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# Estimation of the Multivariate Conditional-Tail-Expectation for extreme risk levels: illustrations on environmental data-sets

Elena Di Bernardino<sup>1</sup>, Clémentine Prieur<sup>2</sup>

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## Abstract

This paper deals with the problem of estimating the Multivariate version of the Conditional-Tail-Expectation introduced in the bivariate framework in Di Bernardino et al. [16], and generalized in Cousin and Di Bernardino [13]. We propose a new semi-parametric estimator for this risk measure, essentially based on statistical extrapolation techniques, well designed for extreme risk levels. Following Cai et al. [9], we prove a central limit theorem. We illustrate the practical properties of our estimator on simulations. The performances of our new estimator are discussed and compared to the ones of the empirical Kendall's process based estimator, previously proposed in Di Bernardino and Prieur [17]. We conclude with two applications on real data-sets: rainfall measurements recorded at three stations located in the south of Paris (France) and the analysis of strong wind gusts in the north west of France.

*Keywords:* Multivariate extreme value theory, multivariate risk measures, central limit theorem, hydrological applications.

*2000 MSC:* 62H12, 62H05, 60G70.

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## Introduction

### *Multivariate risk-measures*

Modeling and quantifying uncertainties related to extreme events is of main interest in environmental sciences. In relation with oceanic and atmospheric domains, e.g., risk assessment is very important for several concerns: hydrological extreme events, cyclonic intensity, storm surges, . . .

Most of the time, environmental risks involve several aleas which are often correlated. A flood, e.g., can be described by three main characteristics: the peak flow, the volume and the duration. As these three quantities are correlated, it is important to define and to estimate the risk in a multivariate setting. For the same reasons, the design of facilities installed alongside of rivers should be based on multivariate extreme value analysis. In the case where the installation lies downstream the confluence of two rivers, neglecting in the risk analysis the correlations between both rivers may lead to an over- or under-estimation of the risk, involving either unnecessary costs or the construction of unsafe dams, with potentially dramatic consequences.

The classical univariate frequency analysis in hydrology, and more generally in environmental sciences, focuses on the estimation of the probably most popular risk measure: the *return level*. A return level with a return period of  $T = 1/p$  years is a threshold  $z_p$  whose probability of exceedance is  $p$ . It is also

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known in other communities as the *Value-at Risk* (VaR). An alternative, that takes into account the intensity of an event above a given threshold, is the mean excess function which is defined as the mean of this event given that it exceeds the threshold, i.e.,  $\mathbb{E}(X \mid X \geq z_p)$ , with  $X$  the variable of interest (flows, rainfall, temperature, ...). This measure is also known as the Conditional-Tail-Expectation (CTE). On the opposite of VaR, it is a coherent risk measure according to Artzner et al. [2].

As already mentioned, it is often insufficient to consider a single real measure to quantify risks, especially when the risk-problem is affected by other external risk factors whose sources cannot be controlled. Note that moreover the evaluation of an individual risk may strongly be affected by the degree of dependence amongst all risks and these risks may also be strongly heterogeneous. This is therefore challenging for practitioners to estimate a multivariate return period and to select a specific design event starting from multivariate dangerous hydrological situations. However, the notion of return period in the multivariate setting is not univalent (see for instance Vandenberghe et al. [37]). From the years 2000 onward, several approaches for the latter problem have evolved over the years, essentially by proposing different multivariate risk measures, following the tendency of describing hydrological phenomena with multiple variables. Recently, level-curves and level sets associated to the multivariate risk vector have been proposed as risk measures in multivariate hydrological models because of their many advantages: they are simple, intuitive, interpretable and probability-based (see Chebana and Ouarda [11], de Haan and Huang [15]). Furthermore, as noticed by Embrechts and Puccetti [21], it can be viewed as a natural multivariate version of the univariate quantile. The interested reader is also referred to Tibiletti [35], Belzunce et al. [6], Nappo and Spizzichino [29].

Recently Cai et al. [9] introduced, in a financial setting, the Marginal Expected Shortfall (MES):

$$\mathbb{E}[X \mid Y > Q_Y(1 - p)], \quad (1)$$

where  $X$  denotes the loss return on the equity of a financial institution,  $Y$  the loss return of the entire market and  $Q_Y$  the quantile function of  $Y$ . The MES in (1) is an important factor when measuring the systemic risk of financial institutions and it can be considered as a bivariate version of the Conditional-Tail-Expectation (CTE) presented above. Some other commonly used multivariate CTE measures, for a  $d$ -dimensional vector of risks  $\mathbf{X} = (X^1, X^2, \dots, X^d)$ , are defined for  $i = 1, \dots, d$  as

$$\mathbb{E}[X^i \mid S > Q_S(t)], \quad \mathbb{E}[X^i \mid X^{(1)} > Q_{X^{(1)}}(t)], \quad \mathbb{E}[X^i \mid X^{(d)} > Q_{X^{(d)}}(t)], \quad (2)$$

with  $S = X^1 + \dots + X^d$  the total risk,  $X^{(1)} = \min\{X^1, \dots, X^d\}$  and  $X^{(d)} = \max\{X^1, \dots, X^d\}$  two extreme risks (see for instance Cai and Li [8], Bargès et al. [4], Landsman and Valdez [27]).

*Statistical inference.* The problem of consistent estimation of the univariate quantile-based risk-measures (VaR and CTE) has received attention in literature essentially in the univariate case. There are less papers on the estimation of multivariate risk-measures, due to a number of theoretical and practical reasons. In the recent literature, some efforts have been done to provide an estimation of the multivariate measures defined by (2) above (see, for instance, Hua and Joe [26], Asimit et al. [3]). In Acharya et al. [1], an estimator for the MES defined in (1) is provided by assuming a specific linear relationship between  $X$  and  $Y$ . A similar setting has been adopted in Brownlees and Engle [7], where a nonparametric kernel estimator of the MES in (1) is proposed. Such a kernel estimation method, however, performs well only if the risk level is not too high (i.e., the level  $p$  should be substantially larger than  $1/n$ , where  $n$  is the size of the considered sample). For a large nonparametric class of bivariate distributions for  $(X, Y)$ , Cai et al. [9] construct an estimator of the risk measure in (1) and they establish the asymptotic normality when  $p = p(n) \rightarrow 0$ , as the sample size  $n \rightarrow \infty$ .

### *Considered multivariate Conditional-Tail-Expectation*

In the following we deal with a version of the multivariate Conditional-Tail-Expectation, previously proposed by Di Bernardino et al. [16] (see also Cousin and Di Bernardino [13], Di Bernardino and Prieur [17]). Let  $I = \{1, \dots, d\}$ . It is constructed as the conditional expectation of a  $d$ -dimensional vector of risks  $\mathbf{X} = (X^1, X^2, \dots, X^d)$  following the distribution function  $F$ , given that the associated multivariate probability integral transformation  $Z := F(\mathbf{X})$  is large. More precisely, for  $i \in I$ , we will consider the multivariate Conditional-Tail-Expectation:

$$\mathbb{E}[X^i | Z > Q_Z(1 - p)], \quad \text{for } p \in (0, 1), \quad (3)$$

and  $p$  *small enough*. Then, we take the conditional expectation of  $\mathbf{X}$  conditionally to the fact that it belongs to the joint risk area  $\{\mathbf{x} \in \mathbb{R}^d : F(\mathbf{x}) \geq 1 - p\}$ .

Remark that our measure is based on the Kendall's distribution function  $K(t) = \mathbb{P}[Z \leq t]$ , for  $t \in [0, 1]$ . For this reason, risk measure in (3) can be used for hydrological risk management in the same vein of the multivariate RP in Salvadori et al. [33] and Salvadori et al. [32]. Furthermore, conversely to risk measures in (2), the multivariate risk measure proposed in (3) is a non-aggregated risk measure. Indeed, hydrological variables can be of different nature (e.g. precipitation, temperature, discharge, ...), prohibiting the aggregation of the various components. For this reason, measure in (3) just considers as conditioning extreme event the behavior of the copula associated to the  $d$  different risks (for further details see Section 1). The interested reader is also referred to Cousin and Di Bernardino [12, 13] for the study of the theoretical properties of the multivariate Conditional-Tail-Expectation in (3).

*Statistical inference for multivariate Conditional-Tail-Expectation in (3).* Some consistent estimator of the multivariate risk measure  $\mathbb{E}[X^i | Z > t]$ , for fixed  $t \in (0, 1)$ , has been provided by Di Bernardino et al. [16], who proposed a plug-in estimator based on the consistent estimation of the whole level sets associated to the  $d$ -dimensional vector of risks  $\mathbf{X} = (X^1, X^2, \dots, X^d)$ . As the level sets are not compact, their estimation procedure requires the choice of an increasing truncation sequence  $(T_n)_{n \geq 1}$ . The non-optimal rate of convergence provided by the authors depends on the rate of convergence of  $(T_n)_{n \geq 1}$  to infinity. Making the "best choice" for  $(T_n)_{n \geq 1}$  is not trivial, and requires the knowledge of the tail behavior of  $\mathbf{X}$ , at least in its generic form. The interested reader is referred to Di Bernardino et al. [16] for further details. Recently, Di Bernardino and Prieur [17] proposed a non-parametric estimator for  $\mathbb{E}[X^i | Z > t]$  based on the estimation of the Kendall's process. For this estimator they provide a functional central limit theorem without requiring the calibration of extra parameters or sequences. However, in Di Bernardino and Prieur [17] a global good performance of the proposed estimator is illustrated only for moderate to high (but not extreme) fixed risk levels  $t$ .

Conversely, risk analysts are frequently interested in extreme risk levels, (i.e. quantiles outside the sample; see, e.g. Embrechts et al. [20]). For this reason, in the present paper, we will develop a consistent estimation procedure to estimate the multivariate Conditional-Tail-Expectation defined in Equation (3) for extreme risk levels (that is for  $p < 1/n$ , where  $n$  is the sample size). For each  $i \in I$ , the approach for estimating  $\mathbb{E}[X^i | Z > Q_Z(1 - p)]$ , is based on the bivariate estimation procedure proposed in Cai et al. [9] for the estimation of the MES in (1). However, a main difference relies on the fact that  $Z$  is a latent variable, which is not observed and has to be estimated. Under reasonable assumptions on the right-upper tail dependence between  $X_i$  and  $Z$ , for  $i = 1, \dots, d$  (see, e.g., Genest and Rivest [23]), we study the asymptotic properties of our plug-in estimator when  $p = p(n)$  decreases to zero as the sample size  $n$  tends to infinity.

## Organization of the paper

The paper is organized as follows. In Section 1, we introduce some notation, tools and preliminary assumptions. In Section 2.1, we propose our estimation procedure for the multivariate Conditional-Tail-Expectation defined in (3), based on Extreme Value Theory. In Section 2.2 we establish the asymptotic normality for the proposed semi-parametric estimator. The practical properties of our estimator are further investigated, and compared to the ones of alternative empirical approaches, on simulated data-sets in Section 3. Finally, in Section 4 we consider first a 3-dimensional rainfall real data-set, then a wind data-set. Technical proofs and auxiliary results are postponed to Appendix.

## 1. Preliminaries and notation

Let  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . Let  $\mathbf{X} = (X^1, X^2, \dots, X^d)$  be a  $d$ -dimensional positive<sup>3</sup> random vector with distribution function  $F$ . Define  $Z = F(\mathbf{X})$  and the associated multivariate Kendall distribution function  $K(t) = \mathbb{P}[Z \leq t]$ , for  $t \in [0, 1]$ . For more details on the multivariate probability integral transformation (or multivariate Kendall distribution), the interested reader is referred to Capéraà et al. [10], Genest and Rivest [23], Nelsen et al. [31], Genest et al. [22] and Belzunce et al. [6].

As a consequence of Sklar's Theorem, the Kendall distribution only depends on the dependence structure or the copula function  $C$  associated with  $\mathbf{X}$  (see Sklar [34]). Thus, we also have  $K(t) = \mathbb{P}[C(\mathbf{V}) \leq t]$ , where  $\mathbf{V} = (V_1, \dots, V_d)$  with uniform marginals  $V_1 = F_{X^1}(X^1), \dots, V_d = F_{X^d}(X^d)$ . The analytical formulation of the Kendall distribution is in general not available. However, for the particular case of multivariate Archimedean copulas, it can be derived explicitly (see Section 3).

