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# $L^p$ -estimates for the heat semigroup on differential forms, and related problems

Jocelyn Magniez and El Maati Ouhabaz

## Abstract

We consider a complete non-compact Riemannian manifold satisfying the volume doubling property and a Gaussian upper bound for its heat kernel (on functions). Let  $\vec{\Delta}_k$  be the Hodge-de Rham Laplacian on differential  $k$ -forms with  $k \geq 1$ . By the Bochner decomposition formula  $\vec{\Delta}_k = \nabla^* \nabla + R_k$ . Under the assumption that the negative part  $R_k^-$  is in an enlarged Kato class, we prove that for all  $p \in [1, \infty]$ ,  $\|e^{-t\vec{\Delta}_k}\|_{p-p} \leq C(t \log t)^{\frac{D}{4}(1-\frac{2}{p})}$  (for large  $t$ ). This estimate can be improved if  $R_k^-$  is strongly sub-critical. In general,  $(e^{-t\vec{\Delta}_k})_{t>0}$  is not uniformly bounded on  $L^p$  for any  $p \neq 2$ . We also prove the gradient estimate  $\|\nabla e^{-t\Delta}\|_{p-p} \leq Ct^{-\frac{1}{p}}$ , where  $\Delta$  is the Laplace-Beltrami operator (acting on functions). Finally we discuss heat kernel bounds on forms and the Riesz transform on  $L^p$  for  $p > 2$ .

## 1 Introduction and main results

Let  $(M, g)$  be a complete non-compact Riemannian manifold of dimension  $N$ , where  $g$  denotes a Riemannian metric on  $M$ , Let  $\rho$  and  $\mu$  be the Riemannian distance and measure associated with  $g$ , respectively. We suppose that  $M$  satisfies the volume doubling property, that is there exists constants  $C, D > 0$  such that

$$v(x, \lambda r) \leq C\lambda^D v(x, r), \quad \forall x \in M, \forall r \geq 0, \forall \lambda \geq 1, \quad (\text{D})$$

where  $v(x, r) = \mu(B(x, r))$  denotes the volume of the ball  $B(x, r)$  of center  $x$  and radius  $r$ . It is a standard fact that this property is equivalent to the following one

$$v(x, 2r) \leq Cv(x, r), \quad \forall x \in M, \forall r \geq 0.$$

Let  $\Delta$  be (the non-negative) Laplace-Beltrami operator and  $(e^{-t\Delta})_{t \geq 0}$  the associated heat semigroup. It is a well known fact that  $(e^{-t\Delta})_{t \geq 0}$  is a submarkovian semigroup. In particular, it acts as a contraction semigroup on  $L^p(M)$  for all  $p \in [1, \infty]$ , i.e.,

$$\|e^{-t\Delta}\|_{p-p} \leq 1 \quad \forall t \geq 0, \forall p \in [1, \infty]. \quad (1)$$

The semigroup is strongly continuous on  $L^p(M)$  for  $1 \leq p < \infty$ .

Now we introduce the Hodge-de Rham Laplacian  $\vec{\Delta}_k := d_k^* d_k + d_{k-1} d_{k-1}^*$  where  $d_k$  denotes the exterior derivative on differential  $k$ -forms and  $d_k^*$  its formal adjoint. For  $k = 1$  we use the notation  $\vec{\Delta}$  instead of  $\vec{\Delta}_1$ . The operator  $\vec{\Delta}_k$  is self-adjoint on  $L^2(\Lambda^k T^* M)$  where  $\Lambda^k T^* M$  denotes the space of smooth differential  $k$ -forms on  $M$ . Since  $\vec{\Delta}_k$  is non-negative,  $(e^{-t\vec{\Delta}_k})_{t \geq 0}$  is a contraction semigroup on  $L^2(\Lambda^k T^* M)$ . One of the main questions we address in this paper concerns  $L^p$ -estimates of this semigroup. We formulate this as follows.

Question: Suppose that  $k \geq 1$ . Does the semigroup  $(e^{-t\vec{\Delta}_k})_{t \geq 0}$  extend from  $L^2(\Lambda^k T^* M) \cap L^p(\Lambda^k T^* M)$  to  $L^p(\Lambda^k T^* M)$  and what is the behavior of its norm  $\|e^{-t\vec{\Delta}_k}\|_{p \rightarrow p}$  on  $L^p(\Lambda^k T^* M)$  for *all or some*  $p \neq 2$ ?

The answer to this question is intimately related to the geometry of the manifold  $M$ . While it is not hard to prove in a quite general setting that  $(e^{-t\vec{\Delta}_k})_{t \geq 0}$  extends to a strongly continuous semigroup on  $L^p(\Lambda^k T^* M)$ , the precise estimate of its  $L^p$ -norm turns to be very complicated. We believe that uniform boundedness (*w.r.t.*  $t$ ), at least when  $k = 1$ , is related to deep questions such as boundedness on  $L^p(M)$  of the Riesz transform (on functions)  $d\Delta^{-1/2}$  on  $L^p(M)$  (with value in  $L^p(\Lambda^1 T^* M)$ ). We shall see that this is indeed the case under an additional assumption on  $M$ , and in general  $(e^{-t\vec{\Delta}})_{t \geq 0}$  is not uniformly bounded on  $L^p(\Lambda^1 T^* M)$  for any  $p \neq 2$ .

According to Bochner's formula,  $\vec{\Delta}_k = \nabla^* \nabla + R_k$  where  $\nabla$  denotes the Levi-Civita connection and  $R_k$  is a symmetric section of  $\text{End}(\Lambda^k T^* M)$ . For differential forms of order 1,  $\vec{\Delta} = \nabla^* \nabla + R$  where  $R$  is identified with the Ricci curvature. The above formula allows us to look at  $\vec{\Delta}_k$  as a Schrödinger operator with the vector-valued "potential"  $R_k$ . Regarding  $L^p$  estimates of  $e^{-t\vec{\Delta}}$  (or  $e^{-t\vec{\Delta}_k}$ ) it is expected that the difficulty occurs in the setting of manifolds whose Ricci curvature has a nontrivial negative part. This is what happens with Schrödinger operators on functions with potentials having a non-trivial negative part. When  $k = 1$ , the link of the previous question to the geometry of  $M$  is promptly done via the negative part of its Ricci curvature. The same observation can be made when  $k \geq 2$ .

Let now  $p(t, x, y)$  be the heat kernel on functions (the heat kernel of the Laplace-Beltrami operator  $\Delta$ ). We assume throughout this paper that  $p(t, x, y)$  satisfies a Gaussian upper bound

$$p(t, x, y) \leq \frac{C}{v(x, \sqrt{t})} \exp\left(-c \frac{\rho^2(x, y)}{t}\right) \quad \forall t > 0, \quad (\text{G})$$

where  $C$  and  $c$  are positive constants. We denote by  $H$  the operator  $\nabla^* \nabla + R_k^+$  where  $R_k^+$  denotes the positive part of  $R_k$ . It is a self-adjoint operator (defined by quadratic form

method). The well known domination property says that (in the pointwise sense)

$$|e^{-tH}\omega| \leq e^{-t\Delta}|\omega| \quad (2)$$

for all  $t > 0$  and  $w \in L^2(\Lambda^k T^*M)$ . Therefore, it follows from (1) that the semigroup  $e^{-tH}$  acts as a contraction semigroup on  $L^p(\Lambda^k T^*M)$  for all  $p \in [1, \infty)$ . It also follows from (G) that the heat kernel  $h_t(x, y)$  of  $H$  satisfies a Gaussian upper bound. The operator  $\vec{\Delta}_k$  can be seen as the perturbation of  $H$  by the negative "potential"  $-R_k^-$ , i.e.,  $\vec{\Delta}_k = H - R_k^-$ . In order to make this precise (using for example the method of quadratic forms to deal with this perturbation) we assume that  $R_k^-$  is in the *enlarged Kato class*  $\hat{K}$  which we introduce now.

**Definition 1.1.** We say that a function  $f \in \hat{K}$  if there exists  $\alpha > 0$  such that

$$\sup_{x \in M} \int_M \int_0^\alpha p_s(x, y) |f(y)| ds d\mu(y) < 1.$$

We say that  $R_k^- \in \hat{K}$  if the function  $x \mapsto |R_k^-(x)|_x$  belongs to  $\hat{K}$ .

Note that  $\hat{K}$  contains the usual Kato class  $K$ , which is defined as the set of functions  $f$  such that

$$\lim_{\alpha \rightarrow 0} \sup_{x \in M} \int_M \int_0^\alpha p_s(x, y) |f(y)| ds d\mu(y) = 0.$$

The Kato class  $K$  plays an important role in the study of Schrödinger operators and their associated semigroups, see Simon [29] and the references there. The class  $\hat{K}$  appears in Voigt [32] who studied properties of semigroups associated to Schrödinger operators with potential in  $\hat{K}$  such as  $L^p - L^q$  properties for instance.

