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| Itai Ben Yaacov, Frank O Wagner. A metric version of Schlichting's Theorem. 2017. hal-01523644v1

HAL Id: hal-01523644

<https://hal.science/hal-01523644v1>

Preprint submitted on 16 May 2017 (v1), last revised 8 Apr 2020 (v3)

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A METRIC VERSION OF SCHLICHTING'S THEOREM

ITAI BEN YAACOV AND FRANK O. WAGNER

ABSTRACT. If \mathfrak{F} is a type-definable family of commensurable subsets, subgroups or sub-vector spaces in a metric structure, then there is an invariant subset, subgroup or sub-vector space commensurable with \mathfrak{F} . This in particular applies to type-definable or hyper-definable objects in a classical first-order structure.

INTRODUCTION

Schlichting's Theorem [4] states that if a subgroup H of a group G is uniformly commensurable with all its G -conjugates, then it is commensurable with a normal subgroup of G . This was generalized by Bergman and Lenstra [1], who showed that if H is uniformly commensurable with all its K -conjugates for some subgroup K of G , then it is commensurable with a K -invariant subgroup. Peter Neumann deduced from this an analogous theorem for sets: A family of subsets of some set Ω invariant under a subgroup K of $\text{Sym}(\Omega)$ with bounded symmetric differences yields a K -invariant subset whose symmetric difference with members of the original family is bounded. This was studied further by Brailovsky, Pasechnik and Praeger [2], Neumann [3] and the second author [5], who proved a version for vector spaces, as well as more general objects.

Meanwhile, a similar theorem was shown for type-definable groups in simple theories, where finite index condition of commensurability is re-interpreted as *bounded* index [6, Theorem 4.5.13] (building on results of Hrushovski for the S1-case). However, simplicity seemed a necessary condition, as the proof is based on the Independence Theorem. Recently, the second author proved a hyperdefinable version of Schlichting's Theorem in [7] without any hypotheses on the ambient theory. In this note we shall rephrase the result in the language of continuous logic and metric structures, and generalize it to families of sub-objects other than groups.

1. CLOSE-KNIT FAMILIES

Definition 1.1. Assume the following data are given:

- A κ -complete lower semi-lattice \mathfrak{L} (i.e. every sub-family of size $< \kappa$ admits an infimum), for some regular cardinal κ .
- A family $\mathfrak{F} = \{f_a : a \in A\} \subseteq \mathfrak{L}$, where the enumeration may have repetitions.
- A compact Hausdorff topological space \mathfrak{T} equipped with a closed partial order relation. We let $\lambda = w(\mathfrak{T})^+ + \aleph_0$, where w denotes the weight of \mathfrak{T} (i.e., least cardinal of an open basis), and require that $\kappa \geq \lambda$.
- A map $\delta: \mathfrak{L} \times A \rightarrow \mathfrak{T}$.
- A map $\mathfrak{L}_0 \times A \rightarrow \mathfrak{L}$ denoted $(s, a) \mapsto s^a$, where $\mathfrak{L}_0 \subseteq \mathfrak{L}$ denotes the family of meets of $< \kappa$ elements of \mathfrak{F} .

We say that \mathfrak{F} (together with the additional data) is a *close-knit family* if the following holds:

- (i) The map δ is monotonous and “upper semi-continuous” in its first argument in the sense that if $S \subseteq \mathfrak{L}$ is closed under finite meet and $|S| < \kappa$, then $\delta(\bigwedge S, a) = \bigwedge \{\delta(s, a) : s \in S\}$.
- (ii) Let $\xi \in \mathfrak{T}$ and let $S \subseteq \mathfrak{L}$ be closed under meet such that $|S| < \kappa$. Assume that for every neighbourhood $\xi \in U \subseteq \mathfrak{T}$ and $s \in S$ there exists $\zeta \in U$ and $a \in A$ such that $\delta(s, a) \geq \zeta$. Then there exists $a \in A$ satisfying $\delta(s, a) \geq \xi$ for all $s \in S$. (In this case, $\delta(\bigwedge S, a) \geq \xi$ as well, by (i).)
- (iii) We have $s \leq s^a$, and whenever $t \leq s$ are in \mathfrak{L}_0 with $\delta(t, a) = \delta(s, a)$, then $t^a = s^a$.
- (iv) For any $s \in \mathfrak{L}_0$ of the form $s = \bigwedge \mathfrak{F}_0$ where $\mathfrak{F}_0 \subset \mathfrak{F}$ with $|\mathfrak{F}_0| < \lambda$, there is some cardinal $\mu_s < \kappa$ such that for all $a \in \mathfrak{F}$, any chain in \mathfrak{L} between s and s^a has cardinal $< \mu_s$.

