise linear finite elements on a uniform simplicial mesh  $\mathcal{T}_h$  of  $\Omega$ . Denoting by  $V_h \subset V$  the corresponding finite element space and by  $\{\phi_k\}_{k=1}^n$  its basis functions,  $n := \dim(V_h)$ , the discretized problem reads: find  $u_h \in V_h$  such that  $a_\varepsilon(u_h, v_h) = F(v_h)$ ,  $\forall v_h \in V_h$ , that is, in matrix form,

$$A_\varepsilon \mathbf{u} = \mathbf{f}, \tag{2}$$

where the coefficients of the matrix  $A_\varepsilon \in \mathbb{C}^{n \times n}$  and the right-hand side  $\mathbf{f} \in \mathbb{C}^n$  are given by  $(A_\varepsilon)_{k,l} = a(\phi_l, \phi_k)$  and  $(\mathbf{f})_k = F(\phi_k)$ . The matrix  $A_\varepsilon$  is complex, symmetric (but not Hermitian), and indefinite if  $\varepsilon = 0$ .

## 2 Two-level domain decomposition preconditioners

In the following we will define the domain decomposition preconditioners for the linear system  $A_0 \mathbf{u} = \mathbf{f}$  resulting from the discretization of the Helmholtz problem without absorption ( $\varepsilon = 0$ ). These are two-level Optimized Restricted Additive Schwarz (ORAS) algorithms, where “optimized” refers to the use of Robin boundary conditions at the interface between subdomains. In the terminology of Graham et al. [2017b], the prefix O is replaced with Imp, which stands for impedance (i.e. Robin) boundary conditions.

First of all, consider a decomposition of the domain  $\Omega$  into a set of overlapping subdomains  $\{\Omega_j\}_{j=1}^{N_{\text{sub}}}$ , with each subdomain consisting of a union of elements of the mesh  $\mathcal{T}_h$ . Let  $V_h(\Omega_j) = \{v|_{\Omega_j} : v \in V_h\}$ ,  $1 \leq j \leq N_{\text{sub}}$ , denote the space of functions in  $V_h$  restricted to the subdomain  $\Omega_j$ . Let  $n_j := \#\text{dof}(\Omega_j)$ ,  $1 \leq j \leq N_{\text{sub}}$ , where  $\text{dof}(D) := \{k : \text{supp}(\phi_k) \subset \overline{D}\}$  represents the degrees of freedom (dofs) of  $V_h(\Omega_j)$ . For  $1 \leq j \leq N_{\text{sub}}$ , we define a restriction operator  $\mathcal{R}_j: V_h \rightarrow V_h(\Omega_j)$  by injection, i.e. for  $u \in V_h$  we set  $(\mathcal{R}_j u)(\mathbf{x}_i) = u(\mathbf{x}_i)$  for all  $\mathbf{x}_i \in \Omega_j$ . We denote by  $R_j$  the corresponding Boolean matrix in  $\mathbb{R}^{n_j \times n}$  that maps coefficient vectors of functions in  $V_h$  to coefficient vectors of functions in  $V_h(\Omega_j)$ . Let  $D_j \in \mathbb{R}^{n_j \times n_j}$  be a diagonal ma-

trix corresponding to a partition of unity in the sense that  $\sum_{i=1}^{N_{\text{sub}}} \tilde{R}_i^T R_i = I$ , where  $\tilde{R}_j := D_j R_j$ . Then the *one-level ORAS* preconditioner (which is also the one-level ImpHRAS of Graham et al. [2017b]) reads

$$M_{1,\varepsilon}^{-1} := \sum_{j=1}^{N_{\text{sub}}} \tilde{R}_j^T A_{j,\varepsilon}^{-1} R_j. \quad (3)$$

