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SUB-RIEMANNIAN INTERPOLATION INEQUALITIES: IDEAL STRUCTURES

DAVIDE BARILARI^b AND LUCA RIZZI[#]

ABSTRACT. We prove that ideal sub-Riemannian manifolds (i.e., admitting no non-trivial abnormal minimizers) support interpolation inequalities for optimal transport. A key role is played by sub-Riemannian Jacobi fields and distortion coefficients, whose properties are remarkably different with respect to the Riemannian case. As a byproduct, we characterize the cut locus as the set of points where the squared sub-Riemannian distance fails to be semiconvex, answering to a question raised by Figalli and Rifford in [FR10].

As an application, we deduce sharp and intrinsic Borell-Brascamp-Lieb and geodesic Brunn-Minkowski inequalities in the aforementioned setting. For the case of the Heisenberg group, we recover in an intrinsic way the results recently obtained by Balogh, Kristály and Sipos in [BKS16], and we extend them to the class of generalized H -type Carnot groups. Our results do not require the distribution to have constant rank, yielding for the particular case of the Grushin plane a sharp measure contraction property and a sharp Brunn-Minkowski inequality.

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1. INTRODUCTION

In the seminal paper [CEMS01] it is proved that some natural inequalities holding in the Euclidean space generalize to the Riemannian setting, provided that the geometry of the ambient space is taken into account through appropriate distortion coefficients. The prototype of these inequalities in \mathbb{R}^n is the Brunn-Minkowski one, or its functional counterpart in the form of Borell-Brascamp-Lieb inequality.

The main results of [CEMS01], which are purely geometrical, were originally formulated in terms of optimal transport. The theory of optimal transport (with quadratic cost) is nowadays well understood in the Riemannian setting, thanks to the works of McCann [McC01], who adapted to manifolds the theory of Brenier in the Euclidean space [Bre99]. We refer to [Vil09] for references, including a complete historical account of the theory and its subsequent developments.

Let then μ_0 and μ_1 be two probability measures on an n -dimensional Riemannian manifold (M, g) . We assume μ_0, μ_1 to be compactly supported, and absolutely continuous with respect to the Riemannian measure \mathfrak{m}_g , so that $\mu_i = \rho_i \mathfrak{m}_g$ for some $\rho_i \in L^1(M, \mathfrak{m}_g)$. Under these assumptions, there exists a unique optimal transport map $T : M \rightarrow M$, such that $T_{\#}\mu_0 = \mu_1$ and which solves the Monge problem:

$$\int_M d^2(x, T(x)) d\mathfrak{m}_g(x) = \inf_{S_{\#}\mu_0 = \mu_1} \int_M d^2(x, S(x)) d\mathfrak{m}_g(x).$$

Furthermore, for μ_0 -a.e. $x \in M$, there exists a unique constant-speed geodesic $T_t(x)$, with $0 \leq t \leq 1$, such that $T_0(x) = x$ and $T_1(x) = T(x)$. The map $T_t : M \rightarrow M$ defines the dynamical interpolation $\mu_t = (T_t)_{\#}\mu_0$, a curve in the space of probability measures joining μ_0 with μ_1 . Roughly speaking, if we think at μ_0 and μ_1 as the initial and final states of a distribution of mass, then μ_t represents the evolution at time t of the process that moves, in an optimal way, μ_0 to μ_1 . More precisely, $(\mu_t)_{0 \leq t \leq 1}$ is the unique Wasserstein geodesic between μ_0 and μ_1 , with respect to the quadratic transportation cost.

By a well known regularity result, μ_t is absolutely continuous with respect to \mathfrak{m}_g , that is $\mu_t = \rho_t \mathfrak{m}_g$ for some $\rho_t \in L^1(M, \mathfrak{m}_g)$. The fundamental result of [CEMS01] is that the concentration $1/\rho_t$ during the transportation process can be estimated with respect to its initial and final values. More precisely, for all $t \in [0, 1]$, the following *interpolation inequality* holds:

$$(1) \quad \frac{1}{\rho_t(T_t(x))^{1/n}} \geq \frac{\beta_{1-t}(T(x), x)^{1/n}}{\rho_0(x)^{1/n}} + \frac{\beta_t(x, T(x))^{1/n}}{\rho_1(T(x))^{1/n}}, \quad \mu_0 - \text{a.e. } x \in M.$$

Here, $\beta_s(x, y)$, for $s \in [0, 1]$, are *distortion coefficients* which depend only on the geometry of the underlying Riemannian manifold, and can be computed once the Riemannian structure is given. Furthermore, if $\text{Ric}_g(M) \geq \kappa g$, then $\beta_t(x, y)$ are controlled from below by their analogues on the Riemannian space forms of constant curvature equal to κ and dimension n . More precisely, we have

$$(2) \quad \beta_t(x, y) \geq \beta_t^{(K,n)}(x, y) = \begin{cases} +\infty & \text{if } K > 0 \text{ and } \alpha > \pi, \\ t \left(\frac{\sin(t\alpha)}{\sin(\alpha)} \right)^{n-1} & \text{if } K > 0 \text{ and } \alpha \in [0, \pi], \\ t^n & \text{if } K = 0, \\ t \left(\frac{\sinh(t\alpha)}{\sinh(\alpha)} \right)^{n-1} & \text{if } K < 0, \end{cases}$$

where

$$\alpha = \sqrt{\frac{|K|}{n-1}} d(x, y).$$

In particular, on reference spaces, the distortion $\beta_t(x, y)$ is controlled only by the distance $d(x, y)$ between the two point.

Remark 1. Notice that $\beta_t(x, y) \sim t^n$. This universal asymptotics, valid in the Riemannian case, led [CEMS01] to extract a factor t^n in (1), expressing it in terms of the modified distortion coefficients $v_t(x, y) := \beta_t(x, y)/t^n$.

Inequality (1), when expressed in terms of the reference coefficients (2), is one of the incarnations of the so-called curvature-dimensions $\text{CD}(K, N)$ condition, which allows to generalize the concept of Ricci curvature bounded from below and dimension bounded from above to more general metric measure spaces. This is the beginning of the synthetic approach propugated by Lott-Villani and Sturm [LV09, Stu06a, Stu06b] and extensively developed subsequently.

The main tools used in [CEMS01] are the properties of the Riemannian cut locus and Jacobi fields, the nature of which changes dramatically in the sub-Riemannian setting (see Section 2 for definitions). For this reason the extension of the above inequalities to the sub-Riemannian world has remained elusive. For example, it is now well known that the Heisenberg group equipped with a left-invariant measure, which is the simplest sub-Riemannian structure, does not satisfy any form of $\text{CD}(K, N)$, as proved in [Jui09].

On the other hand, it has been recently proved in [BKS16] that the Heisenberg group actually supports interpolation inequalities as (1), with different distortion coefficients whose properties are quite different with respect to the Riemannian case. The techniques in [BKS16] consist in employing a one-parameter family of Riemannian extension of the Heisenberg structure, converging to the latter as $\varepsilon \rightarrow 0$. Starting from the Riemannian interpolation inequalities, a fine analysis is required to obtain a meaningful limit for $\varepsilon \rightarrow 0$. It is important to stress that the Ricci curvature of the Riemannian extensions tends to $-\infty$ as $\varepsilon \rightarrow 0$.

The results of [BKS16] and the subsequent extension to the corank 1 case obtained in [BKS17] suggest that a sub-Riemannian theory of interpolation inequalities which parallels the Riemannian one could actually exist. We recall that the Heisenberg group is the sub-Riemannian analogue of the Euclidean plane in Riemannian geometry, hence it is likely that such a general theory requires substantially different techniques. In this paper, we answer to the following question:

Do sub-Riemannian manifolds support weighted interpolation inequalities à la [CEMS01]? How to recover the correct weights and what are their properties?

We obtain a satisfying and positive answer, at least for the so-called *ideal* structures, that is admitting no non-trivial abnormal minimizing geodesics (this is a generic

assumption, see Proposition 14). In this case, the sub-Riemannian transportation problem is well posed (see Section 5.1 for the state of the art).

1.1. Interpolation inequalities. To introduce our results, let (\mathcal{D}, g) be a sub-Riemannian structure on a smooth manifold M , and fix a smooth reference (outer) measure \mathfrak{m} . Moreover, let us introduce the (sub-)Riemannian distortion coefficients.

Definition 2. Let $A, B \subset M$ be measurable sets, and $t \in [0, 1]$. The set $Z_t(A, B)$ of *t-intermediate points* is the set of all points $\gamma(t)$, where $\gamma : [0, 1] \rightarrow M$ is a minimizing geodesic such that $\gamma(0) \in A$ and $\gamma(1) \in B$.

Compare the next definition with the one in [Vil09, Def. 14.17, Prop. 14.18]. The important difference is that here we do not extract a factor $1/t^n$ since, as we will see, the topological dimension does not describe the correct asymptotic behavior in the sub-Riemannian case (cf. also Remark 1). Let $\mathcal{B}_r(x)$ denote the sub-Riemannian ball of center $x \in M$ and radius $r > 0$.

Definition 3 (Distortion coefficient). Let $x, y \in M$. The *distortion coefficient* from x to y at time $t \in [0, 1]$ is

$$\beta_t(x, y) := \limsup_{r \downarrow 0} \frac{\mathfrak{m}(Z_t(x, \mathcal{B}_r(y)))}{\mathfrak{m}(\mathcal{B}_r(y))}.$$

Notice that $\beta_0(x, y) = 0$ and $\beta_1(x, y) = 1$.

Despite the lack of a canonical Levi-Civita connection and curvature, in this paper we develop a suitable theory of sub-Riemannian (or rather Hamiltonian) Jacobi fields, which is powerful enough to derive interpolation inequalities. Our techniques are based on the approach initiated in [AZ02a, AZ02b, ZL09], and subsequently developed in a language that is more close to our presentation, in [ABR13, BR15, BR16]. Our first main result is the extension of (1) to the ideal sub-Riemannian setting.

Theorem 4 (Interpolation inequality). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on M , and $\mu_0, \mu_1 \in \mathcal{P}_c^{ac}(M)$. Let $\rho_s = d\mu_s/d\mathfrak{m}$. For all $t \in [0, 1]$, it holds*

$$(3) \quad \frac{1}{\rho_t(T_t(x))^{1/n}} \geq \frac{\beta_{1-t}(T(x), x)^{1/n}}{\rho_0(x)^{1/n}} + \frac{\beta_t(x, T(x))^{1/n}}{\rho_1(T(x))^{1/n}}, \quad \mu_0 - \text{a.e. } x \in M.$$

If μ_1 is not absolutely continuous, an analogous result holds, provided that $t \in (0, 1)$, and that in (3) the second term on the right hand side is omitted.

A key role in our proof is played by a positivity lemma (cf. Lemma 31) inspired by [Vil09, Ch. 14, Appendix: Jacobi fields forever]. At a technical level, the non positive definiteness of the sub-Riemannian Hamiltonian presents some non-trivial difficulties. Moreover, with respect to previous approaches, we stress that we do not make use of any canonical frame, playing the role of a parallel transported frame.

Concerning the sub-Riemannian distortion coefficients, it is interesting to observe that they can be explicitly computed in terms of the aforementioned sub-Riemannian Jacobi fields. In this regard, the main result is given by Lemma 50, which then is used in Section 7 to yield explicit formulas in different examples.

Thanks to this relation, we are able to deduce general properties of sub-Riemannian distortion coefficients, which are remarkably different with respect to their Riemannian counterpart. For example, even in the most basic examples, $\beta_t(x, y)$ does *not* depend on the distance between x and y , but rather on the covector joining them. Moreover, their asymptotics is not related with the topological dimension (but with the geodesic one). These properties of distortion coefficients are discussed in Section 8. To better highlight the difference with respect to the Riemannian case, we anticipate the following statement.

Theorem 5 (Asymptotics of sub-Riemannian distortion). *Let (\mathcal{D}, g) be a sub-Riemannian structure on M , and $x \in M$. Then, there exists $\mathcal{N}(x) \in \mathbb{N}$ such that*

$$\lim_{t \rightarrow 0^+} \frac{\log \beta_t(x, \exp_x(\lambda))}{\log t} \geq \mathcal{N}(x), \quad \forall \lambda \in T_x^*M.$$

*The equality is attained on a Zariski non-empty open and dense set $\mathcal{A}_x \subseteq T_x^*M$. In particular, $\mathcal{N}(x)$ is the largest number such that, for $t \rightarrow 0^+$, one has*

$$\beta_t(x, y) = O\left(t^{\mathcal{N}(x)}\right), \quad \forall y \notin \text{Cut}(x).$$

The number $\mathcal{N}(x)$ is called the geodesic dimension of the sub-Riemannian structure at x . Finally, the following inequality holds

$$\mathcal{N}(x) \geq \dim(M),$$

with equality if and only if the structure is Riemannian at x , that is $\mathcal{D}_x = T_xM$.

1.2. Regularity of distance. These problems are also related with the properties of the regularity of the distance and the structure of cut locus. In Riemannian geometry, it is well known that for almost-every geodesic γ involved in the transport, $\gamma(1) \notin \text{Cut}(\gamma(0))$. In particular, this implies (in a non-trivial way), that the sub-Riemannian cut-locus, which defined as the set of points where the squared distance is not smooth, can be characterized actually as the set of points where the squared distance fails to be semiconvex. This was indeed another main result of [CEMS01].

Here, we extend the latter to the sub-Riemannian setting, answering affirmatively to the open problem raised by Figalli and Rifford in [FR10, Sec. 5.8], at least when non-trivial abnormal geodesics are not present.

Theorem 6 (Failure of semiconvexity at the cut locus). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on M . Let $y \neq x$. Then $x \in \text{Cut}(y)$ if and only if the squared sub-Riemannian distance from y fails to be semiconvex at x , that is, in local coordinates around x , we have*

$$\inf_{0 < |v| < 1} \frac{d_{SR}^2(x+v, y) + d_{SR}^2(x-v, y) - 2d_{SR}^2(x, y)}{|v|^2} = -\infty.$$

The characterization of Theorem 6 is false in the non-ideal case, as we discuss in Section 4.2. Some related open problems are proposed in Section 4.2.1.

1.3. Geometric inequalities. The classical consequences of interpolation inequalities hold follow from standard arguments, and they are the object of Section 6. The typical examples are the geodesic Brunn-Minkowski inequality (Theorem 54) and its analytic counterpart, the p -mean inequality (Theorem 53). Notice that the p -mean inequality follows from the more fundamental Borell-Brascamp-Lieb inequality. The latter can be recovered as a particular case of the former for $p = -1/n$ (Theorem 52).

To this purpose, for $p \in \mathbb{R} \cup \{\pm\infty\}$, $t \in [0, 1]$ and $a, b \geq 0$, introduce the p -mean

$$\mathcal{M}_t^p(a, b) := \begin{cases} ((1-t)a^p + tb^p)^{1/p} & \text{if } ab \neq 0 \\ 0 & \text{if } ab = 0 \end{cases}, \quad p \neq 0, +\infty.$$

The limit cases are defined as follows

$$\mathcal{M}_t^0(a, b) := a^{1-t}b^t, \quad \mathcal{M}_t^{+\infty}(a, b) := \max\{a, b\}, \quad \mathcal{M}_t^{-\infty}(a, b) := \min\{a, b\}.$$

Theorem 7 (Sub-Riemannian p -mean inequality). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on a n -dimensional manifold M , equipped with a smooth measure m . Fix $p \geq -1/n$ and $t \in [0, 1]$. Let $f, g, h : M \rightarrow \mathbb{R}$ be non-negative and $A, B \subset M$*

be Borel subsets such that $\int_A f \, dm = \|f\|_{L^1(M)}$ and $\int_B g \, dm = \|g\|_{L^1(M)}$. Assume that for every $(x, y) \in (A \times B) \setminus \text{Cut}(M)$ and $z \in Z_t(x, y)$,

$$h(z) \geq \mathcal{M}_t^p \left(\frac{(1-t)^n f(x)}{\beta_{1-t}(y, x)}, \frac{t^n g(y)}{\beta_t(x, y)} \right).$$

Then,

$$\int_M h \, dm \geq \mathcal{M}_t^{p/(1+np)} \left(\int_M f \, dm, \int_M g \, dm \right),$$

with the convention that if $p = +\infty$ then $p/(1+np) = 1/n$, and if $p = -1/n$ then $p/(1+np) = -\infty$.

To introduce the geodesic Brunn-Minkowski inequality, we define for any pair of Borel subsets $A, B \subset M$ the following quantity:

$$\beta_t(A, B) := \inf \{ \beta_t(x, y) \mid (x, y) \in (A \times B) \setminus \text{Cut}(M) \}.$$

with the convention that $\inf \emptyset = 0$. Notice that $0 \leq \beta_t(A, B) < +\infty$, as a consequence of Lemma 50.

Theorem 8 (Sub-Riemannian Brunn-Minkowski inequality). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on a n -dimensional manifold M , equipped with a smooth measure \mathfrak{m} . Let $A, B \subset M$ be Borel subsets. Then we have*

$$\mathfrak{m}(Z_t(A, B))^{1/n} \geq \beta_{1-t}(B, A)^{1/n} \mathfrak{m}(A)^{1/n} + \beta_t(A, B)^{1/n} \mathfrak{m}(B)^{1/n}.$$

A particular role is played by structures where the distortion coefficients are controlled by a power law, that is $\beta_t(x, y) \geq t^N$, for all $t \in [0, 1]$ and $(x, y) \notin \text{Cut}(M)$. By Theorem 8, this implies the so-called *measure contraction property* MCP(0, N), first introduced in [Oht07] (see also [Stu06b] for a similar formulation). The MCP was first investigated in Carnot groups in [Jui09, Rif13a]. In the ideal, sub-Riemannian context, we are able to state the following equivalence result.

Theorem 9. *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on a n -dimensional manifold M , equipped with a smooth measure \mathfrak{m} . Let $N \geq 0$. Then, the following properties are equivalent:*

- (i) $\beta_t(x, y) \geq t^N$, for all $(x, y) \notin \text{Cut}(M)$ and $t \in [0, 1]$;
- (ii) the Brunn-Minkowski inequality holds: for all non-empty Borel sets A, B

$$\mathfrak{m}(Z_t(A, B))^{1/n} \geq (1-t)^{N/n} \mathfrak{m}(A)^{1/n} + t^{N/n} \mathfrak{m}(B)^{1/n}, \quad \forall t \in [0, 1].$$

- (iii) the measure contraction property MCP(0, N) is satisfied: for all non-empty Borel sets B and $x \in M$

$$\mathfrak{m}(Z_t(x, B)) \geq t^N \mathfrak{m}(B), \quad \forall t \in [0, 1];$$

We stress that on a n -dimensional sub-Riemannian manifold that is not Riemannian, the MCP(0, n) is never satisfied (see [Riz16, Thm. 6]).

This clarifies the fact that an Euclidean Brunn-Minkowski inequality with linear weights (that is $N = n$), is not adapted for generalizations to genuine sub-Riemannian situations, as well as the classical curvature-dimension condition.

1.4. Old and new examples. In Section 7, we discuss some key examples, where the distortion coefficients can be explicitly obtained and analyzed. In particular, we consider the following cases:

- **The Heisenberg group \mathbb{H}_3 .** This is particularly important, as it constitutes the most basic sub-Riemannian structure. In this case we recover, in an intrinsic way, the results of [BKS16], with the same distortion coefficients. See Section 7.1.
- **Generalized H -type groups.** This is a class of Carnot groups (which has been introduced in [BR17], and extends the class of Kaplan H -type groups), for which the optimal synthesis is known, and where distortion coefficients can be computed explicitly. It includes all corank 1 Carnot groups but, most importantly, Carnot groups of arbitrary large corank. In the ideal case, we obtain sharp interpolation inequalities for general measures (Corollary 65). These structures are not all ideal, but they are the product of an ideal generalized H -type Carnot group, and an Euclidean space. Exploiting recent results of [RY17] for product structures, we are able to prove in the general case, i.e. not necessarily ideal, sharp Brunn-Minkowski inequality (which implies sharp measure contraction properties, see Corollary 67). To our best knowledge, these are the first results of this kind for sub-Riemannian structures with corank larger than one. See Section 7.2.
- **Grushin plane \mathbb{G}_2 .** Our techniques work also for sub-Riemannian distributions \mathcal{D} whose rank is not constant. In this setting we are able to obtain for the first time interpolation inequalities (Corollary 71), sharp Brunn-Minkowski inequalities (Corollary 72), and sharp measure-contraction properties (Corollary 73). See Section 7.3.

In all the above cases, we are able to prove that the distortion coefficients satisfy

$$(4) \quad \beta_t(x, y) \geq t^{\mathcal{N}}, \quad \forall (x, y) \notin \text{Cut}(M), \forall t \in [0, 1],$$

for some optimal (smallest) \mathcal{N} , given by the *geodesic dimension* of the sub-Riemannian structure. The geodesic dimension is an invariant initially discovered for sub-Riemannian structures in [ABR13], and subsequently generalized to metric measure spaces in [Riz16]. Here, we only mention that, in the sub-Riemannian case, the geodesic dimension is strictly larger than the Hausdorff or the topological dimension, and all three invariants coincide if and only if the structure is actually Riemannian. The interpolation inequalities take hence a very pleasant sharp form, in terms of the geodesic dimension \mathcal{N} . For example in the case of the Brunn-Minkowski inequality, for all non-empty Borel sets A, B , we have

$$\mathfrak{m}(Z_t(A, B))^{1/n} \geq (1-t)^{\mathcal{N}/n} \mathfrak{m}(A)^{1/n} + t^{\mathcal{N}/n} \mathfrak{m}(B)^{1/n}, \quad \forall t \in [0, 1],$$

and similarly for the more general interpolation inequalities.

We notice that the distortion bound (4) was known for the Heisenberg and generalized H -type groups, as a consequence of the sharp measure contraction properties of these structures [Jui09, Riz16, BR17]. Furthermore, these results are new and particularly relevant for the case of the Grushin plane (see Section 7.3). As an example, we state here explicitly the geodesic Brunn-Minkowski inequality.