Date: 16th May 2017.

2000 Mathematics Subject Classification. 03G10, 05E15, 06A12, 06B35, 12L12, 20D30.

Key words and phrases. Schlichting's Theorem, uniformly commensurable, close-knit family, continuous logic, invariant. Partially supported by ANR-13-BS01-0006 ValCoMo.

If Γ is a group of automorphisms of \mathfrak{L} , also acting on A , and all the data (namely, the maps $a \rightarrow f_a$, δ and $(s, a) \mapsto s^a$) are invariant under Γ , then we say that \mathfrak{F} is a Γ -close-knit family.

We should think of $\delta(s, a)$ as a measure of how much $s \not\leq f_a$. In particular, in most applications $s \leq f_a$ if and only if $\delta(s, a) = 0$.

Theorem 1.2. *Let \mathfrak{L} be a κ -complete lower semi-lattice, Γ a group of automorphisms of \mathfrak{L} , and \mathfrak{F} a Γ -close-knit family in \mathfrak{L} . Then Γ has a fixed point in \mathfrak{L} .*

Proof. For $s \in \mathfrak{L}$ define

$$m(s) = \{\xi \in \mathfrak{J} : \xi \leq \delta(s, a) \text{ for some } a \in A\}.$$

It follows directly from [Definition 1.1\(ii\)](#) that $m(s)$ is closed. If $s, t \in \mathfrak{L}$ with $t \leq s$, then $\delta(t, a) \leq \delta(s, a)$ for all $a \in A$, whence $m(t) \subseteq m(s)$. Also, if $s = \bigwedge \mathfrak{F}_1$, where $\mathfrak{F}_1 \subseteq \mathfrak{F}$, $|\mathfrak{F}_1| < \kappa$, and $\xi \notin m(s)$, then by [Definition 1.1\(ii\)](#) there exists a finite $\mathfrak{F}_0 \subseteq \mathfrak{F}_1$ such that $\xi \notin m(\bigwedge \mathfrak{F}_0)$. Since $\kappa \geq \lambda = w(\mathfrak{J})^+ + \aleph_0$, and each $m(s)$ is closed, there exists $s_0 \in \mathfrak{L}_0$ such that $m(s_0)$ is minimal with respect to inclusion, and moreover, s_0 is the meet of $< \lambda$ members of \mathfrak{F} . Put $m = m(s_0)$, and call $s \in \mathfrak{L}_0$ *strong* if $m(s) = m$. For strong s we define

$$A(s) = \{a \in A : \delta(s, a) \text{ is maximal in } m\}, \quad \text{and} \\ n(s) = \bigwedge \{s^a : a \in A(s)\}.$$

Since m is closed, it contains maximal elements, so $A(s)$ is non-empty. Note that if $t \in \mathfrak{L}_0$ with $t \leq s$, then $m(t) \subseteq m(s) = m$, so t is also strong by minimality of m , and $A(t) \subseteq A(s)$. If, in addition, we have $a \in A(t)$, then $\delta(t, a) = \delta(s, a)$ (since both are maximal in m , and they are comparable), whence $t^a = s^a$. In particular this means that for any $a \in A(s \wedge s_0)$ we have $s_0 \leq s_0^a = (s \wedge s_0)^a = s^a$, so by [Definition 1.1\(iv\)](#) applied to s_0 , the meet $n(s)$ is indeed defined.

Suppose there is no greatest $n(s)$ for strong s , and let $\mu = \mu_{s_0}$ as per [Definition 1.1\(iv\)](#). Since $\mu < \kappa$, we may then construct by induction a sequence $(s_\alpha)_{\alpha \leq \mu}$ of strong elements, starting with s_0 . At successor stages take some strong t such that $n(t) \not\leq n(s_\alpha)$ and let $s_{\alpha+1} = s_\alpha \wedge t$, so $s_{\alpha+1} < s_\alpha$ and $n(s_{\alpha+1}) > n(s_\alpha)$. At limit stages put $s_\alpha = \bigwedge_{\beta < \alpha} s_\beta$. If $a \in A(s_\mu)$ then for every $\alpha \leq \mu$ we have $a \in A(s_\alpha)$ and

$$s_0 \leq n(s_0) \leq n(s_\alpha) \leq s_\alpha^a = s_0^a.$$

This produces a chain of length μ between s_0 and s_0^a , contradicting the choice of μ .