Let  $U_Z = (\frac{1}{1-K})^\leftarrow$  be the tail quantile function of  $Z$ , where  $\leftarrow$  denotes the left-continuous inverse. In this paper, for  $i \in I$ , we aim to estimate the quantity

$$\theta_p^i := \mathbb{E}[X^i \mid Z > U_Z(1/p)] \quad \text{for } p \in (0, 1),$$

on independent and identically distributed (i.i.d.)  $d$ -dimensional observations,  $(\mathbf{X}_j)_{j \in I}$  from  $F$ , for small values of  $p$  in a sense detailed below.

In the whole paper we will suppose that, for all  $(x, z) \in [0, \infty]^2 \setminus \{(\infty, \infty)\}$ , and for all  $i \in I$ , the following limits exist:

$$\lim_{t \rightarrow \infty} t \mathbb{P} \left[ 1 - F_i(X^i) \leq \frac{x}{t}, 1 - K(Z) \leq \frac{z}{t} \right] := R_{(X^i, Z)}(x, z). \quad (4)$$

For  $i \in I$ , the function  $R_{(X^i, Z)}$  completely determines the so-called stable tail dependence function  $l_{(X^i, Z)}$ , as for all  $x, z \geq 0$ ,

$$l_{(X^i, Z)}(x, z) = x + z - R_{(X^i, Z)}(x, z),$$

(see, e.g., Drees and Huang [18], Beirlant et al. [5]; Chapter 8.2).

Furthermore, for the marginal distribution  $F_i$  we assume that  $X^i$  follows a distribution with a heavy right tail, i.e., there exists  $\gamma^i > 0$  such that for all  $x > 0$ ,

$$\lim_{t \rightarrow \infty} \frac{U_i(tx)}{U_i(t)} = x^{\gamma^i}, \quad (5)$$

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<sup>3</sup>In the following, we restrict ourselves to  $\mathbb{R}_+^d$ . This choice is motivated essentially by our applications in environmental risk theory, where random variables consist in rainfall measurements (in *mm*), thus defined on a positive support. However the results in this paper can be adapted also in  $\mathbb{R}^d$ .

where  $U_i = (\frac{1}{1-F_i})^\leftarrow$ , for  $i \in I$   $\gamma^i$  is the extreme value index associated to  $F_i$ .

We now introduce different Gaussian processes that will be useful in the following to state the asymptotic normality for the estimator of  $\theta_p^i$ . Let  $i \in I$ . Let  $W_R$  be a zero mean Gaussian process on  $[0, \infty]^2 \setminus \{(\infty, \infty)\}$  with covariance structure

$$\mathbb{E}[W_{R_{(X^i, Z)}}(x_1, z_1) W_{R_{(X^i, Z)}}(x_2, z_2)] = R_{(X^i, Z)}(x_1 \wedge x_2, z_1 \wedge z_2).$$

Let

$$\Theta^i = (\gamma^i - 1) W_{R_{(X^i, Z)}}(\infty, 1) + \left( \int_0^\infty R_{(X^i, Z)}(s, 1) ds^{-\gamma^i} \right)^{-1} \int_0^\infty W_{R_{(X^i, Z)}}(s, 1) ds^{-\gamma^i}, \quad (6)$$

and, for  $q \in (0, +\infty)$ ,

$$\Gamma^i(q) = \frac{\gamma^i}{\sqrt{q}} \left( -W_{R_{(X^i, Z)}}(q, \infty) + \int_0^q s^{-1} W_{R_{(X^i, Z)}}(s, \infty) ds \right). \quad (7)$$

For  $q = 0$  or  $q = +\infty$ ,  $\Gamma^i(q)$  is a  $N(0, \gamma^{i2})$  random variable independent of  $W_R$ .

## 2. Main results

### 2.1. Estimation procedure

Let  $n_1$  and  $n_2 \in \mathbb{N}^*$  and the sample size  $n := n_1 + n_2$ . We consider  $(\mathbf{X}_j)_{j \in I}$  a  $d$ -dimensional i.i.d. sample of  $\mathbf{X}$ . For all  $t \in \mathbb{R}_+^d$  we define the  $d$ -dimensional empirical distribution function of  $\mathbf{X}$  based on  $n_2$  observations of this sample as,

$$F_{n_2}(t) = \frac{1}{n_2} \sum_{j=n_1+1}^{n_1+n_2} 1_{\{\mathbf{X}_j \leq t\}}.$$

For all  $j = 1, \dots, n_1$  we define  $Z_j = F(\mathbf{X}_j)$  and  $\tilde{Z}_j = F_{n_2}(\mathbf{X}_j)$ .

Following classical extrapolation techniques of EVT, we construct an estimator of  $\theta_p^i$ , for  $i \in I$ , by a two-stage approach. Let  $k = k(n_1)$  be an intermediate sequence of integers which satisfies  $k \rightarrow \infty$  and  $k/n_1 \rightarrow 0$ , as  $n_1 \rightarrow \infty$ . Firstly, we consider the estimation of  $\theta_{\frac{k}{n_1}}^i$ , i.e. the Conditional-Tail-Expectation at an intermediate (not extreme) probability level  $\frac{k}{n_1}$ . We can estimate non-parametrically  $\theta_{\frac{k}{n_1}}^i$  by taking the empirical average of the  $X^i$  of those selected observations:

$$\hat{\theta}_{\frac{k}{n_1}, n_2}^i = \frac{1}{k} \sum_{j=1}^{n_1} X_j^i 1_{\{\tilde{Z}_j > \tilde{Z}_{n_1-k, n_1}\}}, \quad (8)$$

where  $\tilde{Z}_{n_1-k, n_1}$  is the  $(n_1 - k)$ -th order statistic of  $\tilde{Z}_1, \dots, \tilde{Z}_{n_1}$ .

Secondly, using an extrapolation method based on Equation (4), Proposition 1 in Cai et al. [9] applied to the bivariate vector  $(X^i, Z)$  and a second order strengthening of Equation (5) (see Assumption (b) below), we have that, for  $n_1 \rightarrow \infty$ ,

$$\theta_p^i \sim \frac{U_i(1/p)}{U_i(n_1/k)} \theta_{\frac{k}{n_1}}^i \sim \left( \frac{k}{n_1 p} \right)^{\gamma^i} \theta_{\frac{k}{n_1}}^i. \quad (9)$$

In order to apply the asymptotic approximation in Equation (9), we need to estimate the tail index  $\gamma^i$ . To this aim, we will consider the Hill estimator (see Hill [25]), i.e.

$$\hat{\gamma}^i = \frac{1}{k_i} \sum_{j=1}^{k_i} \ln(X_{n_1-j+1, n_1}^i) - \ln(X_{n_1-k_i, n_1}^i), \quad (10)$$

where  $k_i = k_i(n_1)$  is an intermediate sequence of integers and  $X_{j, n_1}^i$ , for  $j = 1, \dots, n_1$ , is the  $j$ -th order statistic of  $X_1^i, \dots, X_{n_1}^i$ .

Finally, using Equations (8), (9) and (10), we estimate  $\theta_p^i$  by

$$\hat{\theta}_{p(n_1), n_2}^i = \left( \frac{k}{n_1 p} \right)^{\hat{\gamma}^i} \hat{\theta}_{\frac{k}{n_1}, n_2}^i. \quad (11)$$

The asymptotic normality of our estimator is stated in Theorem 2.1 below, for  $n_1, n_2 \rightarrow \infty$ . The limit process can be written as a combination of both processes  $\Theta^i$  and  $\Gamma^i(q)$  (see Equations (6) and (7)). In particular, the process  $\Theta^i$  plays a central role in Proposition 2.1 which describes the asymptotic behavior of  $\hat{\theta}_{\frac{k}{n_1}, n_2}^i$ . The process  $\Gamma^i(q)$  is related to the asymptotic behavior of  $\hat{\gamma}^i$ .

To conclude this section, we remark that the split-up of the data in two samples of sizes  $n_1$  and  $n_2$ , respectively, seems artificial. One would like to consider the estimator

$$\check{\theta}_{\frac{k}{n}}^i = \frac{1}{k} \sum_{j=1}^n X_j^i 1_{\{\hat{Z}_j > \hat{Z}_{n-k, n}\}}, \quad (12)$$

where  $\hat{Z}_{n-k, n}$  is the  $(n-k)$ -th order statistic of  $\hat{Z}_1, \dots, \hat{Z}_n$ , with for  $j = 1, \dots, n$  and  $\hat{Z}_j = F_n(\mathbf{X}_j)$ . However, the theoretical study of its asymptotic properties still remains an open issue, even with the knowledge of recent studies such as the one in [36].

## 2.2. Asymptotic normality

In this section we characterize the limit distribution of  $\hat{\theta}_{p(n_1), n_2}^i$  in Equation (11). The proof of our main result, i.e., Theorem 2.1 below, requires the following conditions:

(a.1) There exist  $\beta > \max_{i \in I} \gamma^i$  and  $\tau < 0$  such that, for any  $i \in I$ , as  $t \rightarrow \infty$ ,

$$\sup_{\{0 < x < \infty, 1/2 \leq z \leq 2\}} \frac{|t \mathbb{P} [1 - F_i(X^i) \leq \frac{x}{t}, 1 - K(Z) \leq \frac{z}{t}] - R_{(X^i, Z)}(x, z)|}{x^\beta \wedge 1} = O(t^\tau).$$

(a.2) The Kendall distribution function  $K(t)$ ,  $t \in [0, 1]$  of  $Z = F(\mathbf{X})$  admits a continuous density  $K'(t)$  on  $(0, 1]$ .

(a.3) There exist  $p_0 < \frac{1}{\max_{i \in I} \gamma^i}$ ,  $1/p_0 + 1/q_0 = 1$  and  $\epsilon > 0$  such that  $\sup \left\{ \frac{n_1}{\sqrt{k}} n_2^{\frac{-1+\epsilon}{2}}, \sqrt{n_1} \left( \frac{k}{n_1} \right)^{\frac{1}{p_0} - \frac{1}{2}} n_2^{\frac{-1+\epsilon}{2q_0}} \right\} \rightarrow 0$ , as  $n_1 \rightarrow \infty$  and  $n_2 \rightarrow \infty$ .

(b) For  $i \in I$ , there exist  $\rho_i < 0$  and an eventually positive or negative function  $A_i$  such that as  $t \rightarrow \infty$ ,  $A_i(tx)/A_i(t) \rightarrow x^{\rho_i}$  for all  $x > 0$  and

$$\sup_{x > 1} |x^{-\gamma^i} \frac{U_i(tx)}{U_i(t)} - 1| = O(A_i(t)).$$



- (c) For  $i \in I$ , as  $n_1 \rightarrow \infty$ ,  $\sqrt{k_i} A_i(n_1/k_i) \rightarrow 0$ , where  $k_i(n_1)$  is the intermediate sequence of integers in Equation (10).
- (d) For  $i \in I$ , as  $n_1 \rightarrow \infty$ ,  $k = O(n_1^\alpha)$  for some  $\alpha < \min\left(\frac{-2\tau}{-2\tau+1}, \frac{2\gamma^i \rho_i}{2\gamma^i \rho_i + \rho_i - 1}\right)$ , where  $k(n_1)$  is the intermediate sequence of integers in Equation (8).

Under assumptions presented above, we now introduce the central limit theorem for our estimator  $\hat{\theta}_{p(n_1), n_2}^i$ .