Theorem 10 (Grushin Brunn-Minkowski inequality). *The Grushin plane \mathbb{G}_2 equipped with the Lebesgue measure satisfies the following inequality: for all non-empty Borel sets $A, B \subset \mathbb{G}_2$, we have*

$$\mathcal{L}^2(Z_t(A, B))^{1/2} \geq (1-t)^{5/2} \mathcal{L}^2(A) + t^{5/2} \mathcal{L}^2(B), \quad \forall t \in [0, 1].$$

The above inequality is sharp, in the sense that if one replaces the exponent 5 with a smaller one, the inequality fails for some choice of A, B . Moreover, \mathbb{G}_2 satisfies the MCP(K, N) if and only if $N \geq 5$ and $K \leq 0$.

We conclude the paper with some general properties of sub-Riemannian distortion coefficients, proved in Section 8.

1.5. Afterwords. In this work we focused in laying the groundwork for interpolation inequalities in sub-Riemannian geometry. It remains to understand which is the correct class of *models* whose distortion coefficients constitute the reference spaces, playing the role of Riemannian space forms in Riemannian geometry. This will be the object of a subsequent work. We anticipate here that the natural reference spaces *do not belong* to the category of sub-Riemannian structures. In the spirit of [BR16], the unifying framework that we propose is the one of optimal control problems. This setting is sufficiently large to include infinitesimal models for all of the three great classes of geometries: Riemannian, sub-Riemannian, Finslerian, providing the first step of the “great unification” auspicated in [Vil17, Sec. 9].

Another challenging problem is understand how to include abnormal minimizers in this picture. Abnormal geodesics, as [BKS17] suggests for the case of corank 1 Carnot groups, are not *a priori* an obstacle to interpolation inequalities. These remarkable results are the consequence of the special structure of corank 1 Carnot groups, which are the metric product of an (ideal) contact Carnot group and a suitable copy of a flat \mathbb{R}^n . In general, an organic theory of transport and Jacobi fields along abnormal geodesics is still lacking. In this paper, we discuss some aspects of the non-ideal case and some open problems in Section 4.2.

2. PRELIMINARIES

We start by recalling some basic facts in sub-Riemannian geometry. For a comprehensive introduction, we refer to [ABB16b, Rif14, Mon02].

2.1. Sub-Riemannian geometry. A sub-Riemannian structure on a smooth, connected n -dimensional manifold M , where $n \geq 3$, is defined by a set of m global smooth vector fields X_1, \dots, X_m , called a *generating frame*. The *distribution* is the family of subspaces of the tangent spaces spanned by the vector fields at each point

$$\mathcal{D}_x = \text{span}\{X_1(x), \dots, X_m(x)\} \subseteq T_x M, \quad \forall x \in M.$$

The generating frame induces an inner product g_x on \mathcal{D}_x as follows: given $v, w \in T_x M$ the inner product $g_x(v, w)$ is defined by polarization

$$g_x(v, w) := \frac{1}{4}(g_x(v+w, v+w) - g_x(v-w, v-w)),$$

where

$$g_x(v, v) := \inf \left\{ \sum_{i=1}^m u_i^2 \mid \sum_{i=1}^m u_i X_i(x) = v \right\}.$$

We assume that the distribution is *bracket-generating*, i.e., the tangent space $T_x M$ is spanned by the vector fields X_1, \dots, X_m and their iterated Lie brackets evaluated at x . A *horizontal curve* $\gamma : [0, 1] \rightarrow M$ is an absolutely continuous path such that there exists $u \in L^2([0, 1], \mathbb{R}^m)$ satisfying

$$\dot{\gamma}(t) = \sum_{i=1}^m u_i(t) X_i(\gamma(t)), \quad \text{a.e. } t \in [0, 1].$$

This implies that $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for almost every t . If γ is horizontal, the map $t \mapsto \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}$ is measurable on $[0, 1]$, hence integrable [ABB16a, Lemma 3.11]. We define the *length* of an horizontal curve as follows

$$\ell(\gamma) = \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

The *sub-Riemannian distance* is defined by:

$$(5) \quad d_{SR}(x, y) = \inf\{\ell(\gamma) \mid \gamma(0) = x, \gamma(1) = y, \gamma \text{ horizontal}\}.$$

We denote by $\mathcal{B}_r(x)$ the sub-Riemannian ball of center x and radius $r > 0$.

Remark 11. The above definition includes rank-varying sub-Riemannian structures on M , see [BBS16, Ch. 1]. When $\dim \mathcal{D}_x$ is constant, then \mathcal{D} is a vector distribution in the classical sense. If $m \leq n$ and the vector fields X_1, \dots, X_m are linearly independent everywhere, they form a basis of \mathcal{D} and g coincides with the inner product on \mathcal{D} for which X_1, \dots, X_m is an orthonormal frame.

By Chow-Rashevskii theorem, the bracket-generating condition implies that $d_{SR} : M \times M \rightarrow \mathbb{R}$ is finite and continuous. If the metric space (M, d_{SR}) is complete, then for any $x, y \in M$ the infimum in (5) is attained. In place of the length ℓ , it is often convenient to consider the *energy functional*

$$J(\gamma) = \frac{1}{2} \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t)) dt.$$

On the space of horizontal curves defined on a fixed interval and with fixed endpoints, the minimizers of J coincide with the minimizers of ℓ parametrized with constant speed. Since ℓ is invariant by reparametrization (and every horizontal curve is the reparametrization of a horizontal curves with constant speed), we do not lose generality in defining *geodesics* as horizontal curves that locally minimize the energy between their endpoints.

The *Hamiltonian* of the sub-Riemannian structure $H : T^*M \rightarrow \mathbb{R}$ is defined by

$$H(\lambda) = \frac{1}{2} \sum_{i=1}^m \langle \lambda, X_i \rangle^2, \quad \lambda \in T^*M,$$

where X_1, \dots, X_m is the generating frame. Here $\langle \lambda, \cdot \rangle$ denotes the dual action of covectors on vectors. Different generating frames defining the same distribution and scalar product at each point, yield the same Hamiltonian function. The Hamiltonian vector field \vec{H} is the unique vector field such that $\sigma(\cdot, \vec{H}) = dH$, where σ is the canonical symplectic form of the cotangent bundle $\pi : T^*M \rightarrow M$. In particular, the *Hamilton equations* are

$$(6) \quad \dot{\lambda}(t) = \vec{H}(\lambda(t)), \quad \lambda(t) \in T^*M.$$

If (M, d_{SR}) is complete, solutions of (6) are defined for all times.

2.2. End-point map and Lagrange multipliers. Given the generating frame X_1, \dots, X_m , let $\gamma_u : [0, 1] \rightarrow M$ be an horizontal curve joining x and y , where $u \in L^2([0, 1], \mathbb{R}^m)$ is a *control* such that

$$\dot{\gamma}_u(t) = \sum_{i=1}^m u_i(t) X_i(\gamma_u(t)), \quad \text{a.e. } t \in [0, 1].$$

Let $\mathcal{U} \subset L^2([0, 1], \mathbb{R}^m)$ be the neighborhood of u such that, for $v \in \mathcal{U}$, the equation

$$\dot{\gamma}_v(t) = \sum_{i=1}^m v_i(t) X_i(\gamma_v(t)), \quad \gamma_v(0) = x,$$

has a well defined solution for a.e. $t \in [0, 1]$. We define the *end-point map* with base point x as the map $E_x : \mathcal{U} \rightarrow M$, which sends v to $\gamma_v(1)$. The end-point map is smooth on \mathcal{U} .

We can consider $J : \mathcal{U} \rightarrow \mathbb{R}$ as a smooth functional on \mathcal{U} . Let γ_u be a minimizing geodesic, that is a solution of the constrained minimum problem

$$\min\{J(v) \mid v \in \mathcal{U}, E_x(v) = y\}.$$

By the Lagrange multipliers rule, there exists a non-trivial pair (λ_1, ν) , such that

$$(7) \quad \lambda_1 \circ D_u E_x = \nu D_u J, \quad \lambda_1 \in T_y^* M, \quad \nu \in \{0, 1\},$$

where \circ denotes the composition of linear maps and D the (Fréchet) differential. If $\gamma_u : [0, 1] \rightarrow M$ with control $u \in \mathcal{U}$ is an horizontal curve (not necessarily minimizing), we say that a non-zero pair $(\lambda_1, \nu) \in T_y^* M \times \{0, 1\}$ is a *Lagrange multiplier* for γ_u if (7) is satisfied. The multiplier (λ_1, ν) and the associated curve γ_u are called *normal* if $\nu = 1$ and *abnormal* if $\nu = 0$. Observe that Lagrange multipliers are not unique, and a horizontal curve may be both normal *and* abnormal. Observe also that γ_u is an abnormal curve if and only if u is a critical point for E_x . In this case, γ_u is also called a *singular curve*. The following characterization is a consequence of the Lagrange multipliers rule, and can also be seen as a specification of the Pontryagin Maximum Principle to the sub-Riemannian length minimization problem.

Theorem 12. *Let $\gamma_u : [0, 1] \rightarrow M$ be an horizontal curve joining x with y . A non-zero pair $(\lambda_1, \nu) \in T_y^* M \times \{0, 1\}$ is a Lagrange multiplier for γ_u if and only if there exists a Lipschitz curve $\lambda(t) \in T_{\gamma_u(t)}^* M$ with $\lambda(1) = \lambda_1$, such that*

- (N) if $\nu = 1$ then $\dot{\lambda}(t) = \vec{H}(\lambda(t))$, i.e. it is a solution of Hamilton equations,
- (A) if $\nu = 0$ then $\sigma(\lambda(t), T_{\lambda(t)} \mathcal{D}^\perp) = 0$,

where $\mathcal{D}^\perp \subset T^* M$ is the sub-bundle of covectors that annihilate the distribution.

In the first (resp. second) case, $\lambda(t)$ is called a *normal* (resp. *abnormal*) *extremal*. Normal extremals are integral curves $\lambda(t)$ of \vec{H} . As such, they are smooth, and characterized by their *initial covector* $\lambda = \lambda(0)$. A geodesic is normal (resp. abnormal) if admits a normal (resp. abnormal) extremal. On the other hand, it is well-known that the projection $\gamma_\lambda(t) = \pi(\lambda(t))$ of a normal extremal is locally minimizing, hence it is a normal geodesic. The *exponential map* at $x \in M$ is the map

$$\exp_x : T_x^* M \rightarrow M,$$

which assigns to $\lambda \in T_x^* M$ the final point $\pi(\lambda(1))$ of the corresponding normal geodesic. The curve $\gamma_\lambda(t) := \exp_x(t\lambda)$, for $t \in [0, 1]$, is the normal geodesic corresponding to λ , which has constant speed $\|\dot{\gamma}_\lambda(t)\| = \sqrt{2H(\lambda)}$ and length $\ell(\gamma|_{[t_1, t_2]}) = \sqrt{2H(\lambda)}(t_2 - t_1)$.

Definition 13. A sub-Riemannian structure (\mathcal{D}, g) on M is *ideal* if the metric space (M, d_{SR}) is complete and there exists no non-trivial abnormal minimizers.

The above terminology was introduced in [Rif13b, Rif14]. All fat sub-Riemannian structures admit no non-trivial abnormal curves [Mon02, Sec. 5.6]. In particular, complete fat structures are ideal. Moreover, the ideal assumption is generic, when the rank of the distribution is at least 3, in the following sense.

Proposition 14 ([CJT06, Thm. 2.8]). *Let $k \geq 3$ be a positive integer, \mathcal{G}_k be the set of sub-Riemannian structures (\mathcal{D}, g) on M with $\text{rank } \mathcal{D} = k$, endowed with the Whitney C^∞ topology. There exists an open dense subset \mathcal{W}_k of \mathcal{G}_k such that every element of \mathcal{W}_k does not admit non-trivial abnormal minimizers.*

Next, we recall the definition of conjugate points.

Definition 15. Let $\gamma : [0, 1] \rightarrow M$ be a normal geodesic with initial covector $\lambda \in T_x^* M$, that is $\gamma(t) = \exp_x(t\lambda)$. We say that $y = \exp_x(\bar{t}\lambda)$ is a *conjugate point* to x along γ if $\bar{t}\lambda$ is a critical point for \exp_x .

Given a normal geodesic $\gamma : [0, 1] \rightarrow M$ and $0 \leq s < t \leq 1$, we say that $\gamma(s)$ and $\gamma(t)$ are *conjugate* if $\gamma(t)$ is conjugate to $\gamma(s)$ along $\gamma|_{[s, t]}$.

In the Riemannian setting, conjugate points along a geodesic are isolated, and geodesics cease to be minimizers after the first conjugate point. In the general sub-Riemannian setting, the picture is more complicated, but this result remains valid for ideal structures.

Theorem 16 (Conjugate points and minimality). *Let $\gamma : [0, 1] \rightarrow M$ be a minimizing geodesic, which does not contain abnormal segments. Then $\gamma(s)$ is not conjugate to $\gamma(s')$ for every $s, s' \in [0, 1]$ with $|s - s'| < 1$.*

Theorem 16 is a consequence of the second variation formula for the sub-Riemannian energy. This fact is not new, and well-known to experts. An explicit statement can be found in the preprint version of [FR10, Prop. 5.15], and is proved in [Sar80]. For self-containedness, we provide a proof in Appendix A, following the arguments of [ABB16b]. Notice that, as in the Riemannian case, it remain possible that $\gamma(1)$ is conjugate to $\gamma(0)$ along γ .

2.3. Regularity of sub-Riemannian distance. We recall now some basic regularity properties of the sub-Riemannian distance.

Definition 17. Let (\mathcal{D}, g) be a complete sub-Riemannian structure on M , and $x \in M$. We say that $y \in M$ is a *smooth point* (with respect to x) if there exists a unique minimizing geodesic joining x with y , which is not abnormal, and the two points are not conjugate along such a curve. The *cut locus* $\text{Cut}(x)$ is the complement of the set of smooth points with respect to x . The *global cut-locus* of M is

$$\text{Cut}(M) := \{(x, y) \in M \times M \mid y \in \text{Cut}(x)\}.$$

We have the following fundamental result [Agr09, RT05].

Theorem 18. *The set of smooth points is open and dense in M , and the squared sub-Riemannian distance is smooth on $M \times M \setminus \text{Cut}(M)$.*

3. JACOBI FIELDS AND SECOND DIFFERENTIAL

Let $f : M \rightarrow \mathbb{R}$ be a smooth function. Its first differential at $x \in M$ is the linear map $d_x f : T_x M \rightarrow \mathbb{R}$. Let $x \in M$ be a critical point for f , that is $d_x f = 0$. In this case, one can define the *second differential* (or Hessian) of f via the formula

$$\text{Hess}(f)|_x : T_x M \times T_x M \rightarrow \mathbb{R}, \quad \text{Hess}(f)|_x(v, w) = V(W(f))(x),$$

where V, W are local vector fields such that $V(x) = v$ and $W(x) = w$. Since x is a critical point, the definition is well posed, and $\text{Hess}(f)|_x$ is a symmetric bilinear map. The quadratic form associated with the second differential of f at x which, for simplicity, we denote by the same symbol $\text{Hess}(f)|_x : T_x M \rightarrow \mathbb{R}$, is

$$\text{Hess}(f)|_x(v) = \left. \frac{d^2}{dt^2} \right|_{t=0} f(\gamma(t)), \quad \gamma(0) = x, \quad \dot{\gamma}(0) = v.$$

When $x \in M$ is not a critical point, we define the second differential of f as the differential of df , thought as a smooth section of T^*M .

Definition 19 (Second differential at non-critical points). Let $f \in C^\infty(M)$, and

$$df : M \rightarrow T^*M, \quad df : x \mapsto d_x f.$$

The *second differential* of f at $x \in M$ is the linear map

$$d_x^2 f := d_x(df) : T_x M \rightarrow T_\lambda(T^*M),$$

where $\lambda = d_x f \in T^*M$.

Remark 20. The image of the differential $df : M \rightarrow T^*M$ is a Lagrangian¹ submanifold of T^*M . Thus, by definition, the image of the second differential $d_x^2 f(T_x M)$ at a point x is the tangent space of $df(M)$ at $\lambda = d_x f$, which is an n -dimensional Lagrangian subspace of $T_\lambda(T^*M)$ transverse to the vertical subspace $T_\lambda(T_x M)$. Letting $\pi : T^*M \rightarrow M$ be the cotangent bundle projection, and since $\pi \circ df = \text{id}_M$, we have that $\pi_* \circ d_x^2 f = \text{id}_{T_x M}$.

Lemma 21. *Let $\lambda \in T_x^*M$. The set $\mathcal{L}_\lambda := \{d_x^2 f \mid f \in C^\infty(M), d_x f = \lambda\}$ is an affine space over the vector space of quadratic forms on $T_x M$.*

The above lemma follows from the fact that if $f_1, f_2 \in C^\infty(M)$ satisfy $d_x f_1 = d_x f_2 = \lambda$, then x is a critical point for $f_1 - f_2$ and one can define the difference between $d_x^2 f_1$ and $d_x^2 f_2$ as the quadratic form $\text{Hess}(f_1 - f_2)|_x$.

Remark 22. When $\lambda = 0 \in T_x^*M$, \mathcal{L}_λ is the space of the second derivatives of the functions with a critical point at x . In this case we can fix a canonical origin in \mathcal{L}_λ , namely the second differential of any constant function. This provides the identification of \mathcal{L}_λ with the space of quadratic forms on $T_x M$, recovering the standard notion of Hessian at a critical point.

Remark 23. Definition 19 can be extended to any $f : M \rightarrow \mathbb{R}$ twice differentiable at x . In this case, fix local coordinates around x , and let $b(x) \in \mathbb{R}^n$ and $A(x) \in \text{Sym}(n \times n)$ such that

$$\lim_{t \downarrow 0} \frac{f(x + tv) - f(x) - tb(x) \cdot v - \frac{t^2}{2} v \cdot A(x)v}{t^2} = 0, \quad \forall v \in \mathbb{R}^n.$$

Letting $(q, p) \in \mathbb{R}^{2n}$ denote canonical coordinates around $d_x f \in T^*M$, we define

$$d_x^2 f(\partial_{q_i}) := \partial_{q_i}|_{d_x f} + \sum_{j=1}^n A_{ij} \partial_{p_j}|_{d_x f}, \quad \forall i = 1, \dots, n.$$

This definition is well posed, i.e., it does not depend on the choice of coordinates.

3.1. Sub-Riemannian Jacobi fields. Let $\lambda_t = e^{t\vec{H}}(\lambda_0)$, $t \in [0, 1]$ be an integral curve of the Hamiltonian flow. For any smooth vector field $\xi(t)$ along λ_t , the dot denotes the Lie derivative in the direction of \vec{H} , namely

$$\dot{\xi}(t) := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} e_*^{-\varepsilon\vec{H}} \xi(t + \varepsilon).$$

A vector field $\mathcal{J}(t)$ along λ_t is a *Jacobi field* if it satisfies the equation

$$(8) \quad \dot{\mathcal{J}} = 0.$$

Jacobi fields along λ_t are of the form $\mathcal{J}(t) = e_*^{t\vec{H}} \mathcal{J}(0)$, for some unique initial condition $\mathcal{J}(0) \in T_{\lambda_0}(T^*M)$, and the space of solutions of (8) is a $2n$ -dimensional vector space. On T^*M we define the smooth sub-bundle with Lagrangian fibers:

$$\mathcal{V}_\lambda := \ker \pi_*|_\lambda = T_\lambda(T_{\pi(\lambda)}^*M) \subset T_\lambda(T^*M), \quad \lambda \in T^*M,$$

which we call the *vertical subspace*. In this formalism, letting

$$\gamma(t) = \exp_x(t\lambda_0) = \pi \circ e^{t\vec{H}}(\lambda_0), \quad t \in [0, 1],$$

we have that $\gamma(s)$ is conjugate with $\gamma(0)$ along the normal geodesic γ if and only if the Lagrangian subspace $e_*^{s\vec{H}} \mathcal{V}_{\lambda_0} \subset T_{\lambda_s}(T^*M)$ intersects \mathcal{V}_{λ_s} non-trivially.

¹A Lagrangian submanifold of T^*M is a submanifold such that its tangent space is a Lagrangian subspace of the symplectic space $T_\lambda(T^*M)$. A subspace $L \subset \Sigma$ of a symplectic vector space (Σ, σ) is Lagrangian if $\dim L = \dim \Sigma/2$ and $\sigma|_L = 0$.

The next statement generalizes the well known Riemannian fact that, in absence of conjugate points, Jacobi fields are either determined by their value and the value of the covariant derivative in the direction of the given geodesic at the initial time, or by their value at the final and initial times.

Lemma 24. *Assume that, for $s \in (0, 1]$, $\gamma(0)$ is not conjugate to $\gamma(s)$ along γ . Let $\mathcal{H}_{\lambda_i} \subset T_{\lambda_i}(T^*M)$ be transverse to \mathcal{V}_{λ_i} , for $i = 0, s$. Then for any pair $(J_0, J_s) \in \mathcal{H}_{\lambda_0} \times \mathcal{H}_{\lambda_s}$, there exists a unique Jacobi field $\mathcal{J}(t)$ along λ_t , $t \in [0, 1]$, such that the projection of $\mathcal{J}(i)$ on \mathcal{H}_{λ_i} is equal to J_i , for $i = 0, s$.*

Proof. The condition at $t = 0$ implies that $\mathcal{J}(0) \in J_0 + \mathcal{V}_{\lambda_0}$ (an affine space). By definition of Jacobi field, $\mathcal{J}(t) = e_*^{t\vec{H}} \mathcal{J}(0)$, in particular $\mathcal{J}(s) \in e_*^{s\vec{H}} J_0 + e_*^{s\vec{H}} \mathcal{V}_{\lambda_0}$. By the non-conjugate assumption and since $T_{\lambda_s}(T^*M) = \mathcal{V}_{\lambda_s} + \mathcal{H}_{\lambda_s}$, the projection of the affine space $e_*^{s\vec{H}} J_0 + e_*^{s\vec{H}} \mathcal{V}_{\lambda_0}$ on \mathcal{H}_{λ_s} is a bijection, yielding the statement. \square

3.2. Jacobi matrices. We introduce a formalism to describe families of subspaces generated by Jacobi fields. Let $\gamma : [0, 1] \rightarrow M$ be a normal geodesic, projection of $\lambda_t = e^{t\vec{H}}(\lambda_0)$, for some $\lambda_0 \in T^*M$. Consider the family of n -dimensional subspaces generated by a set of independent Jacobi fields $\mathcal{J}_1(t), \dots, \mathcal{J}_n(t)$ along λ_t , that is

$$\mathcal{L}_t = \text{span}\{\mathcal{J}_1(t), \dots, \mathcal{J}_n(t)\} \subset T_{\lambda_t}(T^*M).$$

Since $\mathcal{L}_t = e_*^{t\vec{H}} \mathcal{L}_0$, then \mathcal{L}_t is Lagrangian if and only if it is Lagrangian at time $t = 0$.