Note that the assumption that the Ricci curvature is bounded from below means that the negative part  $R_-$  is bounded. In this case,  $R_- \in \hat{K}$ . Indeed,

$$\int_M \int_0^\alpha p_s(x, y) |R_-(y)|_y ds d\mu(y) \leq \|R_-\|_\infty \int_0^\alpha \int_M p_s(x, y) d\mu(y) ds \leq \alpha \|R_-\|_\infty.$$

The last inequality follows from (1) (with  $p = \infty$ ).

Let  $\vec{p}_k(t, x, y)$  denote the heat kernel of the operator  $\vec{\Delta}_k$ . The following is our first main result.

**Theorem 1.2.** *Suppose that the manifold  $M$  satisfies the volume doubling condition (D), the Gaussian upper bound (G) and  $R_k^- \in \hat{K}$ . Then*

(i) *the semigroup  $(e^{-t\vec{\Delta}_k})$  acts on  $L^p(\Lambda^k T^*M)$  for all  $p \in [1, \infty]$  and*

$$\|e^{-t\vec{\Delta}_k}\|_{p \rightarrow p} \leq C_p (t \log t)^{\left|\frac{1}{2} - \frac{1}{p}\right| \frac{D}{2}}, \quad t > e.$$

(ii) For all  $t \geq 1$  and  $p \geq 2$

$$\|\nabla e^{-t\Delta}\|_{p-p} \leq C_p t^{-\frac{1}{p}}.$$

(iii) There exists  $C > 0$  such that for all  $t > 0$  and  $x, y \in M$

$$|\vec{p}_k(t, x, y)| \leq C \frac{\left(1 + t + \frac{\rho^2(x, y)}{t}\right)^{\frac{D}{2}}}{v(x, \sqrt{t})^{\frac{1}{2}} v(y, \sqrt{t})^{\frac{1}{2}}} \exp\left(-\frac{\rho^2(x, y)}{4t}\right).$$

(iv) There exist  $C, c > 0$  such that for all  $t \geq 1$  and  $x, y \in M$

$$|\vec{p}_k(t, x, y)| \leq C \min\left(1, \frac{t^{\frac{D}{2}}}{v(x, \sqrt{t})}\right) \exp\left(-c \frac{\rho^2(x, y)}{t}\right).$$

Here and throughout this paper we use the notation  $|\vec{p}_k(t, x, y)|$  for the norm from  $\Lambda^k T_y^* M$  to  $\Lambda^k T_x^* M$  of the linear map  $\vec{p}_k(t, x, y)$  between these two spaces. Thus this norm depends on the fixed points  $x$  and  $y$ . We make this dependence implicit.

Note that property (iii) obviously gives

$$|\vec{p}_k(t, x, y)| \leq C_\epsilon \frac{(1 + t)^{\frac{D}{2}}}{v(x, \sqrt{t})^{\frac{1}{2}} v(y, \sqrt{t})^{\frac{1}{2}}} \exp\left(-\frac{\rho^2(x, y)}{(4 + \epsilon)t}\right), \quad t > 0$$

for every  $\epsilon > 0$  and In particular,

$$|\vec{p}_k(t, x, y)| \leq \frac{C}{v(x, \sqrt{t})} \exp\left(-c \frac{\rho^2(x, y)}{t}\right)$$

for all  $t \in (0, 1]$  and some constant  $c > 0$ . This is the reason why we formulate (iv) and several other estimates in this paper for  $t > 1$  only.

Similar results as in this theorem (when  $k = 1$ ) have been proved by Coulhon and Zhang [14] under additional assumptions. More precisely, it is assumed in [14] a "non collapsing" property on the volume  $v(x, r)$ , the Ricci curvature is bounded from below and that the negative part  $V(x)$  of the lowest eigenvalue of the Ricci tensor is strongly sub-critical (see below). The latter is rather a strong assumption. We do not make any of these assumptions and our condition  $R_k^- \in \hat{K}$  is fairly general.

Assertion (i) gives a partial answer to the question addressed above. We do not know whether the behavior  $(t \log t)^{\left|\frac{1}{2} - \frac{1}{p}\right| \frac{D}{2}}$  is optimal in general. If  $M = \mathbb{R}^2 \sharp \mathbb{R}^2$  (connected sum of two copies of  $\mathbb{R}^2$ ) then  $(e^{-t\vec{\Delta}})_{t \geq 0}$  is not uniformly bounded on  $L^p(\Lambda^1 T^* M)$  for any  $p \neq 2$ . The gradient estimate in assertion (ii) is formulated for  $p > 2$  since for  $p \in (1, 2]$  the Riesz transform  $d\Delta^{-1/2}$  is bounded on  $L^p$  (cf. Coulhon and Duong [11]). Therefore  $\|de^{-t\Delta}\|_{p-p} \leq C_p t^{-\frac{1}{2}}$ . The case  $p > 2$  is complicated and this latter estimate is actually equivalent to

the boundedness of the Riesz transform (at least under some additional assumptions on  $M$ , see Auscher et al. [3]). Hence in the general setting of our paper, we cannot hope for gradient estimates of the semigroup in terms of  $t^{-\frac{1}{2}}$  (up to a constant) for all  $p > 2$ .

If in addition to the assumptions made in [14] (mentioned above) it is proved in the same paper that if  $V \in L^p(M)$  then

$$|\vec{p}_1(t, x, y)| \leq C \min \left( 1, \frac{t^{\alpha_p}}{v(x, \sqrt{t})} \right) \exp \left( -c \frac{\rho^2(x, y)}{t} \right) \quad (3)$$

for some constant  $\alpha_p$  depending on  $p$  and the sub-critical constant of  $V$ . In particular,  $\alpha_p \rightarrow \infty$  (as  $p \rightarrow \infty$ ) and hence our estimate is more precise than (3). In addition, we do not make any sub-criticality assumption and no summability condition on  $V$ . We point out that recently, Coulhon, Devyver and Sikora [10] were able to prove the full Gaussian bound

$$|\vec{p}_k(t, x, y)| \leq \frac{C}{v(x, \sqrt{t})} \exp \left( -c \frac{\rho^2(x, y)}{t} \right), \quad t > 0 \quad (4)$$

under additional assumptions on the Ricci curvature among which  $R_k^-$  is strongly sub-critical and *small at infinity* in a precise sense (see the next section). The estimate (4) was proved previously by Devyver [19] under the additional assumption that  $M$  satisfies a global Sobolev inequality.

**Corollary 1.3.** *Suppose the assumptions of the previous theorem are satisfied. Then for every  $\varepsilon > 0$ , the local Riesz transform  $d(\Delta + \varepsilon)^{-\frac{1}{2}}$  is bounded on  $L^p(M)$  (into  $L^p(\Lambda^1 T^* M)$ ) for all  $p \in (1, \infty)$ .*

As mentioned above, if  $p \in (1, 2]$  then the Riesz transform  $d\Delta^{-1/2}$  is bounded on  $L^p(M)$ , see [11]. The novelty here is the case  $p > 2$  although we do not prove boundedness of the Riesz transform (we only treat the local one). A similar result for local Riesz transforms was proved by Bakry [4] for Riemannian manifolds with Ricci curvature bounded from below (i.e.,  $R_- \in L^\infty$ ). We already mentioned that  $\hat{K}$  is larger than  $L^\infty$ .

In order to prove the corollary we use follow the ideas in [12]. By Theorem 1.2, the heat kernel of  $\varepsilon + \vec{\Delta}$  has a full Gaussian bound (as in the RHS of (4)). This and the doubling condition imply that  $d^*(\varepsilon + \vec{\Delta})^{-1/2}$  is bounded from  $L^q(\Lambda^1 T^* M)$  to  $L^q(M)$  for  $q \in (1, 2)$ . Therefore by duality,  $(\varepsilon + \vec{\Delta})^{-1/2} d$  is bounded from  $L^p(M)$  to  $L^p(\Lambda^1 T^* M)$  for all  $p \in (2, \infty)$ . The commutation property  $\vec{\Delta} d = d\Delta$  gives the desired result. We refer to [12] for additional details and to [3], [7], [8], [9], [19], [24] and the references therein for further results on the Riesz transform on  $L^p$  for  $p > 2$ .

The next result is an improvement of Theorem 1.2 under an additional assumption on  $R_k^-$ . We say that  $R_k^-$  is  $\epsilon$ -sub-critical (or  $\epsilon$ - $H$ -sub-critical or strongly sub-critical) for some  $\epsilon \in [0, 1]$  if

$$0 \leq (R_k^- \omega, \omega) \leq \epsilon (H\omega, \omega) \quad \forall \omega \in \mathcal{C}_0^\infty(\Lambda^1 T^* M), \quad (5)$$

where  $H := \nabla^* \nabla + R_k^+$  and  $(\cdot, \cdot)$  denotes the scalar product of  $L^2(\Lambda^1 T^* M)$ . For a scalar potential  $V$ , the definition of sub-criticality is the same, one replaces  $H$  by  $\Delta$ , the scalar product is then taken in  $L^2(M)$ . For further information on sub-criticality in the Euclidean space, see [17].

