From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on 2N – 2 parameters

Pierre Gaillard

From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on $2N-2$ parameters, the structure of the solutions and the case of fourth order.

P. Gaillard,
Université de Bourgogne,
Institut de mathématiques de Bourgogne,
9 avenue Alain Savary BP 47870
21078 Dijon Cedex, France :
E-mail : Pierre.Gaillard@u-bourgogne.fr

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Abstract

We have already constructed solutions to the Kadomtsev-Petviashvili equation (KPI) in terms of Fredholm determinants and wronskians of order $2N$. These solutions have been called solutions of order N and they depend on $2N-1$ parameters. We construct here N-order rational solutions. We prove that they can be written as a quotient of 2 polynomials of degree $2N(N+1)$ in x, y and t depending on $2N-2$ parameters. We explicitly construct the expressions of the rational solutions of order 4 depending on 6 real parameters and we study the patterns of their modulus in the plane (x, y) and their evolution according to time and parameters a_1, a_2, a_3, b_1, b_2, b_3.

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1 Introduction

We consider the Kadomtsev-Petviashvili equation (KP) which can be written in the form

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{1}$$

subscripts x, y and t denoting partial derivatives.
Kadomtsev and Petviashvili [1] first proposed that equation in 1970. That equation is considered as a model, for example, for surface and internal water waves [2], and in nonlinear optics [3]. In 1974 Dryuma showed how the KP equation could be written in Lax form [4]. The inverse scattering transform (IST)for the KPI equation is due to Manakov [5].
The first rational solutions were found in 1977 by Manakov, Zakharov, Bordag and Matveev [6].
From the end of the seventies, a lot of methods have been carried out to solve that equation. Dubrovin constructed for the first time in 1981 [7] the solutions to KPI given in terms of Riemann theta functions in the frame of algebraic geometry.
Various researches were conducted and more general rational solutions of the KPI equation were obtained. We can mention the following studies of Krichever in 1978 [8, 9], Satsuma and Ablowitz in 1979 [10], Matveev in 1979 [11], Freeman and Nimmo in 1983 [12, 13], Pelinovsky and Stepanyants in 1993 [14], Pelinovsky in 1994 [15], Ablowitz and Villarroel [16, 17] in 1997-1999, Biondini and Kodama [18, 19, 20] in 2003-2007.
In particular in 1999, the inverse scattering method has been applied by Prinari to obtain solutions to the KPI and KPII equations [21].

In the following, we recall the results of the author about the representations of solutions to the KPI equation. We have expressed the solutions in terms of Fredholm determinants of order $2N$ depending on $2N-1$ parameters. We have also given another representation in terms of wronskians of order $2N$ with $2N-1$ parameters. These representations allow to obtain an infinite hierarchy of solutions to the KPI equation, depending on $2N-1$ real parameters .
We have used these results to build rational solutions to the KPI equation, making a parameter to 0 tend to 0.
Here we construct rational solutions of order N depending on $2N-2$ parameters without the presence of a limit. That gives a new method to build rational solutions.
We prove that these families depending on $2N-2$ parameters for the N-th order can be written as a ratio of two polynomials of x, y and t of degree $2N(N+1)$. That provides an effective method to build an infinite hierarchy of rational solutions of order N depending on $2N-2$ real parameters. We present here only the explicit rational solutions of order 4, depending on 6 real parameters, and the representations of their modulus in the plane of the coordinates (x, y) according to the real parameters a_1, b_1, a_2, b_2, a_3, b_3 and time t.

2 Rational solutions to KPI equation of order N depending on $2N-2$ parameters

2.1 Families of rational solutions of order N depending on $2N-2$ parameters

We need to define some notations. First one defines real numbers λ_j such that $-1 < \lambda_\nu < 1$, $\nu = 1, \dots, 2N$ depending on a parameter ϵ which will be intended to tend towards 0; they can be written as

$$\lambda_j = 1 - 2\epsilon^2 j^2, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N, \tag{2}$$

The terms κ_ν, δ_ν, γ_ν, τ_ν and $x_{r,\nu}$ are functions of λ_ν, $1 \leq \nu \leq 2N$; they are defined by the formulas :

$$\begin{array}{l} \kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \quad \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}};, \\ x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, \quad r = 1, 3, \quad \tau_j = -12i\lambda_j^2\sqrt{1-\lambda_j^2} - 4i(1-\lambda_j^2)\sqrt{1-\lambda_j^2}, \\ \kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = \gamma_j^{-1}, \\ x_{r,N+j} = -x_{r,j},, \quad \tau_{N+j} = \tau_j \quad j = 1, \dots, N. \end{array} \tag{3}$$

e_ν $1 \leq \nu \leq 2N$ are defined in the following way :

$$\begin{array}{l} e_j = 2i\left(\sum_{k=1}^{1/2\,M-1} a_k (je)^{2\,k+1} - i\sum_{k=1}^{1/2\,M-1} b_K (je)^{2\,k+1}\right), \\ e_{N+j} = 2i\left(\sum_{k=1}^{1/2\,M-1} a_k (je)^{2\,k+1} + i\sum_{k=1}^{1/2\,M-1} b_k (je)^{2\,k+1}\right), \quad 1 \leq j \leq N, \\ a_k, b_k \in \mathbf{R}, \quad 1 \leq k \leq N-1. \end{array} \tag{4}$$

ϵ_ν, $1 \leq \nu \leq 2N$ are real numbers defined by :

$$\epsilon_j = 1, \quad \epsilon_{N+j} = 0 \quad 1 \leq j \leq N. \tag{5}$$

Let I be the unit matrix and $D_r = (d_{jk})_{1 \leq j,k \leq 2N}$ the matrix defined by :

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu}\right) \exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \tag{6}$$

Then we recall the following result[1] :

Theorem 2.1 *The function v defined by*

$$v(x,y,t) = -2\,\frac{|n(x,y,t)|^2}{d(x,y,t)^2} \tag{7}$$

where

$$n(x,y,t) = \det(I + D_3(x,y,t)), \tag{8}$$

[1]The proof of this result has been given by the author [50, 53]

$$d(x,y,t) = \det(I + D_1(x,y,t)), \tag{9}$$

and $D_r = (d_{jk})_{1\leq j,k\leq 2N}$ the matrix

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta\neq\mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu}\right) \exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \tag{10}$$

is a solution to the KPI equation (1), dependent on $2N-1$ parameters a_k, b_k, $1 \leq k \leq N-1$ and ϵ.

We recall a second result on the solutions to KPI equation obtained recently by the author in terms of wronskians. We need to define the following notations :

$$\phi_{r,\nu} = \sin\Theta_{r,\nu}, \quad 1\leq\nu\leq N, \quad \phi_{r,\nu} = \cos\Theta_{r,\nu}, \quad N+1\leq\nu\leq 2N, \quad r=1,3, \tag{11}$$

with the arguments

$$\Theta_{r,\nu} = \tfrac{\kappa_\nu x}{2} + i\delta_\nu y - i\tfrac{x_{r,\nu}}{2} - i\tfrac{\tau_\nu}{2}t + \gamma_\nu w - i\tfrac{e_\nu}{2}, \quad 1\leq\nu\leq 2N. \tag{12}$$

We denote $W_r(w)$ the wronskian of the functions $\phi_{r,1},\ldots,\phi_{r,2N}$ defined by

$$W_r(w) = \det[(\partial_w^{\mu-1}\phi_{r,\nu})_{\nu,\,\mu\in[1,\ldots,2N]}]. \tag{13}$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\,\mu\in[1,\ldots,2N]}$ defined in (10). Then we have the following statement[2]

Theorem 2.2 *The function v defined by*

$$v(x,y,t) = -2\ \frac{|W_3(\phi_{3,1},\ldots,\phi_{3,2N})(0)|^2}{(W_1(\phi_{1,1},\ldots,\phi_{1,2N})(0))^2}$$

is a solution to the KPI equation depending on $2N-1$ real parameters a_k, b_k $1\leq k\leq N-1$ and ϵ, with ϕ_ν^r defined by

$$\begin{array}{l}\phi_{r,\nu}(w) = \sin(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}), \quad 1\leq\nu\leq N,\\ \phi_{r,\nu}(w) = \cos(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}), \quad N+1\leq\nu\leq 2N, \quad r=1,3,\end{array}$$

κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν being defined in(3), (2) and (4).

From those two preceding results, we construct rational solutions to the KPI equation as a quotient of two determinants.
We use the following notations :

$$X_\nu = \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{3,\nu}}{2} - i\frac{\tau_\nu}{2}t - i\frac{e_\nu}{2},$$

[2]The proof of this result has been given [50, 53].

$$Y_\nu = \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{1,\nu}}{2} - i\frac{\tau_\nu}{2}t - i\frac{e_\nu}{2},$$

for $1 \leq \nu \leq 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in (3) and parameters e_ν defined by (4).
We define the following functions :

$$\begin{array}{ll} \varphi_{4j+1,k} = \gamma_k^{4j-1} \sin X_k, & \varphi_{4j+2,k} = \gamma_k^{4j} \cos X_k, \\ \varphi_{4j+3,k} = -\gamma_k^{4j+1} \sin X_k, & \varphi_{4j+4,k} = -\gamma_k^{4j+2} \cos X_k, \end{array} \tag{14}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll} \varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos X_{N+k}, & \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\ \varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos X_{N+k}, & \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin X_{N+k}, \end{array} \tag{15}$$

for $1 \leq k \leq N$.
We define the functions $\psi_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, the term X_k is only replaced by Y_k.

$$\begin{array}{ll} \psi_{4j+1,k} = \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} = \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} = -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} = -\gamma_k^{4j+2} \cos Y_k, \end{array} \tag{16}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll} \psi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin Y_{N+k}, \end{array} \tag{17}$$

for $1 \leq k \leq N$.
The following ratio

$$q(x,t) := \frac{W_3(0)}{W_1(0)}$$

can be written as

$$q(x,t) = \frac{\Delta_3}{\Delta_1} = \frac{\det(\varphi_{j,k})_{j,\, k\in[1,2N]}}{\det(\psi_{j,k})_{j,\, k\in[1,2N]}}. \tag{18}$$

The terms λ_j depending on ϵ are defined by $\lambda_j = 1 - 2j\epsilon^2$. All the functions $\varphi_{j,k}$ and $\psi_{j,k}$ and their derivatives depend on ϵ. They can all be prolonged by continuity when $\epsilon = 0$.
We use the following expansions

$$\varphi_{j,k}(x,y,t,\epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!}\varphi_{j,1}[l]k^{2l}\epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,1}[l] = \frac{\partial^{2l}\varphi_{j,1}}{\partial\epsilon^{2l}}(x,y,t,0),$$

$$\varphi_{j,1}[0] = \varphi_{j,1}(x,y,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1,$$

$$\varphi_{j,N+k}(x,y,t,\epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!}\varphi_{j,N+1}[l]k^{2l}\epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,N+1}[l] = \frac{\partial^{2l}\varphi_{j,N+1}}{\partial\epsilon^{2l}}(x,y,t,0),$$

$$\varphi_{j,N+1}[0]=\varphi_{j,N+1}(x,y,t,0),\quad 1\leq j\leq 2N,\quad 1\leq k\leq N,\quad 1\leq l\leq N-1.$$