Notice that \mathcal{L}_t can be regarded as a smooth curve in a suitable (Lagrange) Grassmannian bundle over T^*M . We do not pursue this approach here, and we opt for an extrinsic formulation based on Darboux frames. To this purpose, and in order to exploit the symplectic structure of T^*M , fix a Darboux moving frame along λ_t , that is a collection of smooth vector fields $E_1(t), \dots, E_n(t), F_1(t), \dots, F_n(t)$ such that

$$\sigma(E_i, F_j) - \delta_{ij} = \sigma(E_i, E_j) = \sigma(F_i, F_j) = 0, \quad \forall i, j = 1, \dots, n,$$

and such that the $E_1(t), \dots, E_n(t)$ generate the vertical subspace $\mathcal{V}_{\lambda_t} = \ker \pi_*|_{\lambda_t}$:

$$\mathcal{V}_{\lambda_t} = \text{span}\{E_1(t), \dots, E_n(t)\}.$$

We also denote with $X_i(t) := \pi_* F_i(t)$ the corresponding moving frame along the geodesic γ . In this case, we say that $E_i(t), F_i(t)$ is a *Darboux lift* of $X_i(t)$. Notice that any smooth moving frame along a normal geodesic admits a Darboux lift along a corresponding normal extremal.

We identify $\mathcal{L}_t = \text{span}\{\mathcal{J}_1(t), \dots, \mathcal{J}_n(t)\}$ with a smooth family of $2n \times n$ matrices

$$\mathbf{J}(t) = \begin{pmatrix} M(t) \\ N(t) \end{pmatrix}, \quad t \in [0, 1],$$

such that, with respect to the given Darboux frame, we have

$$(9) \quad \mathcal{J}_i(t) = \sum_{j=1}^n E_j(t) M_{ji}(t) + F_j(t) N_{ji}(t), \quad \forall i = 1, \dots, n.$$

We call $\mathbf{J}(t)$ a *Jacobi matrix*, while the $n \times n$ matrices $M(t)$ and $N(t)$ represent respectively its ‘‘vertical’’ and ‘‘horizontal’’ components with respect to the decomposition induced by the Darboux moving frame

$$T_{\lambda_t}(T^*M) = \mathcal{H}_{\lambda_t} \oplus \mathcal{V}_{\lambda_t}, \quad \text{with} \quad \mathcal{H}_{\lambda_t} := \text{span}\{F_1(t), \dots, F_n(t)\}.$$

The following property is fundamental for the following.

Lemma 25. *There exist smooth families of matrices $A(t), B(t), R(t)$, $t \in [0, 1]$, with $B(t), R(t)$ symmetric and $B(t) \geq 0$, such that for any Jacobi matrix $\mathbf{J}(t)$, we have*

$$(10) \quad \frac{d}{dt} \begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} -A(t) & -R(t) \\ B(t) & A(t)^* \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}.$$

On any interval $I \subseteq [0, 1]$ such that $M(t)$ is non-degenerate, the matrix $W(t) := N(t)M(t)^{-1}$ satisfies the Riccati equation

$$\dot{W} = B(t) + A(t)W + WA(t)^* + WR(t)W.$$

The family of subspaces associated with $\mathbf{J}(t)$ is Lagrangian if and only if $W(t)$ is symmetric.

Proof. By completeness of the frame, there exist smooth matrices $A(t), B(t), C(t)$ such that, for all $t \in [0, 1]$, it holds

$$(11) \quad \dot{E} = E \cdot A(t) - F \cdot B(t), \quad \dot{F} = E \cdot R(t) - F \cdot C(t)^*.$$

The notation in (11) means that $\dot{E}_i = \sum_{j=1}^n E_j A(t)_{ji} - F_j B(t)_{ji}$, and similarly for \dot{F}_i . For n -tuples V, W , the pairing $\sigma(V, W)$ denotes the matrix $\sigma(V_i, W_j)$. In this notation, $\sigma(V, W)^* = -\sigma(W, V)$. Thanks to the Darboux condition, we obtain

$$C(t) = \sigma_{\lambda_t}(\dot{F}, E) = -\sigma_{\lambda_t}(F, \dot{E}) = A(t).$$

The symmetry of $R(t)$ and $B(t)$ follows similarly. Moreover, we have

$$(12) \quad B(t) = \sigma_{\lambda_t}(\dot{E}, E) = 2H(E, E) \geq 0.$$

Here, H is the Hamiltonian seen as a fiber-wise bilinear form on T^*M , and we identify $T_{\gamma_t}^*M \simeq T_{\lambda_t}(T_{\gamma_t}^*M)$. The second equality in (12) follows from a direct computation in canonical coordinates on T^*M . Observe that $B(t)$ has a non-trivial kernel if and only if the structure is not Riemannian. Finally, equation (10) follows from (11), (9) and the Jacobi equation $\dot{\mathcal{J}}_i(t) = 0$. The claim about Riccati equation is proved by direct verification.

Using (9), the Jacobi fields $\mathcal{J}_1(t), \dots, \mathcal{J}_n(t)$ associated with the Jacobi matrix $\mathbf{J}(t)$ generate a family of Lagrangian subspaces if and only if

$$0 = \sigma_{\lambda_t}(\mathcal{J}, \mathcal{J}) = M(t)^*N(t) - N(t)^*M(t).$$

The above identity is equivalent to the symmetry of $W(t)$. □

Remark 26. In Riemannian geometry, standard tensorial calculus and Jacobi fields along γ are sufficient for the forthcoming manipulations. This correspond to a very particular class of Darboux frames, such that $A(t) = 0$, $B(t) = \mathbb{1}$ and $R(t)$ represents the Riemannian sectional curvature of all 2-planes containing $\dot{\gamma}(t)$ [BR16, BR15]. In the sub-Riemannian case, such a convenient frame and Levi-Civita connection are not available in full generality. To circumvent this problem we “lift” the problem on the cotangent bundle and avoid to pick some particular frame.

3.3. Special Jacobi matrices. Fix a normal geodesic $\gamma : [0, 1] \rightarrow M$, and let $\lambda : [0, 1] \rightarrow T^*M$ be the corresponding extremal. Let $E_1(t), \dots, E_n(t), F_1(t), \dots, F_n(t)$ be a smooth moving frame along λ . Denote with $X_i(t) := \pi_* F_i(t)$ the corresponding smooth frame along γ . Any Jacobi matrix is uniquely defined by its value at some intermediate time $\mathbf{J}(s)$. The following special Jacobi matrices will play a prominent role in the forthcoming statements. Let $s \in [0, 1]$. We define the Jacobi matrices:

$$\mathbf{J}_s^V(t) = \begin{pmatrix} M_s^V(t) \\ N_s^V(t) \end{pmatrix}, \quad \text{such that} \quad \mathbf{J}_s^V(s) = \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}, \quad (\text{“vertical” at time } s),$$

$$\mathbf{J}_s^H(t) = \begin{pmatrix} M_s^H(t) \\ N_s^H(t) \end{pmatrix}, \quad \text{such that} \quad \mathbf{J}_s^H(s) = \begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix}, \quad (\text{“horizontal” at time } s),$$

representing, respectively, the families of Lagrange subspaces

$$e_*^{(t-s)\bar{H}} \operatorname{span}\{E_1(s), \dots, E_n(s)\} \quad \text{and} \quad e_*^{(t-s)\bar{H}} \operatorname{span}\{F_1(s), \dots, F_n(s)\}.$$

Remark 27 (Reading conjugate points from Jacobi matrices). Let $s_1, s_2 \in [0, 1]$. Then $\gamma(s_1)$ is conjugate to $\gamma(s_2)$ along γ if and only if at least one (and then both) the matrices $N_{s_1}^V(s_2)$ and $N_{s_2}^V(s_1)$ are degenerate.

4. MAIN JACOBIAN ESTIMATE

The next result answers affirmatively to a question raised by Figalli and Rifford [FR10, Sec. 5], at least when non-trivial singular minimizing curves are not allowed (see Corollary 32). It follows from the Jacobian estimate of Theorem 29, and for this reason they will be proved together. Notice that the sub-Riemannian structure is not necessarily ideal.

Theorem 28. *Let (\mathcal{D}, g) be a sub-Riemannian structure on M . Let $x \neq y \in M$ and assume that there exists a function $\phi : M \rightarrow \mathbb{R}$, twice differentiable at x , such that*

$$(13) \quad \frac{1}{2}d_{SR}^2(x, y) = -\phi(x), \quad \text{and} \quad \frac{1}{2}d_{SR}^2(z, y) \geq -\phi(z), \quad \forall z \in M.$$

Assume moreover that the minimizing curve joining x with y , which is unique and given by $\gamma(t) = \exp_x(td_x\phi)$, does not contain abnormal segments. Then $x \notin \operatorname{Cut}(y)$.

We will usually apply Theorem 28 to situations in which ϕ is actually (twice) differentiable almost everywhere, in such a way that the map

$$T_t(z) = \exp_z(td_z\phi), \quad \mathfrak{m} - \text{a.e. } z \in M,$$

is well defined. The next result is an estimate for its Jacobian determinant at x .

Theorem 29 (Main Jacobian estimate). *Under the same hypotheses of Theorem 28, let $\gamma(t) = \exp_x(td_x\phi)$, with $t \in [0, 1]$, be the unique minimizing curve joining x with y , which does not contain abnormal segments. Then, the linear maps*

$$d_x T_t : T_x M \rightarrow T_{\gamma(t)} M, \quad d_x T_t := \pi_* \circ e_*^{t\bar{H}} \circ d_x^2 \phi,$$

satisfy the following estimate, for all fixed $s \in (0, 1]$:

$$(14) \quad \det(d_x T_t)^{1/n} \geq \left(\frac{\det N_s^V(t)}{\det N_s^V(0)} \right)^{1/n} + \left(\frac{\det N_0^V(t)}{\det N_0^V(s)} \right)^{1/n} \det(d_x T_s)^{1/n}, \quad \forall t \in [0, s],$$

where the determinant is computed with respect to some smooth moving frame along γ , and the matrices $N_s^V(t)$ are defined as in Section 3.3, with respect to some Darboux lift along the corresponding extremal $e^{t\bar{H}}(d_x\phi)$.

Both terms in the right hand side of (14) are non-negative for $t \in [0, s]$ and, for $t \in [0, s)$, the first one is positive. In particular $\det(d_x T_t) > 0$ for all $t \in [0, 1)$.

Remark 30. As a matter of fact, $\det(d_x T_1)$ can be zero. This is the case arising when one transports a measure $\mu_0 \in \mathcal{P}_c^{ac}(M)$ to a delta mass $\mu_1 = \delta_y \in \mathcal{P}_c(M)$.

More precisely, fix $y \in M$ with $x \notin \operatorname{Cut}(y)$. Then there exists a neighborhood \mathcal{O}_x of x separated from $\operatorname{Cut}(y)$, where $z \mapsto d_{SR}^2(z, y)/2$ is smooth. The assumptions of Theorem 29 are satisfied for $\phi(z) := -d_{SR}^2(z, y)/2$. In particular, for all $z \in \mathcal{O}_x$ we have that $\exp_z(d_z\phi) = \pi \circ e^{\bar{H}}(d_z\phi) = y$, and thus $d_x T_1 = 0$.

We first discuss the strategy of the proof of Theorems 28 and 29. It is well known that, if (13) holds and ϕ is differentiable at x , there exists a unique minimizing curve joining x with y , which is the normal geodesic $\gamma(t) = \exp_x(td_x\phi)$, $t \in [0, 1]$, see e.g. [Rif14, Lemma 2.15]. By Theorem 16, there are no conjugate points along γ , except possibly the pair $\gamma(0)$ and $\gamma(1)$. Thanks to this observation, we first prove that (14)

holds for all $s < 1$. Then, we prove that if $\gamma(1)$ is conjugate to $\gamma(0)$, the right hand side of (14) tends to $+\infty$ for $s \uparrow 1$ and any fixed $t > 0$, hence $\det(d_x T_t)^{1/n} = +\infty$, leading to a contradiction. This implies that $\gamma(1)$ is not conjugate to $\gamma(0)$, yields the validity of (14) for all $s \in (0, 1]$, and proves $y \notin \text{Cut}(x)$.

Proof of Theorems 28 and 29. Let $\lambda(t) := e^{t\vec{H}}(d_x \phi)$, and $\gamma(t) = \pi(\lambda(t))$ the corresponding minimizing geodesic, with $t \in [0, 1]$. Let $E_1(t), \dots, E_n(t), F_1(t), \dots, F_n(t)$ be a Darboux lift along $\lambda(t)$ of a smooth moving frame $X_1(t), \dots, X_n(t)$ along $\gamma(t)$, that is satisfying

$$\sigma(E_i, F_j) - \delta_{ij} = \sigma(E_i, E_j) = \sigma(F_i, F_j) = 0, \quad \forall i, j = 1, \dots, n,$$

with $X_i(t) = \pi_* F_i(t)$ and $\pi_* E_i(t) = 0$ for $i = 1, \dots, n$ and $t \in [0, 1]$.

Since ϕ is twice differentiable at x , the family of Lagrangian subspaces $e_*^{t\vec{H}} \circ d_x^2 \phi(T_x M) \subset T_{\lambda(t)}(T^* M)$ is well defined for all $t \in [0, 1]$, and is associated via the given Darboux frame to the Jacobi matrix

$$\mathbf{J}(t) = \begin{pmatrix} M(t) \\ N(t) \end{pmatrix}, \quad \text{such that} \quad e_*^{t\vec{H}} \circ d_x^2 \phi(X(0)) = E(t) \cdot M(t) + F(t) \cdot N(t).$$

In particular, $d_x T_t(X(0)) = X(t) \cdot N(t)$. Let now $s \in (0, 1)$, and consider the Jacobi matrices \mathbf{J}_0^V and \mathbf{J}_s^V of Section 3.2. Since $\gamma(0)$ is not conjugate to $\gamma(s)$, we have

$$(15) \quad e_*^{s\vec{H}} \mathcal{V}_{\lambda(0)} \cap \mathcal{V}_{\lambda(s)} = \{0\}, \quad \forall s \in (0, 1).$$

Equivalently, $N_s^V(0)$ and $N_0^V(s)$ are invertible. One can verify that

$$(16) \quad \mathbf{J}(t) = \mathbf{J}_s^V(t) N_s^V(0)^{-1} N(0) + \mathbf{J}_0^V(t) N_0^V(s)^{-1} N(s), \quad t \in [0, 1], \quad s \in (0, 1).$$

In fact, Lemma 24 implies that a Jacobi matrix is uniquely specified by its horizontal component $N(0)$ and $N(s)$, from which (16) follows. By construction $N(0) = \mathbb{1}$, and the horizontal component of (16) reads

$$(17) \quad N(t) = N_s^V(t) N_s^V(0)^{-1} + N_0^V(t) N_0^V(s)^{-1} N(s), \quad t \in [0, 1], \quad s \in (0, 1).$$

The next crucial lemma is a consequence of two facts: the non-negativity of the Hamiltonian, and assumption (13). We postpone its proof to Appendix B.

Lemma 31 (Positivity). *Under the assumptions of Theorem 28, there exists a smooth family of $n \times n$ matrices $K(t) = N_0^V(t)^{-1}$, defined for $t \in (0, 1)$, such that, for all $s \in (0, 1)$, we have*

- (a) $\det K(t) > 0$ for all $t \in (0, 1)$,
- (b) $K(t) N_s^V(t) N_s^V(0)^{-1} \geq 0$, for all $t \in (0, s]$,
- (c) $K(t) N_0^V(t) N_0^V(s)^{-1} N(s) \geq 0$, for all $t \in (0, 1)$.

If, furthermore, $\gamma(1)$ is not conjugate to $\gamma(0)$, the above properties hold for all $s \in (0, 1]$ and $t \in (0, 1]$.

Minkowski determinant theorem [MM92, 4.1.8] states that the function $A \mapsto (\det A)^{1/n}$ is concave on the set of $n \times n$ non-negative symmetric matrices. Thus, by multiplying from the left (17) by the matrix $K(t)$ of Lemma 31, we obtain

$$(18) \quad \det(d_x T_t)^{1/n} \geq \left(\frac{\det N_s^V(t)}{\det N_s^V(0)} \right)^{1/n} + \left(\frac{\det N_0^V(t)}{\det N_0^V(s)} \right)^{1/n} \det(d_x T_s)^{1/n}, \quad t \in [0, s].$$

Notice that we do not use Lemma 31 to prove (18) for $t = 0$, but in this case the inequality holds since $d_x T_0 = \text{id}|_{T_x M}$ and $N_0^V(0) = 0$. Hence, we obtain (18) for all $t \in [0, s]$ and $s \in (0, 1)$ and, if $\gamma(0)$ is not conjugate with $\gamma(1)$, also for $s = 1$. We claim that, under the assumptions of Theorem 28, the latter case never occurs.

By contradiction, assume that $\gamma(1)$ is conjugate to $\gamma(0)$. As we already remarked, $N_0^V(s)$ and $N_s^V(0)$ are non-degenerate for $s \in (0, 1)$, but now $\det N_0^V(1) = \det N_1^V(0) = 0$. We claim that, for fixed $t \in (0, 1)$, the right hand side of (18) tends to $+\infty$ for $s \uparrow 1$. To prove this claim, notice that both terms in the right hand side of (18) are non-negative thanks to Lemma 31, and therefore

$$(19) \quad \det(d_x T_t)^{1/n} \geq \left(\frac{\det N_s^V(t)}{\det N_s^V(0)} \right)^{1/n} \geq 0, \quad t \in [0, s].$$

By Theorem 16, $\gamma(t)$ is not conjugate to $\gamma(1)$ for any fixed $t \in (0, 1)$. Hence $N_1^V(t)$ is not degenerate. On the other hand $\gamma(0)$ and $\gamma(1)$ are conjugate by our assumption, and $N_1^V(0)$ is degenerate. Taking the limit for $s \uparrow 1$, and since the left hand side of (19) does not depend on s , we obtain $\det(d_x T_t)^{1/n} = +\infty$, leading to a contradiction. Thus $\gamma(1)$ cannot be conjugate to $\gamma(0)$.

We have so far proved that there is a unique minimizing geodesic joining x with y , which is not abnormal, and y is not conjugate to x . This means that $y \notin \text{Cut}(x)$, and concludes the proof of Theorem 28. Moreover, (15)-(17) hold for all $s \in (0, 1]$ and $t \in [0, 1]$. Therefore we can apply Lemma 31 also for $s = 1$, which completes the proof of (14) for all $s \in (0, 1]$ and $t \in [0, s]$.

We already proved that both terms in the right hand side of (14) are non-negative for $t \in [0, s]$ (actually, the second one is non-negative for all $t \in [0, 1]$, by part (c) of Lemma 31). Now that we also proved that $\gamma(t)$ is not conjugate to $\gamma(s)$ for all possible $0 < |s - t| \leq 1$, we obtain that the first term cannot be zero except for $t = s$, and hence it is strictly positive for all $t \in [0, s]$. \square

4.1. Failure of semiconvexity at the cut locus. For definitions and alternative characterizations of locally semiconvex and semiconcave function see [CS04]. Here we only need the following notions.

We say that a continuous function $f : M \rightarrow \mathbb{R}$ fails to be semiconvex at $x \in M$ if, in any set of local coordinates around x , we have

$$(20) \quad \inf_{0 < |v| < 1} \frac{f(x+v) + f(x-v) - 2f(x)}{|v|^2} = -\infty.$$

Similarly, we say that f fails to be semiconcave at $x \in M$ if

$$\sup_{0 < |v| < 1} \frac{f(x+v) + f(x-v) - 2f(x)}{|v|^2} = +\infty.$$

Denote by $d_y^2 : M \rightarrow \mathbb{R}$ the sub-Riemannian squared distance from $y \in M$, that is

$$d_y^2(z) := d_{SR}^2(z, y), \quad \forall z \in M.$$

Corollary 32. *Let (\mathcal{D}, g) be a sub-Riemannian structure on M . Let $y \neq x$. Assume that there exists a neighborhood Ω of x such that all minimizing geodesics joining y with points of Ω do not contain abnormal segments. Then $x \in \text{Cut}(y)$ if and only if the function d_y^2 fails to be semiconvex at x .*

Remark 33. For ideal structures, one can take $\Omega = M \setminus \{y\}$, and Corollary 32 reduces to Theorem 6 discussed in Section 1.

Proof. We prove the contrapositive of the above statement. First, if $x \notin \text{Cut}(y)$ then $f(z) := d_{SR}^2(z, y)$ is smooth in a neighborhood of x by Theorem 18, and hence the infimum in (20) is finite.

To prove the opposite implication, observe that that by [CR08, Thm. 1, Thm. 5] f is locally semiconcave in a neighborhood of x (for this property it suffices that no minimizing geodesic joining y with points of Ω is abnormal). By standard properties

of semiconcave functions (see [CR08, Prop. 3.3.1] or [Can05, Prop. 2.3(a)]), there exist local charts around x , and $p \in \mathbb{R}^n$, $C \in \mathbb{R}$ such that

$$(21) \quad f(x+v) - f(x) \leq p \cdot v + C|v|^2, \quad \forall |v| < 1.$$

Hence, assume that the infimum in (20) is finite, that is there exists $K \in \mathbb{R}$ such that, in some local charts around x , we have

$$(22) \quad f(x+v) + f(x-v) - 2f(x) \geq K|v|^2, \quad \forall |v| < 1.$$

Equations (21)–(22) yield that there exists a function $\phi : M \rightarrow \mathbb{R}$, twice differentiable at x , such that $f(z) \geq -\phi(z)$ for all $z \in M$, and $f(x) = -\phi(x)$. Our assumptions imply that the unique geodesic joining x with y does not contain abnormal segments. Hence, by Theorem 28, $y \notin \text{Cut}(x)$. \square

Remark 34. For later purposes, we observe the following general fact. For any sub-Riemannian structure, the infimum in (20) for $f = d_y^2$ is always finite for $x = y$. On the other hand (when the structure is not Riemannian) d_y^2 fails to be semiconcave at y . The first statement follows trivially observing that $d_y^2(y) = 0$ and $d_y^2(z) \geq 0$. The second statement is a classical consequence of the Ball-Box theorem [Bel96, Jea14].