**Theorem 1.4.** *Suppose that the manifold  $M$  satisfies the volume doubling condition (D) with some  $D > 2$  and the Gaussian upper bound (G). Suppose in addition that  $R_k^- \in \hat{K}$  and  $R_k^-$  is  $\epsilon$ -sub-critical for some  $\epsilon \in [0, 1)$ . Let  $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$  and fix  $\tilde{p}_0 < p_0$ . Denote by  $p'_0$  the conjugate number of  $p_0$ . Then*

(i) *We have*

$$\begin{aligned} \|e^{-t\vec{\Delta}_k}\|_{p-p} &\leq C_p \text{ if } p \in (p'_0, p_0), \quad t \geq 0 \\ \|e^{-t\vec{\Delta}_k}\|_{p-p} &\leq C_p (t \log t)^{\left|\frac{1}{\tilde{p}_0} - \frac{1}{p}\right| \frac{D}{2}} \text{ if } p \in [p_0, \infty], \quad t > e \\ \|e^{-t\vec{\Delta}_k}\|_{p-p} &\leq C_p (t \log t)^{\left|\frac{1}{\tilde{p}_0} + \frac{1}{p} - 1\right| \frac{D}{2}} \text{ if } p \in [1, p'_0], \quad t > e. \end{aligned}$$

(ii) *For all  $t \geq 1$*

$$\|\nabla e^{-t\Delta}\|_{p-p} \leq \frac{C_p}{\sqrt{t}} \text{ if } p \in (1, p_0)$$

*and*

$$\|\nabla e^{-t\Delta}\|_{p-p} \leq C_p \inf \left( t^{(\frac{1}{\tilde{p}_0} - \frac{1}{p})D - \frac{1}{2}}, t^{-\frac{\tilde{p}_0}{2p}} \right) \text{ if } p \geq p_0.$$

(iii) *There exists  $c, C > 0$  such that for all  $t > 0$  and  $x, y \in M$*

$$|\vec{p}_k(t, x, y)| \leq \frac{C(1+t)^{\frac{D}{\tilde{p}_0}}}{v(x, \sqrt{t})} \exp \left( -c \frac{\rho^2(x, y)}{t} \right).$$

(iv) *For all  $t \geq 1$  and  $x, y \in M$*

$$|\vec{p}_k(t, x, y)| \leq C \min \left( 1, \frac{t^{\frac{D}{\tilde{p}_0}}}{v(x, \sqrt{t})} \right) \exp \left( -c \frac{\rho^2(x, y)}{t} \right).$$

The bounds given in this result are better than those in Theorem 1.2. For example, taking  $p = 1$  in the first assertion we obtain the following  $L^1 - L^1$  estimate (or  $L^\infty - L^\infty$  by duality)

$$\|e^{-t\vec{\Delta}_k}\|_{1-1} \leq C_\delta (t \log t)^{\frac{D-2}{4}(1-\sqrt{1-\epsilon})+\delta}$$

for all  $\delta > 0$  and all large  $t$ . In the case where  $D \leq 2$  it is proved in [24] that the semigroup  $(e^{-t\vec{\Delta}})_{t \geq 0}$  is uniformly bounded on  $L^p(\Lambda^1 T^* M)$  for all  $p \in (1, \infty)$ . The proof given there works for  $\vec{\Delta}_k$  and gives the same result for  $(e^{-t\vec{\Delta}_k})_{t \geq 0}$  for any  $k \geq 2$ . We note however that this uses the assumption that  $R_k^-$  is strongly sub-critical. We already mentioned above that if  $M = \mathbb{R}^2 \# \mathbb{R}^2$  then  $(e^{-t\vec{\Delta}})_{t \geq 0}$  is not uniformly bounded on  $L^p(\Lambda^1 T^* M)$  for any  $p \neq 2$ .

The last result we mention in this introduction is the following.

**Proposition 1.5.** *Suppose that the assumptions of Theorem 1.4 are satisfied. If  $(e^{-t\vec{\Delta}})_{t \geq 0}$  is uniformly bounded on  $L^p(\Lambda^1 T^* M)$  for some fixed  $p$ . Then the Riesz transform  $d\Delta^{-1/2}$  is bounded on  $L^r(M)$  for all  $r \in (1, \max(p, p'))$ .*

The paper is organized as follows. In Section 2, we prove some useful preliminary results on the enlarged Kato class  $\hat{K}$ . Theorem 1.2 will be proved in Section 3. Finally, Section 4 is devoted to the proofs of Theorem 1.4 and Proposition 1.5.

Throughout this paper we use  $C$  and  $c$  for all inessential constants, their value may change from line to line. For a bounded linear operator  $T : L^p \rightarrow L^q$  we use  $\|T\|_{p \rightarrow q}$  to denote the corresponding norm. For simplicity, we use the same notation  $(\cdot, \cdot)$  for the scalar product in  $L^2(M)$  as well as in  $L^2(\Lambda^k T^* M)$  (the difference will be clear from the context). We also use the same notation for the duality  $L^p - L^{p'}$ .

## 2 Preliminaries

We first recall the following very well-known lemma.

**Lemma 2.1.** *Let  $x \in M$ . There exists a constant  $C > 0$  independent of  $x$  such that for all  $t > 0$*

$$\int_M \exp\left(-c \frac{\rho^2(x, y)}{t}\right) d\mu(y) \leq C v(x, \sqrt{t}).$$

*Proof.* We have

$$\begin{aligned} \int_M \exp\left(-c \frac{\rho^2(x, y)}{t}\right) d\mu(y) &\leq \sum_{j=0}^{\infty} \int_{j\sqrt{t} \leq \rho(x, y) \leq (j+1)\sqrt{t}} \exp(-cj^2) d\mu(y) \\ &\leq \sum_{j=0}^{\infty} \exp(-cj^2) v(x, (j+1)\sqrt{t}). \end{aligned}$$

We use the doubling property (D) and obtain the lemma.  $\square$

Let  $R_k$  be an  $L_{loc}^\infty$  section of the vector bundle  $\text{End}(\Lambda^k T^* M)$ . For each  $x \in M$ , the symmetric endomorphism  $R_k(x)$  can be decomposed into a positive and negative parts



$R_k(x)^+$  and  $R_k(x)^-$  (i.e.  $R_k(x) = R_k(x)^+ - R_k(x)^-$ ). A way to do this is to define  $R_k(x)^+w = R_k(x)w$  if  $w$  is an eigenvector corresponding to a positive eigenvalue and 0 if not. The negative part is  $R_k(x)^- = (-R_k(x))^+$ .

**Lemma 2.2.** *Suppose that  $R_k^- \in \hat{K}$  and set  $W(x) := |R_k^-(x)|_x$ . Then there exists constants  $\kappa \in [0, 1)$  and  $C > 0$  such that*

$$\int_M W|f|^2 d\mu \leq \kappa \|df\|_2^2 + C \|f\|_2^2$$

for all  $f \in W^{1,2}(M)$ .

*Proof.* The arguments are the same as in [32], Proposition 4.7. We prove first that there exists a sufficiently large  $\lambda > 0$  such that

$$\|(\Delta + \lambda)^{-1}W\|_{\infty-\infty} < 1.$$

Indeed,

$$\begin{aligned} \|(\Delta + \lambda)^{-1}W\|_{\infty-\infty} &= \left\| \int_0^\infty e^{-\lambda s} e^{-s\Delta} W ds \right\|_{\infty-\infty} \\ &\leq \sum_{n=0}^\infty \left\| \int_{n\alpha}^{(n+1)\alpha} e^{-\lambda s} e^{-s\Delta} W ds \right\|_{\infty-\infty} \\ &\leq \sum_{n=0}^\infty \left\| \int_0^\alpha e^{-\lambda(t+n\alpha)} e^{-(t+n\alpha)\Delta} W dt \right\|_{\infty-\infty} \\ &\leq \sum_{n=0}^\infty e^{-\lambda n\alpha} \|e^{-n\alpha\Delta}\|_{\infty-\infty} \left\| \int_0^\alpha e^{-\lambda t} e^{-t\Delta} W dt \right\|_{\infty-\infty} \\ &\leq \frac{1}{1 - e^{-\lambda\alpha}} \left\| \int_0^\alpha e^{-t\Delta} W dt \right\|_{\infty-\infty}. \end{aligned}$$

Since we assume  $W \in \hat{K}$ , there exists  $\alpha > 0$  such that  $\left\| \int_0^\alpha e^{-t\Delta} W dt \right\|_{\infty-\infty} < 1$ . Taking  $\lambda$  sufficiently large we obtain  $\|(\Delta + \lambda)^{-1}W\|_{\infty-\infty} < 1$ .

Next, it follows by duality that  $\|W(\Delta + \lambda)^{-1}\|_{1-1} < 1$ . Now we apply Stein's interpolation theorem to  $F(z) := W^z(\Delta + \lambda)^{-1}W^{1-z}$  and obtain

$$\|W^{\frac{1}{2}}(\Delta + \lambda)^{-\frac{1}{2}}\|_{2-2}^2 = \|W^{\frac{1}{2}}(\Delta + \lambda)^{-1}W^{\frac{1}{2}}\|_{2-2} < 1.$$

This gives the desired assertion. □

The operator  $\vec{\Delta}_k$  is defined via the quadratic form

$$\vec{\mathfrak{a}}(\omega, \omega) = (d_{k-1}^* \omega, d_{k-1}^* \omega) + (d_k \omega, d_k \omega). \quad (6)$$

On the other hand, it is known by the Bochner formula that

$$\vec{\Delta}_k = \nabla^* \nabla + R_k \quad (7)$$

where  $R_k$  can be expressed in terms of the Riemann curvature (see e.g., [5], Section E). Again, if  $k = 1$  then  $R_1$  is identified with the Ricci curvature  $R$  of  $M$ .