We have the same expansions for the functions $\psi_{j,k}$.

$$\psi_{j,k}(x,y,t,\epsilon)=\sum_{l=0}^{N-1}\frac{1}{(2l)!}\psi_{j,1}[l]k^{2l}\epsilon^{2l}+O(\epsilon^{2N}),\quad \psi_{j,1}[l]=\frac{\partial^{2l}\psi_{j,1}}{\partial\epsilon^{2l}}(x,y,t,0),$$

$$\psi_{j,1}[0]=\psi_{j,1}(x,y,t,0),\quad 1\leq j\leq 2N,\quad 1\leq k\leq N,\quad 1\leq l\leq N-1,$$

$$\psi_{j,N+k}(x,t,\epsilon)=\sum_{l=0}^{N-1}\frac{1}{(2l)!}\psi_{j,N+1}[l]k^{2l}\epsilon^{2l}+O(\epsilon^{2N}),\quad \psi_{j,N+1}[l]=\frac{\partial^{2l}\psi_{j,N+1}}{\partial\epsilon^{2l}}(x,y,t,0),$$

$$\psi_{j,N+1}[0]=\psi_{j,N+1}(x,t,0),\quad 1\leq j\leq 2N,\quad 1\leq k\leq N,\quad N+1\leq k\leq 2N..$$

Then we get the following result :

Theorem 2.3 *The function v defined by*

$$v(x,y,t)=-2\frac{|\det((n_{jk})_{j,k\in[1,2N]})|^2}{\det((d_{jk})_{j,k\in[1,2N]})^2}\tag{19}$$

is a rational solution to the KPI equation (1).

$$(4u_t-6uu_x+u_{xxx})_x-3u_{yy}=0,$$

where

$$\begin{array}{l}
n_{j1}=\varphi_{j,1}(x,y,t,0),\ 1\leq j\leq 2N\quad n_{jk}=\frac{\partial^{2k-2}\varphi_{j,1}}{\partial\epsilon^{2k-2}}(x,y,t,0),\\
n_{jN+1}=\varphi_{j,N+1}(x,y,t,0),\ 1\leq j\leq 2N\quad n_{jN+k}=\frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,y,t,0),\\
d_{j1}=\psi_{j,1}(x,y,t,0),\ 1\leq j\leq 2N\quad d_{jk}=\frac{\partial^{2k-2}\psi_{j,1}}{\partial\epsilon^{2k-2}}(x,y,t,0),\\
d_{jN+1}=\psi_{j,N+1}(x,y,t,0),\ 1\leq j\leq 2N\quad d_{jN+k}=\frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,y,t,0),\\
2\leq k\leq N,\ 1\leq j\leq 2N
\end{array}\tag{20}$$

The functions φ and ψ are defined in (14),(15), (16), (17).

Remark 2.1 *In general solutions of KPI have the form*

$$u(X,Y,T)=-2\frac{\partial^2}{\partial X^2}\ln F(X,Y,T)\tag{21}$$

for some function $F(X,Y,T)$, due to the formulation of solutions through the inverse scattering method.
It is clear that the formulation given in this article is different from those given by (21). If it were the case, then necessarily, F should be of degree 20 in X,and an elementary calculation proves that the expression of u would be the quotient of a polynomial of degree 38 in X on a polynomial of degree 40 in X in the case of order 4 of the article, what is not the case.

Proof : In each column k (and $N+k$) of the determinants appearing in $q(x,t)$, we successively eliminate the powers of ϵ strictly inferior to $2(k-1)$; then each common term in the numerator and denominator is factorized and simplified; in the end, we take the limit when ϵ goes to 0.
First, the components j of the columns 1 and $N+1$ are respectively equal by definition to $\varphi_{j1}[0]+0(\epsilon)$ for C_1, $\varphi_{jN+1}[0]+0(\epsilon)$ for C_{N+1} of Δ_3, and $\psi_{j1}[0]+0(\epsilon)$ for C'_1, $\psi_{jN+1}[0]+0(\epsilon)$ for C'_{N+1} of Δ_1.
At the the first step of the reduction, we replace the columns C_k by $C_k - C_1$ and C_{N+k} by $C_{N+k} - C_{N+1}$ for $2 \leq k \leq N$, for Δ_3; the same changes for Δ_1 are made. Each component j of the column C_k of Δ_3 can be rewritten as $\sum_{l=1}^{N-1} \frac{1}{(2l)!}\varphi_{j,1}[l](k^{2l}-1)\epsilon^{2l}$ and the column C_{N+k} replaced by $\sum_{l=1}^{N-1} \frac{1}{(2l)!}\varphi_{j,N+1}[l](k^{2l}-1)\epsilon^{2l}$ for $2 \leq k \leq N$. For Δ_1, we make the same reductions, each component j of the column C'_k can be rewritten as $\sum_{l=1}^{N-1} \frac{1}{(2l)!}\psi_{j,1}[l](k^{2l}-1)\epsilon^{2l}$ and the column C'_{N+k} replaced by $\sum_{l=1}^{N-1} \frac{1}{(2l)!}\psi_{j,N+1}[l](k^{2l}-1)\epsilon^{2l}$ for $2 \leq k \leq N$.
The term $\frac{k^2-1}{2}\epsilon^2$ for $2 \leq k \leq N$ can be factorized in Δ_3 and Δ_1 in each column k and $N+k$, and so those common terms can be simplified in the numerator and denominator.
If we restrict the developments at order 1 in columns 2 and $N+2$, we respectively get $\varphi_{j1}[1]+0(\epsilon)$ for the component j of C_2, $\varphi_{jN+1}[1]+0(\epsilon)$ for the component j of C_{N+2} of Δ_3, and $\psi_{j1}[1]+0(\epsilon)$ for the component j of C'_2, $\psi_{jN+1}[1]+0(\epsilon)$ for the component j of C'_{N+2} of Δ_1. We can continue this algorithm up to the columns C_N, C_{2N} of Δ_3 and C'_N, C'_{2N} of Δ_1.
Then we take the limit when ϵ tends to 0, and $q(x,y,t)$ can be replaced by $Q(x,y,t)$ defined by :

$$Q(x,y,t) := \frac{\left\| \begin{array}{cccccc} \varphi_{1,1}[0] & \dots & \varphi_{1,1}[N-1] & \varphi_{1,N+1}[0] & \dots & \varphi_{1,N+1}[N-1] \\ \varphi_{2,1}[0] & \dots & \varphi_{2,1}[N-1] & \varphi_{2,N+1}[0] & \dots & \varphi_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{2N,1}[0] & \dots & \varphi_{2N,1}[N-1] & \varphi_{2N,N+1}[0] & \dots & \varphi_{2N,N+1}[N-1] \end{array} \right\|^2}{\left(\left| \begin{array}{cccccc} \psi_{1,1}[0] & \dots & \psi_{1,1}[N-1] & \psi_{1,N+1}[0] & \dots & \psi_{1,N+1}[N-1] \\ \psi_{2,1}[0] & \dots & \psi_{2,1}[N-1] & \psi_{2,N+1}[0] & \dots & \psi_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{2N,1}[0] & \dots & \psi_{2N,1}[N-1] & \psi_{2N,N+1}[0] & \dots & \psi_{2N,N+1}[N-1] \end{array} \right| \right)^2} \quad (22)$$

So the solution to the KPI equation takes the form :

$$v(x,y,t) = -2Q(x,y,t)$$

and we get the result. $\square$

2.2 The structure of the families of rational solutions of order N depending on $2N-2$ parameters

Here, we give a theorem which states, in this representation, the structure of the rational solutions to the KPI equation. In this section we use the notations defined in the previous sections. The functions φ and ψ are defined in (14), (15), (16), (17).

Theorem 2.4 *The function v defined by*

$$v(x,y,t) = -2\frac{\left|\det((n_{jk})_{j,k\in[1,2N]})\right|^2}{(\det((d_{jk})_{j,k\in[1,2N]}))^2} \tag{23}$$

is a rational solution to the KPI equation (1) quotient of two polynomials $n(x,y,t)$ and $d(x,y,t)$ depending on $2N-2$ real parameters a_j and b_j, $1 \leq j \leq N-1$.
n and d are polynomials of degrees $2N(N+1)$ in x, y and t.
The terms n_{jk} and d_{jk} are defined by (20) and the functions φ and ψ are defined in (14), (15), (16), (17).

Proof : From the previous result (22), we have to analyze the functions $\varphi_{k,1}$, $\psi_{k,1}$ and $\varphi_{k,N+1}$, $\psi_{k,N+1}$. The functions $\varphi_{k,j}$ and $\psi_{k,j}$ differ only by the term of the argument $x_{3,k}$, so only the study of functions $\varphi_{k,j}$ will be carried out. Then we study functions $\psi_{k,j}$ and it can be easily deduced from the analysis of $\varphi_{k,j}$. We study the expansions of those functions in ϵ. We denote $(l_{kj})_{k,j\in[1,2N]}$ the matrix defined by

$$l_{kj} = \frac{\partial^{2j-2}}{\partial\epsilon^{2j-2}}\varphi_{k1}, \quad l_{k,j+N} = \frac{\partial^{2j-2}}{\partial\epsilon^{2j-2}}\varphi_{k,1+N}, \quad 1 \leq j \leq N,\ 1 \leq k \leq 2N,$$

$\frac{\partial^0}{\partial x^0}\varphi$ meaning φ. Each coefficient of the matrix $(l_{kj})_{k,j\in[1,2N]}$ must be evaluated, the power of x, y and t in the coefficient of $\epsilon^{2(m-1)}$ for the column $m \in [1,2N]$. We remark that with those notations, the matrix $(l_{kj})_{k,j\in[1,2N]}$ evaluated in $\epsilon = 0$ is exactly $(n_{kj})_{k,j\in[1,2N]}$ defined in (20). There are four cases to study depending on the parity of k.
1. Case l_{k1} for k odd, $k = 2s+1$.

$$\begin{aligned}
l_{k1} = (-1)^s \sin(&2\epsilon(1-\epsilon^2)^{\frac{1}{2}}x + 4i\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)y - (12\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)^2 + 16\epsilon^3(1-\epsilon^2)^{\frac{3}{2}})t \\
&-i\ln\frac{1+i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}}{1-i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}} - e_1) \times \epsilon^{k-2}(1-\epsilon^2)^{-\frac{k-2}{2}} \\
= (-1)^s \sin\epsilon(&\sum_{l=0}^{p} c_{2l}\epsilon^{2l}x + 2i\sum_{l=0}^{p} c_{2l}\epsilon^{2l}(1-2\epsilon^2)y + \sum_{l=0}^{p} h_{2l}\epsilon^{2l}t + 2\sum_{l=0}^{p}(-1)^l\epsilon^{2l}\frac{(1-\epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)} \\
&-\sum_{l=1}^{N-1}\tilde{a}_l\epsilon^{2l} + i\sum_{l=1}^{N-1}\tilde{b}_l\epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{k-2}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))
\end{aligned}$$

$$= (-1)^s \sin \epsilon(\sum_{l=0}^{p}(c_{2l}x+d_{2l}y+h_{2l}t+f_{2l}+O(\epsilon^{p+1}))\epsilon^{2l})\times\epsilon^{k-2}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l}+O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q} \frac{(-1)^{l+s}\epsilon^{2l}}{(2l+1)!}(\sum_{n=0}^{p}(c_{2n}x+d_{2n}y+h_{2l}t+f_{2n}+O(\epsilon^{p+1}))\epsilon^{2n})^{2l+1}\times\epsilon^{k-1}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l}+O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q} \frac{(-1)^{l+s}\epsilon^{2l}}{(2l+1)!}(\sum_{n=0}^{p} P_n(x,y,t)\epsilon^{2n})^{2l+1} \times \epsilon^{k-1}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t)$$

where $P_n(x,y,t)$ is a polynomial of order 1 in x, y and t.