4.2. Regularity versus optimality: the non-ideal case. The characterization of Corollary 32 is false in the non-ideal case. In fact, consider the standard left-invariant sub-Riemannian structure on the product $\mathbb{H} \times \mathbb{R}$ of the three-dimensional Heisenberg group and the Euclidean line (which has a rank 3 distribution). Denoting points $x = (q, s) \in \mathbb{H} \times \mathbb{R}$, one has

$$(23) \quad d_{SR}^2((q, s), (q', s')) = d_{\mathbb{H}}^2(q, q') + |s - s'|^2.$$

Without loss of generality, fix $(q', s') = (0, 0)$. The set of points reached by abnormal minimizers from the origin is $\text{Abn}(0) = \{(0, s) \mid s \in \mathbb{R}\}$. Here, the squared distance $d_0^2(q, s) := d_{SR}^2((q, s), (0, 0))$ is not smooth, but the infimum in (20) is finite. In fact, the loss of smoothness is due to the failure of semiconcavity. These two properties follows from (23) and Remark 34.

Notice that abnormal geodesics joining the origin with points in $\text{Abn}(0)$ are straight lines $t \mapsto (0, t)$, which are optimal for all times. Hence it seems likely that the failure of semiconvexity is related with the loss of optimality, while the failure of semiconcavity is related with the presence of abnormal minimizers. In the conclusion of this section, we formalize this latter statement.

4.2.1. On the definition of cut locus. In this paper, following [FR10], we define the cut locus $\text{Cut}(x)$ as the set of points y where the squared distance from x is not smooth. Classically, the cut locus is related with the loss of optimality of geodesics. Hence, one could consider the set of points where geodesics from x lose optimality. To give a precise definition, taking in account the presence of possibly branching abnormal geodesics, we proceed as follows. First, we say that a geodesic $\gamma : [0, T] \rightarrow M$ (horizontal curve which locally minimizes the energy between its endpoints) is maximal if it is not the restriction of a geodesic defined on a larger interval $[0, T']$. The cut time of a maximal geodesic is

$$t_{\text{cut}}(\gamma) := \sup\{t > 0 \mid \gamma|_{[0,t]} \text{ is a minimizing geodesic}\}.$$

Assuming (M, d_{SR}) to be complete, we define the *optimal cut locus* of $x \in M$ as

$$\text{CutOpt}(x) := \{\gamma(t_{\text{cut}}(\gamma)) \mid \gamma \text{ is a maximal geodesic starting at } x\}.$$

In the ideal case, which includes the Riemannian case, it is well known that

$$(24) \quad \text{CutOpt}(x) = \text{Cut}(x) \setminus \{x\}.$$

For a general, complete sub-Riemannian structure (\mathcal{D}, g) on M , let $x \in M$ and define the following sets:

$$\text{SC}^-(x) := \{y \in M \mid d_x^2 \text{ fails to be semiconcave at } y\},$$

$$\text{SC}^+(x) := \{y \in M \mid d_x^2 \text{ fails to be semiconvex at } y\},$$

$$\text{Abn}(x) := \{y \in M \mid \exists \text{ abnormal minimizing geodesic joining } x \text{ to } y\}.$$

Open questions. Are the following equalities true?

$$(25) \quad \text{CutOpt}(x) = \text{SC}^+(x),$$

$$(26) \quad \text{Abn}(x) = \text{SC}^-(x).$$

In the ideal case, (24) holds and $\text{Abn}(x) = \{x\}$. Hence (25) follows from Corollary 32, (26) follows from the results of [CR08] (where the general inclusion $\text{Abn}(x) \supseteq \text{SC}^-(x)$ is proved). In particular, the following identities are true in the ideal case:

$$(27) \quad \text{Cut}(x) = \text{CutOpt}(x) \cup \text{Abn}(x), \quad \text{Cut}(x) = \text{SC}^+(x) \cup \text{SC}^-(x).$$

We do not know whether (27) remain true in general. However, if (25) and (26) are true, then the two statements in (27) are equivalent. Notice that the first union in (27), in general, is not disjoint [RS16].

We mention that, in [MM17], the authors proved the inclusion $\text{CutOpt}(x) \subseteq \text{SC}^+(x)$ for the free Carnot group of step 2 and rank 3.

5. OPTIMAL TRANSPORT AND INTERPOLATION INEQUALITIES

In this section we apply the main Jacobian inequality to the Monge problem. We briefly review the concepts of optimal transport in sub-Riemannian geometry that we need, following [Rif14, Ch. 3]. See also [Vil09, Ch. 5].

5.1. Sub-Riemannian optimal transport. In this paper, the reference (outer) measure \mathfrak{m} is always assumed to be smooth. A measure \mathfrak{m} on a smooth manifold M is *smooth* if it is locally defined by a positive definite tensor density. In particular \mathfrak{m} is Borel regular and locally finite, hence Radon [EG15].

The space of compactly supported probability measures on M is denoted by $\mathcal{P}_c(M)$, while $\mathcal{P}_c^{ac}(M)$ is the subset of the absolutely continuous ones w.r.t. \mathfrak{m} . We denote by $\pi_i : M \times M \rightarrow M$, for $i = 1, 2$, the canonical projection on the i -th factor. Furthermore, D denotes the diagonal in $M \times M$ that is

$$D := \{(x, y) \in M \times M \mid x = y\}.$$

For given probability measures μ_0, μ_1 on M , we look for *transport maps* between μ_0 and μ_1 , that is measurable maps $T : M \rightarrow M$, such that $T_{\#}\mu_0 = \mu_1$. Furthermore, for a given cost function $c : M \times M \rightarrow \mathbb{R}$, we want to minimize the transport cost among all transport maps, that is solve the Monge problem:

$$(28) \quad C_{\mathcal{M}}(\mu_0, \mu_1) = \min_{T_{\#}\mu_0 = \mu_1} \int_M c(x, T(x)) d\mathfrak{m}(x).$$

Optimal transport maps, i.e. solutions of (28), may not always exist. For this reason Kantorovich proposed, instead, to consider optimal transport *plans*, that is probability measures $\alpha \in \mathcal{P}(M \times M)$ whose marginals satisfy $(\pi_i)_{\#}\alpha = \mu_i$. Letting $\Pi(\mu_0, \mu_1)$ be the set of all such measures, one defines the following Kantorovich relaxation of Monge problem:

$$(29) \quad C_{\mathcal{K}}(\mu_0, \mu_1) = \min_{\alpha \in \Pi(\mu_0, \mu_1)} \int_{M \times M} c(x, y) d\alpha(x, y).$$

The advantage of (29) is that, at least when c is continuous and the supports μ_0, μ_1 are compact, it always admits a solution. Furthermore, for a given transport map $T :$

$M \rightarrow M$, we have that $(\text{id} \times T)_\# \mu_0$ is an optimal transport plan, hence $C_{\mathcal{K}}(\mu_0, \mu_1) \leq C_{\mathcal{M}}(\mu_0, \mu_1)$. For this reason, a technique to prove existence and uniqueness of a solution for the primal problem (28) is to show that all solutions of the dual problem (29) are concentrated on the graph of some function $T : M \rightarrow M$. We now introduce some standard terminology related with the transportation problem.

Definition 35. Let $\psi : M \rightarrow \mathbb{R} \cup \{+\infty\}$. Its c -transform is the function $\psi^c : M \rightarrow \mathbb{R} \cup \{-\infty\}$, defined by

$$\psi^c(y) = \inf_{x \in M} \{\psi(x) + c(x, y)\}, \quad \forall y \in M.$$

A function $\psi : M \rightarrow \mathbb{R} \cup \{+\infty\}$ is c -convex if it is not identically $+\infty$ and

$$\psi(x) = \sup_{y \in M} \{\psi^c(y) - c(x, y)\}, \quad \forall x \in M.$$

If ψ is c -convex, we define the c -subdifferential of ψ at x as

$$\partial_c \psi(x) := \{y \in M \mid \psi(x) + \psi^c(y) = c(x, y)\},$$

and the c -subdifferential of ψ as

$$\partial_c \psi := \{(x, y) \in M \times M \mid y \in \partial_c \psi(x)\}.$$

Remark 36. If $M = \mathbb{R}^n$ and $c(x, y) = |y - x|$, then c -convex functions are exactly 1-Lipschitz functions. If $c(x, y) = |y - x|^2/2$, then c -convex functions are the functions $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ such that $x \mapsto \psi(x) + \frac{1}{2}|x|^2$ is convex, that is ψ is semiconvex.

The following well known result is the cornerstone of most of the results of existence and uniqueness of optimal transport maps.

Theorem 37. *Let $\mu_0, \mu_1 \in \mathcal{P}_c(M)$, and c be continuous. Then there exists a c -convex function $\psi : M \rightarrow \mathbb{R}$ such that any transport plan $\alpha \in \Pi(\mu_0, \mu_1)$ is optimal if and only if $\alpha(\partial_c \psi) = 1$ (that is, is concentrated on the c -subdifferential of ψ). Moreover, we can assume that*

$$\begin{aligned} \psi(x) &= \max\{\psi^c(y) - c(x, y) \mid y \in \text{supp}(\mu_1)\}, & \forall x \in M, \\ \psi^c(y) &= \min\{\psi(x) + c(x, y) \mid x \in \text{supp}(\mu_0)\}, & \forall y \in M. \end{aligned}$$

Moreover, both ψ and ψ^c are continuous.

The strategy to prove existence and uniqueness of an optimal transport map is to show that, outside of a \mathfrak{m} -negligible set $N \subset M$, then $\partial_c \psi(x)$ is a singleton, that is $\partial_c \psi$ is the graph in $M \times M \setminus N \times M$ of a given map $T : M \rightarrow M$. The main difficulty with respect to the Riemannian case is that the squared sub-Riemannian distance is not locally Lipschitz on the diagonal.

Definition 38. For a c -convex function $\psi : M \rightarrow \mathbb{R}$, we define the following sets

$$\begin{aligned} \mathcal{M}^\psi &:= \{x \in M \mid x \notin \partial_c \psi(x)\}, & (\text{moving set}) \\ \mathcal{S}^\psi &:= \{x \in M \mid x \in \partial_c \psi(x)\}, & (\text{static set}) \end{aligned}$$

We take from [FR10, Thm. 3.2] the main result about well-posedness of the Monge problem (with quadratic cost) on ideal sub-Riemannian structures.

Theorem 39 (Well posedness of Monge problem). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on M , $\mu_0 \in \mathcal{P}_c^{ac}(M)$, and $\mu_1 \in \mathcal{P}_c(M)$. Then, the c -convex function $\psi : M \rightarrow \mathbb{R}$ of Theorem 37 satisfies:*

- (i) \mathcal{M}^ψ is open and ψ is locally semiconvex in a neighborhood of $\mathcal{M}^\psi \cap \text{supp}(\mu_0)$. In particular ψ is twice differentiable μ_0 -a.e. in \mathcal{M}^ψ ;
- (ii) For μ_0 -a.e. $x \in \mathcal{S}^\psi$, $\partial_c \psi(x) = \{x\}$.

Hence, there exists a unique transport map $T_1 : M \rightarrow M$ such $(T_1)_\# \mu_0 = \mu_1$, optimal with respect to the quadratic cost

$$c(x, y) = \frac{1}{2} d_{SR}^2(x, y), \quad \forall x, y \in M,$$

where, for all $t \in [0, 1]$, the map $T_t : M \rightarrow M$ is defined μ_0 -a.e. by

$$T_t(x) := \begin{cases} \exp_x(td_x\psi) & x \in \mathcal{M}^\psi, \\ x & x \in \mathcal{S}^\psi, \end{cases} \quad t \in [0, 1].$$

Moreover, for μ_0 -a.e. $x \in M$, there exists a unique minimizing geodesic between x and $T_1(x)$ given by $T_t(x)$.

For what concerns the important issue of regularity, in [FR10, Thm. 3.7], Figalli and Rifford obtained a formula for the differential of the transport map akin the classical one of [CEMS01]. To this purpose, they ask additional assumptions on the sub-Riemannian cut-locus. More precisely, they demand that if $x \in \text{Cut}(y)$, then there exist at least two distinct minimizing geodesics joining x with y . Thanks to Corollary 32, the aforementioned result holds with no assumption on the cut locus. It is interesting to notice that, by employing the second differential of maps as defined in Section 3, the proof is an immediate consequence of Alexandrov's second differentiability theorem, which states that locally semiconcave functions are two times differentiable almost everywhere (see [FR10, Appendix A.2]). Finally, thanks to this differentiability result and the estimate of Theorem 29, we are able to give an independent and alternative proof of the absolute continuity of the displacement interpolation $\mu_t := (T_t)_\# \mu_0$ (see below). The differentiability result is most easily expressed in terms of approximate differential (see e.g. [AGS08, Sec. 5.5]).

Definition 40 (Approximate differential). We say that $f : M \rightarrow \mathbb{R}$ has an approximate differential at $x \in M$ if there exists a function $g : M \rightarrow \mathbb{R}$ differentiable at x such that the set $\{f = g\}$ has density 1 at x with respect to \mathfrak{m} .² In this case, the approximate value of f at x is defined as $\tilde{f}(x) = g(x)$, and the approximate differential of f at x is defined as $\tilde{d}_x f := d_x g : T_x M \rightarrow T_{g(x)} M$.

Theorem 41 (Regularity of optimal transport). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on M , $\mu_0 \in \mathcal{P}_c^{ac}(M)$, and $\mu_1 \in \mathcal{P}_c(M)$. The map T_t is differentiable μ_0 -a.e. on $\mathcal{M}^\psi \cap \text{supp}(\mu_0)$, and it is approximately differentiable μ_0 -a.e. Its approximate differential is given by*

$$(30) \quad \tilde{d}_x T_t = \begin{cases} \pi_* \circ e_*^{t\tilde{H}} \circ d_x^2 \psi & \mu_0 - \text{a.e. } x \in \mathcal{M}^\psi \cap \text{supp}(\mu_0), \\ \text{id}|_{T_x M} & \mu_0 - \text{a.e. } x \in \mathcal{S}^\psi \cap \text{supp}(\mu_0). \end{cases}$$

Finally, $\det(\tilde{d}_x T_t) > 0$ for all $t \in [0, 1]$ and μ_0 -a.e. $x \in M$.

Remark 42. If \mathcal{S}^ψ is empty, which is the case for example when $\text{supp}(\mu_0) \cap \text{supp}(\mu_1)$ is empty, then T_t is differentiable, and not only approximately differentiable, μ_0 -a.e.

Proof. The closed set \mathcal{S}^ψ is measurable, $\mu_0 \ll \mathfrak{m}$, and \mathfrak{m} is smooth. Then by applying Lebesgue density theorem we obtain that T_t is approximately differentiable μ_0 -a.e. on $\mathcal{S}^\psi \cap \text{supp}(\mu_0)$, with approximate differential given by the identity map.

Furthermore, since local semiconvexity is invariant by diffeomorphisms, and since \mathfrak{m} is smooth, Alexandrov theorem in \mathbb{R}^n (see, e.g. [FR10, Thm. A.5]) yields that

²We compute the density using Euclidean balls in local charts around x . Since \mathfrak{m} is smooth, it has positive density with respect to the Lebesgue measure in charts, hence the concept of density does not depend on the choice of charts. In particular, Lebesgue density theorem holds.

ψ is twice differentiable \mathfrak{m} -a.e. on M , and for such x , its second differential can be computed according to Definition 19 and Remark 23. Hence, since $\mu_0 \ll \mathfrak{m}$, then

$$x \mapsto T_t(x) = \exp_x(td_x\psi) = \pi \circ e^{t\tilde{H}} \circ d_x\psi$$

is differentiable for μ_0 -a.e. $x \in M$, and its differential is computed as in (30).

Clearly, $\det(d_x T_t) = 1$ for $x \in \mathcal{S}^\psi$, hence we focus on \mathcal{M}^ψ . By construction, $y = T_1(x)$ if and only if $y \in \partial_c\psi(x)$ which is a singleton for μ_0 -a.e. $x \in M$ (see the proof of [FR10, Thm. 3.2]). Thus, by definition of c -subdifferential, we have

$$y \in \partial_c\psi(x) \iff \psi(z) + c(z, y) - \psi(x) - c(x, y) \geq 0, \quad \forall z \in M.$$

In particular, one can apply Theorem 28 to the function $\phi(z) := \psi(z) - \psi(x) - c(x, T(x))$, at any point x where ψ is twice differentiable, i.e. μ_0 -almost everywhere on \mathcal{M}^ψ . For all such points, $\det(d_x T_t) > 0$ for all $t \in [0, 1)$ by Theorem 29. \square

As a consequence of Theorem 41, we obtain an independent proof of [FR10, Thm. 3.5] about the absolute continuity of the Wasserstein geodesic between μ_0 and μ_1 .

Theorem 43 (Absolute continuity of Wasserstein geodesic). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on M , $\mu_0 \in \mathcal{P}_c^{ac}(M)$, and $\mu_1 \in \mathcal{P}_c(M)$. Then there exists a unique Wasserstein geodesic joining μ_0 with μ_1 , given by $\mu_t := (T_t)_\# \mu_0$, for $t \in [0, 1]$. Moreover, $\mu_t \in \mathcal{P}_c^{ac}(M)$ for all $t \in [0, 1)$.*

Proof. The existence and uniqueness part is standard and is done as in [FR10, Sec. 6.3, first paragraph]. Since \mathfrak{m} is a smooth positive measure, the absolute continuity statement is equivalent to the absolute continuity of $\mu_t = (T_t)_\# \mu_0$ with respect to Lebesgue measure \mathcal{L}^d in all local coordinate charts, where $\mu_0 = \rho \mathcal{L}^d$, for some $\rho \in L^1(\mathbb{R}^d)$. Thanks to Theorem 41, in these charts $T_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is approximately differentiable $\rho \mathcal{L}^d$ -almost everywhere. Hence, the statement follows from the next lemma, which is a relaxed version of [AGS08, Lemma 5.5.3] (weaker in the sense that we do not require injectivity of f , at least for the first part of the statement). Its proof for completeness is in Appendix C.

Lemma 44. *Let $\rho \in L^1(\mathbb{R}^d)$ be a non-negative function. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a measurable function and let Σ_f be the set where it is approximately differentiable. Assume there exists a Borel set $\Sigma \subseteq \Sigma_f$ such that the difference $\{\rho > 0\} \setminus \Sigma$ is \mathcal{L}^d -negligible. Then $f_\#(\rho \mathcal{L}^d) \ll \mathcal{L}^d$ if and only if $|\det(\tilde{d}_x f)| > 0$ for \mathcal{L}^d -a.e. $x \in \Sigma$.*

In this case, letting $f_\#(\rho \mathcal{L}^d) = \rho_f \mathcal{L}^d$, we have

$$\rho_f(y) = \sum_{x \in \tilde{f}^{-1}(y) \cap \Sigma} \frac{\rho(x)}{|\det(\tilde{d}_x f)|}, \quad y \in \mathbb{R}^n,$$

with the convention that the r.h.s. is zero if $y \notin \tilde{f}(\Sigma)$. In particular, if we further assume that $\tilde{f}|_\Sigma$ is injective, then we have

$$\rho_f(\tilde{f}(x)) = \frac{\rho(x)}{|\det(\tilde{d}_x f)|}, \quad \forall x \in \Sigma.$$

Notice that, in order to prove Theorem 43, we need only the first implication of Lemma 44, that is if $|\det(\tilde{d}_x f)| > 0$ for \mathcal{L}^d -a.e. $x \in \Sigma$, then $f_\#(\rho \mathcal{L}^d) \ll \mathcal{L}^d$. \square

Thanks to the above result, for all $t \in [0, 1)$, and also $t = 1$ if $\mu_1 \in \mathcal{P}_c^{ac}(M)$, let $\rho_t := d\mu_t/d\mathfrak{m}$. Then we have the following Jacobian identity.

Theorem 45 (Jacobian identity). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on M , $\mu_0 \in \mathcal{P}_c^{ac}(M)$, and $\mu_1 \in \mathcal{P}_c(M)$. For all $t \in [0, 1)$, and also $t = 1$ if*

$\mu_1 \in \mathcal{P}_c^{ac}(M)$, let $\mu_t = \rho_t \mathbf{m}$. Then,

$$(31) \quad \frac{\rho_0(x)}{\rho_t(T_t(x))} = \begin{cases} \det(d_x T_t) \frac{\mathbf{m}(X_1(t), \dots, X_n(t))}{\mathbf{m}(X_1(0), \dots, X_n(0))} > 0 & \mu_0 - \text{a.e. } x \in \mathcal{M}^\psi, \\ 1 & \mu_0 - \text{a.e. } x \in \mathcal{S}^\psi, \end{cases}$$

where $X_1(t), \dots, X_n(t)$ is some smooth moving frame along the geodesic $\gamma(t) = T_t(x)$, and the determinant of the linear map $d_x T_t : T_x M \rightarrow T_{T_t(x)} M$ is computed with respect to the given frame, that is

$$d_x T_t(X_i(0)) = \sum_{j=1}^n N_{ji}(t) X_j(t), \quad \det(d_x T_t) := \det N(t).$$

Remark 46. In the Riemannian case, when $\mathbf{m} = \mathbf{m}_g$ is the Riemannian volume, one can compute the determinant in (31) with respect to orthonormal frames, eliminating any dependence on the frame and obtaining the classical Monge-Ampère equation.