The previous lemma allows us to look at  $\vec{\Delta}_k$  as the perturbation of  $H := \nabla^* \nabla + R_K^+$  by  $-R_k^-$  in the quadratic form sense. That is,  $\vec{\Delta}_k$  is the operator associated with the quadratic form

$$\vec{a}(\omega, \omega) = (H\omega, \omega) - (R_k^- \omega, \omega). \quad (8)$$

Indeed, by the domination property (see [23], [5] or [27]) we have

$$\int_M |d\omega|^2 \leq \int_M |\nabla \omega|^2. \quad (9)$$

Thus by Lemma 2.2,

$$\begin{aligned} |(R_k^- \omega, \omega)| &\leq \kappa \|d\omega\|_2^2 + C \|\omega\|_2^2 \\ &\leq \kappa \int_M |\nabla \omega|^2 + C \|\omega\|_2^2 \\ &\leq \kappa (H\omega, \omega) + C \|\omega\|_2^2. \end{aligned}$$

We can now apply the well known KLMN theorem (see e.g. [25], Chapter 1).

Following [10], a non-negative function  $W \in K^\infty$  (it is called *small at infinity*) if there exists  $A > 0$  such that

$$\sup_{x \in M} \int_{M \setminus B(o, A)} g(x, y) W(y) d\mu(y) < 1, \quad (10)$$

where  $o \in M$  is a fixed point and  $g(x, y)$  is the Green function of the Laplace-Beltrami operator (we assume that  $g(x, y)$  exists). As mentioned in the introduction, one of the main assumptions in [10] in order to prove a Gaussian upper bound for the heat kernel of  $\vec{\Delta}_k$  is  $R_k^- \in K^\infty$  (i.e.,  $x \mapsto |R_k^-(x)| \in K^\infty$ ). We point out that this class is smaller than the enlarged Kato class  $\hat{K}$ .

**Proposition 2.3.** *We have  $L_{loc}^\infty(M) \cap K^\infty \subset \hat{K}$ . In particular, if  $R_k^- \in K^\infty$  then  $R_k^- \in \hat{K}$ .*

*Proof.* Let  $W \in L_{loc}^\infty(M) \cap K^\infty$ . For  $A > 0$

$$\begin{aligned} \int_M \int_0^t p_s(x, y) W(y) ds d\mu(y) &\leq \int_{B(o, A)} \int_0^t p_s(x, y) W(y) ds d\mu(y) \\ &\quad + \int_{M \setminus B(o, A)} \int_0^\infty p_s(x, y) W(y) ds d\mu(y) \end{aligned}$$

$$\begin{aligned}
&\leq \|W\|_{L^\infty(B(o,A))} \int_0^t \int_M p_s(x,y) d\mu(y) ds \\
&+ \int_{M \setminus B(o,A)} g(x,y) W(y) d\mu(y) \\
&\leq t \|W\|_{L^\infty(B(o,A))} + \int_{M \setminus B(o,A)} g(x,y) W(y) d\mu(y).
\end{aligned}$$

Taking  $t$  small enough and using the definition of  $W \in K^\infty$  we see that the last term is less than 1. This means that  $W \in \hat{K}$ .  $\square$

### 3 Proof of Theorem 1.2

In this section we prove the four statements of Theorem 1.2. We divide the proof into several steps.

In order to avoid repetition, we assume throughout this section that  $M$  satisfies the doubling property (D) and the Gaussian upper bound (G). We also assume that  $R_k^- \in \hat{K}$ . We denote by  $V(x)$  the lowest eigenvalue of the Riemann curvature (the Ricci curvature if  $k = 1$ )  $R_k(x)$  and  $V_-$  is the negative part of the function  $V(x)$ .

**Lemma 3.1.** *There exist positive constants  $M$  and  $\nu$  such that*

$$\|e^{-t(\Delta - V_-)}\|_{1 \rightarrow 1} \leq M e^{\nu t}$$

*Proof.* The arguments of the proof are mainly borrowed from [32]). First, since  $R_k^- \in \hat{K}$  then by Lemma 2.2  $V_-$  satisfies

$$\int_M V_- |f|^2 d\mu \leq \kappa \|df\|_2^2 + C \|f\|_2^2$$

for all  $f \in W^{1,2}(M)$  and for some constant  $\kappa < 1$ . As explained in the previous section, by the KLMN theorem, the operator  $\Delta - V_-$  is self-adjoint (with an appropriate domain) and hence the semigroup  $e^{-t(\Delta - V_-)}$  acts on  $L^2(M)$ . On the other hand, since  $V_- \in \hat{K}$  there exist  $\alpha > 0$  and  $\kappa' < 1$  such that for  $V_n := \inf(V_-, n)$

$$\int_0^\alpha \|V_n e^{-s\Delta} f\|_1 ds \leq \kappa' \|f\|_1.$$

The constants  $\alpha$  and  $\kappa'$  are independent of  $n$ . Therefore, by Miyadera perturbation theorem (cf. [32], Theorem 1.1)  $e^{-t(\Delta - V_n)}$  is a strongly continuous semigroup on  $L^1(M)$  and

$$\|e^{-t(\Delta - V_n)} f\|_1 \leq M e^{\nu t} \|f\|_1 \tag{11}$$

with constants  $M$  and  $\nu$  independent of  $n$ . On the other hand, by classical monotone convergence theorems (for quadratic forms),  $e^{-t(\Delta - V_n)}$  converges strongly in  $L^2(M)$  to  $e^{-t(\Delta - V_-)}$  as  $n \rightarrow \infty$ . An application of Fatou's lemma in (11) with  $f \in L^1 \cap L^2$  gives (11) for  $e^{-t(\Delta - V_-)}$  and  $f \in L^1 \cap L^2$ . Finally, we argue by density to extend the estimate to all  $f \in L^1$ .  $\square$

**Lemma 3.2.** *The heat kernel  $\vec{p}_k(t, x, y)$  of  $\vec{\Delta}_k$  has the following upper bound*

$$|\vec{p}_k(t, x, y)| \leq \frac{C}{v(x, \sqrt{t})} \exp\left(-c \frac{\rho^2(x, y)}{t}\right) e^{\nu t},$$

where  $C, c$  and  $\nu$  are positive constants.

*Proof.* By the well known domination (cf. [23], [5] or [27]) we have the pointwise estimate

$$|e^{-t\vec{\Delta}_k}\omega(x)|_x \leq e^{-t(\Delta-V_-)}|\omega(x)|_x. \quad (12)$$

Thus,  $|\vec{p}_k(t, x, y)|$  is bounded by the heat kernel  $p^{V_-}(t, x, y)$  of the Schrödinger operator  $\Delta - V_-$ . It is then enough to prove the above upper bound for  $p^{V_-}(t, x, y)$ .

Since the heat kernel  $p(t, x, y)$  of  $\Delta$  has a Gaussian upper bound (G) it follows that the following Gagliardo-Nirenberg type inequality holds

$$\|v(\cdot, \sqrt{t})^{\frac{1}{2}-\frac{1}{q}}u\|_q \leq C(\|u\|_2 + \sqrt{t}\|\nabla u\|_2), \quad (13)$$

for all  $q \in (2, \infty)$  such that  $\frac{q-2}{q}D < 2$  (and  $u \in W^{1,2}(M)$ ). See [2], Proposition 2.1 and p. 1125 or [6], Theorem 1.2.1. On the other hand, using Lemma 2.2 we see that for  $\nu > 0$  large enough,

$$\|\nabla u\|_2^2 \leq C\|(\Delta - V_- + \nu)^{1/2}u\|_2^2.$$

Therefore, (13) implies

$$\|v(\cdot, \sqrt{t})^{\frac{1}{2}-\frac{1}{q}}u\|_q \leq C(\|u\|_2 + \sqrt{t}\|(\Delta - V_- + \nu)^{1/2}u\|_2). \quad (14)$$

This together with the fact that the semigroup  $(e^{-t(\Delta-V_-+\nu)})_{t \geq 0}$  is uniformly bounded on  $L^1(M)$  (cf. Lemma 3.1) allows us to apply [6], Theorem 1.2.1 and conclude that  $p^{V_-}(t, x, y)e^{-\nu t}$  satisfies a full Gaussian upper bound. One missing thing before applying the result from [6] is that the semigroup  $T_t := e^{-t(\Delta-V_-+\nu)}$  has to satisfy  $L^2 - L^2$  Davies-Gaffney estimates

$$\|T_t u\|_{L^2(F)} \leq e^{-\frac{\rho^2(E,F)}{ct}} \|u\|_2,$$

where  $E, F$  are two closed subsets of  $M$  and  $u$  has support in  $E$ . This is indeed true and it is a standard fact for Schrödinger operators.  $\square$

In order to improve the Gaussian upper bound given here and obtain the bound (iii) of Theorem 1.2 we take advantage of the fact that  $(e^{-t\vec{\Delta}_k})_{t \geq 0}$  is uniformly bounded on  $L^2(\Lambda^k T^*M)$  (it is even a contraction semigroup since  $\vec{\Delta}_k$  is non-negative). This strategy was used in [26] and we follow the same arguments which we adapt to the setting of differential forms.