$$l_{k,1} = \sum_{l=0}^{q} \epsilon^{2l} \sum_{\alpha_0+\ldots+\alpha_p=2l+1} \beta_{\alpha_0,\ldots,\alpha_p} P_0(x,y,t)^{\alpha_0}\ldots P_p(x,y,t)^{\alpha_p}\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)}\times\epsilon^{2s}\sum_{l=1}^{r} g_{2l}\epsilon^{2l}+O(\epsilon^t)$$

$$= \sum_{l=0}^{q} \epsilon^{2l} \sum_{\alpha_0+\ldots+\alpha_p=2l+1} Q_{\alpha_0,\ldots,\alpha_p}(x,y,t)\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{2s}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t),$$

where $Q_{\alpha_0,\ldots,\alpha_p}(x,y,t)$ is a polynomial of order $2l+1$ in x, y and t.
Terms in ϵ^0 are obtained for $l=0$ in the two summations with $\alpha_0 = 1$.
For the column m, we search the terms in ϵ^{2m-2} with the maximal power in x, y and t. It is obtained for $2l+k-1 = 2m-2$, which gives $l = m-s-1$.
We get the following result

Proposition 2.1

$$\deg(n_{2s+1,m}) = 2(m-s)-1 \text{ for } s \le m-1, \quad n_{2s+1,m} = 0 \text{ for } s \ge m. \quad (24)$$

2. Case l_{k1} for k even, $k = 2s$.

$$l_{k1} = (-1)^{s+1}\cos(2\epsilon(1-\epsilon^2)^{\frac{1}{2}}x+4i\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)y-(12\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)^2+16\epsilon^3(1-\epsilon^2)^{\frac{3}{2}})t$$
$$-i\ln\frac{1+i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}}{1-i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}} - e_1) \times \epsilon^{k-2}(1-\epsilon^2)^{-\frac{k-2}{2}}$$

$$= (-1)^{s+1}\cos\epsilon(\sum_{l=0}^{p} c_{2l}\epsilon^{2l}x+2i\sum_{l=0}^{p} c_{2l}\epsilon^{2l}(1-2\epsilon^2)y+\sum_{l=0}^{p} h_{2l}\epsilon^{2l}t+2\sum_{l=0}^{p}(-1)^l\epsilon^{2l}\frac{(1-\epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)}$$
$$-\sum_{l=1}^{N-1}\tilde{a}_l\epsilon^{2l} + i\sum_{l=1}^{N-1}\tilde{b}_l\epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{k-2}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= (-1)^{s+1}\cos\epsilon(\sum_{l=0}^{p}(c_{2l}x+d_{2l}y+h_{2y}t+f_{2l}+O(\epsilon^{p+1}))\epsilon^{2l})\times\epsilon^{k-2}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l}+O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q} \frac{(-1)^{l+d+1}\epsilon^{2l}}{(2l)!}(\sum_{n=0}^{p}(c_{2n}x+d_{2n}y+h_{2l}t+f_{2n}+O(\epsilon^{p+1}))\epsilon^{2n})^{2l}\times\epsilon^{k-2}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l}+O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q} \frac{(-1)^{l+s+1}\epsilon^{2l}}{(2l)!} (\sum_{n=0}^{p} P_n(x,y,t)\epsilon^{2n})^{2l} \times \epsilon^{2s-2} \sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t)$$

where $P_n(x,y,t)$ is a polynomial of order 1 in x, y and t.

$$l_{k,1} = \sum_{l=0}^{q} \epsilon^{2l} \sum_{\alpha_0+\ldots+\alpha_p=2l} \beta_{\alpha_0,\ldots,\alpha_p} P_0(x,y,t)^{\alpha_0} \ldots P_p(x,y,t)^{\alpha_p} \epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{2s-2} \sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t)$$

$$= \sum_{l=0}^{q} \epsilon^{2l} \sum_{\alpha_0+\ldots+\alpha_p=2l} Q_{\alpha_0,\ldots,\alpha_p}(x,y,t)\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{2s-2} \sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t),$$

where $Q_{\alpha_0,\ldots,\alpha_p}(x,y,t)$ is a polynomial of order $2l$ in x, y and t.
Terms in ϵ^0 are obtained for $l=0$ in the two summations with $\alpha_0 = 1$.
For the column m, we search the terms in ϵ^{2m-2} with the maximal power in x and t. It is obtained for $2l + k - 2 = 2m - 2$, which gives $l = m - s$.
So we have :

Proposition 2.2

$$\deg(n_{2s,m}) = 2(m-s) \text{ for } s \leq m, \quad n_{2s,m} = 0 \text{ for } s > m. \tag{25}$$

3. Case $l_{k\frac{M}{2}+1}$ for k odd, $k = 2s+1$.

$$l_{k\frac{M}{2}+1} = (-1)^s \cos(2\epsilon(1-\epsilon^2)^{\frac{1}{2}}x - 4i\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)y + i\ln\frac{1+i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}}{1-i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}} - e_{\frac{M}{2}+1}$$

$$-(12\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)^2 + 16\epsilon^3(1-\epsilon^2)^{\frac{3}{2}})t) \times \epsilon^{M-k-1}(1-\epsilon^2)^{-\frac{M-k-1}{2}}$$

$$= (-1)^s(\cos\epsilon(\sum_{l=0}^{p} c_{2l}\epsilon^{2l}x - 2i\sum_{l=0}^{p} c_{2l}\epsilon^{2l}(1-2\epsilon^2)y + \sum_{l=0}^{p} h_{2l}\epsilon^{2l}t - 2\sum_{l=0}^{p}(-1)^l\epsilon^{2l}\frac{(1-\epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)}$$

$$-\sum_{l=1}^{N-1}\tilde{a}_l\epsilon^{2l} + i\sum_{l=1}^{N-1}\tilde{b}_l\epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{M-k-1}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= (-1)^s(\cos\epsilon(\sum_{l=0}^{p}(c_{2l}x + d_{2l}y + h_{2l}t + f_{2l})\epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{M-k-1}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q}\frac{(-1)^{l+s}\epsilon^{2l}}{(2l)!}(\sum_{n=0}^{p}(c_{2n}x + d_{2n}y + h_{2l}t + f_{2n} + O(\epsilon^{p+1}))\epsilon^{2n})^{2l} \times \epsilon^{M-k-1}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q}\frac{(-1)^{l+s}\epsilon^{2l}}{(2l)!}(\sum_{n=0}^{p} P_n(x,y,t)\epsilon^{2n} + O(\epsilon^{p+1}))^{2l} \times \epsilon^{M-2s-2}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

where $P_n(x,y,t)$ is a polynomial of order 1 in x and t.

$$l_{k,\frac{M}{2}+1} = \sum_{l=0}^{q}\epsilon^{2l}\sum_{\alpha_0+\ldots+\alpha_p=2l}\beta_{\alpha_0,\ldots,\alpha_p}P_0(x,y,t)^{\alpha_0}$$

$$\dots P_p(x,y,t)^{\alpha_p}\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{M-2s-2}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t)$$

$$= \sum_{l=0}^{q}\epsilon^{2l}\sum_{\alpha_0+\ldots+\alpha_p=2l} Q_{\alpha_0,\ldots,\alpha_p}(x,y,t)\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{M-2s-2}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t),$$

where $Q_{\alpha_0,\ldots,\alpha_p}(x,y,t)$ is a polynomial of order $2l$ in x, y and t.
The terms in ϵ^0 (column $\frac{M}{2}+1$) are obtained for $l=0$ in the two summations with $\alpha_0 = 1$.
For the column $\frac{M}{2}+m$, we search the terms in ϵ^{2m-2} with the maximal power in x and t. It is obtained for $2l+2(N-s-1) = 2m-2$, which gives $l = m+s-N$.
So we obtain the following result

Proposition 2.3

$$\deg(n_{2s+1,m+\frac{M}{2}}) = 2m+2s-M \ \mathit{for}\ s \geq \frac{M}{2}-m, \quad n_{2s+1,m} = 0 \ \mathit{for}\ s < \frac{M}{2}-m. \tag{26}$$

4. Case $l_{k,1+\frac{M}{2}}$ for k even, $k=2s$.

$$l_{k\frac{M}{2}+1} = (-1)^s \sin(2\epsilon(1-\epsilon^2)^{\frac{1}{2}}x - 4i\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)y + i\ln\frac{1+i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}}{1-i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}} - e_{\frac{M}{2}+1}$$

$$-(12\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)^2 + 16\epsilon^3(1-\epsilon^2)^{\frac{3}{2}})t) \times \epsilon^{M-k-1}(1-\epsilon^2)^{-\frac{M-k-1}{2}}$$

$$= (-1)^s \sin\epsilon(\sum_{l=0}^{p} c_{2l}\epsilon^{2l}x - 2i\sum_{l=0}^{p} c_{2l}\epsilon^{2l}(1-2\epsilon^2)y + \sum_{l=0}^{p} h_{2l}t - 2\sum_{l=0}^{p}(-1)^l\epsilon^{2l}\frac{(1-\epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)}$$

$$-\sum_{l=1}^{N-1}\tilde{a}_l\epsilon^{2l} + i\sum_{l=1}^{N-1}\tilde{b}_l\epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{M-k-1}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= (-1)^s \sin\epsilon(\sum_{l=0}^{p}(c_{2l}x + d_{2l}y + h_{2l}t + f_{2l})\epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{M-k-1}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q}\frac{(-1)^{l+s}\epsilon^{2l}}{(2l+1)!}(\sum_{n=0}^{p}(c_{2n}x + d_{2n}y + h_{2l}t + f_{2n} + O(\epsilon^{p+1}))\epsilon^{2n})^{2l+1} \times \epsilon^{M-k}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

$$= \sum_{l=0}^{q}\frac{(-1)^{l+s}\epsilon^{2l}}{(2l+1)!}(\sum_{n=0}^{p} P_n(x,y,t)\epsilon^{2n} + O(\epsilon^{p+1}))^{2l+1} \times \epsilon^{M-2s}(\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^{r+1}))$$

where $P_n(x,y,t)$ is a polynomial of order 1 in x, y and t.