Proof. By Theorem 43, $\mu_t = (T_t)_\# \mu_0 \ll \mu_0$, hence one can repeat the arguments in the last paragraph of [FR10, Sec. 6.4]. Since $\mu_t \in \mathcal{P}_c^{ac}(M)$, there are optimal transport maps T_t, S_t such that $(T_t)_\# \mu_0 = \mu_t$ and $(S_t)_\# \mu_t = \mu_0$. By uniqueness of the transport map, we obtain that T_t is μ_0 -a.e. injective. Hence we can use the second part of Lemma 44, and in particular for μ_0 -a.e. $x \in \mathcal{M}^\psi$ we have

$$\frac{\rho_0(x)}{\rho_t(T_t(x))} = \det(d_x T_t) \frac{\mathbf{m}(X_1(t), \dots, X_n(t))}{\mathbf{m}(X_1(0), \dots, X_n(0))}.$$

The extra term in the right hand side is due to the fact that we are not computing $d_x T_t$ in a set of local coordinates, but with respect to a smooth frame. \square

5.2. Distortion coefficients and interpolation inequalities. Let (\mathcal{D}, g) be a sub-Riemannian structure on M , not necessarily ideal, and fix a smooth reference measure \mathbf{m} .

Definition 47. Let $A, B \subset M$ be measurable sets, and $t \in [0, 1]$. The set $Z_t(A, B)$ of t -intermediate points is the set of all points $\gamma(t)$, where $\gamma : [0, 1] \rightarrow M$ is a minimizing geodesic such that $\gamma(0) \in A$ and $\gamma(1) \in B$.

Compare the next definition with the one in [Vil09, Def. 14.17, Prop 14.18] and [BKS16, Pag. 12]. The main difference is that here we do not extract a factor $1/t^n$, since the topological dimension does not describe the correct asymptotic behavior in the sub-Riemannian case.

Definition 48 (Distortion coefficient). Let $x, y \in M$ and $t \in [0, 1]$. The *distortion coefficient* from x to y at time $t \in [0, 1]$ is

$$\beta_t(x, y) := \limsup_{r \downarrow 0} \frac{\mathbf{m}(Z_t(x, \mathcal{B}_r(y)))}{\mathbf{m}(\mathcal{B}_r(y))}.$$

Notice that $\beta_0(x, y) = 0$ and $\beta_1(x, y) = 1$.

The next lemma provides a general bound for the distortion coefficient.

Lemma 49 (On-diagonal distortion bound). *Let \mathbf{m} be a smooth measure on M . Then, for any $x \in M$, there exists $Q(x) \geq \dim(M)$ such that*

$$\beta_t(x, x) \leq t^{Q(x)}, \quad \forall t \in [0, 1].$$

Proof. The proof is based on privileged coordinates and dilations in sub-Riemannian geometry, see [Bel96] for reference. Fix $x \in M$, and let z denote a system of privileged coordinates on a neighborhood \mathcal{U} of x (which we identify from now on with a relatively compact open set of \mathbb{R}^n , where x corresponds to the origin). We

claim that there exists $Q(x) \geq \dim(M)$ and a constant $C(x) > 0$ such that, for sufficiently small ε , we have

$$\mathfrak{m}(\mathcal{B}_\varepsilon(x)) = \varepsilon^{Q(x)} C(x) (1 + O(\varepsilon)).$$

This claim, together with the observation that $Z_t(x, \mathcal{B}_r(x)) \subseteq \mathcal{B}_{tr}(x)$, implies the statement. In order to prove the claim, in the given set of privileged coordinates, let $\mathfrak{m} = \mathfrak{m}(z) d\mathcal{L}(z)$ for some smooth, strictly positive function \mathfrak{m} . Assume ε sufficiently small such that $\mathcal{B}_\varepsilon(x) \subset \mathcal{U}$. Let δ_ε be the non-homogeneous dilation defined by the given system of privileged coordinates at x , with non-holonomic weights $w_i(x)$, for $i = 1, \dots, n$, that is $\delta_\varepsilon(z_1, \dots, z_n) = (\varepsilon^{w_1} z_1, \dots, \varepsilon^{w_n} z_n)$. Notice that, in coordinates, $\det(d_z \delta_\varepsilon) = \varepsilon^{Q(x)}$, where $Q(x) = \sum_{i=1}^n w_i(x)$.

As a consequence of the classical distance estimates in [Bel96, Thm. 7.32]³, for all sufficiently small $\varepsilon > 0$ there exists $\alpha(\varepsilon) \downarrow 0$ such that

$$(32) \quad \hat{\mathcal{B}}_{(1-\alpha(\varepsilon))\varepsilon} \subseteq \mathcal{B}_\varepsilon(x) \subseteq \hat{\mathcal{B}}_{(1+\alpha(\varepsilon))\varepsilon},$$

where $\hat{\mathcal{B}}$ denotes the ball of the nilpotent structure, centered at the origin, in this set of privileged coordinates. By the homogeneity with respect to δ_ε , we have

$$\hat{\mathcal{B}}_{1-\alpha(\varepsilon)} \subseteq \delta_{1/\varepsilon}(\mathcal{B}_\varepsilon(x)) \subseteq \hat{\mathcal{B}}_{1+\alpha(\varepsilon)}.$$

The above relation, and the monotonicity of the Lebesgue measure as a function of the domain, imply that there exists a constant $B(x) = \mathcal{L}^n(\hat{\mathcal{B}}_1) > 0$, such that

$$\lim_{\varepsilon \rightarrow 0} \mathcal{L}^n(\delta_{1/\varepsilon}(\mathcal{B}_\varepsilon(x))) = B(x).$$

Hence, since δ_ε and \mathfrak{m} are smooth, we have

$$\begin{aligned} \mathfrak{m}(\mathcal{B}_\varepsilon) &= \int_{\mathcal{B}_\varepsilon} \mathfrak{m}(z) \mathcal{L}^n(dz) \\ &= \int_{\delta_{1/\varepsilon}(\mathcal{B}_\varepsilon)} \mathfrak{m}(\delta_\varepsilon(z)) \det(d_z \delta_\varepsilon) \mathcal{L}^n(dz) \\ &= \varepsilon^{Q(x)} \mathfrak{m}(0) \int_{\delta_{1/\varepsilon}(\mathcal{B}_\varepsilon)} (1 + \varepsilon R(\delta_\varepsilon(z))) \mathcal{L}^n(dz) \\ &= \varepsilon^{Q(x)} \mathfrak{m}(0) B(x) (1 + O(\varepsilon)), \end{aligned}$$

where $R(z)$ is a smooth remainder, and the remainder term $O(\varepsilon)$ possibly depends on x . This concludes the proof of the claim. \square

Lemma 50 (Computation of distortion coefficients). *Let $x, y \in M$, with $x \notin \text{Cut}(y)$. Let X_1, \dots, X_n be a smooth frame along the unique geodesic from y and x . Then, in terms of the Jacobi matrices defined in Section 3.3 (with respect to a Darboux lift of X_1, \dots, X_n), we have*

$$(33) \quad \begin{aligned} \beta_t(x, y) &= \frac{\det N_0^Y(t)}{\det N_0^Y(1)} \frac{\mathfrak{m}(X_1(t), \dots, X_n(t))}{\mathfrak{m}(X_1(1), \dots, X_n(1))}, & \forall t \in [0, 1], \\ \beta_{1-t}(y, x) &= \frac{\det N_1^Y(t)}{\det N_1^Y(1)} \frac{\mathfrak{m}(X_1(t), \dots, X_n(t))}{\mathfrak{m}(X_1(0), \dots, X_n(0))}, & \forall t \in [0, 1]. \end{aligned}$$

Moreover, $\beta_t(x, y) > 0$, for all $t \in (0, 1]$.

³The estimate in [Bel96, Thm. 7.32] contains a typo, as one can check by setting $p = q$ in [Bel96, Eq. 70]. The statement and its proof are however correct, provided that in the latter one correctly applies [Bel96, Prop. 7.29], yielding a term $M_p(q, q') := \max\{\hat{d}_p(p, q), \hat{d}_p(p, q')\}$ in place of $\hat{d}_p(p, q)$ in [Bel96, Eq. 71], and giving, in place of [Bel96, Eq. 70], the correct formula

$$-C_p M_p(q, q') d(q, q')^{1/r} \leq d(q, q') - \hat{d}_p(q, q') \leq C_p M_p(q, q') \hat{d}_p(q, q')^{1/r}, \quad q, q' \in \mathcal{B}_\varepsilon(p),$$

where in this notation p is the center of privileged coordinates. The correct estimate appears in the literature in [GJ15, Eq. 21]. Setting $q = p$ in the above estimate yields (32).

Proof of Lemma 50. We prove first (33). For $t = 0$ both sides are zero, hence let $t \in (0, 1]$. Let λ_0 be the initial covector of the unique minimizing geodesic such that $\exp_x(\lambda_0) = y$. Since $x \notin \text{Cut}(y)$, there exists an open neighborhood \mathcal{U} of y and $U \subset T_x^*M$ such that $\exp_x : U \rightarrow \mathcal{U}$ is a smooth diffeomorphism, and for all $\lambda' \in U$, the geodesic $t \mapsto \exp_x(t\lambda')$ is the unique minimizing geodesic joining x with $y' = \exp_x(\lambda')$, and y' is not conjugate with x along such a geodesic. Assuming r sufficiently small such that $\mathcal{B}_r(y) \subset \mathcal{U}$, let $A_r \subset U$ be the relatively compact set such that $\exp_x(A_r) = \mathcal{B}_r(y)$. By uniqueness of the minimizing geodesics, which do not contain conjugate points, the map \exp_x^t is a smooth diffeomorphism from A_r onto $Z_t(x, \mathcal{B}_r(y))$. In particular, we have

$$(34) \quad \beta_t(x, y) = \lim_{r \downarrow 0} \frac{\int_{A_r} \exp_x^{t*} \mathbf{m}}{\int_{A_r} \exp_x^{1*} \mathbf{m}} = \frac{(\exp_x^{t*} \mathbf{m})(\lambda_0)}{(\exp_x^{1*} \mathbf{m})(\lambda_0)},$$

where $\exp_x^t : T_x^*M \rightarrow M$ is the exponential map at time $t \in (0, 1]$, which is a shorthand for $\exp_x^t(\lambda) := \exp_x(t\lambda)$. The right hand side of (34) is the ratio of two smooth tensor densities computed at λ_0 . To compute it, we evaluate both factors on a n -tuple of independent vectors of T_x^*M . Thus, pick a Darboux frame $E_1(t), \dots, E_n(t), F_1(t), \dots, F_n(t) \in T_{\lambda(t)}(T^*M)$ such that $\pi_* E_i(t) = 0$ and $\pi_* F_i(t) = X_i(t)$ for all $t \in [0, 1]$, $i = 1, \dots, n$. Then,

$$(\exp_x^{t*} \mathbf{m})(E_1(0), \dots, E_n(0)) = \mathbf{m}(\pi_* \circ e_*^{t\tilde{H}} E_1(0), \dots, \pi_* \circ e_*^{t\tilde{H}} E_n(0)).$$

The n -tuple $\mathcal{J}_i(t) = e_*^{t\tilde{H}} E_i(0)$, $i = 1, \dots, n$, corresponds to the Jacobi matrix $\mathbf{J}_0^V(t)$, such that $M_0^V(0) = \mathbb{1}$ and $N_0^V(0) = \mathbb{0}$, defined in Section 3.3. Thus,

$$(\exp_x^{t*} \mathbf{m})(E_1(0), \dots, E_n(0)) = \det N_0^V(t) \mathbf{m}(X_1(t), \dots, X_n(t)).$$

By replacing the above formula in (34), we obtain $\beta_t(x, y)$. Since $\gamma(t)$ is not conjugate to $\gamma(0)$ for all $t \in (0, 1]$, we have $\beta_t(x, y) > 0$ on that interval.

The formula for $\beta_{1-t}(y, x)$ is deduced in a similar way and with some additional care, following the geodesic backwards starting from the final point. We sketch the proof for this case. Let $\gamma : [0, 1] \rightarrow M$ be the unique minimizing geodesic from x to y , with extremal $\lambda : [0, 1] \rightarrow T^*M$. Of course, the unique minimizing geodesic from y to x is $\tilde{\gamma}(t) = \gamma(1-t)$. The corresponding normal extremal is $\tilde{\lambda}(t) = -\lambda(1-t)$. Consider the inversion map $\iota : T^*M \rightarrow T^*M$, such that $\iota(\lambda) = -\lambda$. In particular if $E_i(t), F_i(t)$ are a Darboux frame along $\lambda(t)$, then $\tilde{E}_i(t) := -\iota_* E_i(1-t)$ and $\tilde{F}_i(t) := -\iota_* F_i(1-t)$ are a Darboux frame along $\tilde{\lambda}(t)$. Furthermore, the n -tuple

$$\tilde{\mathcal{J}}_i(t) = e_*^{(1-t)\tilde{H}} \tilde{E}_i(t) = e_*^{(t-1)\tilde{H}} E_i(1-t), \quad i = 1, \dots, n,$$

corresponds to the Jacobi matrix $\mathbf{J}_1^V(t) = \begin{pmatrix} M_1^V(t) \\ N_1^V(t) \end{pmatrix}$. A computation similar to the previous one yields

$$\begin{aligned} \beta_{1-t}(y, x) &= \frac{\mathbf{m}(\pi_* \circ e_*^{(1-t)\tilde{H}} \tilde{E}_1(0), \dots, \pi_* \circ e_*^{(1-t)\tilde{H}} \tilde{E}_n(0))}{\mathbf{m}(\pi_* \circ e_*^{(1-t)\tilde{H}} \tilde{E}_1(1), \dots, \pi_* \circ e_*^{(1-t)\tilde{H}} \tilde{E}_n(1))} \\ &= \frac{\mathbf{m}(\pi_* \circ e_*^{(t-1)\tilde{H}} E_1(1), \dots, \pi_* \circ e_*^{(t-1)\tilde{H}} E_n(1))}{\mathbf{m}(\pi_* \circ e_*^{(t-1)\tilde{H}} E_1(0), \dots, \pi_* \circ e_*^{(t-1)\tilde{H}} E_n(0))}, \quad \forall t \in [0, 1], \end{aligned}$$

concluding the proof. \square

Theorem 51 (Interpolation inequality). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on M , and $\mu_0, \mu_1 \in \mathcal{P}_c^{ac}(M)$. Let $\rho_s = d\mu_s/d\mathbf{m}$. For all $t \in [0, 1]$, it holds*

$$(35) \quad \frac{1}{\rho_t(T_t(x))^{1/n}} \geq \frac{\beta_{1-t}(T(x), x)^{1/n}}{\rho_0(x)^{1/n}} + \frac{\beta_t(x, T(x))^{1/n}}{\rho_1(T(x))^{1/n}}, \quad \mu_0 - \text{a.e. } x \in M.$$

If μ_1 is not absolutely continuous, an analogous result holds, provided that $t \in [0, 1)$, and that in (35) the second term on the right hand side is omitted.

Proof. For μ_0 -a.e. $x \in \mathcal{S}^\psi$, by Theorems 39-41 we have $T_t(x) = x$ and $\rho_t(x) = \rho_0(x)$ for all $t \in [0, 1]$. In this case the inequality follows from Lemma 49, which implies that for some $Q(x) \geq n$, it holds

$$\beta_{1-t}(x, x)^{1/n} + \beta_t(x, x)^{1/n} \leq (1-t)^{Q(x)/n} + t^{Q(x)/n} \leq 1, \quad \forall t \in [0, 1].$$

Fix now $x \in \mathcal{M}^\psi$, such that (i) $\psi : M \rightarrow T^*M$ is twice differentiable, (ii) the Jacobian identity of Theorem 41 holds. By the absolute continuity of μ_0 w.r.t. \mathfrak{m} , properties (i)-(ii) are satisfied μ_0 -a.e. in \mathcal{M}^ψ . Letting $X_i(t)$ be a moving frame along the geodesic $T_t(x) = \exp_x(td_x\psi(x))$, we have

$$(36) \quad \frac{\rho_0(x)}{\rho_t(T_t(x))} = \det(d_x T_t) \frac{\mathfrak{m}(X_1(t), \dots, X_n(t))}{\mathfrak{m}(X_1(0), \dots, X_n(0))} > 0.$$

Recall that, by construction, $y = T_1(x)$ if and only if y belongs to the c -subdifferential of the Kantorovich potential ψ of the transport problem, which is a singleton for μ_0 -a.e. $x \in M$. By definition of c -subdifferential of a c -convex function, one has

$$y \in \partial_c \psi(x) \quad \Leftrightarrow \quad \psi(z) + c(z, y) - \psi(x) - c(x, y) \geq 0, \quad \forall z \in M.$$

One can apply Theorem 28 with $\phi(z) := \psi(z) - \psi(x) - c(x, T(x))$ at the point x , where ψ is twice differentiable. In this way, we obtain an estimate for the determinant of the linear map $\pi_* \circ e_*^{t\bar{H}} \circ d_x^2 \phi : T_x M \rightarrow T_{T_t(x)} M$, which by definition of ϕ coincides with the linear map $\pi_* \circ e_*^{t\bar{H}} \circ d_x^2 \psi$ (thus justifying the notation we used in Theorem 28). In particular, $T(x) \notin \text{Cut}(x)$ and we can use the expressions for the distortion coefficients $\beta_t(x, T(x))$ of Lemma 50.

If $\mu_1 \in \mathcal{P}_c^{ac}(M)$, the statement for all $t \in [0, 1]$ follows from the estimate (14) and the change of variable formula (36).

If $\mu_1 \in \mathcal{P}_c(M) \setminus \mathcal{P}_c^{ac}(M)$, we omit the second term from (14) (which is non-negative), and we obtain

$$\det(d_x T_t)^{1/n} \geq (\det N_1^Y(t))^{1/n}, \quad \forall t \in [0, 1].$$

Then we conclude as in the previous case, using (36) only when it is well defined, that is for $t \in [0, 1)$. \square

6. GEOMETRIC AND FUNCTIONAL INEQUALITIES

In this section we discuss some consequences of interpolation inequalities. The first result is a sub-Riemannian Borell-Brascamp-Lieb inequality, that is Theorem 52. Its proof follows, without any modification, as in [CEMS01]. Notice that in the proof of this theorem, cf. [CEMS01, Sec. 6], one only uses assumption (37) for triple of points (x, y, z) satisfying $y = T_1(x)$ and $z = T_t(x)$, for some transport map T . This justifies removing $\text{Cut}(M)$ from $A \times B$.

Theorem 52 (Sub-Riemannian Borell-Brascamp-Lieb inequality). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on a n -dimensional manifold M , equipped with a smooth measure \mathfrak{m} . Fix $t \in [0, 1]$. Let $f, g, h : M \rightarrow \mathbb{R}$ be non-negative and $A, B \subset M$ Borel subsets such that $\int_A f \, d\mathfrak{m} = \int_B g \, d\mathfrak{m} = 1$. Assume that for every $(x, y) \in (A \times B) \setminus \text{Cut}(M)$ and $z \in Z_t(x, y)$,*

$$(37) \quad \frac{1}{h(z)^{1/n}} \leq \left(\frac{\beta_{1-t}(y, x)}{f(x)} \right)^{1/n} + \left(\frac{\beta_t(x, y)}{g(y)} \right)^{1/n}.$$

Then $\int_M h \, d\mathfrak{m} \geq 1$.

Let $p \in \mathbb{R} \cup \{\pm\infty\}$, $t \in [0, 1]$ and $a, b \geq 0$, and introduce the p -mean

$$\mathcal{M}_t^p(a, b) := \begin{cases} ((1-t)a^p + tb^p)^{1/p} & \text{if } ab \neq 0 \\ 0 & \text{if } ab = 0 \end{cases}, \quad p \neq 0, +\infty,$$

The limit cases are defined as follows

$$\mathcal{M}_t^0(a, b) := a^{1-t}b^t, \quad \mathcal{M}_t^{+\infty}(a, b) := \max\{a, b\}, \quad \mathcal{M}_t^{-\infty}(a, b) := \min\{a, b\}.$$

The next result follows in a standard way from Theorem 52, by elementary properties of \mathcal{M}_t^p . Theorem 52 can be recovered from Theorem 53 by setting $p = -1/n$.

Theorem 53 (Sub-Riemannian p -mean inequality). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on a n -dimensional manifold M , equipped with a smooth measure \mathbf{m} . Fix $p \geq -1/n$ and $t \in [0, 1]$. Let $f, g, h : M \rightarrow \mathbb{R}$ be non-negative and $A, B \subset M$ be Borel subsets such that $\int_A f \, d\mathbf{m} = \|f\|_{L^1(M)}$ and $\int_B g \, d\mathbf{m} = \|g\|_{L^1(M)}$. Assume that for every $(x, y) \in (A \times B) \setminus \text{Cut}(M)$ and $z \in Z_t(x, y)$,*

$$h(z) \geq \mathcal{M}_t^p \left(\frac{(1-t)^n f(x)}{\beta_{1-t}(y, x)}, \frac{t^n g(y)}{\beta_t(x, y)} \right).$$

Then,

$$\int_M h \, d\mathbf{m} \geq \mathcal{M}_t^{p/(1+np)} \left(\int_M f \, d\mathbf{m}, \int_M g \, d\mathbf{m} \right),$$

with the convention that if $p = +\infty$ then $p/(1+np) = 1/n$, and if $p = -1/n$ then $p/(1+np) = -\infty$.

For any pair of Borel subsets $A, B \subset M$, we define

$$\beta_t(A, B) := \inf \{ \beta_t(x, y) \mid (x, y) \in (A \times B) \setminus \text{Cut}(M) \}.$$

with the convention that $\inf \emptyset = 0$. Notice that $0 \leq \beta_t(A, B) < +\infty$, as a consequence of Lemma 50.

Theorem 53 immediately yields the following Brunn-Minkowski inequality, of which we give a proof for completeness.