**Proposition 3.3.** *We have*

$$|\vec{p}_k(t, x, y)| \leq \frac{C \left(1 + t + \frac{\rho^2(x, y)}{t}\right)^{\frac{D}{2}}}{v(x, \sqrt{t})^{\frac{1}{2}} v(y, \sqrt{t})^{\frac{1}{2}}} \exp\left(-\frac{\rho^2(x, y)}{4t}\right), \quad (15)$$

for some positive constant  $C$ .

*Proof.* We apply Davies's perturbation method. Let  $\lambda \in \mathbb{R}$  and  $\phi \in \mathcal{C}_0^\infty(M)$  such that  $|\nabla \phi| \leq 1$  on  $M$ . We consider the semigroup  $\vec{T}_{t, \lambda} := e^{-\lambda \phi} e^{-t \vec{\Delta}_k} e^{\lambda \phi}$  and its integral kernel

$$\vec{k}_\lambda(t, x, y) = e^{-\lambda(\phi(x) - \phi(y))} \vec{p}_k(t, x, y).$$

Step 1. As a consequence of  $|\nabla \phi| \leq 1$  and Lemma 3.2 we have

$$\begin{aligned} |\vec{k}_\lambda(t, x, y)| &\leq \frac{C e^{\nu t} e^{|\lambda| |\phi(x) - \phi(y)|}}{v(x, \sqrt{t})} \exp\left(-c \frac{\rho^2(x, y)}{t}\right) \\ &\leq \frac{C e^{\nu t}}{v(x, \sqrt{t})} \exp\left(-c \frac{\rho^2(x, y)}{t} + |\lambda| \rho(x, y)\right) \\ &\leq \frac{C e^{\nu t}}{v(x, \sqrt{t})} \exp\left(\frac{1}{2c} \lambda^2 t\right) \exp\left(-\frac{c}{2} \frac{\rho^2(x, y)}{t}\right). \end{aligned}$$

Step 2. We prove that there exists a constant  $C$  independent of  $\lambda$  and  $\phi$  such that for all  $t > 0$ ,  $x, y \in M$  and  $\lambda \in \mathbb{R}$

$$\int_M |\vec{k}_\lambda(\frac{t}{2}, x, y)|^2 d\mu(y) \leq \frac{C e^{\lambda^2 t}}{v\left(x, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)}, \quad (16)$$

where  $\beta := (\frac{1}{2c} - 1)\lambda^2 + \nu$ . One can obviously take the constant  $c$  small enough in the estimate of Step 1 so that  $\frac{1}{2c} > 1$ . Note that if  $\beta = 0$ , then  $\lambda = \nu = 0$  and (16) follows from the estimate in Step 1 and Lemma 2.1. Hence we may assume in the sequel that  $\beta > 0$ .

We fix  $t > 0$ . According to Step 1, if  $t \leq \frac{1}{\beta}$

$$\begin{aligned} \int_M |\vec{k}_\lambda(t, x, y)|^2 d\mu(y) &\leq \frac{C e^{(\frac{1}{c} \lambda^2 + 2\nu)t}}{v(x, \sqrt{t})^2} \int_M e^{-c \frac{\rho^2(x, y)}{t}} d\mu(y) \\ &\leq \frac{C e^{2\beta t}}{v(x, \sqrt{t})} e^{2t\lambda^2} \leq \frac{C e^2}{v(x, \sqrt{t})} e^{2t\lambda^2}. \end{aligned}$$

Thus (16) holds for all  $t \leq \frac{1}{\beta}$ .

Next, we suppose  $t > \frac{1}{\beta}$ . The semigroup property implies

$$\int_M |\vec{k}_\lambda(t, x, y)|^2 d\mu(y) \leq \left\| \vec{T}_{t - \frac{1}{\beta}, \lambda} \vec{k}_\lambda\left(\frac{1}{\beta}, x, \cdot\right) \right\|_2^2 \leq e^{2(t - \frac{1}{\beta})\lambda^2} \left\| \vec{k}_\lambda\left(\frac{1}{\beta}, x, \cdot\right) \right\|_2^2. \quad (17)$$

The last inequality uses

$$\|\vec{T}_{t,\lambda}\|_{2,2} \leq e^{\lambda^2 t}, \forall t \geq 0,$$

which follows from that fact that the operator  $\vec{A}_\lambda + \lambda^2$  is positive, where  $-\vec{A}_\lambda$  denotes the generator of the semigroup  $(\vec{T}_{t,\lambda})_{t \geq 0}$ . For more details see the proof of [24] Proposition 3.7 in the case  $k = 1$ , the arguments works for general  $k \geq 1$ . Now we use the inequality

$$\left\| \vec{k}_\lambda\left(\frac{1}{\beta}, x, \cdot\right) \right\|_2^2 \leq \frac{C}{v(x, \sqrt{\frac{1}{\beta}})} e^{\frac{2\lambda^2}{\beta}},$$

proved above (in the case  $t \leq \frac{1}{\beta}$ ) to obtain

$$\left\| \vec{k}_\lambda(t, x, \cdot) \right\|_2^2 \leq \frac{C}{v(x, \sqrt{\frac{1}{\beta}})} e^{2t\lambda^2}.$$

This proves (16).

Step 3. We prove that for all  $t > 0$  and  $x, y \in M$

$$|\vec{k}_\lambda(t, x, y)| \leq \frac{Ce^{\lambda^2 t}}{v\left(x, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}} v\left(y, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}}}. \quad (18)$$

First, changing  $\lambda$  into  $-\lambda$  in Step 2 gives by duality

$$\int_M |\vec{k}_\lambda(\frac{t}{2}, x, y)|^2 d\mu(x) \leq \frac{Ce^{\lambda^2 t}}{v\left(y, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)}. \quad (19)$$

The semigroup property implies

$$|\vec{k}_\lambda(t, x, y)| \leq \int_M |\vec{k}_\lambda(\frac{t}{2}, x, z)| |\vec{k}_\lambda(\frac{t}{2}, z, y)| d\mu(z).$$

Using the Cauchy-Schwarz inequality, (16) and (19), we obtain

$$\begin{aligned} |\vec{k}_\lambda(t, x, y)| &\leq \left( \int_M |\vec{k}_\lambda(\frac{t}{2}, x, z)|^2 d\mu(z) \right)^{\frac{1}{2}} \left( \int_M |\vec{k}_\lambda(\frac{t}{2}, z, y)|^2 d\mu(z) \right)^{\frac{1}{2}} \\ &\leq \frac{Ce^{\lambda^2 t}}{v\left(x, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}} v\left(y, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})}\right)^{\frac{1}{2}}}. \end{aligned}$$

Step 4. We claim that for all  $t > 0$  and  $x, y \in M$

$$|\vec{p}_k(t, x, y)| \leq \frac{C}{v(x, \sqrt{r})^{\frac{1}{2}} v(y, \sqrt{r})^{\frac{1}{2}}} \exp\left(-\frac{\rho^2(x, y)}{4t}\right), \quad (20)$$

where

$$r := \min \left( \frac{t}{2}, \left[ \left( \frac{1}{2c} - 1 \right) \frac{\rho^2(x, y)}{4t^2} + \nu \right]^{-1} \right).$$

The estimate (18) and the definition of  $\vec{k}_{t, \lambda}(x, y)$  give

$$|\vec{p}_k^\rightarrow(t, x, y)| \leq \frac{C e^{\lambda^2 t}}{v \left( x, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})} \right)^{\frac{1}{2}} v \left( y, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})} \right)^{\frac{1}{2}}} e^{\lambda(\phi(x) - \phi(y))}.$$

Choosing  $\lambda = \frac{\phi(y) - \phi(x)}{2t}$ , we obtain

$$|\vec{p}_k^\rightarrow(t, x, y)| \leq \frac{C}{v \left( x, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})} \right)^{\frac{1}{2}} v \left( y, \sqrt{\min(\frac{t}{2}, \frac{1}{\beta})} \right)^{\frac{1}{2}}} \exp \left( -\frac{|\phi(x) - \phi(y)|^2}{4t} \right),$$

with

$$\beta = \left( \frac{1}{2c} - 1 \right) \frac{|\phi(x) - \phi(y)|^2}{4t^2} + \nu.$$

Since  $|\nabla \phi| \leq 1$ , we have  $|\phi(x) - \phi(y)| \leq \rho(x, y)$ . We deduce that

$$|\vec{p}_k^\rightarrow(t, x, y)| \leq \frac{C}{v(x, \sqrt{r})^{\frac{1}{2}} v(y, \sqrt{r})^{\frac{1}{2}}} \exp \left( -\frac{|\phi(x) - \phi(y)|^2}{4t} \right),$$

with  $r$  as above. We optimize over  $\phi$  and obtain (20).