Therefore there exists a greatest $n(s) \in \mathfrak{L}$, which is a fixed point of Γ . ■

If \mathfrak{J} is finite then $\lambda = \aleph_0$; if all μ_s are finite, we can also take $\kappa = \aleph_0$ and [Theorem 1.2](#) is a re-statement of the classical case (in particular, condition [\(ii\)](#) of [Definition 1.1](#) holds automatically).

2. ALMOST INVARIANT FAMILIES IN CONTINUOUS LOGIC

Recall that in a κ -saturated metric structure \mathfrak{M} with domain M , a subset of M^n is *type-definable* if it is the intersection of the zero-sets of fewer than κ many formulas. Given a set $F \subseteq M^\alpha \times M^\beta$ and $a \in M^\beta$, we define $F_a = \{b : (b, a) \in F\}$. A family \mathfrak{F} of subsets of M^α is *type-definable* if there are type-definable sets $F \subseteq M^\alpha \times M^\beta$ and $A \subseteq M^\beta$ such that $\mathfrak{F} = F_A = \{F_a : a \in A\}$.

Definition 2.1. We shall a type-definable ambient object X of one of three kinds: sets, groups or vector-spaces over a definable field K . For two type-definable sub-objects of the same kind $S, S' \subseteq X$, we say that S is *commensurably contained* in S' if

- $|S \setminus S'|$, or
- $[S : S \cap S']$, or
- $\text{codim}_S(S \cap S')$, respectively,

is strictly less than κ . We say that S and S' are *commensurable* if either is commensurably contained in the other.

Note that by κ -saturation this implies that the difference/index/co-dimension does not increase when we replace \mathfrak{M} by an elementary extension; we say that it is *bounded*. In the following, whenever we talk about type-definable sets, we shall assume that \mathfrak{M} is sufficiently saturated.

Definition 2.2. Let X be a type-definable set/group/vector space over K in a metric structure \mathfrak{M} , and Γ a group of automorphisms of X . A type-definable family of subsets/subgroups/subspaces of X is *almost invariant* if F and F' are commensurable for all $F, F' \in \mathfrak{F}$. For an almost invariant family \mathfrak{F} , we shall say that S is commensurable with \mathfrak{F} if S is commensurable with some (equivalently, all) $F \in \mathfrak{F}$.

Theorem 2.3. *Let X be a type-definable set/group/vector space over a definable field K in a metric structure \mathfrak{M} , and Γ a type-definable group of automorphisms of X . Suppose \mathfrak{F} is a Γ -invariant almost invariant type-definable family of subsets/subgroups/subspaces of X . Then there is a Γ -invariant subset/subgroup/subspace N of X commensurable with all $F \in \mathfrak{F}$, which is moreover type-definable over the same parameters.*

Proof. We may assume that X , Γ and $\mathfrak{F} = \{F_a : a \in A\}$ are type-defined over \emptyset , and K is definable over \emptyset . Notice that then we may also enumerate the family \mathfrak{F} as $\{F_{a,\gamma} : a \in A, \gamma \in \Gamma\}$ where $F_{a,\gamma} = \gamma F_a$. Let $S \subseteq X$ be type-definable. Suppose F_a is defined (for $a \in A$) by $\Phi(x, a) = 0$, where Φ is a family of $[0, 1]$ -valued formulae closed under the connective max, and $|\Phi| < \kappa$. For $\gamma \in \Gamma$, $n \in \mathbb{N}$ and $\varphi \in \Phi$ define

- in the set case: $\delta_{\varphi,n}(S, a, \gamma) = \sup_{x \in S^n} (\min_{i < j < n} d(\gamma^{-1}x_i, \gamma^{-1}x_j) \wedge \min_{i < n} \varphi(\gamma^{-1}x_i, a))$,
- in the group case: $\delta_{\varphi,n}(S, a, \gamma) = \sup_{x \in S^n} \min_{i < j < n} \varphi(\gamma^{-1}(x_i^{-1}x_j), a)$,
- in the vector space case: $\delta_{\varphi,n}(S, a, \gamma) = \sup_{x \in S^n} \inf_{\eta \in K^n \setminus \bar{0}} \varphi(\gamma^{-1}(\sum \eta_i x_i), a)$.

Let $\mathfrak{J} = [0, 1]^{\Phi \times \mathbb{N}}$, so $\lambda = |\Phi|^+ + \aleph_0$. For type-definable $S \subseteq X$ and $(a, \gamma) \in A \times \Gamma$ define

$$\delta(S, a, \gamma) = (\delta_{\varphi,n}(S, a, \gamma) : (\varphi, n) \in \Phi \times \mathbb{N}) \in \mathfrak{J}.$$

Let \mathfrak{L} be the lower semi-lattice of type-definable subsets/subgroups/subspaces commensurable with \mathfrak{F} . Then conditions (i) and (ii) of Definition 1.1 hold by compactness.