$$l_{k,1} = \sum_{l=0}^{q}\epsilon^{2l}\sum_{\alpha_0+\ldots+\alpha_p=2l+1}\beta_{\alpha_0,\ldots,\alpha_p}P_0(x,y,t)^{\alpha_0}$$

$$\dots P_p(x,y,t)^{\alpha_p}\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{M-2s}\sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t)$$

$$= \sum_{l=0}^{q} \epsilon^{2l} \sum_{\alpha_0+\ldots+\alpha_p=2l+1} Q_{\alpha_0,\ldots,\alpha_p}(x,y,t)\epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{M-2s} \sum_{l=1}^{r} g_{2l}\epsilon^{2l} + O(\epsilon^t),$$

where $Q_{\alpha_0,\ldots,\alpha_p}(x,y,t)$ is a polynomial of order $2l+1$ in x,y and t.
The terms in ϵ^0 are obtained for $l=0$ in the two summations with $\alpha_0 = 1$.
For the column $\frac{M}{2}+m$, we search the terms in ϵ^{2m-2} with the maximal power in x and t. It is obtained for $2l+M-k = 2m-2$, which gives $l = m+s-N-1$.
We get the following result

Proposition 2.4

$$\begin{array}{l} \deg(n_{2s,m+\frac{M}{2}}) = 2m+2s-M-1 \textit{ for } s \geq \frac{M}{2}+1-M, \\ n_{2s,m+\frac{M}{2}} = 0 \textit{ for } s < \frac{M}{2}+1-m. \end{array} \tag{27}$$

These results can be rewritten in the following way

Proposition 2.5

$$\begin{array}{l} \deg(n_{j,k}) = 2k-j \textit{ for } j \leq 2k, \\ n_{j,k} = 0 \textit{ for } j > 2k, \\ \deg(n_{j,k}) = 2k+j-2M-1 \textit{ for } j \geq 2M+1-2k, \\ n_{j,k} = 0 \textit{ for } j < 2M+1-2k. \end{array} \tag{28}$$

Now we can evaluate the degree of the determinant of the matrix $(n_{kj})_{k,j\in[1,2N]}$.
From the previous analysis, x, y and t have necessarily the same power in each n_{kj}. The maximal power in x, y and t, is successively taken in each column. It is realized by the following product

$$\prod_{j=1}^{N} n_{j,j} \prod_{j=1}^{N} n_{N+j,2N+1-j}.$$

Applying the result given in (28) we get

$$\deg(\det(n_{kj})_{k,j\in[1,2N]}) = \sum_{j=1}^{N} \deg(n_{j,j}) + \sum_{j=1}^{N} \deg(n_{N+j,2N+1-j})$$

$$= \sum_{j=1}^{N} 2j - j + \sum_{j=1}^{N} 2(M+1-j) - 2M - 1 + \frac{M}{2} + j$$

$$= \sum_{j=1}^{N} j + \sum_{j=1}^{N} N+1-j = N(N+1).$$

We have the same argument for the determinant $\det(d_{kj})_{k,j\in[1,2N]}$, we have $\deg(\det(d_{kj})_{k,j\in[1,2N]}) = N(N+1)$.
Thus the quotient

$$\frac{\det((n_{kj})_{j,k\in[1,2N]})}{\det((d_{kj})_{j,k\in[1,2N]})}$$

defines a quotient of two polynomials, each of them of degree $N(N+1)$. As the expression of the solution v is given by $\frac{n(x,y,t)n^*(x,y,t)}{d(x,y,t)^2}$, it gives the result on the degrees of the polynomials $|n|^2$ and d^2. So we obtain

$$\frac{\left|\det((n_{kj})_{j,k\in[1,2N]})\right|^2}{\left(\det((d_{kj})_{j,k\in[1,2N]})\right)^2}$$

is a quotient of two polynomials of degree $2N(N+1)$.
□

3 Explicit expression of rational solutions of order 4 depending on 6 parameters

In the following, we explicitly construct rational solutions to the KPI equation of order 4 depending on 6 parameters.
Because of the length of the expression, it cannot be given in that paper. We only give the expression without parameters in the appendix.

We give patterns of the modulus of the solutions in the plane (x, y) of coordinates in function of the parameters a_1, a_2, a_3, b_1, b_2, b_3, and time t.
When at least one parameter is not equal to 0, we observe the presence of ten peaks. The maximum of modulus of those solutions is checked equal in this case $N = 4$ to $2(2N+1)^2 = 2\times 9^2 = 162$.

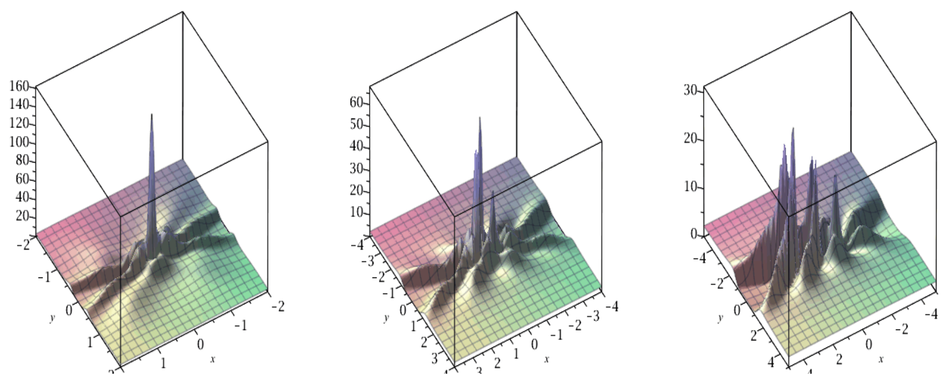

Figure 1. Solution of order 4 to KPI, on the left for $t = 0$; in the center for $t = 0,01$; on the right for $t = 0,1$; all the parameters to equal to 0.

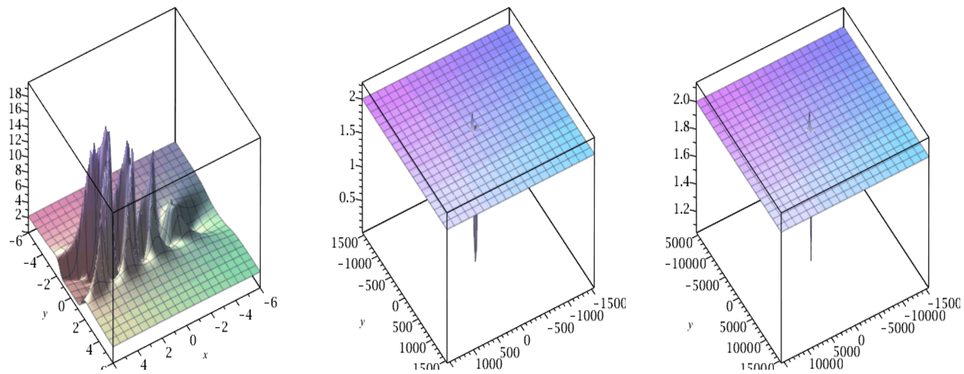

Figure 2. Solution of order 4 to KPI, on the left for $t = 0, 2$; in the center for $t = 10$; on the right for $t = 50$; all the parameters to equal to 0.

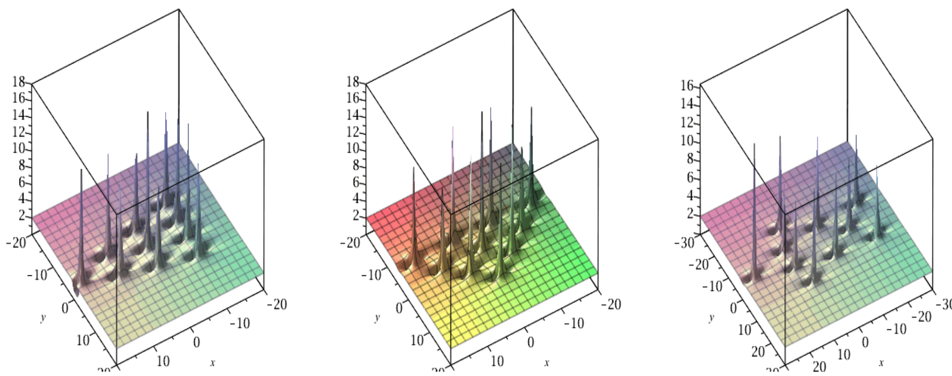

Figure 3. Solution of order 4 to KPI for $t = 0$, on the left for $a_1 = 10^3$; in the center for $b_1 = 10^3$; on the right for $a_2 = 10^6$; all the other parameters to equal to 0.

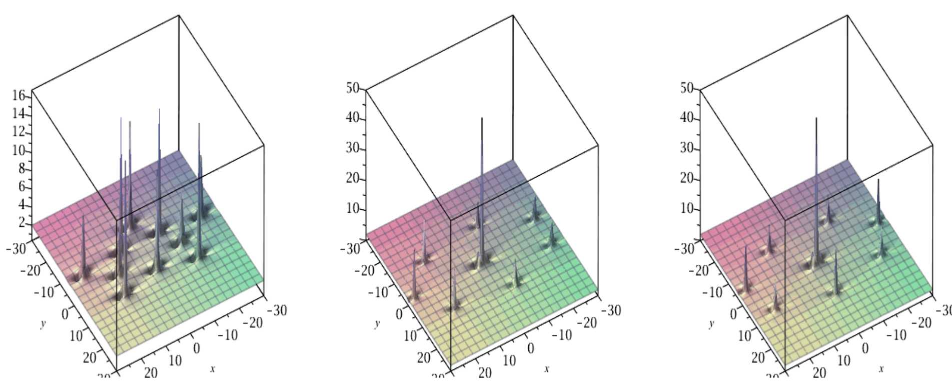

Figure 4. Solution of order 4 to KPI for $t = 0$, on the left for $b_2 = 10^6$; in the center for $a_3 = 10^9$; on the right for $b_3 = 10^9$; all the other parameters to equal to 0.

4 Conclusion

From the previous representations of the solutions to the KPI equation given by the author in terms of Fredholm determinants of order $2N$ depending on $2N-1$ real parameters and in terms of wronskians of order $2N$ depending on $2N-1$ real parameters, we succeed in obtaining rational solutions to the KPI equation depending on $2N-2$ real parameters. These solutions can be expressed in terms of a ratio of two polynomials of degree $2N(N+1)$ in x, y and t. That gives a new approach to find explicit solutions for higher orders and try to describe the structure of those rational solutions.
In the (x, y) plane of coordinates, different structures appear. For a given t, when one parameter grows and the other ones are equal to 0 we obtain triangles or rings; for $a_1 \neq 0$ or $b_1 \neq 0$ and the other parameters equal to zero, we obtain a triangle with 10 peaks; for $a_2 \neq 0$ or $b_2 \neq 0$, and other parameters equal to zero, we obtain two concentric rings of 5 peaks on each of them; in the last case, when $a_3 \neq 0$ or $b_3 \neq 0$, and the other parameters equal to zero, we obtain one ring with 7 peaks.
It will be relevant to go on this study for higher orders to try to understand the structure of those rational solutions.