Step 5. We deduce (15) using assumption (D). Indeed, noting that

$$v(x, \sqrt{t}) \leq v(x, \sqrt{r}) \left( \frac{t}{r} \right)^{\frac{D}{2}}$$

and

$$\frac{t}{r} \leq 2 + \left( \frac{1}{2c} - 1 \right) \frac{\rho^2(x, y)}{4t} + \nu t,$$

we obtain for all  $t > 0$  and  $x, y \in M$

$$|\vec{p}_k^\rightarrow(t, x, y)| \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{2}} v(y, \sqrt{t})^{\frac{1}{2}}} \exp \left( -\frac{\rho^2(x, y)}{4t} \right) \left( 1 + \nu t + \frac{\rho^2(x, y)}{t} \right)^{\frac{D}{2}}.$$

□

The following result was proved in [14] under some additional assumptions. Since the ideas of the proof are the same we just sketch them.

**Proposition 3.4.** *With the same assumptions as in the previous theorem, there exists  $c, C > 0$  such that for all  $t \geq 1$  and  $x, y \in M$*

$$|\vec{p}_k(t, x, y)| \leq C \exp \left( -c \frac{\rho^2(x, y)}{t} \right).$$

*Proof.* Following the proof of [14] Theorem 4.1, Step 1, we find for  $A > 0$  sufficiently large and  $t \geq 1$

$$\int_M |\vec{p}_k(t, x, y)|^2 e^{\frac{\rho^2(x, y)}{At}} d\mu(x) \leq \int_M |\vec{p}_k(1, x, y)|^2 e^{\frac{\rho^2(x, y)}{A}} d\mu(x).$$

The estimate (15) gives

$$|\vec{p}_k(1, x, y)| \leq \frac{C}{v(x, 1)} e^{-c\rho^2(x, y)}.$$

We deduce that for  $A > 0$  sufficiently large and all  $t \geq 1$

$$\int_M |\vec{p}_k(t, x, y)|^2 e^{\frac{\rho^2(x, y)}{At}} d\mu(x) \leq C.$$

Then the rest of the proof is the same as in [14] Theorem 4.1, Step 1.  $\square$

We deduce from Propositions 3.3 and 3.4 the following estimate which is the estimate (iv) of Theorem 1.2

$$|\vec{p}_k(t, x, y)| \leq C \min \left( 1, \frac{t^{\frac{D}{2}}}{v(x, \sqrt{t})} \right) \exp \left( -c \frac{\rho^2(x, y)}{t} \right). \quad (21)$$

Now we prove assertion (ii) of Theorem 1.2. This result can be found in [12] where the authors suppose that  $\vec{p}_k(t, x, y)$  satisfies a full Gaussian upper bound instead of (15). We give the proof for the sake of completeness.

**Proposition 3.5.** *For all  $2 \leq p \leq \infty$  and  $t \geq 1$ ,*

$$|\nabla_x p(t, x, y)| \leq \frac{C t^{\frac{D}{2}}}{\sqrt{t} v(x, \sqrt{t})} e^{-c \frac{\rho^2(x, y)}{t}}, \quad \forall x, y \in M, \forall t > 1.$$

and

$$\|\nabla e^{-t\Delta}\|_{p,p} \leq C t^{-\frac{1}{p}}.$$



*Proof.* The Gaussian upper bound (G) and the doubling condition (D) imply that there exist  $\nu > 0$  small enough such that

$$\int_M |\nabla_x p(t, x, y)|^2 e^{\nu \frac{\rho^2(x, y)}{t}} d\mu(x) \leq \frac{C}{t v(y, \sqrt{t})}, \quad y \in M, t > 0 \quad (22)$$

This is proved in [20] (see also [11]). In addition, by the Cauchy-Schwarz inequality we have for  $\gamma > 0$  small enough,

$$\begin{aligned} & \int_M |\nabla_x p(t, x, y)| e^{\gamma \frac{\rho^2(x, y)}{t}} d\mu(x) \\ & \leq \left( \int_M |\nabla_x p(t, x, y)|^2 e^{4\gamma \frac{\rho^2(x, y)}{t}} d\mu(x) \right)^{\frac{1}{2}} \left( \int_M e^{-2\gamma \frac{\rho^2(x, y)}{t}} d\mu(x) \right)^{\frac{1}{2}} \\ & \leq \left( \frac{C_\gamma}{t v(y, \sqrt{t})} \right)^{\frac{1}{2}} (C'_\gamma v(y, \sqrt{t}))^{\frac{1}{2}} \\ & = \frac{C''_\gamma}{\sqrt{t}}, \end{aligned}$$

where we used (22) and Lemma 2.1. Hence

$$\int_M |\nabla_x p(t, x, y)| e^{\gamma \frac{\rho^2(x, y)}{t}} d\mu(x) \leq \frac{C_\gamma}{\sqrt{t}}. \quad (23)$$

Now we prove that there exist  $c, C > 0$  such that

$$|\nabla_x p(t, x, y)| \leq \frac{C t^{\frac{D}{2}}}{\sqrt{t} v(x, \sqrt{t})} e^{-c \frac{\rho^2(x, y)}{t}}, \quad \forall x, y \in M, \forall t > 1. \quad (24)$$

First notice that

$$\nabla_x p(t, x, y) = \int_M \nabla_x p\left(\frac{t}{2}, x, z\right) p\left(\frac{t}{2}, z, y\right) d\mu(z) = \nabla_x e^{-\frac{t}{2}\Delta} f_{y,t}(x),$$

with  $f_{y,t}(z) = p(\frac{t}{2}, z, y)$ . According to the commutation formula  $\vec{\Delta} d = d\Delta$  (recall that  $\vec{\Delta} = \vec{\Delta}_1$ ) we have

$$|\nabla_x e^{-\frac{t}{2}\Delta} f_{y,t}(x)| = |e^{-\frac{t}{2}\vec{\Delta}} df_{y,t}(x)|.$$

Therefore using the estimate (15) (in the case  $k = 1$ ) and (D), we obtain for all  $t > 1$  and some constant  $c > 0$

$$\begin{aligned} |\nabla_x e^{-\frac{t}{2}\Delta} f_{y,t}(x)| & \leq \frac{C t^{\frac{D}{2}}}{v(x, \sqrt{t})} \int_M e^{-c \frac{\rho^2(x, z)}{t}} |df_{y,t}(z)| d\mu(z) \\ & = \frac{C t^{\frac{D}{2}}}{v(x, \sqrt{t})} \int_M e^{-c \frac{\rho^2(x, z)}{t}} |\nabla_z f_{y,t}(z)| d\mu(z). \end{aligned}$$

Hence

$$|\nabla_x p(t, x, y)| \leq \frac{Ct^{\frac{D}{2}}}{v(x, \sqrt{t})} \int_M e^{-c\frac{\rho^2(x, z)}{t}} |\nabla_z f_{y, t}(z)| d\mu(z).$$

It suffices then to prove that for some  $\gamma < c$ , there exist constants  $c', C' > 0$  such that

$$\int_M e^{-\gamma\frac{\rho^2(x, z)}{t}} |\nabla_z p(\frac{t}{2}, z, y)| d\mu(z) \leq \frac{C'}{\sqrt{t}} e^{-c'\frac{\rho^2(x, y)}{t}}.$$

Using

$$e^{-\gamma\frac{\rho^2(x, z)}{t}} \leq e^{-\gamma\frac{\rho^2(x, y)}{2t}} e^{\gamma\frac{\rho^2(z, y)}{t}},$$

we have

$$\int_M e^{-\gamma\frac{\rho^2(x, z)}{t}} |\nabla_z p(\frac{t}{2}, z, y)| d\mu(z) \leq e^{-\gamma\frac{\rho^2(x, y)}{2t}} \int_M e^{\gamma\frac{\rho^2(z, y)}{t}} |\nabla_z p(\frac{t}{2}, z, y)| d\mu(z).$$

According to (23), we know that there exists  $\gamma > 0$  small enough such that

$$\int_M e^{\gamma\frac{\rho^2(z, y)}{t}} |\nabla_z p(\frac{t}{2}, z, y)| d\mu(z) \leq \frac{C'}{\sqrt{t}}.$$

This shows (24).

It follows from Lemma 2.1 and (24) that

$$\|\nabla e^{-t\Delta}\|_{\infty-\infty} \leq \frac{Ct^{\frac{D}{2}}}{\sqrt{t}}. \quad (25)$$

We write for  $t \geq 1$ ,  $\nabla e^{-t\Delta} = \nabla e^{-\Delta} e^{-(t-1)\Delta}$ . We apply (25) with  $t = 1$  and the standard estimate

$$\|\nabla e^{-t\Delta}\|_{2-2} \leq \frac{C}{\sqrt{t}}, \quad (26)$$

to obtain

$$\|\nabla e^{-t\Delta}\|_{p-p} \leq Ct^{-\frac{1}{p}}.$$

□

Finally using the estimate in assertion (iii) we prove assertion (i) in Theorem 1.2 as in [26], Théorème 7. In order to avoid repetition, we do not give details here and we come back to this in the proof of Theorem 1.4, assertion (i).