We define the matrices  $A_{j,\varepsilon}$  in (3) to be the matrices stemming from the discretization of the following local Robin problems with absorption

$$\begin{aligned} -\Delta u_j - (k^2 + \mathbf{i}\varepsilon)u_j &= f && \text{in } \Omega_j, \\ \frac{\partial u_j}{\partial n_j} - \mathbf{i}\eta u_j &= 0 && \text{on } \partial\Omega_j. \end{aligned}$$

In order to achieve weak dependence on the wavenumber  $k$  and number of subdomains, we add a coarse component to (3). The *two-level* preconditioner can be written in a generic way as follows

$$M_{2,\varepsilon}^{-1} = Q M_{1,\varepsilon}^{-1} P + Z E^{-1} Z^*, \quad (4)$$

where  $*$  denotes the conjugate transpose,  $M_{1,\varepsilon}^{-1}$  is the one-level preconditioner (3),  $Z$  is a rectangular matrix with full column rank,  $E = Z^* A_\varepsilon Z$  is the so-called coarse grid matrix,  $\Xi = Z E^{-1} Z^*$  is the so-called coarse grid correction matrix. If  $P = Q = I$  this is an *additive* two-level preconditioner (which would be called two-level ImpRAS in Graham et al. [2017b]). If  $P = I - A_\varepsilon \Xi$  and  $Q = I - \Xi A_\varepsilon$ , this is a *hybrid* two-level preconditioner (ImpHRAS in Graham et al. [2017b]), also called the *Balancing* Neumann Neumann (BNN) preconditioner. Preconditioner (4) is characterized by the choice of  $Z$ , whose columns span the *coarse space* (CS). We will consider the following two cases:

**The grid coarse space** The most natural coarse space would be one based on a coarser mesh, we subsequently call it “grid coarse space”. Let us consider  $\mathcal{T}_{H_{\text{coarse}}}$  a simplicial mesh of  $\Omega$  with mesh diameter  $H_{\text{coarse}}$  and  $V_{H_{\text{coarse}}} \subset V$  the corresponding finite element space. Let  $\mathcal{I}_0: V_{H_{\text{coarse}}} \rightarrow V_h$  be the nodal interpolation operator and define  $Z$  as the corresponding matrix. Then in this case  $E = Z^* A_\varepsilon Z$  is really the stiffness matrix of the problem (with absorption) discretized on the coarse mesh. Related preconditioners without absorption are used in Kimm and Sarkis [2007].

**The DtN coarse space** This coarse space (see Conen et al. [2014]) is based on local Dirichlet-to-Neumann (DtN) eigenproblems on the subdomains interfaces. For a subdomain  $\Omega_i$ , first of all consider  $a^{(i)}: H^1(\Omega_i) \times H^1(\Omega_i) \rightarrow \mathbb{R}$

$$a^{(i)}(v, w) = \int_{\Omega_i} (\nabla v \cdot \overline{\nabla w} - (k^2 + \mathbf{i}\varepsilon)v\bar{w}) - \int_{\partial\Omega_i \cap \partial\Omega} \mathbf{i}\eta v\bar{w}.$$

Let  $(A^{(i)})_{kl} = a^{(i)}(\phi_k, \phi_l)$ , and let  $I$  and  $\Gamma_i$  be the sets of indices corresponding, resp., to the interior and boundary dofs on  $\Omega_i$ , with  $n_I$  and  $n_{\Gamma_i}$  their cardinalities. With the usual block notation, the subscripts  $I$  and  $\Gamma_i$  for the matrices  $A$  and  $A^{(i)}$  denote the entries of these matrices associated with the respective dofs. Let  $M_{\Gamma_i} = \left( \int_{\Gamma_i} \phi_k \phi_l \right)_{k,l \in \Gamma_i}$  be the mass matrix on the interface  $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$  of subdomain  $\Omega_i$ . We need to solve the following eigenproblem: find  $(\mathbf{u}, \lambda) \in \mathbb{C}^{n_{\Gamma_i}} \times \mathbb{C}$ , s.t.