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Appendix : Because of the length of the complete expression, we only give in this appendix the explicit expression of the rational solution of order 4 to KPI equation without parameters. They can be written as

$$v_4(x,y,t) = -2\frac{|n_4(x,y,t)|^2}{(d_4(x,y,t))^2}$$

with

$$\frac{n_4(x,y,t)}{d_4(x,y,t)} = \frac{F_4(2x,4y,4t) - iH_4(2x,4y,4t)}{Q_4(2x,4y,4t)}$$

with

$$\begin{aligned} F_4(X,Y,T) &= \textstyle\sum_{k=0}^{20} f_k(Y,T)X^k, \\ H_4(X,Y,T) &= \textstyle\sum_{k=0}^{20} h_k(Y,T)X^k, \\ Q_4(X,Y,T) &= \textstyle\sum_{k=0}^{20} q_k(Y,T)X^k. \end{aligned}$$

$\mathbf{f}_{20} = 1$, $\mathbf{f}_{19} = -60\,T$, $\mathbf{f}_{18} = 1710\,T^2 + 10\,Y^2 - 30$, $\mathbf{f}_{17} = -30780\,T^3 + (-540\,Y^2 + 2340)T$, $\mathbf{f}_{16} = 392445\,T^4 + 45\,Y^4 + (13770\,Y^2 - 78030)T^2 - 1350\,Y^2 - 675$, $\mathbf{f}_{15} = -3767472\,T^5 + (-220320\,Y^2 + 1542240)T^3 + (-2160\,Y^4 + 67680\,Y^2 + 23760)T$, $\mathbf{f}_{14} = 28256040\,T^6 + 120\,Y^6 + (2478600\,Y^2 - 20655000)T^4 - 9000\,Y^4 + (48600\,Y^4 - 1587600\,Y^2 - 210600)T^2 + 16200\,Y^2 - 19800$, $\mathbf{f}_{13} = -169536240\,T^7 + (-20820240\,Y^2 + 201262320)T^5 + (-680400\,Y^4 + 23133600\,Y^2 - 3402000)T^3 + (-5040\,Y^6 + 378000\,Y^4 - 680400\,Y^2 + 831600)T$, $\mathbf{f}_{12} = 826489170\,T^8 + 210\,Y^8 + (135331560\,Y^2 - 1488647160)T^6 - 27720\,Y^6 + (6633900\,Y^4 - 234397800\,Y^2 + 112776300)T^4 + 224700\,Y^4 + (98280\,Y^6 - 7371000\,Y^4 + 14175000\,Y^2 - 18937800)T^2 + 163800\,Y^2 - 330750$, $\mathbf{f}_{11} = -3305956680\,T^9 + (-695990880\,Y^2 + 8583887520)T^7 + (-47764080\,Y^4 + 1751349600\,Y^2 - 1512529200)T^5 + (-1179360\,Y^6 + 88452000\,Y^4 - 191872800\,Y^2 + 297410400)T^3 + (-7560\,Y^8 + 977760\,Y^6 - 7182000\,Y^4 - 4989600\,Y^2 + 12209400)T$, $\mathbf{f}_{10} = 10909657044\,T^{10} + 252\,Y^{10} + (2870962380\,Y^2 - 39236485860)T^8 - 49140\,Y^8 + (262702440\,Y^4 - 9982692720\,Y^2 + 12872419560)T^6 + 768600\,Y^6 + (9729720\,Y^6 - 729729000\,Y^4 + 1852389000\,Y^2 - 3381701400)T^4 - 37800\,Y^4 + (124740\,Y^8 - 15800400\,Y^6 + 106255800\,Y^4 - 14288400\,Y^2 - 247268700)T^2 + 4517100\,Y^2 + 5953500$, $\mathbf{f}_9 = -29753610120\,T^{11} + (-9569874600\,Y^2 + 143548119000)T^9 + (-1125867600\,Y^4 + 44284125600\,Y^2 - 77684864400)T^7 + (-58378320\,Y^6 + 4378374000\,Y^4 - 13269841200\,Y^2 + 28193734800)T^5 + (-1247400\,Y^8 + 154677600\,Y^6 - 967302000\,Y^4 + 1756188000\,Y^2 + 3847095000)T^3 + (-7560\,Y^{10} + 1423800\,Y^8 - 18824400\,Y^6 + 39942000\,Y^4 - 114345000\,Y^2 - 353241000)T$, $\mathbf{f}_8 = 66945622770\,T^{12} + 210\,Y^{12} + (25838661420\,Y^2 - 422031469860)T^{10} - 54180\,Y^{10} + (3799803150\,Y^4 - 154525328100\,Y^2 + 348315288750)T^8 + 1086750\,Y^8 + (262702440\,Y^6 - 19702683000\,Y^4 + 71839013400\,Y^2 - 174023526600)T^6 - 9991800\,Y^6 + (8419950\,Y^8 - 1021620600\,Y^6 + 6070018500\,Y^4 - 25969167000\,Y^2 - 43117373250)T^4 + 16584750\,Y^4 + (102060\,Y^{10} - 18540900\,Y^8 + 201511800\,Y^6 - 995085000\,Y^4 - 579190500\,Y^2 + 8419099500)T^2 + 196465500\,Y^2 + 36713250$, $\mathbf{f}_7 = -123591918960\,T^{13} + (-56375261280\,Y^2 + 995962949280)T^{11} + (-10132808400\,Y^4 + 425577952800\,Y^2 - 1185538582800)T^9 + (-900694080\,Y^6 + 67552056000\,Y^4 - 296189784000\,Y^2 + 801733204800)T^7 + (-40415760\,Y^8 + 4796003520\,Y^6 - 27813391200\,Y^4 + 202397227200\,Y^2 + 326806326000)T^5 + (-816480\,Y^{10} + 142884000\,Y^8 - 1227441600\,Y^6 + 11498760000\,Y^4 + 23249268000\,Y^2 - 113429484000)T^3 + (-5040\,Y^{12} + 1239840\,Y^{10} - 17917200\,Y^8 + 205329600\,Y^6 + 445662000\,Y^4 - 7613676000\,Y^2 - 1166886000)T$, $\mathbf{f}_6 = 185387878440\,T^{14} + 120\,Y^{14} + (98656707240\,Y^2 - 1874477437560)T^{12} - 37800\,Y^{12} + (21278897640\,Y^4 - 922085564400\,Y^2 + 3085440157800)T^{10} + 550200\,Y^{10} + (2364321960\,Y^6 - 177324147000\,Y^4 + 930269755800\,Y^2 - 2766559811400)T^8 - 11428200\,Y^8 + (141455160\,Y^8 - 16408798560\,Y^6 + 95803722000\,Y^4 - 996272978400\,Y^2 - 1664434283400)T^6 - 61651800\,Y^6 + (4286520\,Y^{10} - 721564200\,Y^8 + 4615153200\,Y^6 - 80658018000\,Y^4 - 184165569000\,Y^2 + 1033444251000)T^4 + 1215837000\,Y^4 + (52920\,Y^{12} - 12383280\,Y^{10} + 106936200\,Y^8 - 2327421600\,Y^6 - 11188611000\,Y^4 + 112283010000\,Y^2 + 14967099000)T^2 - 662823000\,Y^2 + 178605000$, $\mathbf{f}_5 = -222465454128\,T^{15} + (-136601594640\,Y^2 + 2777565757680)T^{13} + (-34820014320\,Y^4 + 1555293972960\,Y^2 - 6116715848880)T^{11} + (-4728643920\,Y^6 + 354648294000\,Y^4 - 2209731678000\,Y^2 + 7123883936400)T^9 + (-363741840\,Y^8 + 41224075200\,Y^6 - 250320520800\,Y^4 + 3293296574400\,Y^2 + 5725976281200)T^7 + (-15431472\,Y^{10} + 2494754640\,Y^8 - 10716300000\,Y^6 + 377813872800\,Y^4 + 687000560400\,Y^2 - 6652893366000)T^5 + (-317520\,Y^{12} + 70489440\,Y^{10} - 181666800\,Y^8 + 18397108800\,Y^6 + 95136930000\,Y^4 - 961466436000\,Y^2 - 46461114000)T^3 + (-2160\,Y^{14} + 640080\,Y^{12} - 1920240\,Y^{10} + 265582800\,Y^8 + 2820106800\,Y^6 - 30077082000\,Y^4 - 9787554000\,Y^2 - 6453594000)T$, $\mathbf{f}_4 = 208561363245\,T^{16} + 45\,Y^{16} + (146358851400\,Y^2 - 3171108447000)T^{14} - 16200\,Y^{14} + (43525017900\,Y^4 - 2002150823400\,Y^2 + 9096728741100)T^{12} - 195300\,Y^{12} + (7092965880\,Y^6 - 531972441000\,Y^4 + 3903859297800\,Y^2 - 13501187746200)T^{10} - 16291800\,Y^{10} + (682015950\,Y^8 - 75476431800\,Y^6 + 491671498500\,Y^4 - 7437280599000\,Y^2 - 13220948081250)T^8 - 277877250\,Y^8 + (38578680\,Y^{10} - 5979695400\,Y^8 + 13759729200\,Y^6 - 1219729266000\,Y^4 - 1319712345000\,Y^2 + 29637106443000)T^6 + 3163293000\,Y^6 + (1190700\,Y^{12} - 250047000\,Y^{10} - 941503500\,Y^8 - 99137682000\,Y^6 - 431777587500\,Y^4 + 5344754625000\,Y^2 - 429098512500)T^4 - 803722500\,Y^4 + (16200\,Y^{14} - 4498200\,Y^{12} - 42751800\,Y^{10} - 3134943000\,Y^8 - 18928161000\,Y^6 + 324168075000\,Y^4 + 203431095000\,Y^2 + 188428275000)T^2 - 1369305000\,Y^2 - 602791875$, $\mathbf{f}_3 = -147219785820\,T^{17} + (-117087081120\,Y^2 + 2693002865760)T^{15} + (-40176939600\,Y^4 + 1901708474400\,Y^2 - 9843350202000)T^{13} + (-7737780960\,Y^6 + 580333572000\,Y^4 - 4973012301600\,Y^2 + 18299851970400)T^{11} + (-909354600\,Y^8 + 98210296800\,Y^6 - 705163158000\,Y^4 + 11330558316000\,Y^2 + 19880558271000)T^9 + (-66134880\,Y^{10} + 9810007200\,Y^8 - 4188542400\,Y^6 + 2694996360000\,Y^4 + 1242417204000\,Y^2 - 87870659436000)T^7 + (-2857680\,Y^{12} + 565820640\,Y^{10} + 5909274000\,Y^8 + 344579054400\,Y^6 + 1171720242000\,Y^4 - 19326632724000\,Y^2 + 4056976854000)T^5 + (-64800\,Y^{14} + 16783200\,Y^{12} + 388735200\,Y^{10} + 19525212000\,Y^8 + 49294980000\,Y^6 - 2027603340000\,Y^4 - 1354778460000\,Y^2 - 1753901100000)T^3 + (-540\,Y^{16} + 180000\,Y^{14} + 6274800\,Y^{12} + 337125600\,Y^{10} - 1266489000\,Y^8 -$