**Theorem 2.1** *Let  $i \in I$  and  $p = p(n_1) \rightarrow 0$ , for  $n_1 \rightarrow \infty$ . Assume that Assumptions (a.1)-(d) hold true and  $\gamma^i \in (0, 1/2)$ . Assume  $d_{n_1} := \frac{k}{n_1 p} \geq 1$ ,  $r := \lim_{n_1 \rightarrow \infty} \frac{\sqrt{k} \ln(d_{n_1})}{\sqrt{k_i}} \in [0, +\infty]$  and  $q = \lim_{n \rightarrow \infty} \frac{k_i}{k} \in [0, +\infty]$ . If  $\lim_{n_1 \rightarrow \infty} \frac{\ln(d_{n_1})}{\sqrt{k_i}} = 0$ , then for  $n_1, n_2 \rightarrow \infty$ ,*

$$v_{n_1} \left( \frac{\hat{\theta}_{p(n_1), n_2}^i}{\theta_{p(n_1)}^i} - 1 \right) \rightarrow \begin{cases} \Theta^i + r \Gamma^i(q), & \text{if } r \leq 1, \\ \frac{1}{r} \Theta^i + \Gamma^i(q), & \text{if } r > 1, \end{cases} \quad (13)$$

where  $v_{n_1} = \min(\sqrt{k}, \frac{\sqrt{k_i}}{\ln(d_{n_1})})$ , and for  $q \in (0, +\infty)$ ,  $\text{Var}(\Theta^i) = ((\gamma^i)^2 - 1) - b^2 \int_0^\infty R_{(X^i, Z)}(s, 1) ds^{-2\gamma^i}$ ,  $\text{Var}(\Gamma^i(q)) = (\gamma^i)^2$ ,  $\text{Cov}(\Gamma^i(q), \Theta^i) = \frac{\gamma^i}{\sqrt{q}}(1 - \gamma^i + \frac{b}{q\gamma^i}) R_{(X^i, Z)}(q, 1) - \frac{\gamma^i(1-\gamma^i)b}{\sqrt{q}} \int_0^q R_{(X^i, Z)}(s, 1) s^{-1-\gamma^i} ds - \frac{\gamma^i(1-\gamma^i)}{\sqrt{q}} \int_0^q R_{(X^i, Z)}(s, 1) s^{-1} ds - \frac{\gamma^{i2}b}{\sqrt{q}} \int_0^q R_{(X^i, Z)}(s, 1) \ln(\frac{q}{s}) s^{-1-\gamma^i} ds$ , with  $b = \left( \int_0^\infty R_{(X^i, Z)}(s, 1) ds^{-\gamma^i} \right)^{-1}$ .

**Remark 1** We now comment Assumptions (a.1), (a.2), (a.3), (b), (c) and (d) introduced above.

- Assumption (a.1) is a second order strengthening of the condition in (4) (see also Condition (7.2.8) in de Haan and Ferreira [14]). It is classically required in EVT to derive central limit theorems. Note that this second order assumption is required on the bivariate vectors  $(X^i, Z)$ , for  $i \in I$ . Moreover, the constants  $\beta$  and  $\tau$  do not depend on  $i \in I$ .
- Assumption (a.2) is a regularity assumption on the Kendall density  $K'(t)$ . Note that this assumption is satisfied for a large class of multivariate distributions, as the class of Archimedean copulas, bivariate extreme copulas, Farlie-Gumbel-Morgenstern class of distributions.
- Assumption (a.3) describes the relationship between the sample sizes  $n_1$  and  $n_2$  (see Proposition A.1).
- Assumption (b) is a second order strengthening of the tail behaviour condition for  $X^i$  in Equation (5) (see also Condition (3.2.4) in de Haan and Ferreira [14]).
- Assumptions (c) and (d) deal with the intermediate sequences  $k_i$  and  $k$  respectively (see also Cai et al. [9]).

The proof of Theorem 2.1 above is mainly based on arguments in de Haan and Ferreira [14], Cai et al. [9], as far as on Proposition 2.1 below, whose proof is postponed to Appendix for sake of clarity.

**Proposition 2.1** *Let  $i \in I$ . Under conditions of Theorem 2.1, we get for  $n_1, n_2 \rightarrow \infty$ ,*

$$\sqrt{k} \left( \frac{\hat{\theta}_{\frac{k}{n_1}, n_2}^i}{\theta_{\frac{k}{n_1}}^i} - 1 \right) \xrightarrow{\mathbb{P}} \Theta^i,$$

where  $\Theta^i$  is defined in Equation (6).



The proof of Proposition 2.1 is postponed to Appendix.

*Proof of Theorem 2.1:* We write

$$\frac{\widehat{\theta}_{p(n_1),n_2}^i}{\theta_{p(n_1)}^i} = \frac{\widehat{d}_{n_1}^i}{d_{n_1}^i} \times \frac{\widehat{\theta}_{\frac{k}{n_1},n_2}^i}{\theta_{\frac{k}{n_1}}^i} \times \frac{d_{n_1}^i \theta_{\frac{k}{n_1}}^i}{\theta_{p(n_1)}^i} := L_1^{n_1} \times L_2^{n_1,n_2} \times L_3^{n_1}.$$

We now analyse these three factors separately.

Under Assumptions (b) and (c), since  $L_1^{n_1}$  does not depend on  $n_2$  then, as in the proof of Theorem 4.3.8 in de Haan and Ferreira [14], we get for  $n_1 \rightarrow \infty$ ,

$$\frac{\sqrt{k_i}}{\ln(d_{n_1})} (L_1^{n_1} - 1) - \Gamma^i(q) \xrightarrow{\mathbb{P}} 0, \quad (14)$$

where  $\Gamma^i(q)$  is defined in Equation (7).

The asymptotic behaviour of  $L_2^{n_1,n_2}$ , for  $n_1, n_2 \rightarrow \infty$  is stated in Proposition 2.1 above.

Finally for  $L_3^{n_1}$ , by Equations (4), (5), and their second order strengthening given by Assumptions (b) and (d), we know from the proof of Theorem 1 in Cai et al. [9] that

$$L_3^{n_1} = 1 + o\left(\frac{1}{\sqrt{k}}\right). \quad (15)$$

Now, combining (14), (15) and Proposition 2.1 we obtain the convergence result of Theorem 2.1, where the covariance matrix of  $(\Theta^i, \Gamma^i(q))$  follows from straightforward computation.  $\square$

**Remark 2 (Asymptotic independence and range of  $\gamma^i$ )** We discuss here two problematic points in the assumptions of our main result (see Theorem 2.1).

1. Assumption (a.1) excludes asymptotic independence, i.e., the case  $R_{(X^i, Z)} \equiv 0$ . However, the robustness with respect to this assumption will be illustrated by an example in Section 3.
2. Let  $i \in I$ . The assumption  $\gamma^i \in (0, 1/2)$  is necessary for Theorem 2.1, i.e., the result does not hold true when  $\gamma^i = 1/2$  (see proof of Proposition A.2). For the consistency of  $\widehat{\theta}_{p(n_1),n_2}^i$  this assumption can be relaxed to  $\gamma^i \in (0, 1)$ . Indeed, if we assume that  $(X^i, Z)$  satisfies condition in (4) and (b),  $R_{(X^i, \bar{Z})}(1, 1) > 0$ ,  $\lim_{n_1 \rightarrow \infty} \frac{\log(d_{n_1})}{\sqrt{k_i}} = 0$  and  $\gamma^i \in (0, 1)$ , then

$$\frac{\widehat{\theta}_{p(n_1),n_2}^i}{\theta_{p(n_1)}^i} \xrightarrow{\mathbb{P}} 1,$$

for  $n_1, n_2 \rightarrow \infty$ . To illustrate this situation, in our simulation study we provide an example with  $\gamma^i \notin (0, 1/2)$  (see Section 3).

### 3. Simulation Study

In this section, a simulation and comparison study is implemented to investigate the finite sample performance of our estimator of the multivariate Conditional-Tail-Expectation in (11). We focus in the following on different 2-dimensional distribution functions. Note that in Section 4 we will provide a multivariate analysis for a real data-set in larger dimensions.

To illustrate the finite sample properties of our estimator, we draw 500 samples from each considered distribution. Based on each sample, we estimate  $\theta_{p(n_1)}^i$  for different values of  $n_1$  and  $p(n_1)$ . We present the boxplots of the ratio between the estimates and the true values, i.e.,  $\widehat{\theta}_{p(n_1), n_2}^i / \theta_{p(n_1)}^i$ .

The asymptotic normality proved in Theorem 2.1, will be also investigated. For  $r < \infty$ , Theorem 2.1 can be expressed as

$$\sqrt{k} \left( \frac{\widehat{\theta}_{p(n_1), n_2}^i}{\theta_{p(n_1)}^i} - 1 \right) \rightarrow \Theta^i + r \Gamma^i(q),$$

for  $n_1, n_2 \rightarrow \infty$ . Notice that the limit distribution is a centered normal distribution. Let  $(\sigma_p^i)^2 := \frac{1}{k} \text{Var}(\Theta^i + r \Gamma^i(q))$ . Then, we compare using Q-Q plots, the distribution of  $\frac{1}{\sigma_p^i} \ln \left( \widehat{\theta}_{p(n_1), n_2}^i / \theta_{p(n_1)}^i \right)$  with the limit distribution  $N(0, 1)$ . Specific values for  $k(n_1)$  and  $k_i(n_1)$  are chosen for each sample size  $n_1$ , accordingly to a selection procedure similar to that described in the real data application (see Section 4).

Finally in this section, we compare the performance of our estimator with the empirical one, i.e.,

$$\hat{\theta}_{p, emp}^i = \frac{1}{[np]} \sum_{j=1}^n X_j^i 1_{\{\tilde{Z}_j > \tilde{Z}_{n-[np], n}\}}, \quad (16)$$

with  $\tilde{Z}_j = F_n(\mathbf{X}_j)$  and  $\tilde{Z}_{n-k, n}$  the associated order statistic. Remark that this estimator is the empirical counterpart of  $\theta_p^i$ .

We also investigate the performance of the proposed estimator when assumptions required in Theorem 2.1 are partially violated (see Section 3.3) and we analyse the performance estimation for some choices of sample sizes  $n_1$  and  $n_2$ .

### 3.1. Copula 4.2.2 in Nelsen [30]

We assume that the bivariate vector  $(X^1, X^2)$  follows Copula 4.2.2 in Nelsen [30]. In this case we have  $C_{(X^1, X^2)}(s, t) = 1 - ((1-s)^\theta + (1-t)^\theta)^{\frac{1}{\theta}}$ , for  $\theta \in [1, +\infty)$ . When  $\theta = 1$ ,  $C_{X^1, X^2} = W$  (i.e., counter-monotonicity copula), when  $\theta = +\infty$ ,  $C_{X^1, X^2} = M$  (i.e., comonotonicity copula). Furthermore, for  $\theta > 1$ , we have

$$R_{(X^1, Z)}(x, z) = \begin{cases} z, & \text{if } x \geq \frac{\theta z}{\theta-1} \\ x - \frac{x^\theta}{\theta^\theta \left(\frac{z}{\theta-1}\right)^{\theta-1}}, & \text{if } x < \frac{\theta z}{\theta-1}. \end{cases} \quad (17)$$

Assume now that  $\theta = 2$ . In this case  $R_{(X^1, Z)}(1, 1) = 0.75$ . Thus the vector  $(X^1, Z)$  presents an asymptotic dependence structure. Furthermore  $X^1$  and  $X^2$  have marginal Pareto distributions with parameter 3, i.e., the tail index is  $\gamma^1 = \gamma^2 = 1/3$  and  $\rho^1 = \rho^2 = -1$  (see Assumption (b) in Section 2.2). Remark that  $\forall \theta \in (1, +\infty)$ ,  $R_{(X^i, Z)}$  in (17) satisfy Assumption (a.1) in Section 2.2, for any  $\beta > \gamma^1$  and any  $\tau < 0$ .

In Figures 1 and 2 show the quality of our estimator for different values of risk level  $p$  and sample size  $n$  (i.e.,  $n_1 + n_2$ ) both in terms of boxplots and of QQ-plots on 500 Monte-Carlo simulations.