For $S \in \mathfrak{L}$ and $F \in \mathfrak{F}$ put $S^F = S \cup F$ in the set case, $S^F = S + F$ in the vector space case, and

$$S^F = \bigcap_{s \in S} (SF)^s$$

in the group case. It is easy to see that S^F is a type-definable supergroup of S with $[S_F : S]$ bounded. Finally, put $S^{a,\gamma} = S^{F_{a,\gamma}}$.

We claim that in all three cases, condition (iii) of Definition 1.1 holds. Clearly $S \subseteq S^F$. So assume that $T \subseteq S$ and $T^F \neq S^F$, where $F = F_{a,\gamma}$.

In the set case, since $T \cup F \subseteq S \cup F$, this means that there exists $x \in S \setminus (T \cup F)$. In particular, $d(\gamma^{-1}x, \gamma^{-1}T) > 0$, so for some $\varphi \in \Phi$ and $0 < e < 1$ we have $d(\gamma^{-1}x, \gamma^{-1}y) \wedge \varphi(\gamma^{-1}x, a) \geq e$ for all $y \in T$. Since T and F are commensurable, there exists n such that $\delta_{\varphi,n}(T, a, \gamma) < e$, and we may assume that n is least such. As $\delta_{\varphi,0} \equiv 1$, we have $n > 0$, and $\delta_{\varphi,n-1}(T, a, \gamma) \geq e$ by minimality of n . Therefore $\delta_{\varphi,n}(S, \gamma) \geq e$, so $\delta(T, F) < \delta(S, F)$.

In the group case, note that if $T \leq S$ with $SF = TF$ and $I \subset T$ is a system of representatives for $S/(S \cap F) \cong SF/F = TF/F \cong T/(T \cap F)$, then

$$S^F = \bigcap_{s \in S} (SF)^s = \bigcap_{s \in I} (SF)^s = \bigcap_{s \in I} (TF)^s = \bigcap_{s \in T} (TF)^s = T^F.$$

It hence suffices to show that $S \subseteq TF$, as then $SF = TF$. Suppose not, and consider $x \in S \setminus TF$. In other words $y^{-1}x \notin F$ for all $y \in T$. By compactness, for some $\varphi \in \Phi$ and $0 < e < 1$ the partial type $y \in T$ implies that $\varphi(\gamma^{-1}(y^{-1}x), a) \geq e$. We conclude as above.

In the vector space case, since $T + F \leq S + F$, this means that there is $x \in S \setminus (T + F)$. That is, $\eta x + y \notin F$ for all $\eta \in K^\times$ and $y \in T$. By compactness, for some $\varphi \in \Phi$ and $0 < e < 1$ the partial type $y \in T$ implies that $\inf_{\eta \in K^\times} \varphi(\gamma^{-1}(\eta x + y), a) \geq e$. Again we conclude as above.

If $F \in \mathfrak{F}$ and $S = \bigwedge \mathfrak{F}_0$ for some subfamily $\mathfrak{F}_0 \subset \mathfrak{F}$ of cardinality strictly less than λ , then S^F is type-definable with strictly less than λ parameters. Since F and S are commensurable, the difference $|S_F \setminus S|$, the index $[S_F : S]$ or the co-dimension $\text{codim}_{S^F}(S)$ are bounded by $2^{<\lambda}$. It follows that any chain between S and S_F has length at most $2^{<\lambda}$. We can thus put $\kappa = (2^{<\lambda})^+$ to satisfy condition (iv) of Definition 1.1.

Define an action of Γ on $(A \times \Gamma)$ by $\gamma' \cdot (a, \gamma) = (a, \gamma'\gamma)$. Then

$$\begin{aligned} \gamma' F_{a,\gamma} &= \gamma' \gamma F_a &= F_{a,\gamma'\gamma} = F_{\gamma' \cdot (a,\gamma)}, \\ \delta_{\varphi,n}(\gamma' S, \gamma' \cdot (a, \gamma)) &= \delta_{\varphi,n}(\gamma' S, a, \gamma'\gamma) &= \delta_{\varphi,n}(S, a, \gamma), \\ (\gamma' S)^{F_{\gamma' \cdot (a,\gamma)}} &= (\gamma' S)^{\gamma' F_{a,\gamma}} &= \gamma' (S^{F_{a,\gamma}}), \end{aligned}$$

so everything is Γ -invariant. Clearly, we also have invariance under $\text{Aut}(\mathfrak{M})$.