$$(A_{\Gamma_i \Gamma_i}^{(i)} - A_{\Gamma_i I} A_{II}^{-1} A_{I \Gamma_i}) \mathbf{u} = \lambda M_{\Gamma_i} \mathbf{u}. \quad (5)$$

Now, the matrix  $Z$  of the DtN coarse space is a rectangular, block-diagonal matrix with blocks  $W_i$ , associated with the subdomain  $\Omega_i$ ,  $1 \leq i \leq N_{\text{sub}}$ , given by Algorithm 2.1. If  $m_i$  is the number of eigenvectors selected by the automatic criterion in Line 2 of Algorithm 2.1, the block  $W_i$  has dimensions  $n_i \times m_i$ , and the matrix  $Z$  has dimensions  $n \times \sum_{j=1}^{N_{\text{sub}}} m_j$ . Due to the overlap in the decomposition, the blocks may share some rows inside the matrix  $Z$ .

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**Algorithm 2.1** Construction of the block  $W_i$  of the DtN CS matrix  $Z$

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- 1: Solve the discrete DtN eigenproblem (5) on subdomain  $\Omega_i$  for the eigenpairs  $(\lambda_j, \mathbf{g}_i^j)$ .
  - 2: Choose the  $m_i$  eigenvectors  $\mathbf{g}_i^j \in \mathbb{C}^{n_{\Gamma_i}}$  such that  $\Re(\lambda_j) < k$ ,  $1 \leq j \leq m_i$ .
  - 3: **for**  $j = 1$  to  $m_i$  **do**
  - 4:   Compute the discrete Helmholtz extension  $\mathbf{u}_i^j \in \mathbb{C}^{n_i}$  to  $\Omega_i$  of  $\mathbf{g}_i^j$  as  $\mathbf{u}_i^j = [-A_{II}^{-1} A_{I \Gamma_i} \mathbf{g}_i^j, \mathbf{g}_i^j]^T$ .
  - 5: **end for**
  - 6: Define the matrix  $W_i \in \mathbb{C}^{n_i \times m_i}$  as  $W_i = (D_i \mathbf{u}_i^1, \dots, D_i \mathbf{u}_i^{m_i})$ .
- 

### 3 Numerical experiments and conclusion

We solve (2) with  $\varepsilon = 0$  on the unit square/cube, with a uniform simplicial mesh of diameter  $h \sim k^{-3/2}$ , which is believed to remove the pollution effect. The right-hand side is given by  $f = -\exp(-100((x-0.5)^2 + (y-0.5)^2))$  for  $d = 2$ ,  $f = -\exp(-400((x-0.5)^2 + (y-0.5)^2 + (z-0.5)^2))$  for  $d = 3$ .

We use GMRES with right preconditioning (with a tolerance  $\tau = 10^{-6}$ ), starting with a *random initial guess*; the stopping criterion is based on the relative residual. We consider a regular decomposition into subdomains (squares/cubes), the overlap for each subdomain is of size  $\mathcal{O}(2h)$  in all directions and the two-level preconditioner (4) is used in the hybrid way. All the computations are done in the open source language FreeFem++ (<http://www.freefem.org/ff++/>). The 3d code is parallelized and run on the TGCC Curie supercomputer. We assign each subdomain to one processor. So in our experiments the number of processors increases if the number

of subdomains increases. To apply the preconditioner, the local problems in each subdomain (with matrices  $A_{j,\varepsilon}$  in (3)) and the coarse space problem (with matrix  $E$  in (4)) are solved with a direct solver.