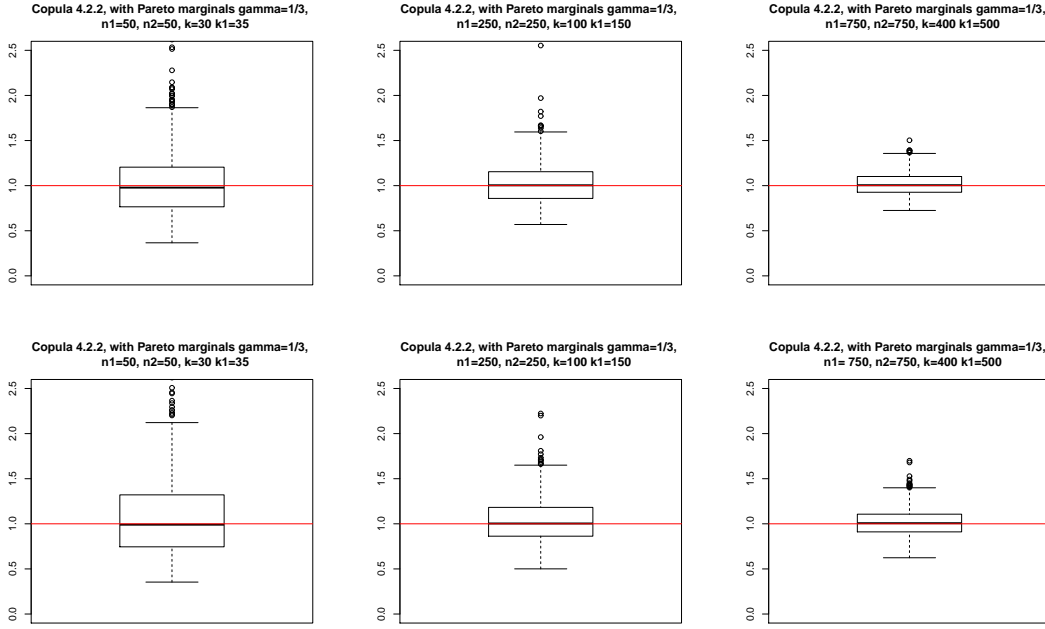


Figure 1: Copula 4.2.2 in Nelsen [30] with parameter  $\theta = 2$  and Pareto marginals. Here we take 500 Monte-Carlo simulations and different values of sample sizes  $n_1$  and  $n_2$  and risk level  $p$ . **First row:**  $p = 1/4 n_1 = 1/2 n$ ; **second row:**  $p = 1/10 n_1 = 1/5 n$ . **First column:**  $n_1 = 50, n_2 = 50$ ; **second column:**  $n_1 = 250, n_2 = 250$ ; **third column:**  $n_1 = 750, n_2 = 750$ .

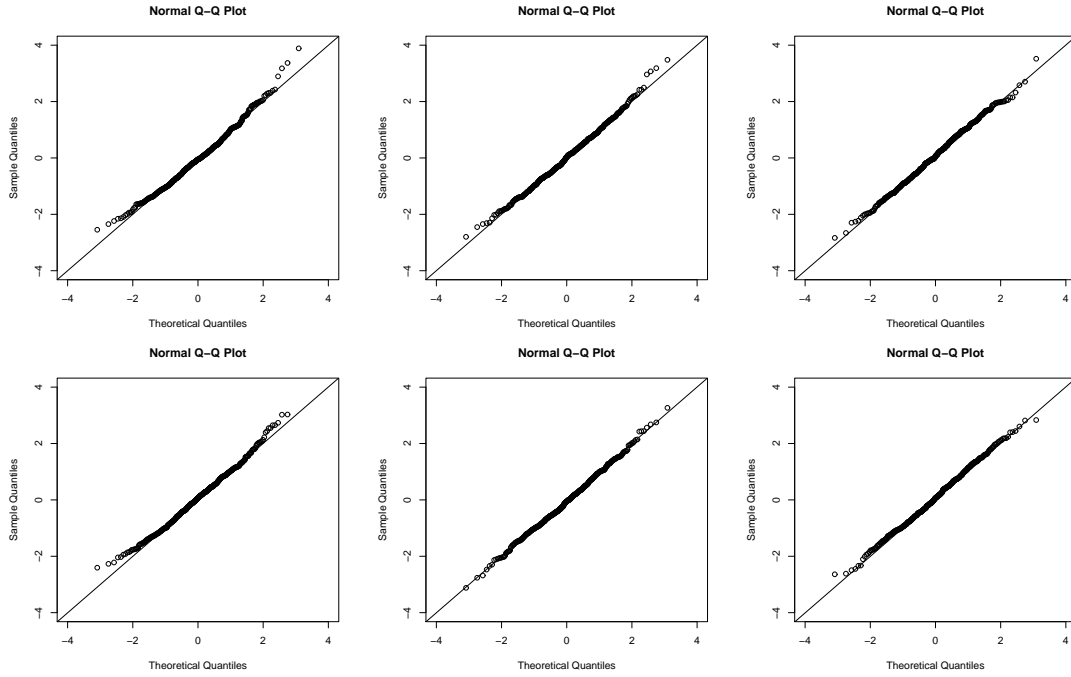


Figure 2: Copula 4.2.2 in Nelsen [30] with parameter  $\theta = 2$  and Pareto marginals. Here we take 500 Monte-Carlo simulations and different values of sample sizes  $n_1$  and  $n_2$  and risk level  $p$ . **First row:**  $p = 1/4 n_1 = 1/2 n$ ; **second row:**  $p = 1/10 n_1 = 1/5 n$ . **First column:**  $n_1 = 50, n_2 = 50$ ; **second column:**  $n_1 = 250, n_2 = 250$ ; **third column:**  $n_1 = 750, n_2 = 750$ .

*Comparison with the empirical estimator.* In the following, we compare the performance of the presented estimator with the Kendall process based one, i.e.,  $\hat{\theta}_{p,emp}^i$  in Equation (16). Clearly,  $\hat{\theta}_{p,emp}^i$  is not applicable when  $np < 1$  and is not expected to perform well if  $np$  is “too small” (the reader is also referred to Di Bernardino and Prieur [17]). Results are gathered in Figure 3. We remark that the empirical estimator  $\hat{\theta}_{p,emp}^i$  underestimates the multivariate CTE and is consistently outperformed by the proposed EVT estimator  $\hat{\theta}_{p(n_1),n_2}^i$ .

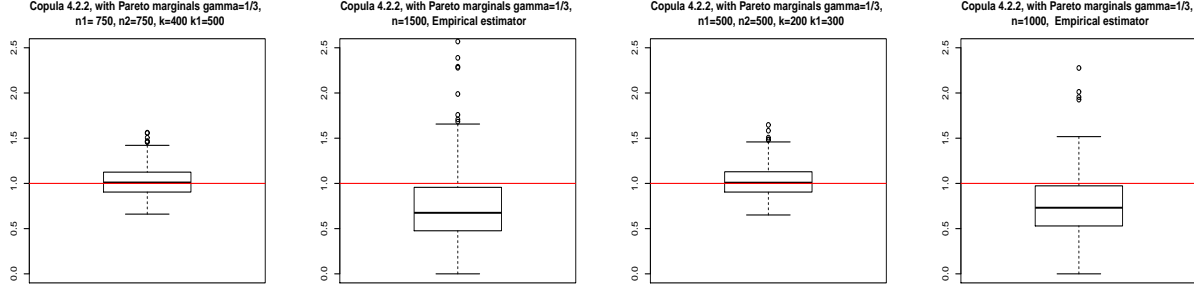


Figure 3: Copula 4.2.2 in Nelsen [30] with parameter  $\theta = 2$  and Pareto marginals. Here we consider 500 Monte Carlo simulations, different values of sample sizes  $n_1$  and  $n_2$  and  $p = 1/2n_1 = 1/n$ .

*Some choices for  $n_1$  and  $n_2$ .* We analyse the performance of our estimator for different choices of the sample sizes  $n_1$  and  $n_2$ . In Figure 4, we consider the boxplots of ratios between estimates and true values for fixed values of  $n_1$  and  $n_2$  varying in a range. We take  $n_1 = 250$  (Figure 4, left panel) and  $n_1 = 50$  (Figure 4, right panel). We remark that  $n_2$  can be chosen particularly much smaller than  $n_1$ . Indeed, we see that with  $n_2 = 15$  (resp.  $n_2 = 10$ ), our estimator has only a very small bias. Furthermore, the part of the bias due to the discrepancy between  $Z$  and  $\tilde{Z}$ , reduces quickly with  $n_2$ . The second remark is that, as expected, the variance in the estimation reduces with  $n_1$ .

A similar simulation study is developed for Copula 4.2.15 in Nelsen [30]. The results are completely analogous to those gathered in this section, then for sake of brevity, they are omitted.

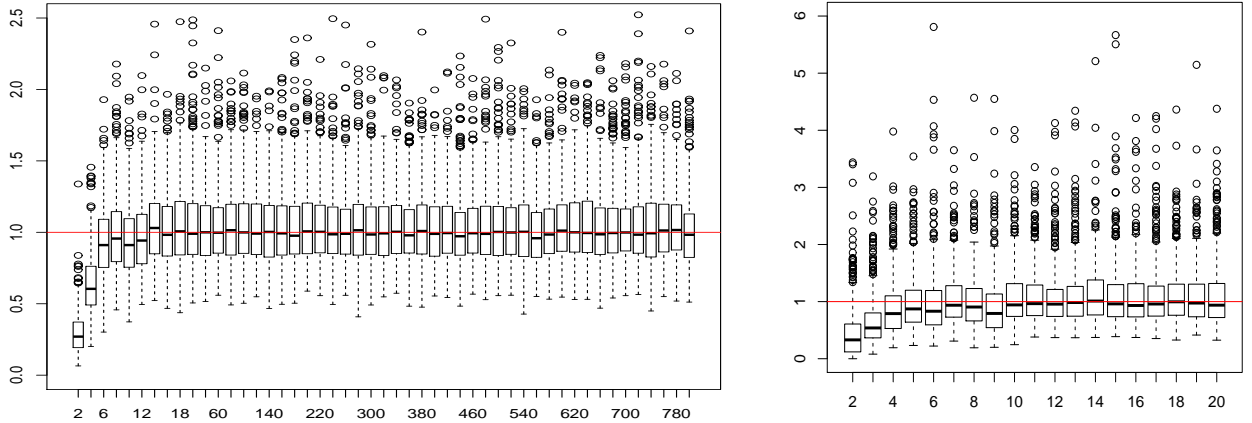


Figure 4: Copula 4.2.2 in Nelsen [30] with parameter  $\theta = 2$  and Pareto marginals with  $\gamma^1 = \gamma^2 = 1/3$ . Boxplots of ratios of estimates and true values, with 500 Monte Carlo samples for risk level  $p = 1/12n_1$ ,  $n_1 = 250$  and  $n_2 \in [2, 800]$  (left panel);  $p = 1/10n_1$ ,  $n_1 = 50$  and  $n_2 \in [2, 20]$  (right panel). We take the auxiliary sequences  $k = 100$  and  $k_i = 150$  (left panel);  $k = 30$  and  $k_i = 35$  (right panel).

### 3.2. Not-Archimedean example: HRT Copula

Let us consider the HRT copula (also called *Clayton survival copula* in the insurance and finance literature), i.e.,

$$C(u, v) = u + v - 1 + ((1 - u)^{-1/\theta} + (1 - v)^{-1/\theta} - 1)^{-\theta}, \quad \text{for } \theta > 0.$$

This copula has low correlation in the left tail but high correlation in the right one, i.e. for large losses. The HRT copula is not an Archimedean copula. It was invented by Gary Venter in 2001 to model the dependence on events of strong intensity (see, e.g., Section 3.8 in Gorge [24]).

Notice that in this case, we can not use the Archimedean generator in order to write the Kendall distribution  $K(t)$ . However, Assumptions (a.1) and (a.2) in Theorem 2.1 can be at least empirically accepted in this HRT copula setting.

We consider here Pareto marginal distributions with  $\gamma^1 = \gamma^2 = 1/4$ . We draw 500 samples from this distribution with sample sizes  $n_1 = n_2 = 750, 250, 50$ . Based on each sample, we estimate  $\theta_{p(n_1)}^i$  for  $p = 1/10 n_1 = 1/5 n$ . Results are gathered in Figure 5 where we present boxplots of the ratio of the estimates and the true values and associated Q-Q plots in order to illustrate the good finite sample performance of our estimator.