**Example.** Let  $M = \mathbb{R}^2 \sharp \mathbb{R}^2$ . It is proved in [10] Proposition 3.3, that if the gluing is made in such a way that  $M$  has genus zero then  $\ker_{L^2}(\vec{\Delta}) = \{0\}$  and  $\ker_{L^p}(\vec{\Delta}) \neq \{0\}$  for all  $p > 2$ . We claim that  $(e^{-t\vec{\Delta}})_{t \geq 0}$  is not uniformly bounded on  $L^p(\Lambda^1 T^* M)$  for all  $p \neq 2$ . Suppose for a contradiction that  $(e^{-t\vec{\Delta}})_{t \geq 0}$  is uniformly bounded on  $L^{p_0}(\Lambda^1 T^* M)$  for some

$p_0 \neq 2$ . Arguing by duality, we may assume that  $p_0 > 2$ . For  $\omega = \vec{\Delta} \eta \in R_{L^2}(\vec{\Delta})$ , the range of  $\vec{\Delta}$  as an operator on  $L^2(\Lambda^1 T^* M)$ , we have

$$\|e^{-t\vec{\Delta}} \omega\|_2 = \|\vec{\Delta} e^{-t\vec{\Delta}} \eta\|_2 \leq \frac{C}{t} \|\eta\|_2.$$

In particular,  $e^{-t\vec{\Delta}} \omega \rightarrow 0$  in  $L^2(\Lambda^1 T^* M)$  as  $t \rightarrow \infty$ . Since  $R_{L^2}(\vec{\Delta})$  is dense in  $L^2(\Lambda^1 T^* M)$  (thanks to the fact that  $\ker_{L^2}(\vec{\Delta}) = \{0\}$ ),  $e^{-t\vec{\Delta}} \omega \rightarrow 0$  (as  $t \rightarrow \infty$ ) for all  $\omega \in L^2(\Lambda^1 T^* M)$ . Let  $p \in (2, p_0)$  and  $\omega \in L^2(\Lambda^1 T^* M) \cap L^{p_0}(\Lambda^1 T^* M)$ . The classical interpolation inequality together with the assumption that  $(e^{-t\vec{\Delta}})_{t \geq 0}$  is uniformly bounded on  $L^{p_0}(\Lambda^1 T^* M)$  imply that  $e^{-t\vec{\Delta}} \omega \rightarrow 0$  in  $L^p(\Lambda^1 T^* M)$  as  $t \rightarrow \infty$ . Uniform boundedness of  $(e^{-t\vec{\Delta}})_{t \geq 0}$  on  $L^p(\Lambda^1 T^* M)$  implies that this holds for all  $\omega \in L^p(\Lambda^1 T^* M)$ . This cannot be the case since  $\ker_{L^p}(\vec{\Delta}) \neq \{0\}$ .

## 4 Proof of Theorem 1.4

Recall that  $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$ . It is shown in [24] that the semigroup  $(e^{-t\vec{\Delta}})_{t \geq 0}$  is uniformly bounded on  $L^p(\Lambda^1 T^* M)$  for all  $p \in (p'_0, p_0)$ . The proof uses perturbation arguments and it works for the semigroup  $(e^{-t\vec{\Delta}_k})_{t \geq 0}$  as well for any  $k \geq 1$ . Hence the first estimate in assertion (i) of Theorem 1.4 is known. In addition, it is also proved in [24] that the Riesz transform  $d\Delta^{-1/2}$  is bounded on  $L^p$  for  $p \in (2, p_0)$  and as we already mentioned before, the Riesz transform is bounded on  $L^p$  for  $p \in (1, 2]$  (cf. [11]). This implies the first estimate in assertion (ii). Now we prove the remaining estimates of Theorem 1.4.

**Proposition 4.1.** *Assume that  $(D)$ ,  $(G)$  and  $R_k^- \in \hat{K}$ . If the semigroup  $(e^{-t\vec{\Delta}_k})_{t \geq 0}$  is uniformly bounded on  $L^p(\Lambda^k T^* M)$  for some  $p \in (1, 2]$  then for all  $t > e$*

$$\|e^{-t\vec{\Delta}_k}\|_{\infty-\infty} \leq C(t \log t)^{\frac{D}{2}(1-\frac{1}{p})}.$$

Since  $(e^{-t\vec{\Delta}_k})_{t \geq 0}$  is uniformly bounded on  $L^p(\Lambda^k T^* M)$  for  $p \in (p'_0, p_0)$  we obtain immediately assertion (i) of Theorem 1.4 for  $p \in [p_0, \infty]$ . The case  $p \in [1, p'_0]$  is handled by duality.

*Proof.* The idea of proof is similar to [26], Théorème 7. Let  $t > e$ . Since the semigroup  $(e^{-t\vec{\Delta}_k})_{t \geq 0}$  is uniformly bounded on  $L^p$ , we have

$$\int_M |\vec{p}_k(t, x, y)|^p d\mu(y) \leq \|e^{-(t-1)\vec{\Delta}_k}\|_{p-p}^p \|\vec{p}_k(1, x, \cdot)\|_p^p \leq C \int_M |\vec{p}_k(1, x, y)|^p d\mu(y).$$

Then using (15) and Lemma 2.1 we obtain

$$\int_M |\vec{p}_k(1, x, y)|^p d\mu(y) \leq \frac{C}{v(x, 1)^p} \int_M \exp(-cp\rho^2(x, y)) \leq \frac{C}{v(x, 1)^{p-1}}.$$

Hence

$$\int_M |\vec{p}_k(t, x, y)|^p d\mu(y) \leq \frac{C}{v(x, 1)^{p-1}}. \quad (27)$$

Let  $\epsilon \in (0, 1)$  and  $p_\epsilon \in (0, 1)$  such that  $\frac{1-\epsilon}{p} + \frac{\epsilon}{p_\epsilon} = 1$ , that is  $p_\epsilon = \frac{cp}{p-1+\epsilon}$ . Using (15) and Lemma 2.1, we have

$$\int_M |\vec{p}_k(t, x, y)|^{p_\epsilon} d\mu(y) \leq \frac{Ct^{\frac{Dp_\epsilon}{2}}}{v(x, \sqrt{t})^{p_\epsilon}} \int_M \exp\left(-cp_\epsilon \frac{\rho^2(x, y)}{t}\right) d\mu(y) \leq \frac{Ct^{\frac{Dp_\epsilon}{2}}}{v(x, \sqrt{t})^{p_\epsilon}} v(x, \sqrt{\frac{t}{p_\epsilon}}).$$

Using (D) we it follows that

$$\int_M |\vec{p}_k(t, x, y)|^{p_\epsilon} d\mu(y) \leq Ct^{\frac{Dp_\epsilon}{2}} v(x, \sqrt{t})^{1-p_\epsilon} p_\epsilon^{-\frac{D}{2}}. \quad (28)$$

From (27), (28) and the Hölder inequality, we deduce that

$$\begin{aligned} \int_M |\vec{p}_k(t, x, y)| d\mu(y) &\leq \left( \int_M |\vec{p}_k(t, x, y)|^p d\mu(y) \right)^{\frac{1-\epsilon}{p}} \left( \int_M |\vec{p}_k(t, x, y)|^{p_\epsilon} d\mu(y) \right)^{\frac{\epsilon}{p_\epsilon}} \\ &\leq C \left( \frac{v(x, \sqrt{t})}{v(x, 1)} \right)^{\frac{(p-1)(1-\epsilon)}{p}} t^{\frac{\epsilon D}{2}} p_\epsilon^{-\frac{D(p-1+\epsilon)}{2p}} \\ &\leq Ct^{\frac{D(p-1)}{2p}} t^{\frac{\epsilon D}{2}} p_\epsilon^{-\frac{D(p-1+\epsilon)}{2p}}. \end{aligned}$$

We have from the definition of  $p_\epsilon$  that  $p_\epsilon^{-(p-1+\epsilon)} \leq C\epsilon^{1-p}$ . Hence

$$\int_M |\vec{p}_k(t, x, y)| d\mu(y) \leq Ct^{\frac{D(p-1)}{2p}} \left[ t^\epsilon \epsilon^{1-p} \right]^{\frac{D}{2p}}. \quad (29)$$

Noticing that the RHS has its minimum for  $\epsilon = \frac{p-1}{\log t} \in (0, 1)$  (since  $t > e$ ), we conclude that

$$\int_M |\vec{p}_k(t, x, y)| d\mu(y) \leq C(t \log t)^{\frac{D(p-1)}{2p}},$$

which is the desired result.  $\square$

In the rest of this section we assume (D), (G),  $R_- \in \hat{K}$ . In addition, we suppose that  $R_-$  is  $\epsilon$ -sub-critical, that is there exists  $\epsilon \in [0, 1)$  such that

$$0 \leq (R_- \omega, \omega) \leq \epsilon (H \omega, \omega), \forall \omega \in \mathcal{C}_0^\infty(\Lambda^1 T^* M), \quad (\text{S-C})$$

where  $H = \nabla^* \nabla + R_+$ . Set  $p_0 := \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$  and fix  $\tilde{p}_0 < p_0$ .