By Theorem 1.2 there is some $N \in \mathfrak{L}$ invariant under the group of automorphisms of \mathfrak{L} generated by $\Gamma \cup \text{Aut}(\mathfrak{M})$. In particular N is commensurable with \mathfrak{F} , type-definable over \emptyset , and Γ -invariant. \blacksquare

Remark 2.1. The usual metrisation of quotients modulo type-definable equivalence relations shows that Theorem 2.3 also holds for hyperdefinable families of commensurable subsets/subgroups/sub-vector spaces. (If the equivalence relation is given by an uncountable partial type, we first express the quotient

as a type-definable subset of an infinite (possibly uncountable) product of hyperimaginary sets modulo countable equivalence relations, which in turn are equivalent to imaginary metric sorts.)

3. FIELDS

For two fields F and K we say that F is commensurably contained in K if the degree $[FK : K]$ is finite. Then commensurable fields form an *upper* semi-lattice which need not be closed under meet. Theorem 1 in [5] implies in particular that if K is a field, Γ a group of automorphisms of K and \mathfrak{F} a family of uniformly commensurable subfields of K such that any finite intersection of elements in \mathfrak{F} is commensurable with \mathfrak{F} , then there is a Γ -invariant subfield of K commensurable with \mathfrak{F} . However, the condition that finite intersections be commensurable with \mathfrak{F} is much stronger than mere pairwise commensurability. In this section, we shall show that in case the extensions FF'/F for $F, F' \in \mathfrak{F}$ are separable, or the Eršov invariant of any field in \mathfrak{F} is finite, there still is a Γ -invariant commensurable subfield.

Theorem 3.1. *Let K be a field, Γ a group of automorphisms of K and \mathfrak{F} a Γ -invariant family of uniformly commensurable subfields of K . If FF'/F is separable for all $F, F' \in \mathfrak{F}$, or if the Eršov invariant $[F : F^p]$ is finite for any $F \in \mathfrak{F}$, there is a Γ -invariant subfield N commensurable with \mathfrak{F} .*

Proof. For $F \in \mathfrak{F}$ let F^s be the separable closure of F in K , and put $\bar{K} = \bigcap_{F \in \mathfrak{F}} F^s$, a Γ -invariant subfield of K . If FF'/F is separable for all $F, F' \in \mathfrak{F}$, then $F \leq \bar{K}$ for all $F \in \mathfrak{F}$. Otherwise, by uniform commensurability of \mathfrak{F} there is a finite power q of p such that $F^q F'/F'$ is separable for all $F, F' \in \mathfrak{F}$, where $p = \text{char}(K) > 0$ is the characteristic. Then $F^q \leq F \cap \bar{K} \leq F$ for all $F \in \mathfrak{F}$. Note that since F has finite Eršov invariant, $[F : F^q]$ is finite, so $\bar{\mathfrak{F}} = \{F \cap \bar{K} : F \in \mathfrak{F}\}$ is a family of uniformly commensurable subfields of \bar{K} .

Let $G = \text{Aut}(\bar{K})$ with the topology of pointwise convergence and the induced action of Γ . For $F \in \mathfrak{F}$ put $H_F = \text{Gal}(\bar{K}/F \cap \bar{K}) \leq G$, so $H_{\gamma F} = H_F^{-1}$ for $\gamma \in \Gamma$. Then uniform commensurability of $\bar{\mathfrak{F}}$ implies that $\mathfrak{H} = \{H_F : F \in \mathfrak{F}\}$ is a Γ -invariant family of uniformly commensurable closed subgroups of G . By Schlichting's Theorem for \mathfrak{H} there is a Γ -invariant subgroup $H \leq G$ commensurable with \mathfrak{H} . Moreover H is closed as it is a finite extension of a finite intersection of groups in \mathfrak{H} . Then $N = \text{Fix}_{\bar{K}}(H)$ is a Γ -invariant subfield of \bar{K} commensurable with $\text{Fix}_{\bar{K}}(H_F)$ for any $H_F \in \mathfrak{H}$. As $F^q \leq \bar{K} \leq F^s$ we have $\text{Fix}_{\bar{K}}(H_F) = F \cap \bar{K}$, which is commensurable with F . ■

[8] gives examples of commensurable fields F and F' such that FF'/F is purely inseparable, FF'/F' is either separable or purely inseparable, and $F \cap F'$ has infinite degree in F and in F' . As we have not been able to deal with this problem, we have not managed to prove **Theorem 3.1** in full generality.

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