Theorem 54 (Sub-Riemannian Brunn-Minkowski inequality). *Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on a n -dimensional manifold M , equipped with a smooth measure \mathbf{m} . Let $A, B \subset M$ Borel subsets. Then we have*

$$(38) \quad \mathbf{m}(Z_t(A, B))^{1/n} \geq \beta_{1-t}(B, A)^{1/n} \mathbf{m}(A)^{1/n} + \beta_t(A, B)^{1/n} \mathbf{m}(B)^{1/n}, \quad \forall t \in [0, 1].$$

Proof. For $t = 0$ or $t = 1$, inequality (38) is trivially verified. Hence let $t \in (0, 1)$. Assume first that $Z_t(A, B)$ is measurable, and set

$$(39) \quad f = \frac{\beta_{1-t}(B, A)}{(1-t)^n} \chi_A, \quad g = \frac{\beta_t(A, B)}{t^n} \chi_B, \quad h = \chi_{Z_t(A, B)},$$

where χ_S is the characteristic function of a set $S \subset M$. The assumption in Theorem 53 is satisfied with $p = +\infty$ since for every $(x, y) \in (A \times B) \setminus \text{Cut}(M)$ and $z \in Z_t(x, y)$,

$$1 = h(z) \geq \max \left\{ \frac{(1-t)^n f(x)}{\beta_{1-t}(y, x)}, \frac{t^n g(y)}{\beta_t(x, y)} \right\} = \max \left\{ \frac{\beta_{1-t}(B, A)}{\beta_{1-t}(y, x)}, \frac{\beta_t(A, B)}{\beta_t(x, y)} \right\}.$$

Then, we have (when $p = +\infty$ it is understood that $p/(1+np) = 1/n$)

$$\begin{aligned} \mathbf{m}(Z_t(A, B)) &= \int_M h \, d\mathbf{m} \geq \mathcal{M}_t^{1/n} \left(\int_M f \, d\mathbf{m}, \int_M g \, d\mathbf{m} \right) \\ &= \left(\beta_{1-t}(B, A)^{1/n} \mathbf{m}(A)^{1/n} + \beta_t(A, B)^{1/n} \mathbf{m}(B)^{1/n} \right)^n, \end{aligned}$$

which proves the required inequality.

Assume now that $Z_t(A, B)$ is not measurable. Since \mathfrak{m} is Borel regular, there exists a measurable set C such that $Z_t(A, B) \subset C$, with $\mathfrak{m}(Z_t(A, B)) = \mathfrak{m}(C)$. We have clearly that $\chi_C \geq \chi_{Z_t(A, B)}$ and χ_C is measurable. The conclusion follows repeating the argument above replacing $h = \chi_C$ in (39). \square

Theorem 54 is a weighted version of the Brunn-Minkowski inequality where the coefficients depends on the sets A, B . In the Riemannian case it is well-known that a control on the curvature implies a control on β . For example, if M is a n -dimensional Riemannian manifold with non-negative Ricci curvature, one has $\beta_t(x, y) \geq t^n$, yielding the geodesic Brunn-Minkowski inequality

$$(40) \quad \mathfrak{m}(Z_t(A, B))^{1/n} \geq (1-t)\mathfrak{m}(A)^{1/n} + t\mathfrak{m}(B)^{1/n}, \quad \forall t \in [0, 1].$$

Inequality (40) reduces to the classical Brunn-Minkowski inequality in the Euclidean space \mathbb{R}^n , where $Z_t(A, B) = (1-t)A + tB$ is the Minkowski sum.

Remark 55. Another generalization of the Euclidean Brunn-Minkowski inequality, at least for left-invariant structures on Lie groups is the *multiplicative* Brunn-Minkowski inequality. The latter is defined by replacing the Minkowski sum $A + B$ of two measurable sets with the group multiplication $A \star B$. For example, for the Heisenberg group \mathbb{H}_3 (see Section 7.1), with group law \star and left-invariant measure \mathfrak{m} , the multiplicative Brunn-Minkowski inequality reads

$$\mathfrak{m}(A \star B)^{1/d} \geq \mathfrak{m}(A)^{1/d} + \mathfrak{m}(B)^{1/d}, \quad A, B \subset \mathbb{H}^3.$$

We stress that the above inequality is true for the topological dimension $d = 3$ [LM05], but false for the Hausdorff dimension $d = 4$ [Mon03].

We end this section with the proof of Theorem 9.

6.1. Proof of Theorem 9. Let (\mathcal{D}, g) be an ideal sub-Riemannian structure on a n -dimensional manifold M , equipped with a smooth measure \mathfrak{m} , and $N \geq 0$. We prove that (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i). First, by plugging (i) in the result of Theorem 54 we obtain (ii). Furthermore, (iii) is a particular case of (ii) by considering only sets of the form $A = \{x\}$. Finally, (iii) implies (i) by choosing in the former $B = \mathcal{B}_r(y)$, and recalling definition 48 of $\beta_t(x, y)$.

Notice that the non-trivial part of the proof is Theorem 54, which allows to pass from a control on $\beta_t(x, y)$ to a global control on $\mathfrak{m}(Z_t(A, B))$, that is (i) \Rightarrow (ii). \square

7. EXAMPLES

In this section we discuss the form of the distortion coefficients for some examples. The first one is the Heisenberg group. In this case, we recover the results of [BKS16].

7.1. Heisenberg group. The Heisenberg group \mathbb{H}_3 is the sub-Riemannian structure on $M = \mathbb{R}^3$ defined by the global set of generating vector fields

$$X_1 = \partial_x - \frac{y}{2}\partial_z, \quad X_2 = \partial_y + \frac{x}{2}\partial_z.$$

The distribution has constant rank equal to two, and the sub-Riemannian structure is left-invariant with respect to the group product

$$(x, y, z) \star (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(x'y - y'x) \right).$$

The Heisenberg group is hence a Lie group and we equip it with the Lebesgue measure $\mathfrak{m} = \mathcal{L}^3$, which is a Haar measure. Thanks to the left-invariance of the sub-Riemannian structure, it is enough to compute the distortion coefficients when one of the two points is the origin.

In dual coordinates (u, v, w, x, y, z) on $T^*\mathbb{R}^3$, the corresponding Hamiltonian is

$$H(u, v, w, x, y, z) = \frac{1}{2} \left(\left(u - \frac{y}{2}w \right)^2 + \left(v + \frac{x}{2}w \right)^2 \right).$$

Hamilton equations can be explicitly integrated. In particular for an initial covector $\lambda_0 = u_0 dx + v_0 dy + w_0 dz \in T_0^*\mathbb{R}^3$, the exponential map from the origin reads $\exp_0^t(u_0, v_0, w_0) = (x(t), y(t), z(t))$, where

$$\begin{aligned} x(t) &= \frac{u_0 \sin(w_0 t) + v_0 (\cos(w_0 t) - 1)}{w_0}, \\ y(t) &= \frac{u_0 (1 - \cos(w_0 t)) + v_0 \sin(w_0 t)}{w_0}, \\ z(t) &= (u_0^2 + v_0^2) \frac{w_0 t - \sin(w_0 t)}{2w_0^2}. \end{aligned}$$

In order to use Lemma 50 for the computation of the distortion coefficient, we choose the global Darboux frame induced by the global sections of $T(T^*\mathbb{R}^3)$:

$$E_1 = \partial_u, \quad E_2 = \partial_v, \quad E_3 = \partial_w, \quad F_1 = \partial_x, \quad F_2 = \partial_y, \quad F_3 = \partial_z.$$

In particular, the horizontal part of the Jacobi matrix $N_0^Y(t)$ is simply the Jacobian of the exponential map $(u, v, w) \mapsto \exp_0^t(u, v, w)$ computed at (u_0, v_0, w_0) in these coordinates. A straightforward computation and Lemma 50 yield the following.

Proposition 56 (Heisenberg distortion coefficient). *Let $q \notin \text{Cut}(0)$. Then*

$$\beta_t(0, q) = t \frac{\sin\left(\frac{tw_0}{2}\right) \sin\left(\frac{tw_0}{2}\right) - \frac{tw_0}{2} \cos\left(\frac{tw_0}{2}\right)}{\sin\left(\frac{w_0}{2}\right) \sin\left(\frac{w_0}{2}\right) - \frac{w_0}{2} \cos\left(\frac{w_0}{2}\right)}, \quad \forall t \in [0, 1],$$

where (u_0, v_0, w_0) is the initial covector of the unique geodesic joining 0 with q .

For the Heisenberg group, it is well-known that $t_{\text{cut}}(u_0, v_0, w_0) = 2\pi/|w_0|$ (see e.g. [ABB12, Lemma 37]). Hence, since $q \notin \text{Cut}(0)$, in the above formula it is understood that $|w_0| < 2\pi$, in which case one can check that $\beta_t(0, q) > 0$ for all $t \in (0, 1]$.

Remark 57. In the above notation, $d_{SR}^2(0, q) = \|\lambda\|^2 = u_0^2 + v_0^2$. We observe that the Heisenberg distortion coefficient *does not* depend on the distance $d_{SR}(0, q)$, but rather on the ‘‘vertical part’’ w_0 of the covector λ . See Section 8.

Lemma 58 (Sharp bound to Heisenberg distortion). *Let $N \in \mathbb{R}$. The inequality*

$$\beta_t(q_0, q) \geq t^N, \quad \forall t \in [0, 1],$$

holds for all points $q_0, q \in \mathbb{H}_3$ with $q \notin \text{Cut}(q_0)$, if and only if $N \geq 5$.

Lemma 58 follows from left invariance and the sharp inequalities of [Riz16, Lemma 18]. We recover the following known results [BKS16].

Corollary 59 (Sharp interpolation inequality). *Let $\mu_0 \in \mathcal{P}_c^{ac}(\mathbb{H}_3)$, and $\mu_1 \in \mathcal{P}_c(\mathbb{H}_3)$. Let $\mu_t = (T_t)_\# \mu_0 = \rho_t \mathcal{L}^3$ be the unique Wasserstein geodesic joining μ_0 with μ_1 . Then,*

$$\frac{1}{\rho_t(T_t(x))^{1/3}} \geq \frac{(1-t)^{5/3}}{\rho_0(x)^{1/3}} + \frac{t^{5/3}}{\rho_1(T(x))^{1/3}}, \quad \mathcal{L}^3 - \text{a.e.}, \quad \forall t \in [0, 1].$$

The above inequality is sharp, in the sense that if one replaces the exponent 5 with a smaller one, the inequality fails for some choice of μ_0, μ_1 .

Corollary 60 (Sharp geodesic Brunn-Minkowski inequality). *For all non-empty Borel sets $A, B \subset \mathbb{H}_3$, we have*

$$\mathcal{L}^3(Z_t(A, B))^{1/3} \geq (1-t)^{5/3} \mathcal{L}^3(A)^{1/3} + t^{5/3} \mathcal{L}^3(B)^{1/3}, \quad \forall t \in [0, 1].$$

The above inequality is sharp, in the sense that if one replaces the exponent 5 with a smaller one, the inequality fails for some choice of A, B .

Notice that, as a consequence of Theorem 9, we recover also the following result originally obtained in [Jui09]: the Heisenberg group \mathbb{H}_3 , equipped with the Lebesgue measure, satisfies the MCP(K, N) if and only if $N \geq 5$ and $K \leq 0$.

7.2. Generalized H-type groups. These structures were introduced in [BR17], and constitute a large class of Carnot groups where the optimal synthesis is known. This class contains Kaplan H -type groups, and some of these structures might admit non-trivial abnormal minimizing geodesics.

We take the definitions directly from [BR17], to which we refer for more details. Let (G, \mathcal{D}, g) be a step 2 Carnot group, with Lie algebra \mathfrak{g} of rank k , dimension n satisfying $\dim \mathfrak{g}_1 = k$, $\dim \mathfrak{g}_2 = n - k$ and

$$[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2, \quad [\mathfrak{g}_i, \mathfrak{g}_2] = 0, \quad i = 1, 2.$$

Any choice g of a scalar product on \mathfrak{g}_1 induces a left-invariant sub-Riemannian structure (\mathcal{D}, g) on G , such that $\mathcal{D}(p) = \mathfrak{g}_1(p)$ for all $p \in G$. Extend the scalar product g on \mathfrak{g}_1 to a scalar product on the whole \mathfrak{g} , which we denote with the same symbol.

For any $V \in \mathfrak{g}_2$, the skew-symmetric operator $J_V : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ is defined by

$$g(X, J_V Y) = g(V, [X, Y]), \quad \forall X, Y \in \mathfrak{g}_1.$$

Definition 61. We say that a step 2 Carnot group is of *generalized H-type* if there exists a symmetric, non-zero and non-negative operator $S : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ such that

$$(41) \quad J_V J_W + J_W J_V = -2g(V, W)S^2, \quad \forall V, W \in \mathfrak{g}_2.$$

Remark 62. The above definition is well posed and does not depend on the choice of the extension of g . More precisely, if (41) is verified for the operators J_V defined by a choice of an extension of g , then the operators \tilde{J}_V defined by a different extension \tilde{g} will verify (41), with the *same* operator S . Moreover, by polarization, it is easy to show that a step 2 Carnot group is of generalized H-type if and only if there exists a symmetric, non-negative and non-zero operator $S : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ such that $J_V^2 = -\|V\|^2 S^2$ for all $V \in \mathfrak{g}_2$.

Remark 63. A generalized H -type group does not admit non-trivial abnormal geodesics, and is thus ideal, if and only if S is invertible. When $n = k + 1$, we are in the case of corank 1 Carnot groups. If S is also non-degenerate (and thus $k = 2d$ is even and $S > 0$), we are in the case of contact Carnot groups. The case $S = \text{Id}_{\mathfrak{g}_1}$ and $k = 2d$ corresponds to classical Kaplan H -type groups.

The next result follows from the explicit expression for the Jacobian determinant of generalized H -type groups [BR17, Lemma 19], which in turn allows to compute explicit distortion coefficients. The latter, in turn, can be bounded by a power law thanks to [BR17, Corollary 26]. In particular, we have the following.

Lemma 64 (Sharp bound to generalized H -type distortion). *Let (G, \mathcal{D}, g) be a generalized H -type group, with dimension n and rank k , equipped with a left-invariant measure \mathfrak{m} . Let $N \in \mathbb{R}$. The inequality*

$$\beta_t(x, y) \geq t^N, \quad \forall t \in [0, 1],$$

holds for all points $x, y \in G$ with $y \notin \text{Cut}(x)$, if and only if $N \geq k + 3(n - k)$, the latter number being the geodesic dimension of the Carnot group.

The same consequences as in Section 7.1 hold, with appropriate exponents. The sharp measure contraction properties of generalized H -type groups were already the object of [BR17]. Here we only state the following new consequences of Lemma 64 and our general theory.

Corollary 65 (Sharp interpolation inequality). *Let (G, \mathcal{D}, g) be an ideal generalized H -type group, with dimension n and rank k , equipped with a left-invariant measure \mathfrak{m} . Let $\mu_0 \in \mathcal{P}_c^{ac}(G)$, and $\mu_1 \in \mathcal{P}_c(G)$. Let $\mu_t = (T_t)_\# \mu_0 = \rho_t \mathfrak{m}$ be the unique Wasserstein geodesic joining μ_0 with μ_1 . Then,*

$$\frac{1}{\rho_t(T_t(x))^{1/n}} \geq \frac{(1-t)^{\frac{k+3(n-k)}{n}}}{\rho_0(x)^{1/n}} + \frac{t^{\frac{k+3(n-k)}{n}}}{\rho_1(T(x))^{1/n}}, \quad \mathfrak{m} - \text{a.e.}, \forall t \in [0, 1].$$

The above inequality is sharp, in the sense that if one replaces the exponent $k + 3(n - k)$ with a smaller one, the inequality fails for some choice of μ_0, μ_1 .

Remark 66. The restriction to ideal structures in the above corollary arises from the requirements of the general theory leading to Theorem 51, while this assumption is not necessary in Lemma 64. However, we remark that abnormal geodesics of generalized H -type groups are very docile (they consists in straight lines, and never lose minimality). Thus, we expect all the above results to hold also for non-ideal generalized H -type groups. This is supported by the positive results obtained for corank 1 Carnot groups obtained in [BKS17] and the forthcoming Corollary 67.

Indeed, the sharp Brunn-Minkowski inequality for ideal generalized H -type groups follows from Theorem 9 and Lemma 64. However, thanks to the results of [RY17] for product structures, we are able to eliminate the ideal assumption.

Corollary 67 (Sharp geodesic Brunn-Minkowski inequality). *Let (G, \mathcal{D}, g) be a generalized H -type group, with dimension n and rank k , equipped with a left-invariant measure \mathfrak{m} . For all non-empty Borel sets $A, B \subset G$, we have*

$$\mathfrak{m}(Z_t(A, B))^{1/n} \geq (1-t)^{\frac{k+3(n-k)}{n}} \mathfrak{m}(A)^{1/n} + t^{\frac{k+3(n-k)}{n}} \mathfrak{m}(B)^{1/n}, \quad \forall t \in [0, 1].$$

The above inequality is sharp, in the sense that if one replaces the exponent $k + 3(n - k)$ with a smaller one the inequality fails for some choice of A, B .

Proof. In this proof, given $N, n \in \mathbb{N}$, we denote $\text{BM}(N, n)$ the following property: for all non-empty Borel sets A, B , we have

$$\mathfrak{m}(Z_t(A, B))^{1/n} \geq (1-t)^{N/n} \mathfrak{m}(A)^{1/n} + t^{N/n} \mathfrak{m}(B)^{1/n}, \quad \forall t \in [0, 1].$$

A generalized H -type group G is the product of an ideal one G_0 with dimension $n_0 = n - d$ and rank $k_0 = k - d$, and d copies of the Euclidean \mathbb{R} , for a unique $d \geq 0$. Furthermore, the left-invariant measure \mathfrak{m} of G is the product of the a left-invariant measure \mathfrak{m}_0 of G_0 and d copies of the Lebesgue measures \mathcal{L} of each factor \mathbb{R} . It follows immediately from Lemma 64 and Theorem 9 that G_0 , equipped with the measure \mathfrak{m}_0 , satisfies $\text{BM}(N_0, n_0)$, with $N_0 = k_0 + 3(n_0 - k_0)$. Furthermore, each copy of \mathbb{R} , equipped with the Lebesgue measure, satisfies the standard linear Brunn-Minkowski inequality $\text{BM}(1, 1)$. It follows from [RY17, Thm. 3.3] that the product $G = G_0 \times \mathbb{R}^d$ equipped with the left-invariant measure $\mathfrak{m} = \mathfrak{m}_0 \times \mathcal{L}^d$ satisfies $\text{BM}(N_0 + d, n_0 + d) = \text{BM}(k + 3(n - k), n)$, which is the desired inequality.

Assume that G satisfies $\text{BM}(k + 3(n - k) - \varepsilon, n)$ for some $\varepsilon > 0$. Let $x \notin \text{Cut}(y)$. Letting $A = x \in G$ and $B = \mathcal{B}_r(y) \subset G$, and taking the limit for $r \downarrow 0$, we obtain that $\beta_t(x, y) \geq t^{k+3(n-k)-\varepsilon}$, contradicting the results of Lemma 64. \square

We can also easily recover the following result proved in [BR17]: a generalized H -type group with dimension n and rank k , equipped with a left-invariant measure \mathfrak{m} , satisfies the MCP(K, N) if and only if $N \geq k + 3(n - k)$ and $K \leq 0$.

7.3. Grushin plane. The Grushin plane \mathbb{G}_2 is the sub-Riemannian structure on \mathbb{R}^2 defined by the global set of generating vector fields

$$X_1 = \partial_x, \quad X_2 = x\partial_y.$$

We stress that the rank of $\mathcal{D} = \text{span}\{X_1, X_2\}$ is not constant. More precisely, the structure is Riemannian on $\{x \neq 0\}$, and it is singular otherwise. We equip the Grushin plane with the Lebesgue measure $\mathfrak{m} = \mathcal{L}^2$ of \mathbb{R}^2 . In canonical coordinates (u, v, x, y) on $T^*\mathbb{R}^2$, the corresponding Hamiltonian is

$$H(u, v, x, y) = \frac{1}{2}(u^2 + x^2v^2).$$

Hamilton equations are easily integrated, and the Hamiltonian flow

$$e^{t\vec{H}}(u_0, v_0, x_0, y_0) = (u(t), v(t), x(t), y(t))$$

with initial covector $\lambda_0 = u_0dx + v_0dy \in T^*_{(x_0, y_0)}\mathbb{R}^2$ reads

$$\begin{aligned} u(t) &= u_0 \cos(tv_0) - x_0v_0 \sin(tv_0), \\ v(t) &= v_0, \\ x(t) &= x_0 \cos(tv_0) + u_0 \frac{\sin(tv_0)}{v_0}, \\ y(t) &= y_0 + \frac{\sin(2tv_0)(v_0^2x_0^2 - u_0^2) + 2v_0(t(v_0^2x_0^2 + u_0^2) + u_0x_0 - u_0x_0 \cos(2tv_0))}{4v_0^2}. \end{aligned}$$

In particular, $\exp^t_{(x_0, y_0)}(u_0, v_0) = (x(t), y(t))$. Notice that the geodesic flow is an analytic function of the initial data, and if $v_0 = 0$ the above equations are understood by taking the limit $v_0 \rightarrow 0$. We always adopt this convention in this section.

To compute the distortion coefficients, fix $q_0 = (x_0, y_0) \in \mathbb{R}^2$, let $q \notin \text{Cut}(q_0)$, and let $\lambda_0 = u_0dx + v_0dy \in T^*_{(x_0, y_0)}\mathbb{R}^2$ the covector of the unique minimizing geodesic $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ joining q_0 with q .

In order to use Lemma 50 for the computation of the distortion coefficient, we choose the global Darboux frame induced by the global sections of $T(T^*\mathbb{R}^2)$:

$$E_1 = \partial_u, \quad E_2 = \partial_v, \quad F_1 = \partial_x, \quad F_2 = \partial_y.$$

In particular, the horizontal part of the Jacobi matrix $N_0^V(t)$ is simply the Jacobian of the exponential map $(u, v) \mapsto \exp^t_{(x_0, y_0)}(u, v)$ in these coordinates, computed at (u_0, v_0) . A straightforward computation and Lemma 50 yield the following.

Proposition 68 (Grushin distortion coefficient). *Let $q \notin \text{Cut}(q_0)$. Then*

$$(42) \quad \beta_t(q_0, q) = t \frac{(u_0^2 + tu_0v_0^2x_0 + v_0^2x_0^2) \sin(tv_0) - tu_0^2v_0 \cos(tv_0)}{(u_0^2 + u_0v_0^2x_0 + v_0^2x_0^2) \sin(v_0) - u_0^2v_0 \cos(v_0)}, \quad \forall t \in [0, 1],$$

where (u_0, v_0) is the initial covector of the unique geodesic joining q_0 with q .