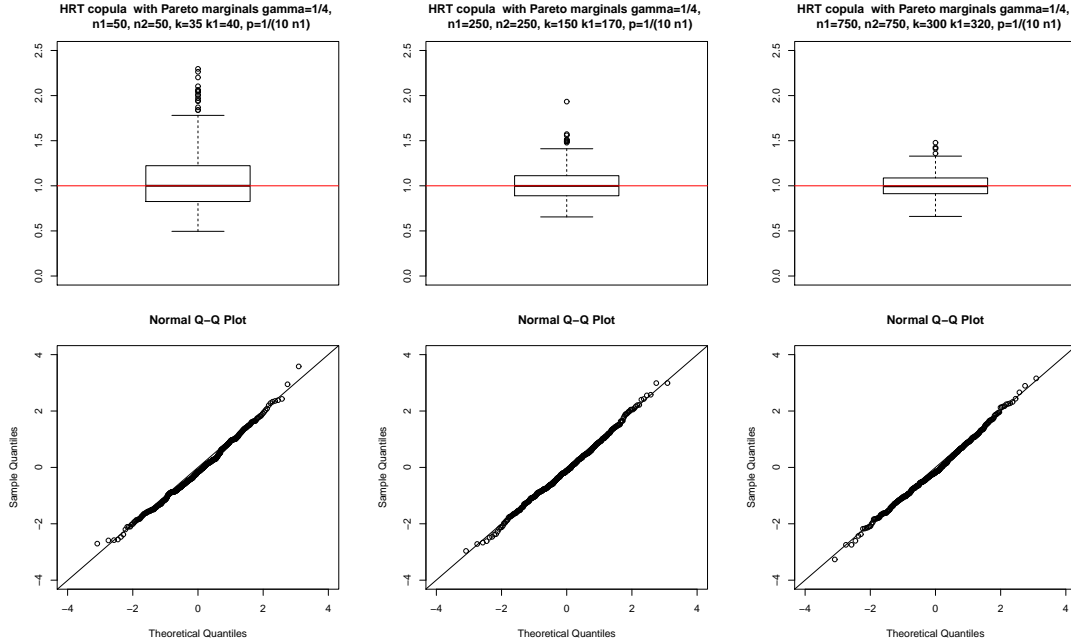


Figure 5: HRT copula with parameter  $\theta = 1$  and Pareto marginals. Here we take 500 Monte-Carlo simulations and  $p = 1/10 n_1 = 1/5 n$ . **First column:**  $n_1 = n_2 = 750$ ; **second column:**  $n_1 = n_2 = 250$ ; **third column:**  $n_1 = n_2 = 50$ .

### 3.3. Estimation when our assumptions are partially violated

In the last part of this section, we investigate the performance of our estimator when assumptions of Theorem 2.1 are partially violated. Firstly, we consider  $(X^1, X^2)$  with Copula 4.2.2 in Nelsen [30] (see Section 3.1) with parameter  $\theta = 2$  and Pareto marginals with  $\gamma^1 = \gamma^2 = 1/2$ . In this case, the estimator  $\hat{\theta}_{p(n_1), n_2}^i$  is still consistent (see Remark 2). Results are gathered in Figure 6, where we present boxplots of the ratio of the estimates and the true values  $\theta_{p(n_1)}^i$  for  $n_1 = 750, n_2 = 750$  and  $p = 1/2 n_1$  (left panel),  $p = 1/4 n_1$  (central panel),  $p = 1/10 n_1$  (right panel).

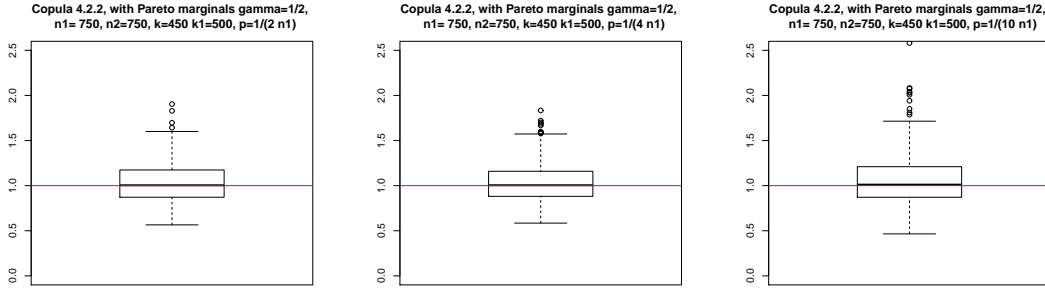


Figure 6: Copula 4.2.2 in Nelsen [30] with parameter  $\theta = 2$  and Pareto marginals with  $\gamma^1 = \gamma^2 = 1/2$ . Here we take  $n_1 = 750$ ,  $n_2 = 750$  and 500 Monte-Carlo simulations. Left panel  $p = 1/2 n_1$ ; center panel  $p = 1/4 n_1$ ; right panel  $p = 1/10 n_1$ .

Secondly, we consider the asymptotically independent bivariate distribution defined by an Independent copula and Pareto distributed marginals with  $\gamma^1 = \gamma^2 = 1/4$ . Notice that  $R_{(X^i, Z)} \equiv 0$ , then this distribution does not satisfy Assumption (a.1) in Theorem 2.1. In this case, the proposed EVT estimator overestimates the theoretical multivariate Conditional-Tail-Expectation  $\theta_{p(n_1)}^i$  (see Figure 7; for a similar behavior the reader is also referred to Figure 3 in Cai et al. [9]).

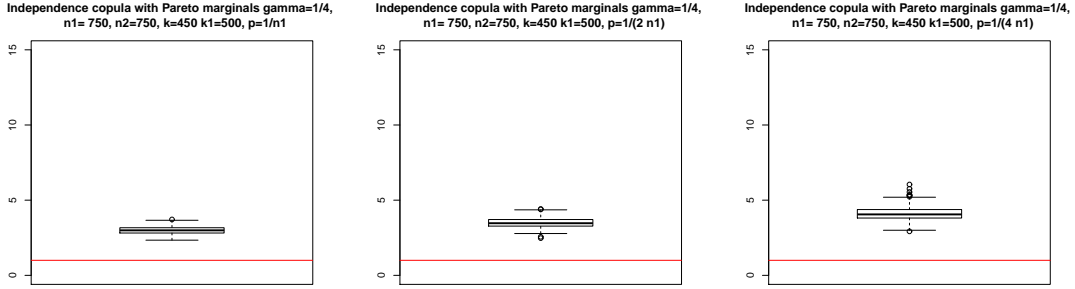


Figure 7: Independence copula and Pareto distributed marginals. Here we take  $n_1 = n_2 = 750$  and 500 Monte-Carlo simulations. Left panel  $p = 1/n_1$ ; center panel  $p = 1/2 n_1$ ; right panel  $p = 1/4 n_1$ .

## 4. Applications to environmental real data

### *Analysis of rainfall measurements*

In this first real application we consider the monthly mean of the rainfall measurements recorded in 3 different stations of the region *Bièvre*, located in the south of Paris (France), from 2003 to 2013 (see also Di Bernardino and Prieur [17]). This data-set was provided by the SIAVB (Syndicat Intercommunal pour l'Assainissement de la Vallée de la Bièvre, <http://www.siavb.fr/>). The unit of measurements is *mm*. The temporal series of monthly mean data are denoted by  $X^1$  at Station 1,  $X^2$  at Station 2 and  $X^3$  at Station 3. The length of the data-set is  $n = 125$ . We take in the following  $n_1 = 63$ ,  $n_2 = 62$ , so that  $n = n_1 + n_2$ . Recall that our estimation of the risk measure  $\theta_p^i$  is based on independent and identically distributed  $d$ -dimensional observations. Di Bernardino and Prieur [17] illustrated the reasonability of the temporal independence assumption for this data set.

The 3-dimensional data-set is represented in Figure 8 (left). We apply our estimation procedure to estimate the multivariate CTE, i.e.,  $\theta_p^i = \mathbb{E}[X^i | Z > U_Z(1-p)]$ , where  $i = 1, 2, 3$  and  $Z = F(X^1, X^2, X^3)$ .

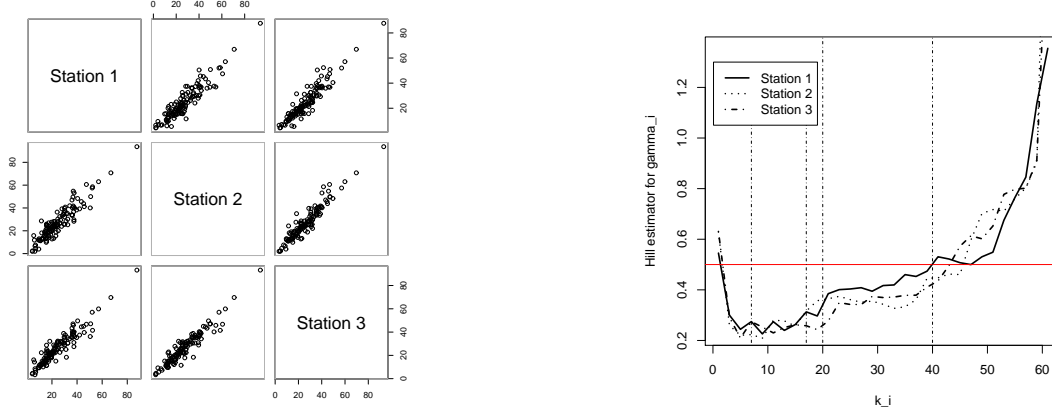


Figure 8: **Left:** Scatterplot of considered 3-dimensional mean monthly rainfall data-set. **Right:** Hill estimates  $\hat{\gamma}^i$  in Equation (10) for  $i = 1, 2, 3$ . Horizontal line represents the upper-bound of  $1/2$ .

Before applying our estimation procedure, let us validate some of the required assumptions. First of all, remark that r.v.s.  $X^i$ , for  $i = 1, 2, 3$ , take positive values. For the assumption  $\gamma^i < 1/2$ , for  $i = 1, 2, 3$ , we plot the Hill estimations  $\hat{\gamma}^i$  against various values of the intermediate sequence of integers  $k_i$  (see Figure 8, right). A usual practice to choose  $k_i$ , for each  $i = 1, 2, 3$ , is then to select a range corresponding to the first stable region of this plot. Here we choose  $k_i \in [7, 17]$  for the first station (i.e.,  $X^1$ ) and  $k_i \in [20, 40]$  for the second and third stations (i.e.,  $X^2$  and  $X^3$ ). Then, to gain in stability, we average the estimations  $\hat{\gamma}^i$  corresponding to  $k_i$  in the selected range. The results are reported in Table 1.

Furthermore, we check the asymptotic dependence between  $(X^i, \tilde{Z})$ , for  $i \in \{1, 2, 3\}$  (see Remark 2), by depicting in Figure 9 (left) the estimated tail dependence coefficient

$$\hat{R}_{(X^i, \tilde{Z})}(1, 1) = \frac{1}{k} \sum_{j=1}^{n_1} 1_{\{X_j^i > X_{n_1 - \lfloor k \rfloor, n_1}^i, \tilde{Z}_j > \tilde{Z}_{n_1 - \lfloor k \rfloor, n_1}\}},$$

where  $\tilde{Z}_j = F_{n_2}(\mathbf{X}_j)$ , for  $j = 1, \dots, n_1$  and  $\lfloor \cdot \rfloor$  denotes the integer part. Figure 9 (left) shows that the estimations are stable around the value 0.8, for  $i \in \{1, 2, 3\}$ . This strongly indicates that right-upper asymptotic dependence is present in this 3-dimensional real data-set.

Using the values  $\hat{\gamma}^i$  gathered in Table 1, we estimate the Multivariate CTE and also plot the estimates against various values of the intermediate sequence  $k$ . Obtained results are gathered in Figure 9 (right). Following the same idea on balancing bias and variance, we choose  $k \in [20, 45]$ . The final estimates based on averaging the estimates from this region are reported in Table 1 for different values of the risk level  $p = p(n_1)$ .

Values gathered in Table 1 represent the averaged monthly precipitations with return period of around 10 years (for  $p = 1/2 n_1$ ), 21 years (for  $p = 1/4 n_1$ ), 52 years (for  $p = 1/10 n_1$ ). We remark an important contribution of the second and third stations (i.e.,  $X^2$  and  $X^3$ ) which strongly contribute to the multivariate stress scenario represented here by the event  $\{Z > U_Z(1/p)\}$  (or equivalently by the event  $\{(X^1, X^2, X^3) \text{ belong to their multivariate upper level set at risk level } U_Z(1/p)\}$ ), for small values of risk level  $p$ .