As in Graham et al. [2017b,a], in the experiments we take the subdomain diameter  $H_{\text{sub}}$  and the coarse mesh diameter  $H_{\text{coarse}}$  constrained by  $k$ :  $H_{\text{sub}} \sim k^{-\alpha}$  and  $H_{\text{coarse}} \sim k^{-\alpha'}$ , for some choices of  $0 < \alpha, \alpha' \leq 1$  detailed in the following; if not differently specified, we take  $\alpha = \alpha'$ , which is the setting of all numerical experiments in Graham et al. [2017b]. Note that  $H_{\text{coarse}}$  does not appear as a parameter in the DtN coarse space. We denote by  $n_{\text{CS}}$  the size of the coarse space. For the grid coarse space  $n_{\text{CS}} = (1/H_{\text{coarse}} + 1)^d$ , the number of dofs for the nodal linear finite elements in the unit square/cube. For the DtN coarse space  $n_{\text{CS}} = \sum_{j=1}^{N_{\text{sub}}} m_j$ , the total number of computed eigenvectors for all the subdomains. While we solve the pure Helmholtz problem without absorption, both the one-level preconditioner (3) and the two-level preconditioner (4) are built from problems which can have non zero absorption given by  $\varepsilon_{\text{prec}} = k^\beta$ . In the experiments we put  $\beta = 1$  or  $\beta = 2$ .

In the following tables we compare the one-level preconditioner, the two-level preconditioners with the grid coarse space and with the DtN coarse space in terms of number of iterations of GMRES and size of the coarse space ( $n_{\text{CS}}$ ), for different values of the wavenumber  $k$  and of the parameters  $\alpha, \beta$ . We also report the number of subdomains  $N_{\text{sub}}$ , which is controlled by  $k$  and  $\alpha$  as mentioned above. Since  $h \sim k^{-3/2}$ , the dimension  $n$  of the linear system matrix is of order  $k^{3d/2}$ ; for 3d experiments we report  $n$  explicitly. Tables 1, 2 concern the 2d problem, Table 3 the 3d problem.

In Table 1, we let the DtN coarse space size be determined by the automatic choice criterion in Line 2 of Algorithm 2.1 (studied in Conen et al. [2014]) and the grid coarse space size by  $H_{\text{coarse}} \sim k^{-\alpha}$ . We see that the DtN coarse space is much larger than the grid coarse space and gives fewer iterations. The preconditioners with absorption  $\varepsilon_{\text{prec}} = k^2$  perform much worse than those with absorption  $\varepsilon_{\text{prec}} = k$  independently of  $n_{\text{CS}}$ . For  $\varepsilon_{\text{prec}} = k$ , when  $\alpha = 1$  the number of iterations grows mildly with the wavenumber  $k$  for both coarse spaces (but at the cost of an increasing coarse space size), while the one-level preconditioner performs poorly. When  $\alpha < 1$ , i.e. for coarser coarse meshes, the growth with  $k$  is higher, and for  $\alpha = 0.6$  the two-level preconditioner is not much better than the one-level preconditioner because the coarse grid problem is too coarse; for  $\alpha = 0.8$  with the DtN coarse space the growth with  $k$  degrades less than with the grid coarse space.

We have seen in Table 1 that the DtN coarse space gives fewer iterations than the grid coarse space, but their sizes differed significantly. Therefore, in Table 2 we compare the two methods forcing  $n_{\text{CS}}$  to be similar. On the left, we force the DtN coarse space to have a smaller size, similar to the one of the grid coarse space, by taking just  $m_i = 2$  eigenvectors for each subdomain. On the right, we do the opposite, we force the grid coarse space to have the size of the DtN coarse space obtained in Table 1, by prescribing a smaller coarse mesh diameter  $H_{\text{coarse}}$ , while keeping the same number of subdomains as in