For the Grushin plane, $t_{\text{cut}}(u_0, v_0) = \pi/|v_0|$ (see [ABS08, Sec. 3.2] or [ABB16b, Ch. 9]). Hence, since $q \notin \text{Cut}(q_0)$, in the above formula it is understood that $|v_0| < \pi$, in which case one can check directly that $\beta_t(q_0, q) > 0$ for all $t \in (0, 1]$.

We have the following non-trivial estimate.

Proposition 69 (Sharp bound to Grushin distortion). *Let $N \in \mathbb{R}$. The inequality*

$$\beta_t(q_0, q) \geq t^N, \quad \forall t \in [0, 1],$$

holds for all points $q_0, q \in \mathbb{G}_2$ with $q \notin \text{Cut}(q_0)$, if and only if $N \geq 5$.

Remark 70. The existence of such a bound is not completely surprising, since the Grushin plane is a quotient of the Heisenberg group. Nevertheless, it is not clear how to deduce a bound for distortion coefficients of \mathbb{G}_2 starting from the knowledge of the corresponding inequality for \mathbb{H}_3 . Actually, the most surprising aspect of Proposition 69 is its sharpness. As it is clear from the proof, the necessity of the condition $N \geq 5$ is due to pairs of points q_0, q , possibly with the same y -coordinate, and located on opposite sides of the singular set $\{x = 0\}$.

Proof. Let $q = \exp_{q_0}(u_0, v_0)$, with $|v_0| < \pi$, and $q_0 = (x_0, y_0) \in \mathbb{R}^2$. If $x_0 = 0$,

$$\beta_t(x_0 = 0) = t \times \frac{\sin(tv_0) - tv_0 \cos(tv_0)}{\sin(v_0) - v_0 \cos(v_0)} \geq t^4, \quad \forall t \in [0, 1],$$

which follows from the inequality of [Riz16, Lemma 18], and $|v_0| < \pi$. We now proceed by assuming $x_0 \neq 0$ (by symmetry we actually assume $x_0 > 0$).

Case $v_0 = 0$. This case, corresponding to straight horizontal lines possibly crossing the singular region, is the one which yields the ‘‘only if’’ part of the theorem, and we will settle it first. In this case the trigonometric terms disappear, and

$$\beta_t(v_0 = 0) = t^2 \times \frac{t^2 u_0^2 + 3t u_0 x_0 + 3x_0^2}{u_0^2 + 3u_0 x_0 + 3x_0^2}.$$

We want to find the best $N \in \mathbb{R}$, such that for all $x_0 > 0$ and $u_0 \in \mathbb{R}$, it holds

$$\frac{t^2 u_0^2 + 3t u_0 x_0 + 3x_0^2}{u_0^2 + 3u_0 x_0 + 3x_0^2} \geq t^{N-2}, \quad \forall t \in [0, 1].$$

Since both sides are strictly positive for all $t \in (0, 1]$, we can take the logarithms and the above inequality is equivalent to

$$\int_{tu}^u \frac{d}{dz} \log f_{x_0}(z) dz \leq (N-2) \int_{tu}^u \frac{d}{dz} \log |z| dz, \quad \forall t \in (0, 1), u \in \mathbb{R},$$

where $f_{x_0}(z) := z^2 + 3zx_0 + 3x_0^2$. This inequality is equivalent to the corresponding inequality for the integrands. After some computations, we obtain the condition

$$(N-4)z^2 + 3x_0(N-3)z + 3x_0^2(N-2) \geq 0, \quad \forall x_0 > 0, z \in \mathbb{R}.$$

One easily checks that the above holds if and only if $N \geq 5$. This proves the ‘‘only if’’ part of the statement.

Case $v_0 \neq 0$. By symmetry, we actually assume $v_0 > 0$. If $u_0 = 0$, then

$$\beta_t(u_0 = 0) = t \frac{\sin(tv_0)}{\sin(v_0)} \geq t^2 \geq t^5, \quad \forall t \in (0, 1).$$

Hence in the following we consider $u_0 \neq 0$. We recall the assumptions made so far:

$$x_0 > 0, \quad v_0 > 0, \quad u_0 \neq 0.$$

In this case we rewrite (42) as

$$\beta_t(q_0, q) = t \times \frac{f_a(tv_0)}{f_a(v_0)}, \quad a := \frac{v_0 x_0}{u_0} \in \mathbb{R}_0 = \mathbb{R} \setminus \{0\},$$

where, for all $a \in \mathbb{R}_0$, we defined

$$f_a(\xi) := (1 + a\xi + a^2) \sin(\xi) - \xi \cos(\xi).$$

It remains to prove that for all $a \in \mathbb{R}_0$ and $N \geq 5$ it holds

$$\frac{f_a(tv_0)}{f_a(v_0)} \geq t^{N-1}, \quad \forall t \in [0, 1].$$

In particular, it is sufficient to prove the case $N = 5$, which we assume from now on. Since both sides are strictly positive on $t \in (0, 1]$, we can take the logarithms and the inequality is equivalent to

$$\int_{tv_0}^{v_0} \frac{d}{dz} \log f_a(z) dz \leq 4 \int_{tv_0}^{v_0} \frac{d}{dz} \log |z| dz, \quad \forall t \in (0, 1), a \in \mathbb{R}_0.$$

The above inequality is equivalent to the corresponding one for the integrands. After some computation, we obtain the equivalent inequality

$$(43) \quad W_a(z) := Q_a(z) \sin(z) - zP_a(z) \cos(z) \geq 0, \quad \forall z \in (0, \pi), a \in \mathbb{R}_0,$$

where we defined the two polynomials

$$P_a(z) = a(a+z) + 4, \quad Q_a(z) = (a+z)(4a-z) + 4.$$

Consider $a \mapsto W_a(z)$. It is easy to check that for all fixed $z \in (0, \pi)$, we have

$$\lim_{a \rightarrow \pm\infty} W_a(z) = +\infty.$$

Moreover, $\partial_a W_a(z)$ is linear, hence the function $a \mapsto W_a(z)$ has a unique minimum. Then (43) is equivalent to the fact that this minimum is non-negative for all $z \in (0, \pi)$. Setting $\partial_a W_a(z) = 0$, we obtain

$$a_{\min} = -\frac{z}{2} \times \frac{3 \sin(z) - z \cos(z)}{4 \sin(z) - z \cos(z)} \leq 0.$$

Hence, (43) is equivalent to $W_{a_{\min}}(z) \geq 0$ for all $z \in (0, \pi)$. Replacing, and after some computations, we have that such a condition is equivalent to

$$(44) \quad \bar{W}(z) := \alpha(z) \sin(z)^2 + \beta(z) \cos(z) \sin(z) + \gamma(z) \cos(z)^2 \geq 0, \quad \forall z \in (0, \pi),$$

where we have defined the following polynomials

$$\begin{aligned} \alpha(z) &= (64 - 25z^2), \\ \beta(z) &= 10z(z^2 - 8), \\ \gamma(z) &= z^2(16 - z^2) > 0. \end{aligned}$$

By looking to the graph of $W(z)$, for $z \in (0, \pi)$, one notices that the inequality (44) is extremely sharp for z close to 0, while it is easier to prove for larger z . Hence, we split the proof of (44) into two parts.

(i) Proof of (44) on $(0, 2.67)$. Notice that $\bar{W}^{(i)}(0) = 0$ for all $i = 0, 1, 2, 3, 4$, and

$$\bar{W}^{(5)}(z) = 8 \left(2z^4 + 8z^2 + 3 \right) \sin(2z) + 80z \cos(2z).$$

This is the first n -th derivative whose polynomial factors multiplying the trigonometric functions are all non-negative. Furthermore, recall that

$$\begin{aligned} \sin(x) &\geq x - \frac{x^3}{6}, \quad x \in [0, +\infty), \\ \cos(x) &\geq 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}, \quad x \in [0, \infty). \end{aligned}$$

Hence, using the explicit form of $\bar{W}^{(5)}$, and the fact that the polynomial factors are non-negative, we obtain

$$\bar{W}^{(5)}(z) \geq 8 \left(2z^4 + 8z^2 + 3 \right) \left(2z - \frac{4z^3}{3} \right) + 80z \left(1 - 2z^2 + \frac{2z^4}{3} - \frac{4z^6}{45} \right)$$

$$= \frac{64}{9}z \left(18 - 9z^2 - 4z^6 \right), \quad \forall z \in (0, \pi).$$

Integrating five times the above inequality on the interval $[0, z]$, and since $\bar{W}^{(i)}(0) = 0$ for all $i = 0, 1, 2, 3, 4$, we obtain

$$\bar{W}(z) \geq z^6 \left(-\frac{4z^6}{13365} - \frac{z^2}{105} + \frac{8}{45} \right), \quad \forall z \in (0, \pi).$$

The term in parenthesis in right hand side of the above is a third order polynomial, for which we can compute explicitly the roots. The first positive root occurs at $z_* \simeq 2.67491$. Hence the inequality $\bar{W}(z) \geq 0$ is proved for $z \in (0, 2.67)$.

(ii) Proof of (44) on $[2.67, \pi)$. On this interval the inequality is easier to verify with rough estimates. Indeed, $\bar{W}(z)$ in (44) is the sum of three terms, and it is sufficient to bound each one of them with the corresponding minimum. The first term

$$(64 - 25z^2) \sin(z)^2 \quad \text{attains its minimum on } [2.67, \pi) \text{ at } z = 2.67,$$

where its value is approximately -23.57 . The second term

$$10z(z^2 - 8) \sin(z) \cos(z) \quad \text{attains its minimum on } [2.67, \pi) \text{ at } z \simeq 3,$$

where its value is approximately -4.20 . The third term

$$z^2(16 - z^2) \cos(z)^2 \quad \text{attains its minimum on } [2.67, \pi) \text{ at } z = 2.67.$$

where its value is approximately 50.19 . Thus $\bar{W}(z)$ is larger than the sum of the three aforementioned values, which is positive. This proves (44), and concludes the proof of the proposition. \square

As in the previous examples, one obtains the following consequences.

Corollary 71 (Sharp interpolation inequality). *Let $\mu_0 \in \mathcal{P}_c^{ac}(\mathbb{G}_2)$, and $\mu_1 \in \mathcal{P}_c(\mathbb{G}_2)$. Let $\mu_t = (T_t)_\# \mu_0 = \rho_t \mathcal{L}^2$ be the unique Wasserstein geodesic joining μ_0 with μ_1 . Then,*

$$\frac{1}{\rho_t(T_t(x))^{1/2}} \geq \frac{(1-t)^{5/2}}{\rho_0(x)^{1/2}} + \frac{t^{5/2}}{\rho_1(T(x))^{1/2}}, \quad \mathcal{L}^2 - \text{a.e.}, \forall t \in [0, 1].$$

The above inequality is sharp, in the sense that if one replaces the exponent 5 with a smaller one, the inequality fails for some choice of μ_0, μ_1 .

Corollary 72 (Sharp geodesic Brunn-Minkowski inequality). *For all non-empty Borel sets $A, B \subset \mathbb{G}_2$, we have*

$$\mathcal{L}^2(Z_t(A, B))^{1/2} \geq (1-t)^{5/2} \mathcal{L}^2(A) + t^{5/2} \mathcal{L}^2(B), \quad \forall t \in [0, 1].$$

The above inequality is sharp, in the sense that if one replaces the exponent 5 with a smaller one, the inequality fails for some choice of A, B .

Finally, by taking $\mu_1 = \delta_y$ for some $y \in \mathbb{G}_2$, and using the fact that the Grushin plane admits a one-parameter group of metric dilations, we obtain the following result (see [Jui09] or [Riz16] for definitions adapted to this context).

Corollary 73 (Sharp measure contraction property). *The Grushin plane, equipped with the Lebesgue measure, satisfies the MCP(K, N) if and only if $N \geq 5$ and $K \leq 0$.*

8. PROPERTIES OF DISTORTION COEFFICIENTS

As we have discussed in Section 7, sub-Riemannian distortion coefficients present major differences with respects to the Riemannian case. In this section, we discuss some of their general properties. Henceforth, let (\mathcal{D}, g) be a fixed ideal sub-Riemannian structure on M , and let $x, y \in M$, with $y \notin \text{Cut}(x)$.

8.1. Dependence on distance. Under the above assumptions, $y = \exp_x(\lambda)$ for a unique $\lambda \in T_x^*M$ such that $\|\lambda\| = \sqrt{2H(\lambda)} = d_{SR}(x, y)$. In particular, one can regard the sub-Riemannian distortion coefficients as a one-parameter family of functions depending on the initial covector $\lambda \in T^*M$ of a minimizing geodesic joining a pair of points $(x, y) \in M \times M \setminus \text{Cut}(M)$. Loosely speaking:

$$\beta_t(x, \exp_x(\lambda)) = f_t(\lambda).$$

The basic Riemannian examples where β_t are explicitly available are space forms (simply connected Riemannian manifolds with constant sectional curvature). In these cases, it is well known that the distortion coefficients depend on λ *only* through its (dual) norm $\|\lambda\| = d(x, y)$. In particular, for the space form with constant curvature $K \in \mathbb{R}$ and dimension $n > 1$, one obtains

$$\beta_t^{(K,n)}(x, y) = \begin{cases} +\infty & \text{if } K > 0 \text{ and } \alpha > \pi, \\ t \left(\frac{\sin(t\alpha)}{\sin(\alpha)} \right)^{n-1} & \text{if } K > 0 \text{ and } \alpha \in [0, \pi], \\ t^n & \text{if } K = 0, \\ t \left(\frac{\sinh(t\alpha)}{\sinh(\alpha)} \right)^{n-1} & \text{if } K < 0, \end{cases}$$

where we have denoted

$$\alpha = \sqrt{\frac{|K|}{n-1}} d(x, y).$$

As we discussed in Section 7.1, in the simplest sub-Riemannian structure, the Heisenberg group, the dependence on λ is fundamentally more complicated, and $\beta_t(x, y)$ is *not* a function of the sub-Riemannian distance between x and y . A similar phenomenon occurs in the case of the Grushin plane, treated in Section 7.3.

8.2. Small time asymptotics. For Riemannian structures, it is well known that $\beta_t(x, y) \sim C(x, y)t^n$, with $n = \dim M$. This is the reason for the presence of a normalization factor t^{-n} in the standard Riemannian distortion coefficients, which we did not include in Definition 48 (compare the latter with [Vil09, Def. 14.17]).

In fact, in the genuinely sub-Riemannian case, the asymptotic is remarkably different. More precisely we have the following statement, which follows from [ABR13, Sec. 5.6].

Theorem 74 (Asymptotics of sub-Riemannian distortion). *Let (\mathcal{D}, g) be a sub-Riemannian structure on M , not necessarily ideal, and let $x \in M$. Then, there exists $\mathcal{N}(x) \in \mathbb{N}$ such that*

$$\lim_{t \rightarrow 0^+} \frac{\log \beta_t(x, \exp_x(\lambda))}{\log t} \geq \mathcal{N}(x), \quad \forall \lambda \in T_x^*M.$$

*The equality is attained on a Zariski non-empty open and dense set $\mathcal{A}_x \subseteq T_x^*M$. In particular, $\mathcal{N}(x)$ is the largest number such that, for $t \rightarrow 0^+$, one has*

$$\beta_t(x, y) = O\left(t^{\mathcal{N}(x)}\right), \quad \forall y \notin \text{Cut}(x).$$

The number $\mathcal{N}(x)$ is called the geodesic dimension of the sub-Riemannian structure at x . Finally, the following inequality holds

$$\mathcal{N}(x) \geq \dim(M),$$

with equality if and only if the structure is Riemannian at x , that is $\mathcal{D}_x = T_xM$.

APPENDIX A. CONJUGATE TIMES AND OPTIMALITY: PROOF OF THEOREM 16

The aim of this appendix is to give a self-contained proof of the fact that geodesics not containing abnormal segments lose minimality after their first conjugate point, that is Theorem 16. This fact is not new, and well-known to experts. An explicit statement can be found in the preprint version of [FR10, Prop. 5.15], and is proved in [Sar80]. The main difference with respect to the proof of the analogue statement in the Riemannian setting is that the explicit formula for the second variation of energy (the index form, see e.g. [Mil63]) is usually expressed in terms of Levi-Civita connection and curvature, which are not available in the sub-Riemannian setting. Hence one has to work with a suitable generalization of the index form on the space of controls. Here, the sub-Riemannian structure is not required to be ideal.

Given a normal geodesic $\gamma : [0, 1] \rightarrow M$ we say that γ *contains no abnormal segments* if for every $0 \leq s_1 < s_2 \leq 1$ the restriction $\gamma|_{[s_1, s_2]}$ is not abnormal.

An horizontal trajectory γ_u parametrized by constant speed realizes the sub-Riemannian distance between x and y if and only if u is a solution of the constrained minimum problem

$$(45) \quad \min\{J(v) \mid v \in \mathcal{U}, E_x(v) = y\}.$$

Here $J : \mathcal{U} \rightarrow \mathbb{R}$ is the energy functional and $E_x : \mathcal{U} \rightarrow M$ is the end-point map based at x , where $\mathcal{U} \subseteq L^2([0, 1], \mathbb{R}^m)$, cf. Section 2.2.

The Lagrange multipliers rule, in the normal case, gives the first order necessary condition for a control (and the corresponding horizontal curve) to be a minimizer: there exists $\lambda \in T_y^*M$, such that

$$(46) \quad \lambda \circ D_u E_x = D_u J.$$

Hence a solution of (45) is a pair (u, λ) satisfying (46). Higher order condition for minimality of γ_u are given by the second variation of J on the level sets of E_x . The second differential of the restriction to the level set is not in general the restriction of the second differential to the tangent space to the level set $T_u E_x^{-1}(y) = \ker D_u E_x$. The following formulas hold (see [ABB16b, Ch. 8], and also [Rif14, Sec. 2.4]).

Proposition 75 (Second variation of the energy). *Let $\gamma_u : [0, 1] \rightarrow M$ be a normal geodesic joining x with y satisfying (46) for some $\lambda \in T_y^*M$. Then, we have*

$$\text{Hess}_u J|_{E_x^{-1}(y)}(v) = D_u^2 J(v) - \lambda \circ D_u^2 E_x(v), \quad \forall v \in \ker D_u E_x.$$

Moreover we have

$$D_u^2 J(v) = \|v\|_{L^2}^2, \quad D_u^2 E_x(v) = \iint_{0 \leq \tau \leq t \leq 1} [(P_{\tau,1})_* X_{v(\tau)}, (P_{t,1})_* X_{v(t)}](y) d\tau dt,$$

where $X_{v(t)} := \sum_{i=1}^m v_i(t) X_i$ and $P_{\tau,t}$ denotes the flow of the non-autonomous vector field $X_{u(t)}$, with initial datum at time τ and final time t .

Given a pair (u, λ) such that γ_u is a normal geodesic satisfying the first order condition (46), we denote by $u^s(t) := su(st)$ the reparametrized control associated with the reparametrized trajectory $\gamma_{u^s}(t) = \gamma_u(st)$, both defined for $t \in [0, 1]$. The covector $\lambda^s = s(P_{s,1}^*)\lambda \in T_{\gamma_u(s)}^*M$, is a Lagrange multiplier associated with u^s .

For normal geodesics containing no abnormal segments, conjugate points (in the sense of Definition 15) can be characterized by the second variation of the energy, as in the Riemannian case, cf. [ABB16b, Ch. 8].

Proposition 76. *Assume that $\gamma_u : [0, 1] \rightarrow M$ contains no abnormal segments. Then $\gamma_u(s)$ is conjugate to $\gamma_u(0)$ if and only if $\text{Hess}_{u^s} J|_{E_x^{-1}(\gamma_u(s))}$ is a degenerate quadratic form.*

The following lemma, proved in [ABB16b, Ch. 8], is crucial. For the reader's convenience, we provide a sketch of the proof.

Lemma 77. *Assume that a normal geodesic $\gamma : [0, 1] \rightarrow M$ contains no abnormal segments. Define the function $\alpha : (0, 1] \rightarrow \mathbb{R}$ as follows*

$$(47) \quad \alpha(s) := \inf \left\{ \|v\|_{L^2}^2 - \lambda_s \circ D_{u^s}^2 E_x(v) \mid \|v\|_{L^2}^2 = 1, v \in \ker D_{u^s} E_x \right\}.$$

Then α is continuous and has the following properties:

- (a) $\alpha(0) := \lim_{s \rightarrow 0} \alpha(s) = 1$;
- (b) $\alpha(s) = 0$ implies that $\text{Hess}_{u^s} J|_{E_x^{-1}(\gamma_u^s(1))}$ is degenerate;
- (c) α is monotone decreasing;
- (d) if $\alpha(\bar{s}) = 0$ for some $\bar{s} > 0$, then $\alpha(s) < 0$ for $s > \bar{s}$.

Proof of Lemma 77. Notice that one can write

$$\|v\|_{L^2}^2 - \lambda_s \circ D_{u^s}^2 E_x(v) = \langle (I - Q_s)(v) | v \rangle_{L^2},$$

where $Q_s : L^2([0, 1], \mathbb{R}^m) \rightarrow L^2([0, 1], \mathbb{R}^m)$ is a compact and symmetric operator (cf. [ABB16b, Lemma 8.29]). As a consequence, one can prove that the infimum in (47) is attained.

Observe that since every restriction $\gamma|_{[0, s]}$ is not abnormal, the rank of $D_{u^s} E_x$ is maximal, equal to n , for all $s \in (0, 1]$. Then, by Riesz representation Theorem, we find a continuous orthonormal basis $\{v_i^s\}_{i \in \mathbb{N}}$ for $\ker D_{u^s} E_x$, yielding a continuous one-parameter family of isometries $\phi_s : \ker D_{u^s} E_x \rightarrow \mathcal{H}$ on a fixed Hilbert space \mathcal{H} . Since also $s \mapsto Q_s$ is continuous (in the norm topology), we reduce (47) to

$$\alpha(s) = 1 - \sup \{ \langle \phi_s \circ Q_s \circ \phi_s^{-1}(w) | w \rangle_{\mathcal{H}} \mid w \in \mathcal{H}, \|w\|_{\mathcal{H}} = 1 \},$$

where the composition $\tilde{Q}_s := \phi_s \circ Q_s \circ \phi_s^{-1}$ is a continuous one-parameter family of symmetric and compact operators on a fixed Hilbert space \mathcal{H} . The supremum coincides with the largest eigenvalue of \tilde{Q}_s , which is well known to be continuous as a function of s if \tilde{Q}_s is (see [Kat95, V Thm. 4.10]). This proves that α is continuous.