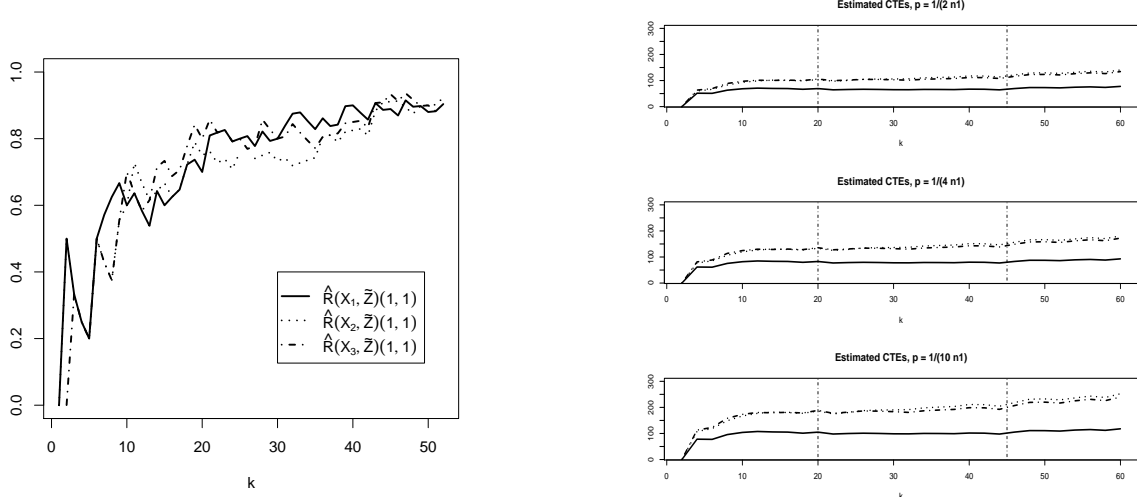


Figure 9: **Left:**  $\hat{R}_{(X^i, \bar{Z})}(1, 1)$  in term of  $k$  for the 3-dimensional mean monthly rainfall data-set, for  $i = 1, 2, 3$ . **Right:** Estimated multivariate CTE against various values of the intermediate sequence  $k$ , for  $i = 1, 2, 3$  and for different values of risk level  $p$ . Full line corresponds to Station 1, dotted to Station 2 and dashed-dotted to Station 3.

Station $i$	$\hat{\gamma}^i$	$\hat{\theta}_{p=1/(2n_1), n_2}^i$	$\hat{\theta}_{p=1/(4n_1), n_2}^i$	$\hat{\theta}_{p=1/(10n_1), n_2}^i$
1	0.259	65.579	78.524	99.637
2	0.363	108.782	139.964	195.307
3	0.359	105.331	135.133	187.845

Table 1: The estimates  $\hat{\gamma}^i$  are computed by taking the average for  $k_i \in [7, 17]$  for the first station (i.e.,  $X^1$ ) and  $k_i \in [20, 40]$  for the second and third stations (i.e.,  $X^2$  and  $X^3$ ). The estimates of the multivariate CTE are based on these values of  $\hat{\gamma}^i$ . We report the average of  $\hat{\theta}_{p(n_1), n_2}^i$  for  $k \in [20, 45]$  and for different values of  $p(n_1)$ .

### Analysis of high wind gusts

In this second real data application we focus on the study of strong wind gusts. We consider the 2 weeks-max wind speed (WS) in meter per second (m/s), wind gust (WG) in m/s and positive increment air pressure (IP) at the sea level in millibar (mbar) recorded in Parcay-Meslay city in the north west of France, from July 2004 to July 2013 (see Figure 10, left). The length of the data-set is  $n = 232$ . We take in the following  $n_1 = n_2 = 116$ . This data set comes down from a large data-set previously analysed in Marcon et al. [28].

Remark that this 3-dimensional data-set is composed by hydrological variables of different nature prohibiting the aggregation of the various components. However, in this situation, our multivariate CTE measure can be useful since it just considers as conditioning extreme event the behavior of the copula associated to the 3 different risk variables (see discussion in the Introduction section).

As illustrated before, we firstly study the Hill estimator against various values of the intermediate sequence of integers  $k_i$  for these 3 positive series (see Figure 10, right). Here we choose  $k_i \in [10, 60]$  for WS and WG (i.e.,  $i = 1, 2$ ) and  $k_3 \in [10, 40]$  for IP. Then, to gain in stability, we average the estimations  $\hat{\gamma}^i$  corresponding to  $k_i$  in the selected range. The results are reported in Table 2.

Then we analyse the asymptotic dependence between  $(WS, \tilde{Z})$ ,  $(WG, \tilde{Z})$  and  $(IP, \tilde{Z})$  (see Remark 2), by

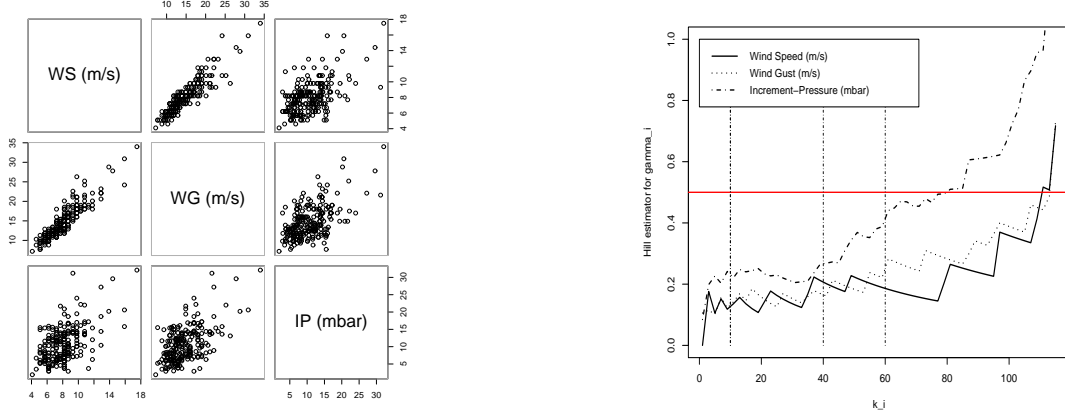


Figure 10: **Left:** Scatterplot of considered 3-dimensional wind gusts data-set. **Right:** Hill estimates  $\hat{\gamma}^i$  in Equation (10). Horizontal line represents the upper-bound of  $1/2$ .

using the estimated tail dependence coefficient  $\hat{R}(1, 1)$ , Figure 11 (left) shows that the estimations are stable around the value 0.8. This strongly indicates that right-upper asymptotic dependence is present in this wind gusts data-set. Using the values  $\hat{\gamma}^i$  gathered in Table 2, we estimate the Multivariate CTE and also plot the estimates against various values of the intermediate sequence  $k$  (see Figure 11, right). Following the same idea on balancing bias and variance, we choose  $k \in [40, 80]$ . The final estimates based on averaging the estimates from this region are reported in Table 2 for different values of the risk level  $p = p(n_1)$ .

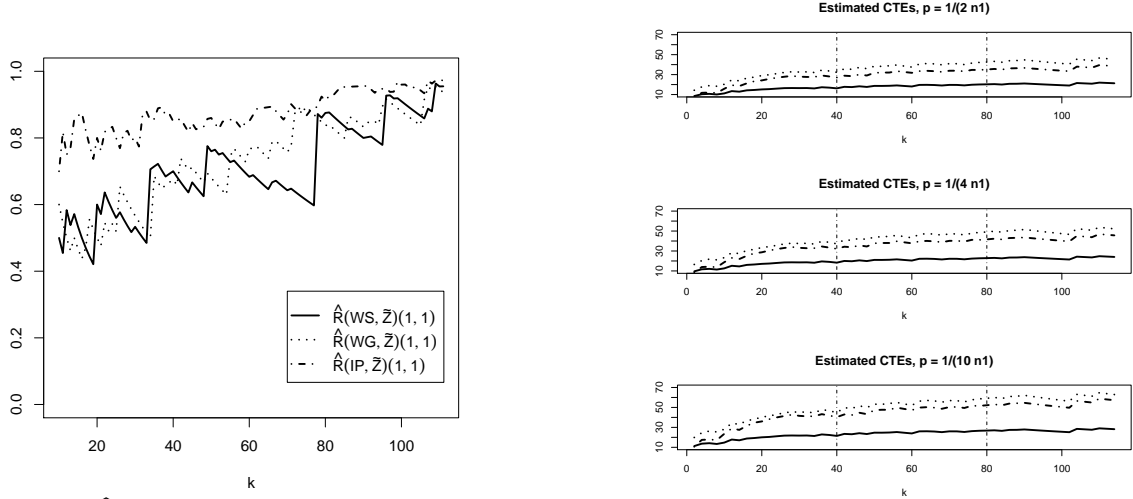


Figure 11: **Left:**  $\hat{R}(1, 1)$  in term of  $k$  for the 3-dimensional wind gusts data-set, for  $i = 1, 2, 3$ . **Right:** Estimated multivariate CTE against various values of the intermediate sequence  $k$  and for different values of risk level  $p$ . Full line corresponds to WS, dotted line to WG and dashed-dotted one to IP.

Variables	$\hat{\gamma}^i$	$\hat{\theta}_{p=1/(2n_1), n_2}^i$	$\hat{\theta}_{p=1/(4n_1), n_2}^i$	$\hat{\theta}_{p=1/(10n_1), n_2}^i$
WS	0.174	18.927	21.346	25.024
WG	0.204	39.127	45.056	54.295
IP	0.246	32.323	38.347	48.067

Table 2: The estimates  $\hat{\gamma}^i$  are computed by taking the average for  $k_i \in [10, 60]$  for  $i = 1, 2$  (i.e., WS and WG) and  $k_3 \in [10, 40]$  (i.e., for IP). The estimates of the multivariate CTE are based on these values of  $\hat{\gamma}^i$ . We report the average of  $\hat{\theta}_{p(n_1), n_2}^i$  for  $k \in [40, 80]$  and for different values of  $p(n_1)$ .

## A. Technical proofs and auxiliary results

This appendix is devoted to the proof of Proposition 2.1 stated in Section 2. This proof requires different preliminary results, introduced and proved below.

**Proposition A.1** *Let  $e_{n_1, n_2} := \frac{n_1}{k}(1 - K(\tilde{Z}_{n_1-k, n_1}))$ , with  $k = k(n_1)$ ,  $\tilde{Z}_{n_1-k, n_1}$  the  $(n_1 - k)$ -th order statistic of  $\tilde{Z}_1, \dots, \tilde{Z}_{n_1}$  and  $\tilde{Z}_j = F_{n_2}(\mathbf{X}_j)$ . Under Assumptions (a.2) and (a.3),  $e_{n_1, n_2} \xrightarrow{\mathbb{P}} 1$ , as  $n_1, n_2 \rightarrow \infty$ .*

*Proof:* Applying the  $d$ -dimensional extension of Kolmogorov-Smirnov Theorem and the properties of ordered statistics, we know that, for all  $\epsilon > 0$

$$\left| Z_{n_1-k, n_1} - \tilde{Z}_{n_1-k, n_1} \right| = |(F(\mathbf{X}))_{n_1-k, n_1} - (F_{n_2}(\mathbf{X}))_{n_1-k, n_1}| = o(\sqrt{n_2}^{-1+\epsilon}), \quad (\text{A.1})$$

where the ordered statistics are defined from the samples  $(Z_j)_{j=1, \dots, n_1} = (F(\mathbf{X}_j))_{j=1, \dots, n_1}$  and  $(\tilde{Z}_j)_{j=1, \dots, n_1} = (F_{n_2}(\mathbf{X}_j))_{j=1, \dots, n_1}$ . Then we write

$$|e_{n_1, n_2} - 1| \leq \left| \frac{n_1}{k}(1 - K(Z_{n_1-k, n_1})) - 1 \right| + \left| \frac{n_1}{k}(K(Z_{n_1-k, n_1}) - K(\tilde{Z}_{n_1-k, n_1})) \right|.$$

Since  $1 - K(Z_{n_1-k, n_1})$  is the  $k$ -th order statistic of a random sample of size  $n_1$  from the standard uniform distribution, we get  $\left| \frac{n_1}{k}(1 - K(Z_{n_1-k, n_1})) - 1 \right| \xrightarrow{\mathbb{P}} 0$ .