		$\beta = 1$					$\beta = 2$				
		$\alpha = 0.6$					$\alpha = 0.6$				
$k$	$N_{\text{sub}}$	1-level	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$	1-level	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$
10	9	22	19	16	11	39	28	27	16	23	40
20	36	48	46	49	26	204	67	56	49	40	220
40	81	78	98	100	37	531	121	114	100	72	578
60	121	109	114	144	94	875	169	165	144	135	758
		$\alpha = 0.8$					$\alpha = 0.8$				
$k$	$N_{\text{sub}}$	1-level	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$	1-level	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$
10	36	35	19	49	10	122	39	27	49	28	86
20	100	71	35	121	13	394	83	51	121	41	362
40	361	158	88	400	22	1440	182	95	400	71	1370
60	676	230	187	729	39	2700	268	150	729	103	2698
		$\alpha = 1$					$\alpha = 1$				
$k$	$N_{\text{sub}}$	1-level	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$	1-level	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$
10	100	65	26	121	11	324	57	30	121	23	324
20	400	122	26	441	14	1120	130	49	441	42	1120
40	1600	286	33	1681	20	4640	296	80	1681	72	4640
60	3600	445	45	3721	29	10560	455	112	3721	101	10560

Table 1: (d=2) Number of iterations (and coarse space size  $n_{\text{CS}}$ ) for the one-level preconditioner and the two-level preconditioners with the grid coarse space/DtN coarse space, with  $H_{\text{sub}} = H_{\text{coarse}} \sim k^{-\alpha}$ ,  $\varepsilon_{\text{prec}} = k^{\beta}$ .

		$n_{\text{CS}}$ forced by grid CS					$n_{\text{CS}}$ forced by DtN CS			
		$\alpha = 0.6$					$\alpha = 0.6$			
$k$	$N_{\text{sub}}$	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$	
10	9	19	16	18	18	17	36	11	39	
20	36	46	49	44	72	24	196	26	204	
40	81	98	100	85	162	50	529	37	531	
60	121	114	144	109	242	104	841	94	875	
		$\alpha = 0.8$					$\alpha = 0.8$			
$k$	$N_{\text{sub}}$	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$	
10	36	19	49	26	72	15	121	10	122	
20	100	35	121	61	200	20	361	13	394	
40	361	88	400	139	722	35	1369	22	1440	
60	676	187	729	191	1352	52	2601	39	2700	
		$\alpha = 1$					$\alpha = 1$			
$k$	$N_{\text{sub}}$	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$	
10	100	26	121	52	200	17	324	11	324	
20	400	26	441	43	800	23	1089	14	1120	
40	1600	33	1681	157	3200	22	4624	20	4640	
60	3600	45	3721	338	7200	26	10404	29	10560	

Table 2: (d=2) Number of iterations (and coarse space size  $n_{\text{CS}}$ ) for the two-level preconditioners with the grid coarse space/DtN coarse space *forcing similar*  $n_{\text{CS}}$ , with  $H_{\text{sub}} \sim k^{-\alpha}$ ,  $\varepsilon_{\text{prec}} = k$ .

Table 1 with  $H_{\text{sub}} \sim k^{-\alpha}$ . We can observe that for smaller coarse space sizes (left) the grid coarse space gives fewer iterations than the DtN coarse space, while for larger coarse space sizes (right) the result is reversed.

			$\alpha = 0.5, \alpha' = 1$				
$k$	$n$	$N_{\text{sub}}$	1-level	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$
10	39304	27	25	12	1331	14	316
20	704969	64	39	17	9261	31	1240
30	5000211	125	55	21	29791	54	2482
40	16194277	216	74	29	68921	80	4318
			$\alpha = 0.6, \alpha' = 0.9$				
$k$	$n$	$N_{\text{sub}}$	1-level	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$
10	39304	27	25	15	512	14	316
20	912673	216	61	24	3375	41	2946
30	4826809	343	73	34	10648	65	6226
40	16194277	729	98	48	21952	108	13653
			$\alpha = 0.7, \alpha' = 0.8$				
$k$	$n$	$N_{\text{sub}}$	1-level	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$
10	46656	125	34	19	343	11	896
20	912673	512	73	35	1331	18	4567
30	5929741	1000	103	57	4096	65	12756
40	17779581	2197	139	89	8000	116	30603
			$\alpha = 0.8, \alpha' = 0.7$				
$k$	$n$	$N_{\text{sub}}$	1-level	grid CS	$n_{\text{CS}}$	DtN CS	$n_{\text{CS}}$
10	50653	216	39	23	216	19	1354
20	1030301	1000	46	86	729	23	7323
30	5929741	3375	137	116	1331	21	26645
40	28372625	6859	189	200	2744	27	54418