$68409684000\,Y^6-25441290000\,Y^4-88826220000\,Y^2+28666102500)T$, $\mathbf{f}_2 = 73609892910\,T^{18}+10\,Y^{18}+(65861483130\,Y^2-1602629422830)T^{16} - 3870\,Y^{16} + (25828032600\,Y^4 - 1256964253200\,Y^2 + 7326551914200)T^{14} - 307800\,Y^{14} + (5803335720\,Y^6 - 435250179000\,Y^4 + 4319021007000\,Y^2 - 16802144944200)T^{12} - 17602200\,Y^{12} + (818419140\,Y^8 - 86206816080\,Y^6 + 697144303800\,Y^4 - 11117107990800\,Y^2 - 18113723617500)T^{10} + 367699500\,Y^{10} + (74401740\,Y^{10} - 10540246500\,Y^8 - 13805656200\,Y^6 - 3899891205000\,Y^4 - 795891946500\,Y^2 + 16411225801 9500)T^8 + 6701089500\,Y^8 + (4286520\,Y^{12} - 797292720\,Y^{10} - 13970993400\,Y^8 - 731023120800\,Y^6 - 1935042291000\,Y^4 + 43337360178000\,Y^2 - 7924239477000)T^6 - 424683000\,Y^6 + (145800\,Y^{14} - 35040600\,Y^{12} - 1340047800\,Y^{10} - 64262079000\,Y^8 - 83670489000\,Y^6 + 6289575075000\,Y^4 + 3558347415000\,Y^2 + 6769665315000)T^4 + 18753525000\,Y^4 + (2430\,Y^{16} - 745200\,Y^{14} - 45019800\,Y^{12} - 2206990800\,Y^{10} + 10958692500\,Y^8 + 417308598000\,Y^6 + 719599545000\,Y^4 + 521883810000\,Y^2 - 352610921250)T^2 - 20494923750\,Y^2 - 669768750$, $\mathbf{f}_1 = -23245229340\,T^{19} + (-23245229340\,Y^2 + 596627553060)T^{17} + (-10331213040\,Y^4 + 516560652000\,Y^2 - 3357644238000)T^{15} + (-2678462640\,Y^6 + 200884698000\,Y^4 - 2290085557200\,Y^2 + 9370155135600)T^{13} + (-446410440\,Y^8 + 45831471840\,Y^6 - 424089918000\,Y^4 + 6312243621600\,Y^2 + 8458733820600)T^{11} + (-49601160\,Y^{10} + 6696156600\,Y^8 + 19344452400\,Y^6 + 3330717894000\,Y^4 + 1392552567000\,Y^2 - 174096351513000)T^9 + (-3674160\,Y^{12} + 639303840\,Y^{10} + 16037708400\,Y^8 + 858945124800\,Y^6 + 1823944878000\,Y^4 - 54338070780000\,Y^2 - 13343293782000)T^7 + (-174960\,Y^{14} + 38782800\,Y^{12} + 2137136400\,Y^{10} + 106066875600\,Y^8 + 163044932400\,Y^6 - 7593784506000\,Y^4 - 1811269026000\,Y^2 - 11259616410000)T^5 + (-4860\,Y^{16} + 1360800\,Y^{14} + 121791600\,Y^{12} + 5942613600\,Y^{10} - 12650337000\,Y^8 - 757618596000\,Y^6 - 2949482970000\,Y^4 - 1110923100000\,Y^2 + 3258558922500)T^3 + (-60\,Y^{18} + 21060\,Y^{16} + 2624400\,Y^{14} + 114987600\,Y^{12} - 1424190600\,Y^{10} - 26868429000\,Y^8 - 149289966000\,Y^6 - 73942470000\,Y^4 + 188696182500\,Y^2 + 32952622500)T$, $\mathbf{f}_0 = 3486784401\,T^{20} + Y^{20} + (3874204890\,Y^2 - 104603532030)T^{18} - 390\,Y^{18} + (1937102445\,Y^4 - 99437925510\,Y^2 + 714790802205)T^{16} - 88275\,Y^{16} + (573956280\,Y^6 - 43046721000\,Y^4 + 559607373000\,Y^2 - 2398180656600)T^{14} - 2968200\,Y^{14} + (111602610\,Y^8 - 11160261000\,Y^6 + 119414792700\,Y^4 - 1561692522600\,Y^2 - 1211446331550)T^{12} + 49640850\,Y^{12} + (14880348\,Y^{10} - 1909644660\,Y^8 - 8184191400\,Y^6 - 1272021748200\,Y^4 - 1409788970100\,Y^2 + 79887008281500)T^{10} + 498865500\,Y^{10} + (1377810\,Y^{12} - 223205220\,Y^{10} - 7419506850\,Y^8 - 428416241400\,Y^6 - 764167871250\,Y^4 + 29016058585500\,Y^2 + 39256780673250)T^8 + 14702735250\,Y^8 + (87480\,Y^{14} - 17758440\,Y^{12} - 1318411080\,Y^{10} - 69297719400\,Y^8 - 174225605400\,Y^6 + 1085239701000\,Y^4 + 2223882297000\,Y^2 - 11175064803000)T^6 + 11728395000\,Y^6 + (3645\,Y^{16} - 923400\,Y^{14} - 113796900\,Y^{12} - 5747679000\,Y^{10} - 5715785250\,Y^8 + 254226357000\,Y^6 + 4465273837500\,Y^4 + 1876602735000\,Y^2 - 13362757261875)T^4 - 44301481875\,Y^4 + (90\,Y^{18} - 28350\,Y^{16} - 4973400\,Y^{14} - 225552600\,Y^{12} + 934548300\,Y^{10} + 25870225500\,Y^8 + 498653253000\,Y^6 + 1033706205000\,Y^4 - 543093153750\,Y^2 - 28532148750)T^2 - 5492103750\,Y^2 + 200930625$

$\mathbf{h}_{20} = 0$, $\mathbf{h}_{19} = 0$, $\mathbf{h}_{18} = 40\,Y$, $\mathbf{h}_{17} = -2160\,TY$, $\mathbf{h}_{16} = 55080\,T^2Y + 360\,Y^3 - 1080\,Y$, $\mathbf{h}_{15} = -881280\,T^3Y + (-17280\,Y^3 + 63360\,Y)T$, $\mathbf{h}_{14} = 9914400\,T^4Y + 1440\,Y^5 - 24000\,Y^3 + (388800\,Y^3 - 1684800\,Y)T^2 - 21600\,Y$, $\mathbf{h}_{13} = -83280960\,T^5Y + (-5443200\,Y^3 + 27216000\,Y)T^3 + (-60480\,Y^5 + 1008000\,Y^3 + 907200\,Y)T$, $\mathbf{h}_{12} = 541326240\,T^6Y + 3360\,Y^7 - 117600\,Y^5 + (53071200\,Y^3 - 300736800\,Y)T^4 + 50400\,Y^3 + (1179360\,Y^5 - 19656000\,Y^3 - 14061600\,Y)T^2 - 554400\,Y$, $\mathbf{h}_{11} = -2783963520\,T^7Y + (-382112640\,Y^3 + 2420046720\,Y)T^5 + (-14152320\,Y^5 + 235872000\,Y^3 + 81648000\,Y)T^3 + (-120960\,Y^7 + 3991680\,Y^5 - 1008000\,Y^3 + 23587200\,Y)T$, $\mathbf{h}_{10} = 11483849520\,T^8Y + 5040\,Y^9 - 262080\,Y^7 + (2101619520\,Y^3 - 14711336640\,Y)T^6 + 1038240\,Y^5 + (116756640\,Y^5 - 1945944000\,Y^3 + 404157600\,Y)T^4 - 2116800\,Y^3 + (1995840\,Y^7 - 61871040\,Y^5 + 25099200\,Y^3 - 514382400\,Y)T^2 - 7938000\,Y$, $\mathbf{h}_9 = -38279498400\,T^9Y + (-9006940800\,Y^3 + 69053212800\,Y)T^7 + (-700539840\,Y^5 + 11675664000\,Y^3 - 11046974400\,Y)T^5 + (-19958400\,Y^7 + 578793600\,Y^5 - 553392000\,Y^3 + 6371568000\,Y)T^3 + (-151200\,Y^9 + 7056000\,Y^7 - 25502400\,Y^5 + 148176000\,Y^3 + 322812000\,Y)T$, $\mathbf{h}_8 = 103354645680\,T^{10}Y + 5040\,Y^{11} - 310800\,Y^9 + (30398425200\,Y^3 - 253320210000\,Y)T^8 + 2872800\,Y^7 + (3152429280\,Y^5 - 52540488000\,Y^3 + 98210296800\,Y)T^6 - 21319200\,Y^5 + (134719200\,Y^7 - 3637418400\,Y^5 + 7246260000\,Y^3 - 46559772000\,Y)T^4 - 24570000\,Y^3 + (2041200\,Y^9 - 84369600\,Y^7 + 322509600\,Y^5 - 1873368000\,Y^3 - 6317514000\,Y)T^2 + 119070000\,Y$, $\mathbf{h}_7 = -225501045120\,T^{11}Y + (-81062467200\,Y^3 + 729562204800\,Y)T^9 + (-10808328960\,Y^5 + 180138816000\,Y^3 - 536259398400\,Y)T^7 + (-646652160\,Y^7 + 16166304000\,Y^5 - 58688582400\,Y^3 + 195857222400\,Y)T^5 + (-16329600\,Y^9 + 587865600\,Y^7 - 2841350400\,Y^5 + 7293888000\,Y^3 + 62297424000\,Y)T^3 + (-120960\,Y^{11} + 6249600\,Y^9 - 54432000\,Y^7 + 97977600\,Y^5 + 2113776000\,Y^3 - 4000752000\,Y)T$, $\mathbf{h}_6 = 394626828960\,T^{12}Y + 3360\,Y^{13} - 194880\,Y^{11} + (170231181120\,Y^3 - 1645568084160\,Y)T^{10} + 2872800\,Y^9 + (28371863520\,Y^5 - 472864392000\,Y^3 + 2018767212000\,Y)T^8 + 11894400\,Y^7 + (2263282560\,Y^7 - 52055498880\,Y^5 + 313773264000\,Y^3 - 355609699200\,Y)T^6 - 291060000\,Y^5 + (85730400\,Y^9 - 2629065600\,Y^7 + 18574920000\,Y^5 + 18860688000\,Y^3 - 410886756000\,Y)T^4 + 1746360000\,Y^3 + (1270080\,Y^{11} - 52920000\,Y^9 + 491702400\,Y^7 + 1384387200\,Y^5 - 39721752000\,Y^3 + 43341480000\,Y)T^2 + 587412000\,Y$, $\mathbf{h}_5 = -546406378560\,T^{13}Y + (-278560114560\,Y^3 + 2878454517120\,Y)T^{11} + (-56743727040\,Y^5 + 945728784000\,Y^3 - 5434303089600\,Y)T^9 + (-5819869440\,Y^7 + 122217258240\,Y^5 - 1148983315200\,Y^3 - 731010873600\,Y)T^7 + (-308629440\,Y^9 + 7818612480\,Y^7 - 88268019840\,Y^5 - 268164691200\,Y^3 + 2365301736000\,Y)T^5 + (-7620480\,Y^{11} + 241315200\,Y^9 - 2906668800\,Y^7 - 8763552000\,Y^5 + 409537296000\,Y^3 - 240616656000\,Y)T^3 + (-60480\,Y^{13} + 2540160\,Y^{11} - 33969600\,Y^9 - 68947200\,Y^7 + 11945102400\,Y^5 - 27878256000\,Y^3 - 24290280000\,Y)T$, $\mathbf{h}_4 = 585435405600\,T^{14}Y + 1440\,Y^{15} - 50400\,Y^{13} + (348200143200\,Y^3 - 3830201575200\,Y)T^{12} + 554400\,Y^{11} + (85115590560\,Y^5 - 1418593176000\,Y^3 + 10508501757600\,Y)T^{10} - 23436000\,Y^9 + (10912255200\,Y^7 - 207332848800\,Y^5 + 2914894836000\,Y^3 + 6320841156000\,Y)T^8 - 1313172000\,Y^7 + (771573600\,Y^9 - 15431472000\,Y^7 + 294741115200\,Y^5 + 992758032000\,Y^3 - 11729204676000\,Y)T^6 + 4302396000\,Y^5 + (28576800\,Y^{11} - 619164000\,Y^9 + 12369672000\,Y^7 - 27147960000\,Y^5 - 2638829340000\,Y^3 + 863733780000\,Y)T^4 + 3095820000\,Y^3 + (453600\,Y^{13} - 11793600\,Y^{11} + 176148000\,Y^9 - 5579280000\,Y^7 - 136073196000\,Y^5 + 205276680000\,Y^3 + 62154540000\,Y)T^2 + 2143260000\,Y$, $\mathbf{h}_3 = -468348324480\,T^{15}Y + (-321415516800\,Y^3 + 3749847696000\,Y)T^{13} + (-92853371520\,Y^5 + 1547556192000\,Y^3 - 14320846915200\,Y)T^{11} + (-14549673600\,Y^7 + 247344451200\,Y^5 - 5059318320000\,Y^3 - 18008527824000\,Y)T^9 + (-1322697600\,Y^9 + 19399564800\,Y^7 - 667521388800\,Y^5 - 1666598976000\,Y^3 + 41423214672000\,Y)T^7 + (-68584320\,Y^{11} + 800150400\,Y^9 - 37133510400\,Y^7 + 379271289600\,Y^5 + 10599135120000\,Y^3 - 2391878160000\,Y)T^5 + (-1814400\,Y^{13} + 18144000\,Y^{11} - 607824000\,Y^9 + 60927552000\,Y^7 + 869433264000\,Y^5 - 775383840000\,Y^3 + 1976562000000\,Y)T^3 + (-17280\,Y^{15} + 201600\,Y^{13} + 3830400\,Y^{11} + 1823472000\,Y^9 + 19362672000\,Y^7 - 67504752000\,Y^5 + 58106160000\,Y^3 - 80967600000\,Y)T$, $\mathbf{h}_2 = 263445932520\,T^{16}Y + 360\,Y^{17} + 4800\,Y^{15} + (206624260800\,Y^3 - 2548365883200\,Y)T^{14} - 756000\,Y^{13} + (69640028640\,Y^5 - 1160667144000\,Y^3 + 13097682309600\,Y)T^{12} - 160574400\,Y^{11} + (13094706240\,Y^7 - 196420593600\,Y^5 + 5751750513600\,Y^3 + 27957197822400\,Y)T^{10} - 1231146000\,Y^9 + (1488034800\,Y^9 - 13888324800\,Y^7 + 973174759200\,Y^5 + 856446696000\,Y^3 - 90858578202000\,Y)T^8 + 8341704000\,Y^7 + (102876480\,Y^{11} - 171460800\,Y^9 + 72748368000\,Y^7 - 1328135356800\,Y^5 - 25493649048000\,Y^3 + 5177258856000\,Y)T^6 - 17860500000\,Y^5 + (4082400\,Y^{13} + 24494400\,Y^{11} + 1639764000\,Y^9 - 242902800000\,Y^7 - 3055860108000\,Y^5 + 323394120000\,Y^3 - 19690129620000\,Y)T^4 +$