By a rescaling one can see that

$$D_{u^s}^2 E_x(v) = s^2 \iint_{0 \leq \tau \leq t \leq 1} [(P_{s\tau, 1})_* X_{v(s\tau)}, (P_{st, 1})_* X_{v(st)}] |_{\gamma_u(s)} d\tau dt.$$

Taking the limit $s \rightarrow 0$, one can show that $Q_s \rightarrow 0$, hence $\tilde{Q}_s \rightarrow 0$, proving (a).

To prove (b), notice that $\alpha(\bar{s}) = 0$ means that $I - Q_{\bar{s}} \geq 0$, and that there exists a sequence $v_n \in \ker D_{u^{\bar{s}}} E_x$ of controls with $\|v_n\| = 1$ and such that $\|v_n\|_{L^2}^2 - \langle Q_{\bar{s}}(v_n) | v_n \rangle_{L^2} \rightarrow 0$ for $n \rightarrow \infty$. Up to extraction of a sub-sequence, we have that v_n is weakly convergent to some \bar{v} . By compactness of $Q_{\bar{s}}$, we deduce that $\langle Q_{\bar{s}}(\bar{v}) | \bar{v} \rangle_{L^2} = 1$. Since $\|\bar{v}\|_{L^2}^2 \leq 1$, we have $\langle (I - Q_{\bar{s}})(\bar{v}) | \bar{v} \rangle_{L^2} = 0$. Being $A_{\bar{s}}$ a bounded, non-negative symmetric operator, and since $\bar{v} \neq 0$, this implies that $A_{\bar{s}}$ is degenerate.

To prove (c) let us fix $0 \leq s \leq s' \leq 1$ and $v \in \ker D_{u^s} E_x$. Define

$$\hat{v}(t) := \begin{cases} \sqrt{\frac{s'}{s}} v \left(\frac{s'}{s} t \right), & 0 \leq t \leq \frac{s}{s'}, \\ 0, & \frac{s}{s'} < t \leq 1. \end{cases}$$

It follows that $\|\hat{v}\|_{L^2}^2 = \|v\|_{L^2}^2$, $\hat{v} \in \ker D_{u^{s'}} E_x$, and $D_{u^s}^2 E_x(v) = D_{u^{s'}}^2 E_x(\hat{v})$. As a consequence, $\alpha(s) \geq \alpha(s')$.

To prove (d), assume by contradiction that there exists $s_1 > \bar{s}$ such that $\alpha(s_1) = 0$. By monotonicity of point (c), $\alpha(s) = 0$ for every $\bar{s} \leq s \leq s_1$. This implies that every point in the image of $\gamma|_{[\bar{s}, s_1]}$ is conjugate to $\gamma(0)$. Thanks to Lemma 78, the segment $\gamma|_{[\bar{s}, s_1]}$ is abnormal, contradicting the assumption on γ . \square

Lemma 78. *Let $\gamma : [0, 1] \rightarrow M$ be a normal geodesic that does not contain abnormal segments. Then the set $\mathcal{T}_c := \{t > 0 \mid \gamma(t) \text{ is conjugate to } \gamma(0)\}$ is discrete.*

Sketch of the proof. Let $\lambda(t)$ be a normal extremal associated with the geodesic $\gamma(t)$, satisfying condition (N) of Theorem 12. Assume that the set \mathcal{T}_c has an accumulation point $\gamma(\bar{t})$. The fact that the Hamiltonian is non-negative, yields the existence of a segment $\gamma|_{[\bar{t}, \bar{t}+\varepsilon]}$ whose all points are conjugate to $\gamma(0)$. A computation in local coordinates on T^*M shows that $\gamma|_{[\bar{t}, \bar{t}+\varepsilon]}$ is an abnormal extremal, namely satisfies characterization (A) of Theorem 12. \square

We can now prove the following fundamental result.

Theorem 79. *Let $\gamma : [0, 1] \rightarrow M$ be a normal geodesic that does not contain abnormal segments. Then,*

- (i) $t_c := \inf\{t > 0 \mid \gamma(t) \text{ is conjugate to } \gamma(0)\} > 0$.
- (ii) For every $s > t_c$ the curve $\gamma|_{[0, s]}$ is not a minimizer.

Proof. Claim (i) follows directly from Proposition 76 and (a)–(b) of Lemma 77 (or also, independently, from Lemma 78). Using also (d) of Lemma 77, one obtains claim (ii). Indeed, since the Hessian has a negative eigenvalue, we can find a variation joining the same end-points and shorter than the original geodesic, contradicting the minimality assumption. \square

By applying Theorem 79 to every restriction $\gamma|_{[s_1, s_2]}$ with $0 \leq s_1 < s_2 < 1$, we obtain Theorem 16 stated in Section 2.

APPENDIX B. PROOF OF THE POSITIVITY LEMMA

Proof of Lemma 31. Recall that $E_1(t), \dots, E_n(t), F_1(t), \dots, F_n(t)$ is a fixed Darboux frame along the normal extremal $\lambda : [0, 1] \rightarrow T^*M$, with initial covector $\lambda(0) = d_x\phi$. For all $s \in [0, 1]$, consider the Jacobi matrices $\mathbf{J}_s^V(t)$ and $\mathbf{J}_s^H(t)$ defined in Section 3.3, representing the family of Lagrange subspaces

$$\begin{aligned} e_*^{(t-s)\vec{H}} \operatorname{span}\{E_1(s), \dots, E_n(s)\}, & \quad \text{for } \mathbf{J}_s^V(t), \\ e_*^{(t-s)\vec{H}} \operatorname{span}\{F_1(s), \dots, F_n(s)\}, & \quad \text{for } \mathbf{J}_s^H(t), \end{aligned}$$

respectively. Notice that $N_0^V(0) = 0$ and, by the assumption of the Lemma, $N_0^V(t)$ is non-degenerate for all $t \in (0, 1)$. We define $K(t) := N_0^V(t)^{-1}$.

We prove (a) for $t \in (0, 1)$. Since no point $\gamma(t)$ is conjugate to $\gamma(0)$ for $t \in (0, 1)$, it is sufficient to prove that $\det K(t) > 0$ for small $t > 0$. By applying Lemma 25 to the Jacobi matrix $\mathbf{J}_0^V(t)$, we obtain that $W(t) := N_0^V(t)M_0^V(t)^{-1}$ is symmetric and satisfies the Riccati equation

$$(48) \quad \dot{W} = B(t) + A(t)W + WA(t)^* + WR(t)W, \quad W(0) = 0.$$

Equation (48) holds provided that $M_0^V(t)$ is non-degenerate which, since $M_0^V(0) = \mathbb{1}$, holds true for sufficiently small $t > 0$. Again by Lemma 25, $B(t) \geq 0$. Hence, a direct application of the matrix Riccati comparison theorem [BR16, Appendix A] yields that $W(t) \geq 0$ for $t \in [0, \varepsilon]$. Moreover, since $M_0^V(0) = \mathbb{1}$, and $N_0^V(t)$ is non-degenerate for $t \in (0, 1)$, we have that $W(t) = N_0^V(t)M_0^V(t)^{-1} > 0$ for small t . In particular $\det N_0^V(t)M_0^V(t)^{-1} > 0$, which implies $\det K(t) > 0$, yielding (a).

To prove (b) and (c), we need a change of basis lemma, and a new ingredient: the S matrix.

Lemma 80 (Change of basis). *For all $s \in (0, 1)$ and $t \in [0, 1]$, we have*

$$(49) \quad \mathbf{J}_s^V(t) = -\mathbf{J}_0^V(t)N_0^V(s)^{-1}N_0^H(s)N_s^V(0) + \mathbf{J}_0^H(t)N_s^V(0).$$

Moreover, for the original Jacobi matrix $\mathbf{J}(t)$, we have

$$(50) \quad \mathbf{J}(t) = \mathbf{J}_0^V(t)M(0) + \mathbf{J}_0^H(t).$$

Proof. To prove (49), let $\mathbf{J}_s^V(t) = \mathbf{J}_0^V(t)C_V + \mathbf{J}_0^H(t)C_H$ for unique $n \times n$ matrices C_H, C_V . These can be computed by evaluating the horizontal component of both sides at times $t = s$ and $t = 0$. To prove (50), let $\mathbf{J}(t) = \mathbf{J}_0^V(t)D_V + \mathbf{J}_0^H(t)D_H$ for unique $n \times n$ matrices D_H, D_V . The latter can be computed by evaluating both the horizontal and vertical components of $\mathbf{J}(t) = \begin{pmatrix} M(t) \\ N(t) \end{pmatrix}$ at time $t = 0$. \square

Lemma 81 (S matrix). *Consider the smooth family of $n \times n$ matrices*

$$S(t) := N_0^V(t)^{-1}N_0^H(t), \quad \forall t \in (0, 1).$$

Such a matrix is symmetric and $\dot{S}(t) \leq 0$.

Proof. In order to prove the lemma, we start by clarifying the geometric interpretation of $S(t)$. Indeed, observe that, letting

$$Z(t) := E(t) \cdot (M_0^V(t)S(t) - M_0^H(t)),$$

we have

$$(51) \quad E(0) \cdot S(t) - F(0) = e_*^{-t\vec{H}} Z(t).$$

In particular, $Z(t)$ represents a n -tuple of vertical vector fields along $\lambda(t)$, and the left hand side of (51) generates the smooth curve of Lagrange subspaces $\Lambda(t) := e_*^{-t\vec{H}} \mathcal{V}_{\lambda(t)} \subset T_{\lambda(0)}(T^*M)$. In particular $S(t)$ is symmetric, since⁴

$$0 = \sigma(E(0) \cdot S(t) - F(0), E(0) \cdot S(t) - F(0)) = S(t) - S(t)^*,$$

and $S(t)$ is non-increasing:

$$\dot{S}(t) = \sigma_{\lambda(0)} \left(e_*^{-t\vec{H}} Z(t), \frac{d}{dt} e_*^{-t\vec{H}} Z(t) \right) = -2H(Z(t)) \leq 0, \quad \forall t \in (0, s],$$

where in the last equality we identified $Z(t) \in \mathcal{V}_{\lambda(t)} \simeq T_{\gamma(t)}^*M$. The above inequality holds for any smooth family of vertical vector fields $Z(t)$ along $\lambda(t)$, and follows from a straightforward computation in local coordinates around $\lambda(t)$. It corresponds to the fact that H is non-negative on the fibers. \square

Using Lemma 80, one can check that (b) and (c) are equivalent to

- (b') $S(t) - S(s) \geq 0$, for all $t \in (0, s]$,
- (c') $M(0) + S(s) \geq 0$, for all $t \in (0, 1)$.

By Lemma 81, $\dot{S}(t) \leq 0$ for $t \in (0, 1)$, thus proving assertion (b'). To prove (c'), which a fortiori does not depend on t , recall that by the assumptions of Theorem 28,

$$(52) \quad \frac{1}{2}d_{SR}^2(z, y) + \phi(z) \geq 0, \quad \forall z \in M,$$

with equality at $z = x$. Equation (52) yields

$$(53) \quad \frac{1}{2s}d_{SR}^2(z, \gamma(s)) \geq \frac{1}{2}d_{SR}^2(z, y) - (1-s)d_{SR}^2(x, y) \quad \forall z \in M, \quad s \in (0, 1].$$

⁴Here, for n -tuples V, W , the pairing $\sigma(V, W)$ denotes the matrix $\sigma(V_i, W_j)$. Moreover, if A is an $n \times n$ matrix, the notation $W = V \cdot A$ denotes the n -tuple W whose i -th element is $W_i = \sum_{j=1}^n A_{ji}V_j$.

See [CEMS01, Claim 2.4] for a proof of (53) in the Riemannian setting. The proof carries over to the sub-Riemannian case, and is solely a consequence of the triangular and the arithmetic-geometric inequalities. In particular, a property similar to (52) holds replacing y with any midpoint $\gamma(s) \in Z_s(x, y)$, that is

$$\frac{1}{2s} d_{SR}^2(z, \gamma(s)) + \phi(z) \geq \text{const}(s, x, y), \quad \forall z \in M, \quad s \in (0, 1),$$

with equality when $z = x$. By Theorem 16, $\gamma(s)$ is not conjugate to $x = \gamma(0)$ along the unique minimizing curve γ joining x with y , which is not abnormal. Hence $\gamma(s) \notin \text{Cut}(x)$, and $c_s(z) := d_{SR}^2(z, \gamma(s))/2s$ is smooth at $z = x$. Furthermore, ϕ is two times differentiable at x by the assumptions of Theorem 28. Hence, $z \mapsto c_s(z) + \phi(z)$ has a critical point at $z = x$ and a well defined non-negative Hessian

$$(54) \quad \text{Hess}(c_s + \phi)|_x \geq 0,$$

as a quadratic form on $T_x M$. We claim that (54) is equivalent to (c'). To prove this claim, we use the next Lemma, which is essentially, a rewording of Lemma 21.

Lemma 82 (Second differential and Hessian). *Let $f, g : M \rightarrow \mathbb{R}$, twice differentiable at $x \in M$, and such that x is a critical point for $f + g$. Then*

$$(55) \quad d_x^2 f - d_x^2(-g) = \text{Hess}(f + g)|_x,$$

where we used the fact that the space of second differentials at $\lambda = d_x f = d_x(-g)$ is an affine space on the space of quadratic forms on $T_x M$.

Remark 83. The difference of second differentials in the left hand side of (55) is a linear map $T_x M \rightarrow T_\lambda(T^* M)$, with values in $\mathcal{V}_\lambda = T_\lambda(T_x^* M) \simeq T_x^* M$, and it is identified with the quadratic form $\text{Hess}(f + g)|_x : T_x M \times T_x M \rightarrow \mathbb{R}$, i.e. the Hessian of $f + g$ at the critical point x .

We intend to apply Lemma 82 to $\phi + c_s$, which has a minimum point at x . Since $d_x(-c_s) = d_x \phi$, both $e_*^{t\vec{H}} \circ d_x^2 \phi(X(0))$ and $e_*^{t\vec{H}} \circ d_x^2(-c_s)(X(0))$ are n -tuples of Jacobi fields along the same extremal $\lambda(t) = e^{t\vec{H}}(d_x \phi)$. We exploit the relation with Jacobi matrices to compute both second differentials of ϕ and $-c_s$ separately.

Since c_s is smooth in a neighborhood \mathcal{O}_x of x , by [Rif14, Lemma 2.15], we have that $\exp_z(sd_z(-c_s)) = \gamma(s)$ for all $z \in \mathcal{O}_x$ and $s \in (0, 1)$. Thus,

$$(56) \quad \pi \circ e^{s\vec{H}}(d_z(-c_s)) = \gamma(s), \quad \forall z \in \mathcal{O}_x \quad \Rightarrow \quad e_*^{s\vec{H}} \circ d_x^2(-c_s)(T_x M) = \mathcal{V}_{\lambda(s)}.$$

Equation (56) implies that the n -tuple of Jacobi fields $e_*^{t\vec{H}} \circ d_x^2(-c_s)(X(0))$ is associated with the Jacobi matrix $\mathbf{J}_s^V(t)L_s$, for some $n \times n$ matrix L_s . Evaluating at $t = 0$, we obtain $L_s = N_s^V(0)^{-1}$. More precisely, for all $s \in (0, 1)$ we have

$$(57) \quad e_*^{t\vec{H}} \circ d_x^2(-c_s)(X(0)) = E(t) \cdot M_s^V(t)N_s^V(0)^{-1} + F(t) \cdot N_s^V(t)N_s^V(0)^{-1}, \quad t \in [0, 1].$$

Furthermore, by definition of the Jacobi matrix $\mathbf{J}(t) = \begin{pmatrix} M(t) \\ N(t) \end{pmatrix}$, we have

$$(58) \quad e_*^{t\vec{H}} \circ d_x^2 \phi(X(0)) = E(t) \cdot M(t) + F(t) \cdot N(t), \quad t \in [0, 1].$$

By evaluating (57) and (58) at $t = 0$, we obtain

$$\begin{aligned} d_x^2(-c_s)(X(0)) &= E(0) \cdot M_s^V(0)N_s^V(0)^{-1} + F(0), \\ d_x^2 \phi(X(0)) &= E(0) \cdot M(0) + F(0). \end{aligned}$$

In particular, from (54) and Lemma 82 we finally prove (c'), since

$$0 \leq \text{Hess}(\phi + c_s)|_x = M(0) - M_s^V(0)N_s^V(0)^{-1}$$

$$= M(0) + \underbrace{M_0^V(0)}_{=1} \underbrace{N_0^V(s)^{-1} N_0^H(s)}_{=S(s)} - \underbrace{M_0^H(0)}_{=0},$$

where, in the second line, we used Lemma 80 to eliminate $M_s^V(0)$. \square

APPENDIX C. PROOF OF THE DENSITY FORMULA

In the proof of Theorem 43 we used a slightly more general reformulation of [AGS08, Lemma 5.5.3] for non-injective maps. The proof is essentially the same as in the aforementioned reference, and we report it here for completeness.

Lemma. *Let $\rho \in L^1(\mathbb{R}^d)$ be a non-negative function. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a measurable function and let Σ_f be the set where it is approximately differentiable. Assume there exists a Borel set $\Sigma \subseteq \Sigma_f$ such that the difference $\{\rho > 0\} \setminus \Sigma$ is \mathcal{L}^d -negligible. Then $f_{\#}(\rho \mathcal{L}^d) \ll \mathcal{L}^d$ if and only if $|\det(\tilde{d}_x f)| > 0$ for \mathcal{L}^d -a.e. $x \in \Sigma$.*

In this case, letting $f_{\#}(\rho \mathcal{L}^d) = \rho_f \mathcal{L}^d$, we have

$$\rho_f(y) = \sum_{x \in \tilde{f}^{-1}(y) \cap \Sigma} \frac{\rho(x)}{|\det(\tilde{d}_x f)|}, \quad y \in \mathbb{R}^n,$$

with the convention that the r.h.s. is zero if $y \notin \tilde{f}(\Sigma)$. In particular, if we further assume that $\tilde{f}|_{\Sigma}$ is injective, then we have

$$\rho_f(\tilde{f}(x)) = \frac{\rho(x)}{|\det(\tilde{d}_x f)|}, \quad \forall x \in \Sigma.$$

Proof. We start by proving that if $\det(\tilde{d}_x f) > 0$ for \mathcal{L}^d -a.e. $x \in \Sigma$, then $f_{\#}(\rho \mathcal{L}^d) \ll \mathcal{L}^d$. For any Borel function $h : \mathbb{R}^d \rightarrow [0, +\infty]$, we have the area formula for approximately differentiable maps [AGS08, Eq. 5.5.2], that is

$$(59) \quad \int_{\Sigma_f} h(x) |\det(\tilde{d}_x f)| dx = \int_{\mathbb{R}^d} \sum_{x \in \tilde{f}^{-1}(y) \cap \Sigma_f} h(x) dy.$$

Since f is measurable, then $f(x) = \tilde{f}(x)$ up to a \mathcal{L}^d -negligible set (see [AGS08, Remark 5.5.2]), and hence $f_{\#}(\rho \mathcal{L}^d) = \tilde{f}_{\#}(\rho \mathcal{L}^d)$. Since $|\det(\tilde{d}_x f)| > 0$ for \mathcal{L}^d -a.e. $x \in \Sigma \subseteq \Sigma_f$, the function $h : \mathbb{R}^d \rightarrow [0, +\infty]$ given by

$$h(x) := \begin{cases} \frac{\rho(x) \chi_{\tilde{f}^{-1}(B) \cap \Sigma}(x)}{|\det(\tilde{d}_x f)|} & x \in \Sigma, \\ 0 & \text{otherwise,} \end{cases}$$

is Borel and well defined. Hence, for any Borel set $B \subset \mathbb{R}^d$, we obtain

$$\begin{aligned} f_{\#}(\rho \mathcal{L}^d)(B) &= (\rho \mathcal{L}^d)(\tilde{f}^{-1}(B) \cap \Sigma) \\ &= \int_{\tilde{f}^{-1}(B) \cap \Sigma} \rho(x) dx \\ &= \int_{\Sigma_f} \rho(x) \chi_{\tilde{f}^{-1}(B) \cap \Sigma}(x) dx \\ &= \int_{\mathbb{R}^d} \sum_{x \in \tilde{f}^{-1}(y) \cap \Sigma_f} \frac{\rho(x) \chi_{\tilde{f}^{-1}(B) \cap \Sigma}(x)}{|\det(\tilde{d}_x f)|} dy \\ &= \int_B \sum_{x \in \tilde{f}^{-1}(y) \cap \Sigma} \frac{\rho(x)}{|\det(\tilde{d}_x f)|} dy, \end{aligned}$$

where in the fourth line we used (59). In particular if $\mathcal{L}^d(B) = 0$, then also $f_{\#}(\rho \mathcal{L}^d)(B) = 0$.

The inverse implication is proved by contradiction. Assume that there exists a Borel set $B \subset \Sigma$ with $\mathcal{L}^d(B) > 0$ and $\det(\tilde{d}_x f) = 0$ on B . Then, the area formula (59) with $h = \chi_{B \cap \Sigma}$ yields

$$0 = \int_{\mathbb{R}^d} \sum_{x \in \Sigma_f \cap \tilde{f}^{-1}(y)} \chi_{B \cap \Sigma}(x) dy \geq \mathcal{L}^d(\tilde{f}(B)).$$

On the other hand, since $f_{\#}(\rho \mathcal{L}^d) = \tilde{f}_{\#}(\rho \mathcal{L}^d)$, we have

$$f_{\#}(\rho \mathcal{L}^d)(\tilde{f}(B)) = \int_{\tilde{f}^{-1}(\tilde{f}(B))} \rho(x) dx \geq \int_B \rho(x) dx = \mathcal{L}^d(B) > 0.$$

Thus $f_{\#}(\rho \mathcal{L}^d)$ cannot be absolutely continuous w.r.t. \mathcal{L}^d . \square

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