Table 3: (d=3) Number of iterations (and coarse space size  $n_{\text{CS}}$ ) for the one-level preconditioner and the two-level preconditioners with the grid coarse space/DtN coarse space, with  $H_{\text{sub}} \sim k^{-\alpha}$ ,  $H_{\text{coarse}} \sim k^{-\alpha'}$ ,  $\varepsilon_{\text{prec}} = k$ .

We have seen that the coarse mesh obtained with  $H_{\text{coarse}} \sim k^{-\alpha'}$ ,  $\alpha' = \alpha$  can be too coarse if  $\alpha = 0.6$ . At the same time, for  $\alpha = 1$  the number of subdomains gets quite large since  $H_{\text{sub}} \sim k^{-\alpha}$ , especially in  $3d$ ; this is not desirable because in our parallel implementation we assign each subdomain to one processor, so communication among them would prevail and each processor would not be fully exploited since the subdomains would become very small. Therefore, to improve convergence with the grid coarse space while maintaining a reasonable number of subdomains, we consider separate  $H_{\text{coarse}}$  and  $H_{\text{sub}}$ , taking  $\alpha' \neq \alpha$ . For load balancing (meant as local problems having the same size as the grid coarse space problem), in  $3d$  we choose  $\alpha' = 3/2 - \alpha$ . In Table 3 we report the results of this experiment. The DtN coarse space size is still determined by the automatic choice criterion (among 20 computed local eigenvectors) in each subdomain. As expected, for the grid coarse space the best iteration counts are obtained for  $\alpha = 0.5$  because then  $\alpha' = 1$  gives

the coarse mesh with the smallest diameter among the experimented ones: the number of iterations grows slowly, with  $\mathcal{O}(k^{0.61}) \cong \mathcal{O}(n^{0.13})$ . With higher  $\alpha$  the iteration counts get worse quickly, and  $\alpha = 0.8$  is not usable. For the DtN coarse space, the larger coarse space size is obtained by taking  $\alpha$  bigger (recall that  $\alpha'$  is not a parameter in the DtN case): for  $\alpha = 0.8$  the number of iterations grows slowly, with  $\mathcal{O}(k^{0.2}) \cong \mathcal{O}(n^{0.04})$ , but this value may be optimistic, there is a decrease in iteration number between  $k = 20$  and  $30$ . We believe that for the other values of  $\alpha$ , where the iteration counts are not much better or worse than with the one-level preconditioner, we did not compute enough eigenvectors in each subdomain to build the DtN coarse space.

To conclude, we tested numerically two different coarse space definitions for two-level domain decomposition preconditioners for the pure Helmholtz equation (discretized with piecewise linear finite elements), both in  $2d$  and  $3d$ , reaching more than 15 million degrees of freedom in the resulting linear systems. The preconditioners built with absorption  $\varepsilon_{\text{prec}} = k^2$  appear to perform much worse than those with absorption  $\varepsilon_{\text{prec}} = k$ . We have seen that in most cases for smaller coarse space sizes the grid coarse space gives fewer iterations than the DtN coarse space, while for larger coarse space sizes the grid coarse space gives generally more iterations than the DtN coarse space. The best iterations counts for the grid coarse space are obtained by separating the coarse mesh diameter  $H_{\text{coarse}} \sim k^{-\alpha'}$  from the subdomain diameter  $H_{\text{sub}} \sim k^{-\alpha}$ , taking  $\alpha' > \alpha$ . Both for the coarse grid space and the DtN coarse space, for appropriate choices of the method parameters we have obtained iteration counts which grow quite slowly with the wavenumber  $k$ . Further experiments to compare the two definitions of coarse space should be carried out in the heterogenous case.

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