We now study the second term. From Assumption (a.2), by applying a first-order Taylor approximation we get

$$\left| \frac{n_1}{k}(K(Z_{n_1-k, n_1}) - K(\tilde{Z}_{n_1-k, n_1})) \right| = \frac{n_1}{k} K'(Z_{n_1-k, n_1}) \left| (\tilde{Z}_{n_1-k, n_1} - Z_{n_1-k, n_1}) + o_{\mathbb{P}}(\tilde{Z}_{n_1-k, n_1} - Z_{n_1-k, n_1}) \right|. \quad (\text{A.2})$$

Since,  $K(Z_{n_1-k, n_1}) \sim 1 - \frac{k}{n_1}$  in probability, for  $n_1 \rightarrow \infty$ , then, from Assumption (a.2),  $K'(Z_{n_1-k, n_1}) = K'(K^{-1}(K(Z_{n_1-k, n_1})))$  is bounded for large value of  $n_1$ . Then, by using (A.1) and (A.2),

$$\left| \frac{n_1}{k}(K(Z_{n_1-k, n_1}) - K(\tilde{Z}_{n_1-k, n_1})) \right| = o\left(\frac{n_1}{k} n_2^{\frac{-1+\epsilon}{2}}\right),$$

which tends to zero as  $n_1$  and  $n_2$  tend to infinity from Assumption (a.3). Hence the result.  $\square$

Lemma A.1 below is a variation of Lemma 1 in Cai et al. [9] in our setting. The interested reader is also referred to Proposition 3.1 in Einmahl et al. [19]. The limit process is characterized by the aforementioned  $W_R$ -process. For convenient presentation, all the limit processes that are involved in

Lemma A.1 are defined on the same probability space, via the Skorohod construction. However, they are only equal in distribution to the original processes.

Define, for  $i \in I$ ,

$$R_{n_1}^i(x, z) := \frac{n_1}{k} \mathbb{P} \left[ 1 - F_i(X^i) < \frac{kx}{n_1}, 1 - K(Z) < \frac{kz}{n_1} \right]. \quad (\text{A.3})$$

A non-parametric pseudo-estimator of  $R_{n_1}^i$  (with unknown margins) is given by

$$T_{n_1, n_2}^i(x, z) := \frac{1}{k} \sum_{j=1}^{n_1} 1_{\{1-F_i(X_j^i) < \frac{kx}{n_1}, 1-K(\tilde{Z}_j) < \frac{kz}{n_1}\}}, \quad (\text{A.4})$$

where  $\tilde{Z}_j = F_{n_2}(\mathbf{X}_j)$ . Its asymptotic behavior is stated in Lemma A.1 below.

**Lemma A.1** *Let  $i \in I$ . Suppose that condition in (4) and Assumptions (a.2) and (a.3) hold true. Let  $T > 0$  and  $\eta \in (\max_{i \in I} \gamma^i, 1/2)$ . Then, with probability 1, for  $n_1, n_2 \rightarrow \infty$ ,*

$$\begin{aligned} \sup_{x, z \in (0, T]} \left| \frac{\sqrt{k} (T_{n_1, n_2}^i(x, z) - R_{n_1}^i(x, z)) - W_{R_{(X^i, Z)}}(x, z)}{x^\eta} \right| &\rightarrow 0, \\ \sup_{x \in (0, T]} \left| \frac{\sqrt{k} (T_{n_1, n_2}^i(x, \infty) - x) - W_{R_{(X^i, Z)}}(x, \infty)}{x^\eta} \right| &\rightarrow 0, \\ \sup_{z \in (0, T]} \left| \frac{\sqrt{k} (T_{n_1, n_2}^i(\infty, z) - z) - W_{R_{(X^i, Z)}}(\infty, z)}{z^\eta} \right| &\rightarrow 0, \end{aligned}$$

where  $R_{n_1}^i(x, z)$  and  $T_{n_1, n_2}^i(x, z)$  are defined by (A.3) and (A.4) respectively.

*Proof:* Let us write

$$T_{n_1, n_2}^i(x, z) = T_{n_1}^i(x, z) + T_{n_1, n_2}^i(x, z) - T_{n_1}^i(x, z),$$

with  $T_{n_1}^i(x, z) := \frac{1}{k} \sum_{j=1}^{n_1} 1_{\{1-F_i(X_j^i) < \frac{kx}{n_1}, 1-K(Z_j) < \frac{kz}{n_1}\}}$  and  $Z_j = F(\mathbf{X}_j)$ . Remark that, by using Lemma 1 in Cai et al. [9], Lemma A.1 above holds true by replacing  $T_{n_1, n_2}^i(x, z)$  by  $T_{n_1}^i(x, z)$ ,  $T_{n_1, n_2}^i(x, \infty)$  by  $T_{n_1}^i(x, \infty)$  and  $T_{n_1, n_2}^i(\infty, z)$  by  $T_{n_1}^i(\infty, z)$ .

Let us thus study the term

$$\begin{aligned} D_{n_1, n_2}^i(x, z) &:= \sqrt{k} (T_{n_1, n_2}^i(x, z) - T_{n_1}^i(x, z)) \\ &= \frac{1}{\sqrt{k}} \sum_{j=1}^{n_1} \left( 1_{\{1-F_i(X_j^i) < \frac{kx}{n_1}, 1-K(\tilde{Z}_j) < \frac{kz}{n_1}\}} - 1_{\{1-F_i(X_j^i) < \frac{kx}{n_1}, 1-K(Z_j) < \frac{kz}{n_1}\}} \right) \\ &= \frac{1}{n_1} \sum_{j=1}^{n_1} L_{n_1, n_2, k, j}, \end{aligned}$$

with

$$L_{n_1, n_2, k, j} = \frac{n_1}{\sqrt{k}} \left( 1_{\{1-F_i(X_j^i) < \frac{kx}{n_1}, 1-K(\tilde{Z}_j) < \frac{kz}{n_1}\}} - 1_{\{1-F_i(X_j^i) < \frac{kx}{n_1}, 1-K(Z_j) < \frac{kz}{n_1}\}} \right).$$

One can deduce that, under Assumptions (a.2) and (a.3),

$$\frac{1}{n_1} \sum_{j=1}^{n_1} (L_{n_1, n_2, k, j} - \mathbb{E}(L_{n_1, n_2, k, j})) \xrightarrow[n_1, n_2]{\text{a.s.}} 0. \quad (\text{A.5})$$

Thus, we now focus on  $\frac{1}{n_1} \sum_{j=1}^{n_1} \mathbb{E}(L_{n_1, n_2, k, j})$ . Using Hölder's Inequality, we get:

$$\begin{aligned} \left| \frac{1}{n_1} \sum_{j=1}^{n_1} \mathbb{E}(L_{n_1, n_2, k, j}) \right| &\leq \frac{n_1}{\sqrt{k}} \left( \mathbb{P} \left[ F_i(X_j^i) > 1 - \frac{kx}{n_1} \right] \right)^{1/p} \\ &\quad \cdot \left( \mathbb{E} \left[ |1_{1-K(\tilde{Z}_j) < \frac{kz}{n_1}, 1-K(Z_j) \geq \frac{kz}{n_1}}| + |1_{1-K(\tilde{Z}_j) \geq \frac{kz}{n_1}, 1-K(Z_j) < \frac{kz}{n_1}}| \right]^q \right)^{1/q} \\ &= o \left( \frac{n_1}{\sqrt{k}} \left( \frac{kx}{n_1} \right)^{1/p} \left( n_2^{\frac{-1+\epsilon}{2}} \right)^{1/q} \right) = o \left( \sqrt{n_1} \left( \frac{k}{n_1} \right)^{\frac{1}{p}-\frac{1}{2}} x^{\frac{1}{p}} n_2^{\frac{-1+\epsilon}{2q}} \right) \quad (\text{A.6}) \end{aligned}$$

Then, from Assumption (a.3) it is possible to choose  $p_0 < \frac{1}{\max_{i \in I} \gamma_i}$  and  $\eta \in (\max_{i \in I} \gamma_i, 1/2)$  such that  $\sup_{x, z \in (0, T]} \frac{|D_{n_1, n_2}^i(x, z)|}{x^\eta} = o \left( \sqrt{n_1} \left( \frac{k}{n_1} \right)^{\frac{1}{p_0}-\frac{1}{2}} n_2^{\frac{-1+\epsilon}{2q_0}} x^{1/p_0-\eta} \right)$  tends to zero.  $\square$

Finally, we define

$$\bar{\theta}_{\frac{kz}{n_1}, n_2}^i := \frac{1}{kz} \sum_{j=1}^{n_1} X_j^i 1_{\{\tilde{Z}_j > U_Z(\frac{n_1}{kz})\}}. \quad (\text{A.7})$$

Proposition A.2 below provides a central limit theorem for  $\bar{\theta}_{\frac{kz}{n_1}, n_2}^i$  in Equation (A.7) and it will be useful below to archive the proof of Proposition 2.1.

**Proposition A.2** *Let  $i \in I$ . Suppose that condition in (4) and Assumptions (a.2) and (a.3) hold with  $\gamma^i \in (0, 1/2)$ . Then, for  $n_1, n_2 \rightarrow \infty$ ,*

$$\sup_{1/2 \leq z \leq 2} \left| \left( \frac{\sqrt{k}}{U_i(\frac{n_1}{k})} \right) (\bar{\theta}_{\frac{kz}{n_1}, n_2}^i - \theta_{\frac{kz}{n_1}}^i) + \frac{1}{z} \int_0^\infty W_{R_{(X^i, Z)}}(s, z) ds^{-\gamma^i} \right| \xrightarrow{\mathbb{P}} 0,$$

with  $\bar{\theta}_{\frac{kz}{n_1}, n_2}^i$  as in Equation (A.7) and  $\theta_{\frac{kz}{n_1}}^i = \mathbb{E} \left[ X^i | Z > Q_Z \left( 1 - \frac{kz}{n_1} \right) \right]$ .

*Proof:* Let  $s_{n_1}(x) := \frac{n_1}{k} (1 - F_i(U_i(\frac{n_1}{k}) x^{-\gamma^i}))$ , for  $x > 0$ . Remark that, from the regular variation condition in (5),  $s_{n_1}(x) \rightarrow x$ , as  $n_1 \rightarrow \infty$ . Furthermore Lemma 3 in Cai et al. [9] states that, when handling proper integrals,  $s_{n_1}(x)$  can be substituted by  $x$  in the limit. We get

$$\begin{aligned} z \theta_{\frac{kz}{n_1}}^i &= \int_0^\infty \frac{n_1}{k} \mathbb{P} \left[ X^i > s, Z > U_Z \left( \frac{n_1}{kz} \right) \right] ds = \int_0^\infty R_{n_1}^i \left( \frac{n_1}{k} (1 - F_i(s)), z \right) ds = \\ &= -U_i \left( \frac{n_1}{k} \right) \int_0^\infty R_{n_1}^i(s_{n_1}(x), z) dx^{-\gamma^i}, \quad (\text{A.8}) \end{aligned}$$

with  $R_{n_1}^i$  as in (A.3). Similarly,  $z\bar{\theta}_{\frac{kz}{n_1}, n_2}^i = -U_i(\frac{n_1}{k}) \int_0^\infty T_{n_1, n_2}^i(s_{n_1}(x), z) dx^{-\gamma^i}$ , with  $T_{n_1, n_2}^i$  as in (A.4) and  $\bar{\theta}_{\frac{kz}{n_1}, n_2}^i$  as in (A.7). For any  $T > 0$

$$\begin{aligned} & \sup_{1/2 \leq z \leq 2} \left| \left( \frac{\sqrt{k}}{U_i(\frac{n_1}{k})} \right) \left( z\bar{\theta}_{\frac{kz}{n_1}, n_2}^i - z\theta_{\frac{kz}{n_1}}^i \right) + \int_0^\infty W_{R_{(X^i, Z)}}(x, z) dx^{-\gamma^i} \right| \\ & \leq \sup_{1/2 \leq z \leq 2} \left| \int_T^\infty W_{R_{(X^i, Z)}}(x, z) dx^{-\gamma^i} \right| + \sup_{1/2 \leq z \leq 2} \left| \int_T^\infty \sqrt{k} (T_{n_1, n_2}^i(s_{n_1}(x), z) - R_{n_1}^i(s_{n_1}(x), z)) dx^{-\gamma^i} \right| \\ & \quad + \sup_{1/2 \leq z \leq 2} \left| \int_0^T \sqrt{k} (T_{n_1, n_2}^i(s_{n_1}(x), z) - R_{n_1}^i(s_{n_1}(x), z)) - W_{R_{(X^i, Z)}}(x, z) dx^{-\gamma^i} \right| \\ & := I_1(T) + I_2^{n_1, n_2}(T) + I_3^{n_1, n_2}(T). \end{aligned}$$

It is sufficient to prove that for any  $\varepsilon > 0$ , there exist  $T_0 = T_0(\varepsilon)$  such that

$$\mathbb{P}[I_1(T_0) > \varepsilon] < \varepsilon \quad (\text{A.9})$$

and  $n_{1,0}, n_{2,0}$  such that for any  $n_1 > n_{1,0}$  and  $n_2 > n_{2,0}$ ,

$$\mathbb{P}[I_2^{n_1, n_2}(T_0) > \varepsilon] < \varepsilon, \quad (\text{A.10})$$

$$\mathbb{P}[I_3^{n_1, n_2}(T_0) > \varepsilon] < \varepsilon. \quad (\text{A.11})$$

(A.9) holds true by application of Lemma 2 in Cai et al. [9] with  $\eta = 0$ .