$17622360000\,Y^3+(77760\,Y^{15}+907200\,Y^{13}-42638400\,Y^{11}-13343400000\,Y^9-143196984000\,Y^7+87159240000\,Y^5-709180920000\,Y^3+441511560000\,Y)T^2-4822335000\,Y,\quad \mathbf{h}_1=-92980917360\,T^{17}Y+(-82649704320\,Y^3+1074446156160\,Y)T^{15}+(-32141551680\,Y^5+535692528000\,Y^3-7231849128000\,Y)T^{13}+(-7142567040\,Y^7+92853371520\,Y^5-3868890480000\,Y^3-23606184067200\,Y)T^{11}+(-992023200\,Y^9+3968092800\,Y^7-823379256000\,Y^5+972182736000\,Y^3+109732646268000\,Y)T^9+(-88179840\,Y^{11}-734832000\,Y^9-82007251200\,Y^7+2064290054400\,Y^5+33473802096000\,Y^3-7175634480000\,Y)T^7+(-4898880\,Y^{13}-107775360\,Y^{11}-2882174400\,Y^9+425516716800\,Y^7+5142966696000\,Y^5+5569618320000\,Y^3+67042030104000\,Y)T^5+(-155520\,Y^{15}-5443200\,Y^{13}+92534400\,Y^{11}+34228656000\,Y^9+398088432000\,Y^7+895977936000\,Y^5+2717653680000\,Y^3-4260800880000\,Y)T^3+(-2160\,Y^{17}-97920\,Y^{15}+6955200\,Y^{13}+936835200\,Y^{11}+8511804000\,Y^9+12982032000\,Y^7-99352008000\,Y^5-254333520000\,Y^3-73942470000\,Y)T,\quad \mathbf{h}_0=15496819560\,T^{18}Y+40\,Y^{19}+3720\,Y^{17}+(15496819560\,Y^3-211789867320\,Y)T^{16}-344160\,Y^{15}+(6887475360\,Y^5-114791256000\,Y^3+1825180970400\,Y)T^{14}-28072800\,Y^{13}+(1785641760\,Y^7-19642059360\,Y^5+1169595352800\,Y^3+8538343682400\,Y)T^{12}-192402000\,Y^{11}+(297606960\,Y^9+396809280\,Y^7+307328787360\,Y^5-1077337195200\,Y^3-55744263666000\,Y)T^{10}-2171610000\,Y^9+(33067440\,Y^{11}+606236400\,Y^9+40011602400\,Y^7-1235068884000\,Y^5-18432066618000\,Y^3+4392182718000\,Y)T^8+24156468000\,Y^7+(2449440\,Y^{13}+93078720\,Y^{11}+2192248800\,Y^9-275464022400\,Y^7-3069405511200\,Y^5-9810129672000\,Y^3-53432186220000\,Y)T^6+51771636000\,Y^5+(116640\,Y^{15}+6804000\,Y^{13}-42184800\,Y^{11}-29930796000\,Y^9-349786836000\,Y^7-1803148452000\,Y^5-663696180000\,Y^3+13747584060000\,Y)T^4+3750705000\,Y^3+(3240\,Y^{17}+250560\,Y^{15}-10432800\,Y^{13}-1537099200\,Y^{11}-17013402000\,Y^9-116271288000\,Y^7+296865324000\,Y^5+1576010520000\,Y^3+853910505000\,Y)T^2-2679075000\,Y,$