**Proposition 4.2.** *For all  $t > 0$  and  $x, y \in M$*

$$|\vec{p}_k(t, x, y)| \leq \frac{C(1+t)^{\frac{D}{\tilde{p}_0}}}{v(x, \sqrt{t})} \exp\left(-c \frac{\rho^2(x, y)}{t}\right). \quad (30)$$

*Proof.* We proceed in three steps.

Step 1. We show the  $L^2 - L^p$  estimates

$$\sup_{t>0} \|e^{-t\vec{\Delta}_k} v(\cdot, \sqrt{t})^{\frac{1}{2}-\frac{1}{p}}\|_{2-p} \leq C \quad (31)$$

for all  $p \in [2, p_0)$ .

Let  $j \in \mathbb{N}$  and let  $A(x, \sqrt{t}, j)$  be the annulus  $B(x, (j+1)\sqrt{t}) \setminus B(x, j\sqrt{t})$ . Following the proof of [24] Theorem 4.1 leads to the following  $L^q - L^2$  off-diagonal estimates

$$\|\chi_{B(x, \sqrt{t})} e^{-t\vec{\Delta}_k} \chi_{A(x, \sqrt{t}, j)}\|_{q-2} \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{q}-\frac{1}{2}}} e^{-cj^2}$$

for all  $q \in (p'_0, 2]$ . This implies that for all  $q \in (p'_0, 2]$

$$\sup_{t>0} \|v(\cdot, \sqrt{t})^{\frac{1}{q}-\frac{1}{2}} e^{-t\vec{\Delta}_k}\|_{q-2} \leq C.$$

See [2] Proposition 2.9.

Step 2. We prove that for all  $t > 0$

$$\|v(\cdot, \sqrt{t})^{\frac{1}{2}} e^{-t\vec{\Delta}_k}\|_{2-\infty} \leq C(1+t)^{\frac{D}{2\tilde{p}_0}}. \quad (32)$$

Let  $0 < t \leq 1$ . Using Theorem 1.2 and Lemma 2.1, we obtain easily

$$\|v(\cdot, \sqrt{t})^{\frac{1}{2}} e^{-t\vec{\Delta}_k}\|_{2-\infty} \leq C \leq C(1+t)^{\frac{D}{2\tilde{p}_0}}.$$

We now consider  $t > 1$ . Since  $\vec{\Delta}_k$  satisfies the  $L^2 - L^2$  Davies-Gaffney estimates (see e.g., [28], Theorem 6), a consequence of [6], Proposition 4.1.6 is

$$\|v(\cdot, \sqrt{t})^{\frac{1}{2}} e^{-t\vec{\Delta}_k}\|_{2-\infty} \leq C \|v(\cdot, \sqrt{t})^{\frac{1}{\tilde{p}_0}} e^{-t\vec{\Delta}_k} v(\cdot, \sqrt{t})^{\frac{1}{2}-\frac{1}{\tilde{p}_0}}\|_{2-\infty},$$

with  $C$  independent of  $t$ . The semigroup property then gives

$$\|v(\cdot, \sqrt{t})^{\frac{1}{2}} e^{-t\vec{\Delta}_k}\|_{2-\infty} \leq C \|v(\cdot, \sqrt{t})^{\frac{1}{\tilde{p}_0}} e^{-\frac{t}{2}\vec{\Delta}_k}\|_{\tilde{p}_0-\infty} \|e^{-\frac{t}{2}\vec{\Delta}_k} v(\cdot, \sqrt{t})^{\frac{1}{2}-\frac{1}{\tilde{p}_0}}\|_{2-\tilde{p}_0}.$$

We use (31) and (D) to obtain

$$\|v(\cdot, \sqrt{t})^{\frac{1}{2}} e^{-t\vec{\Delta}_k}\|_{2-\infty} \leq C \|v(\cdot, \sqrt{t/2})^{\frac{1}{\tilde{p}_0}} e^{-\frac{t}{2}\vec{\Delta}_k}\|_{\tilde{p}_0-\infty}.$$

Using again (D) and the semigroup property leads to

$$\|v(\cdot, \sqrt{t})^{\frac{1}{2}} e^{-t\vec{\Delta}_k}\|_{2-\infty} \leq C t^{\frac{D}{2\tilde{p}_0}} \|v(\cdot, \sqrt{1/2})^{\frac{1}{\tilde{p}_0}} e^{-\frac{1}{2}\vec{\Delta}_k}\|_{\tilde{p}_0-\infty} \|e^{-(\frac{t}{2}-\frac{1}{2})\vec{\Delta}_k}\|_{\tilde{p}_0-\tilde{p}_0}.$$

As in [24] Theorem 3.1, we prove that the semigroup  $(e^{-t\vec{\Delta}_k})_{t \geq 0}$  is uniformly bounded on  $L^{\tilde{p}_0}$  (the statement in [24] is for  $k = 1$ ). Therefore

$$\begin{aligned} \|v(\cdot, \sqrt{t})^{\frac{1}{2}} e^{-t\vec{\Delta}_k}\|_{2-\infty} &\leq C t^{\frac{D}{2\tilde{p}_0}} \|v(\cdot, \sqrt{1/2})^{\frac{1}{\tilde{p}_0}} e^{-\frac{1}{2}\vec{\Delta}_k}\|_{\tilde{p}_0-\infty} \\ &\leq C(1+t)^{\frac{D}{2\tilde{p}_0}}, \end{aligned}$$

where we use Theorem 1.2 and Lemma 2.1 to obtain the last inequality. This concludes the proof of (32).

Step 3. We finish the proof by using [13], Corollary 4.5 with

$$V(x, t) := \frac{v(x, t)}{(1+t)^{2D/\tilde{p}_0}}.$$

□

Assertion (iii) of Theorem 1.4 is exactly the statement of the previous proposition. Assertion (iv) follows from the same proposition and Proposition 3.4. The proof of assertion (ii) for  $p \geq p_0$  is similar to the proof of Proposition 3.5, one has to use (30) instead of (15) and interpolation with

$$\|\nabla e^{-t\Delta}\|_{\tilde{p}_0-\tilde{p}_0} \leq \frac{C}{\sqrt{t}}. \quad (33)$$

This latter estimate is a consequence of the boundedness of the Riesz transform on  $L^p$  for  $p < p_0$  (cf. [24]). The proof of Theorem 1.4 is complete.

Finally, we prove Proposition 1.5.

*Proof of Proposition 1.5.* Under the assumption of the proposition it is proved in [9], Theorem 3.3 that the Riesz transform  $d^*\vec{\Delta}^{-1/2}$  is bounded from  $H_{\vec{\Delta}}^r(\Lambda^1 T^*M)$  to  $L^r(M)$  for all  $r \in (1, 2]$ . The space  $H_{\vec{\Delta}}^p(\Lambda^1 T^*M)$  is a Hardy space associated with  $\vec{\Delta}$ . On the other hand, the semigroup  $(e^{-t\vec{\Delta}})_{t \geq 0}$  satisfies off-diagonal estimates

$$\|\chi_{B(x, \sqrt{t})} e^{-t\vec{\Delta}} \chi_{B(y, \sqrt{t})}\|_{r-2} \leq C v(x, \sqrt{t})^{\frac{1}{2}-\frac{1}{r}} e^{-c\frac{\rho^2(x, y)}{t}}, \quad t > 0, x, y \in M. \quad (34)$$

This estimate is shown in [24] for all  $r \in (p'_0, 2]$ . The limitation  $r > p'_0$  follows from the fact that the semigroup  $(e^{-t\vec{\Delta}})_{t \geq 0}$  is known to be uniformly bounded on  $L^r(\Lambda^1 T^* M)$  only for  $r \in (p'_0, p_0)$ . Now if the semigroup  $(e^{-t\vec{\Delta}})_{t \geq 0}$  is uniformly bounded on  $L^p(\Lambda^1 T^* M)$  (and so by duality on  $L^{p'}(\Lambda^1 T^* M)$ ) then (34) holds for all  $r \in (\min(p, p'), 2]$ . It is known that the validity of (34) implies that  $L^r(\Lambda^1 T^* M)$  and  $H^r_{\vec{\Delta}}(\Lambda^1 T^* M)$  coincide and have equivalent norms (see Proposition 2.5 in [9] and references there). This implies that  $d^* \vec{\Delta}^{-1/2}$  is bounded from  $L^r(\Lambda^1 T^* M)$  to  $L^r(M)$  for all  $r \in (\min(p, p'), 2]$ . By duality and the commutation formula  $d\Delta = \vec{\Delta}d$  the Riesz transform  $d\Delta^{-1/2}$  is bounded from  $L^r(M)$  to  $L^r(\Lambda^1 T^* M)$  for all  $r \in [2, \max(p, p'))$ . The boundedness for  $r \in (1, 2)$  is well known (cf. [11]).  $\square$

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