We now deal with (A.10). Once more we use the decomposition

$$\begin{aligned} T_{n_1, n_2}^i(s_{n_1}(x), z) &= T_{n_1}^i(s_{n_1}(x), z) + T_{n_1, n_2}^i(s_{n_1}(x), z) - T_{n_1}^i(s_{n_1}(x), z) \\ &= T_{n_1}^i(s_{n_1}(x), z) + \frac{1}{\sqrt{k}} D_{n_1, n_2}^i(s_{n_1}(x), z). \end{aligned}$$

Then, we deduce that  $I_2^{n_1, n_2}(T_0)$  is bounded by

$$\sup_{1/2 \leq z \leq 2} \left| \int_{T_0}^\infty D_{n_1, n_2}^i(s_{n_1}(x), z) dx^{-\gamma^i} \right| + \sup_{1/2 \leq z \leq 2} \left| \int_{T_0}^\infty \sqrt{k} (T_{n_1}^i(s_{n_1}(x), z) - R_{n_1}^i(s_{n_1}(x), z)) dx^{-\gamma^i} \right|.$$

Then using the bound in (A.6) for  $p = +\infty$  and  $q = 1$ , we get

$$\sup_{1/2 \leq z \leq 2} \left| \int_{T_0}^\infty D_{n_1, n_2}^i(s_{n_1}(x), z) dx^{-\gamma^i} \right| = O\left(\frac{1}{T_0^{\gamma^i}} \frac{n_1}{\sqrt{k}} n_2^{\frac{-1+\varepsilon}{2}}\right),$$

which tends to zero from Assumption (a.3). We conclude for the term  $I_2^{n_1, n_2}(T_0)$  by using the result of Proposition 2 in Cai et al. [9] for  $\sup_{1/2 \leq z \leq 2} \left| \int_{T_0}^\infty \sqrt{k} (T_{n_1}^i(s_{n_1}(x), z) - R_{n_1}^i(s_{n_1}(x), z)) dx^{-\gamma^i} \right|$ .

It remains to handle the term (A.11). We get

$$\begin{aligned} \mathbb{P}[I_3^{n_1, n_2}(T) > \varepsilon] &= \mathbb{P}\left[\sup_{1/2 \leq z \leq 2} \left| \int_0^T \sqrt{k} (T_{n_1, n_2}^i(s_{n_1}(x), z) - R_{n_1}^i(s_{n_1}(x), z)) - W_{R_{(X^i, Z)}}(x, z) dx^{-\gamma^i} \right| > \varepsilon\right] \\ &\leq \mathbb{P}\left[\sup_{1/2 \leq z \leq 2} \left| \int_0^T \sqrt{k} (T_{n_1, n_2}^i(s_{n_1}(x), z) - R_{n_1}^i(s_{n_1}(x), z)) - W_{R_{(X^i, Z)}}(s_{n_1}(x), z) dx^{-\gamma^i} \right| > \varepsilon/2\right] \\ &\quad + \mathbb{P}\left[\sup_{1/2 \leq z \leq 2} \left| \int_0^T W_{R_{(X^i, Z)}}(s_{n_1}(x), z) - W_{R_{(X^i, Z)}}(x, z) dx^{-\gamma^i} \right| > \varepsilon/2\right] = p_{31}^{n_1, n_2} + p_{32}^{n_1}. \end{aligned}$$

Firstly we consider  $p_{31}^{n_1, n_2}$ . Notice that for any  $T$ , there exists  $\tilde{n}_1 = n_1(T)$ , such that for all  $n_1 > \tilde{n}_1$ ,  $s_{n_1}(T) < T + 1$ . Hence for  $n_1 > \tilde{n}_1$  and for  $\eta_0 \in (\max_{i \in I} \gamma^i, 1/2)$ ,

$$p_{31}^{n_1, n_2} \leq \mathbb{P} \left[ \sup_{\substack{1/2 \leq z \leq 2, \\ 0 < s \leq T+1}} \left| \frac{\sqrt{k} (T_{n_1, n_2}^i(s, z) - R_{n_1}^i(s, z)) - W_{R_{(X^i, Z)}}(s, z)}{s^{\eta_0}} \right| \left| \int_0^T (s_{n_1}(x))^{\eta_0} dx^{-\gamma^i} \right| > \varepsilon/2 \right].$$

Notice that, by Lemma 3 in Cai et al. [9],  $\left| \int_0^T (s_{n_1}(x))^{\eta_0} dx^{-\gamma^i} \right| \rightarrow \frac{\gamma^i}{\eta_0 - \gamma^i} T^{\eta_0 - \gamma^i}$ , as  $n_1 \rightarrow \infty$ . By application of Lemma A.1, we conclude the proof for  $p_{31}^{n_1, n_2}$ . Finally, since  $p_{32}^{n_1}$  does not depend on  $n_2$  and we can conclude using Lemma 2 in Cai et al. [9].  $\square$

We now use the auxiliary results above to prove Proposition 2.1 stated in Section 2.

*Proof of Proposition 2.1:* From Proposition 1 in Cai et al. [9] applied to the bivariate vector  $(X^i, Z)$ , we have that  $\lim_{n_1 \rightarrow \infty} \frac{\theta_{\frac{k}{n_1}}^i}{U_i(n_1/k)} = \int_0^\infty R_{(X^i, Z)}(x^{-1/\gamma^i}, 1) dx$ . Then, to prove Proposition 2.1 it is sufficient to prove that, for  $n_1, n_2 \rightarrow \infty$

$$\frac{\sqrt{k}}{U_i(n_1/k)} \left( \bar{\theta}_{\frac{k}{n_1}, n_2}^i - \theta_{\frac{k}{n_1}}^i \right) \xrightarrow{\mathbb{P}} \Theta^i \int_0^\infty R_{(X^i, Z)}(x^{-1/\gamma^i}, 1) dx.$$

Note that  $\bar{\theta}_{\frac{k}{n_1}, n_2}^i = e_{n_1, n_2} \bar{\theta}_{\frac{k e_{n_1, n_2}}{n_1}}^i$ . From Proposition A.1, we know that  $e_{n_1, n_2} \xrightarrow{\mathbb{P}} 1$ , as  $n_1, n_2 \rightarrow \infty$ .

Using the analytical expression of process  $\Theta^i$  in Equation (6), we can write:

$$\begin{aligned} & \frac{\sqrt{k}}{U_i(n_1/k)} \left( e_{n_1, n_2} \bar{\theta}_{\frac{k e_{n_1, n_2}}{n_1}}^i - \theta_{\frac{k}{n_1}}^i \right) - \Theta^i \int_0^\infty R_{(X^i, Z)}(x^{-1/\gamma^i}, 1) dx \\ &= \left( \frac{\sqrt{k}}{U_i(n_1/k)} \left( e_{n_1, n_2} \bar{\theta}_{\frac{k e_{n_1, n_2}}{n_1}}^i - e_{n_1, n_2} \theta_{\frac{k e_{n_1, n_2}}{n_1}}^i \right) + \int_0^\infty W_{R_{(X^i, Z)}}(s, 1) ds^{-\gamma^i} \right) \\ &+ \left( \frac{\sqrt{k}}{U_i(n_1/k)} \left( e_{n_1, n_2} \theta_{\frac{k e_{n_1, n_2}}{n_1}}^i - \theta_{\frac{k}{n_1}}^i \right) - W_{R_{(X^i, Z)}}(\infty, 1)(\gamma^i - 1) \int_0^\infty R_{(X^i, Z)}(s^{-1/\gamma^i}, 1) ds \right) \\ &=: J_1^{(n_1, n_2)} + J_2^{(n_1, n_2)}. \end{aligned}$$

We prove that both  $J_1^{(n_1, n_2)}$  and  $J_2^{(n_1, n_2)}$  converge to zero in probability as  $n_1, n_2 \rightarrow \infty$ . Using Lemma A.1, since  $T_{n_1, n_2}^i(\infty, e_{n_1, n_2}) = 1$ , we get, as  $n_1, n_2 \rightarrow \infty$ :

$$\sqrt{k}(e_{n_1, n_2} - 1) \xrightarrow{\mathbb{P}} -W_{R_{(X^i, Z)}}(\infty, 1). \quad (\text{A.12})$$

This implies that  $\lim_{n_1, n_2 \rightarrow +\infty} \mathbb{P}(|e_{n_1, n_2} - 1| > k^{-1/4}) = 0$ . Thus

$$\begin{aligned} |J_1^{(n_1, n_2)}| &\leq \sup_{|z-1| < k^{-1/4}} \left| \frac{\sqrt{k}}{U_i(n_1/k)} \left( z \bar{\theta}_{\frac{k z, n_2}{n_1}}^i - z \theta_{\frac{k z}{n_1}}^i \right) + \int_0^\infty W_{R_{(X^i, Z)}}(s, z) ds^{-\gamma^i} \right| \\ &+ \sup_{|z-1| < k^{-1/4}} \left| \int_0^\infty W_{R_{(X^i, Z)}}(s, z) - W_{R_{(X^i, Z)}}(s, 1) ds^{-\gamma^i} \right|. \end{aligned}$$

The first term of the right hand term above converges to zero in probability by Proposition A.2. The second term can be handled as in the proof of Proposition 3 in Cai et al. [9], using Lemma 2 in Cai



et al. [9].

We now focus on  $J_2^{(n_1, n_2)}$ . Firstly recall that, using Assumption (a.1), as  $n_1 \rightarrow \infty$

$$\sup_{1/2 \leq z \leq 2} \sqrt{k} \left| \int_0^\infty R_{n_1}^i(s_{n_1}(x), z) - R_{(X^i, Z)}(x, z) dx^{-\gamma^i} \right| \rightarrow 0, \quad (\text{A.13})$$

(see Equation (27) in Cai et al. [9]). Combining (A.8) and (A.13), we get:

$$\frac{e_{n_1, n_2} \theta_{\frac{k e_{n_1, n_2}}{n_1}}^i}{U_i(n_1/k)} = - \int_0^\infty R_{n_1}^i(s_{n_1}(x), e_{n_1, n_2}) dx^{-\gamma^i} = - \int_0^\infty R_{(X^i, Z)}(x, e_{n_1, n_2}) dx^{-\gamma^i} + o_{\mathbb{P}}(1/\sqrt{k}),$$

where the last term  $o_{\mathbb{P}}(1/\sqrt{k})$  does not depend on  $n_2$ . Using the homogeneity of  $R$  function, we have:

$$e_{n_1, n_2} \theta_{\frac{k e_{n_1, n_2}}{n_1}}^i = e_{n_1, n_2}^{1-\gamma^i} \theta_{\frac{k}{n_1}}^i + o_{\mathbb{P}}\left(\frac{U_i(n_1/k)}{\sqrt{k}}\right),$$

still with the last term not depending on  $n_2$ . By applying (A.12) and Proposition 1 in Cai et al. [9] for the bivariate vector  $(X^i, Z)$ , as far as the Cramér's delta method, we get as  $n_1, n_2 \rightarrow \infty$ ,

$$\begin{aligned} \frac{\sqrt{k}}{U_i(n_1/k)} \left( e_{n_1, n_2} \theta_{\frac{k e_{n_1, n_2}}{n_1}}^i - \theta_{\frac{k}{n_1}}^i \right) &= \sqrt{k} \left( e_{n_1, n_2}^{1-\gamma^i} - 1 \right) \frac{\theta_{\frac{k}{n_1}}^i}{U_i(n_1/k)} + o_{\mathbb{P}}(1) \\ &\xrightarrow{\mathbb{P}} (\gamma^i - 1) W_{R_{(X^i, Z)}}(\infty, 1) \int_0^\infty R_{(X^i, Z)}(s^{-1/\gamma^i}, 1) ds, \end{aligned}$$

uniformly in  $n_2$ . Hence  $J_2^{(n_1, n_2)}$  converges to zero in probability as  $n_1, n_2 \rightarrow \infty$ . Hence the result.  $\square$

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