$\mathbf{q}_{20}=1,\quad \mathbf{q}_{19}=-60\,T,\quad \mathbf{q}_{18}=1710\,T^2+10\,Y^2+10,\quad \mathbf{q}_{17}=-30780\,T^3+(-540\,Y^2+180)T,\quad \mathbf{q}_{16}=392445\,T^4+45\,Y^4+(13770\,Y^2-22950)T^2-270\,Y^2+405,\quad \mathbf{q}_{15}=-3767472\,T^5+(-220320\,Y^2+660960)T^3+(-2160\,Y^4+15840\,Y^2-16560)T,\quad \mathbf{q}_{14}=28256040\,T^6+120\,Y^6+(2478600\,Y^2-10740600)T^4-1800\,Y^4+(48600\,Y^4-421200\,Y^2+437400)T^2+1800\,Y^2+16200,\quad \mathbf{q}_{13}=-169536240\,T^7+(-20820240\,Y^2+117981360)T^5+(-680400\,Y^4+6804000\,Y^2-8845200)T^3+(-5040\,Y^6+75600\,Y^4-75600\,Y^2-680400)T,\quad \mathbf{q}_{12}=826489170\,T^8+210\,Y^8+(135331560\,Y^2-947320920)T^6-4200\,Y^6+(6633900\,Y^4-75184200\,Y^2+130466700)T^4+6300\,Y^4+(98280\,Y^6-1474200\,Y^4+2381400\,Y^2+14175000)T^2+113400\,Y^2+425250,\quad \mathbf{q}_{11}=-3305956680\,T^9+(-695990880\,Y^2+5799924000)T^7+(-47764080\,Y^4+605011680\,Y^2-1385158320)T^5+(-1179360\,Y^6+17690400\,Y^4-50349600\,Y^2-187034400)T^3+(-7560\,Y^8+131040\,Y^6-529200\,Y^4-5594400\,Y^2-16216200)T,\quad \mathbf{q}_{10}=10909657044\,T^{10}+252\,Y^{10}+(2870962380\,Y^2-27752636340)T^8-3780\,Y^8+(262702440\,Y^4-3677834160\,Y^2+10770800040)T^6+63000\,Y^6+(9729720\,Y^6-145945800\,Y^4+684822600\,Y^2+1692721800)T^4+718200\,Y^4+(124740\,Y^8-1829520\,Y^6+16443000\,Y^4+100926000\,Y^2+307030500)T^2+3005100\,Y^2+1644300,\quad \mathbf{q}_9=-29753610120\,T^{11}+(-9569874600\,Y^2+105268620600)T^9+(-1125867600\,Y^4+17263303200\,Y^2-62673296400)T^7+(-58378320\,Y^6+875674800\,Y^4-6264442800\,Y^2-10874833200)T^5+(-1247400\,Y^8+14968800\,Y^6-268758000\,Y^4-1101492000\,Y^2-3879225000)T^3+(-7560\,Y^{10}+63000\,Y^8-3301200\,Y^6-10962000\,Y^4-153657000\,Y^2+30051000)T,\quad \mathbf{q}_8=66945622770\,T^{12}+210\,Y^{12}+(25838661420\,Y^2-318676824180)T^{10}+1260\,Y^{10}+(3799803150\,Y^4-63330052500\,Y^2+277385629950)T^8+255150\,Y^8+(262702440\,Y^6-3940536600\,Y^4+40314720600\,Y^2+50283941400)T^6-567000\,Y^6+(8419950\,Y^8-78586200\,Y^6+2702038500\,Y^4+9241533000\,Y^2+37326318750)T^4+23388750\,Y^4+(102060\,Y^{10}-170100\,Y^8+68153400\,Y^6+345303000\,Y^4+3816193500\,Y^2-2089678500)T^2-31468500\,Y^2+17435250,\quad \mathbf{q}_7=-123591918960\,T^{13}+(-56375261280\,Y^2+770461904160)T^{11}+(-10132808400\,Y^4+182390551200\,Y^2-942351181200)T^9+(-900694080\,Y^6+13510411200\,Y^4-188106494400\,Y^2-166859352000)T^7+(-40415760\,Y^8+269438400\,Y^6-18113608800\,Y^4-64199822400\,Y^2-282124458000)T^5+(-816480\,Y^{10}-4082400\,Y^8-770212800\,Y^6-6627096000\,Y^4-54153036000\,Y^2+39007332000)T^3+(-5040\,Y^{12}-90720\,Y^{10}-8845200\,Y^8-122472000\,Y^6-1350594000\,Y^4+1122660000\,Y^2-540918000)T,\quad \mathbf{q}_6=185387878440\,T^{14}+120\,Y^{14}+(98656707240\,Y^2-1479850608600)T^{12}+5880\,Y^{12}+(21278897640\,Y^4-411392021040\,Y^2+2461259160360)T^{10}+476280\,Y^{10}+(2364321960\,Y^6-35464829400\,Y^4+646551120600\,Y^2+387081941400)T^8+16443000\,Y^8+(141455160\,Y^8-565820640\,Y^6+84487309200\,Y^4+352437674400\,Y^2+1627955998200)T^6+162729000\,Y^6+(4286520\,Y^{10}+50009400\,Y^8+5415303600\,Y^6+64797894000\,Y^4+448524783000\,Y^2-413065737000)T^4-154791000\,Y^4+(52920\,Y^{12}+1587600\,Y^{10}+125987400\,Y^8+2689394400\,Y^6+22277997000\,Y^4-12216582000\,Y^2+11252115000)T^2+130977000\,Y^2+130977000,\quad \mathbf{q}_5=-222465454128\,T^{15}+(-136601594640\,Y^2+2231159379120)T^{13}+(-34820014320\,Y^4+719613629280\,Y^2-4909622019120)T^{11}+(-4728643920\,Y^6+70929658800\,Y^4-1642294407600\,Y^2-580168234800)T^9+(-363741840\,Y^8+484989120\,Y^6-279419868000\,Y^4-1434024648000\,Y^2-6904022202000)T^7+(-15431472\,Y^{10}-282910320\,Y^8-25119007200\,Y^6-368040607200\,Y^4-2224403823600\,Y^2+2815000549200)T^5+(-317520\,Y^{12}-13335840\,Y^{10}-981817200\,Y^8-24201374400\,Y^6-165864510000\,Y^4+94922604000\,Y^2-154100394000)T^3+(-2160\,Y^{14}-146160\,Y^{12}-11234160\,Y^{10}-395766000\,Y^8-3183138000\,Y^6-2357586000\,Y^4-1214514000\,Y^2-4072194000)T,\quad \mathbf{q}_4=208561363245\,T^{16}+45\,Y^{16}+(146358851400\,Y^2-2585673041400)T^{14}+5400\,Y^{14}+(43525017900\,Y^4-957550393800\,Y^2+7355728025100)T^{12}+459900\,Y^{12}+(7092965880\,Y^6-106394488200\,Y^4+3052703392200\,Y^2+411937633800)T^{10}+19845000\,Y^{10}+(682015950\,Y^8+909354600\,Y^6+655355326500\,Y^4+4144590261000\,Y^2+20846781978750)T^8+153798750\,Y^8+(38578680\,Y^{10}+964467000\,Y^8+78571911600\,Y^6+1303316406000\,Y^4+6596632791000\,Y^2-12646412793000)T^6+702513000\,Y^6+(1190700\,Y^{12}+64297800\,Y^{10}+4630972500\,Y^8+118903302000\,Y^6+624135172500\,Y^4-657087795000\,Y^2+1204780027500)T^4-89302500\,Y^4+(16200\,Y^{14}+1398600\,Y^{12}+106936200\,Y^{10}+3512565000\,Y^8+21015855000\,Y^6+36256815000\,Y^4+81979695000\,Y^2+73406655000)T^2+1250235000\,Y^2+111628125,\quad \mathbf{q}_3=-147219785820\,T^{17}+(-117087081120\,Y^2+2224654541280)T^{15}+(-40176939600\,Y^4+937461924000\,Y^2-8021995606800)T^{13}+(-7737780960\,Y^6+116066714400\,Y^4-4044478586400\,Y^2+264870194400)T^{11}+(-909354600\,Y^8-3637418400\,Y^6-1068904998000\,Y^4-8212298724000\,Y^2-43324546545000)T^9+(-66134880\,Y^{10}-2094271200\,Y^8-164675851200\,Y^6-2935286424000\,Y^4-10952854668000\,Y^2+37041227748000)T^7+(-2857680\,Y^{12}-188606880\,Y^{10}-13637257200\,Y^8-346065048000\,Y^6-1081274670000\,Y^4+3184455708000\,Y^2-5368223322000)T^5+(-64800\,Y^{14}-6804000\,Y^{12}-529351200\,Y^{10}-15773940000\,Y^8-51549372000\,Y^6-124070940000\,Y^4-1219514940000\,Y^2-928031580000)T^3+(-540\,Y^{16}-79200\,Y^{14}-6829200\,Y^{12}-211831200\,Y^{10}-235305000\,Y^8-3826116000\,Y^6-57510810000\,Y^4-70727580000\,Y^2+2946982500)T,\quad \mathbf{q}_2=73609892910\,T^{18}+10\,Y^{18}+(65861483130\,Y^2-1339183490310)T^{16}+2250\,Y^{16}+(25828032600\,Y^4-637091470800\,Y^2+6017931595800)T^{14}+225000\,Y^{14}+(5803335720\,Y^6-87050035800\,Y^4+3622620720600\,Y^2-918861489000)T^{12}+4422600\,Y^{12}+(818419140\,Y^8+5456127600\,Y^6+1155459022200\,Y^4+10590343671600\,Y^2+58758154150500)T^{10}-99508500\,Y^{10}+(74401740\,Y^{10}+2852066700\,Y^8+222295865400\,Y^6+4100775903000\,Y^4+7646225485500\,Y^2-67891381078500)T^8-224248500\,Y^8+(4286520\,Y^{12}+334348560\,Y^{10}+24607686600\,Y^8+597112236000\,Y^6+158494077000\,Y^4-9049915350000\,Y^2+19448619939000)T^6+9704205000\,Y^6+$

$(145800\,Y^{14}+18030600\,Y^{12}+1444149000\,Y^{10}+39165525000\,Y^{8}+15205239000\,Y^{6}+349172775000\,Y^{4}+8546427855000\,Y^{2}+$
$6426029295000)T^{4}+15181425000\,Y^{4}+(2430\,Y^{16}+421200\,Y^{14}+37535400\,Y^{12}+1021734000\,Y^{10}-1274899500\,Y^{8}+$
$21265902000\,Y^{6}+672447825000\,Y^{4}+800507610000\,Y^{2}-79702481250)T^{2}-1920003750\,Y^{2}+223256250,\quad \mathbf{q}_1=$
$-23245229340\,T^{19}+(-23245229340\,Y^{2}+503646635700)T^{17}+(-10331213040\,Y^{4}+268611539040\,Y^{2}-2779096307760)T^{15}+$
$(-2678462640\,Y^{6}+40176939600\,Y^{4}-1968670040400\,Y^{2}+852643940400)T^{13}+(-446410440\,Y^{8}-4166497440\,Y^{6}-$
$745505434800\,Y^{4}-8008603293600\,Y^{2}-46789022233800)T^{11}+(-49601160\,Y^{10}-2232052200\,Y^{8}-175092094800\,Y^{6}-$
$3246395922000\,Y^{4}+2404416231000\,Y^{2}+70630811811000)T^{9}+(-3674160\,Y^{12}-330674400\,Y^{10}-24965917200\,Y^{8}-$
$565159291200\,Y^{6}+2013531534000\,Y^{4}+13474430676000\,Y^{2}-53405599590000)T^{7}+(-174960\,Y^{14}-24902640\,Y^{12}-$
$2066102640\,Y^{10}-52142454000\,Y^{8}+82801278000\,Y^{6}-1480349682000\,Y^{4}-26904557106000\,Y^{2}-19968967746000)T^{5}+$
$(-4860\,Y^{16}-972000\,Y^{14}-90493200\,Y^{12}-2510676000\,Y^{10}-5873553000\,Y^{8}-176366484000\,Y^{6}-2792310570000\,Y^{4}-$
$1725324300000\,Y^{2}+866680762500)T^{3}+(-60\,Y^{18}-15660\,Y^{16}-1609200\,Y^{14}-48610800\,Y^{12}-355509000\,Y^{10}-$
$6036849000\,Y^{8}-91945854000\,Y^{6}-86802030000\,Y^{4}-48491257500\,Y^{2}-6697687500)T,\quad \mathbf{q}_0=3486784401\,T^{20}+$
$Y^{20}+(3874204890\,Y^{2}-89106712470)T^{18}+370\,Y^{18}+(1937102445\,Y^{4}-52947466830\,Y^{2}+595981852245)T^{16}+$
$44325\,Y^{16}+(573956280\,Y^{6}-8609344200\,Y^{4}+490732619400\,Y^{2}-297500671800)T^{14}+2208600\,Y^{14}+(111602610\,Y^{8}+$
$1339231320\,Y^{6}+217625089500\,Y^{4}+2697063075000\,Y^{2}+16594378085250)T^{12}+62795250\,Y^{12}+(14880348\,Y^{10}+$
$768817980\,Y^{8}+61257432600\,Y^{6}+1113794047800\,Y^{4}-4832269010100\,Y^{2}-31831172421300)T^{10}+693384300\,Y^{10}+$
$(1377810\,Y^{12}+140536620\,Y^{10}+10932922350\,Y^{8}+225988396200\,Y^{6}-1962862571250\,Y^{4}-8139343684500\,Y^{2}+$
$62846265683250)T^{8}+6641129250\,Y^{8}+(87480\,Y^{14}+14084280\,Y^{12}+1214309880\,Y^{10}+29500443000\,Y^{8}-40829103000\,Y^{6}+$
$2397343473000\,Y^{4}+36134542017000\,Y^{2}+27930142737000)T^{6}+4346055000\,Y^{6}+(3645\,Y^{16}+826200\,Y^{14}+80797500\,Y^{12}+$
$2448419400\,Y^{10}+21248466750\,Y^{8}+335336841000\,Y^{6}+4250709697500\,Y^{4}-2414918205000\,Y^{2}+772042438125)T^{4}+$
$14042818125\,Y^{4}+(90\,Y^{18}+26730\,Y^{16}+2932200\,Y^{14}+114193800\,Y^{12}+2226098700\,Y^{10}+27136903500\,Y^{8}+$
$291852477000\,Y^{6}+188071065000\,Y^{4}+424231526250\,Y^{2}+99527636250)T^{2}+2902331250\,Y^{2}+22325625,$