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# PROPAGATION OF EPISTEMIC UNCERTAINTY IN QUEUEING MODELS WITH UNRELIABLE SERVER USING CHAOS EXPANSIONS

KATIA BACHI, CÉDRIC CHAUVIÈRE, HACÈNE DJELLOUT, AND KARIM ABBAS

ABSTRACT. In this paper, we develop a numerical approach based on Chaos expansions to analyze the sensitivity and the propagation of epistemic uncertainty through a queueing systems with breakdowns. Here, the quantity of interest is the stationary distribution of the model, which is a function of uncertain parameters. Polynomial chaos provide an efficient alternative to more traditional Monte Carlo simulations for modelling the propagation of uncertainty arising from those parameters. Furthermore, Polynomial chaos expansion affords a natural framework for computing Sobol' indices. Such indices give reliable information on the relative importance of each uncertain entry parameters. Numerical results show the benefit of using Polynomial Chaos over standard Monte-Carlo simulations, when considering statistical moments and Sobol' indices as output quantities.

*Key words:* Unreliable queueing model, Epistemic uncertainty, Chaos expansions, Orthogonal polynomial, Sobol' indices.

*AMS 2010 subject classifications.* 90B22, 60K25, 65C05, 65C50

## 1. INTRODUCTION

Queueing systems with breakdowns are widely used to model problems occurring in computer, manufacturing systems, and communication networks. Typically, in queueing models the performances measurements are assessed for deterministic parameters values. However, in practice, the exact values of these parameters are not well known (uncertain), and they are generally estimated empirically from few experimental observations. This lack of precise information propagates uncertainties in the output measure and such study is the purpose of this work.

Since the pioneering work of Thiruvengadam [51] and Avi-Itzhak and Naor [3], many authors have been investigating the queueing systems with server breakdowns, see for example [7, 28, 52] and references therein. In this paper, we choose a functional approach to the analysis of the dependence of performance measures of certain queues, such as the M/M/1 and the M/G/1 queue with breakdowns, with respect to some input parameters. More specifically, denoting the probability of a server breakdown by  $\theta$ , we seek to compute the stationary distribution of the queue-length process, denoted by  $\pi_\theta$ . For a fixed value of  $\theta$  and a finite waiting capacity,  $\pi_\theta$  can be found numerically by solving  $\pi_\theta \Xi_\theta = \pi_\theta$  and  $\sum \pi_\theta = 1$ , where  $\Xi_\theta$  denotes the transition probability matrix of an embedded jump chain. The definition of the embedded jump chain will depend on the type of the queue: for the M/M/1 queue we use the sample-chain, embedded at appropriate Poisson-times, and for the M/G/1 queue we will embed the chain at departure and repair moments; details will be provided later in the paper. In case of a large or infinite waiting capacity,  $\pi_\theta$  can be obtained via a Laplace-Stieltjes transform [1]. Solving for  $\pi_\theta$  involves numerical inversion, which can be computationally demanding [44]. The computation of  $\pi_\theta$  is a challenging problem and a variety of approaches have been proposed in the literature for approximately or indirectly solving the stationary distribution. The predominant approach is to obtain either the generating function of  $\pi_\theta$  or an analytical expression for  $\pi_\theta$  containing a Laplace-Stieltjes transform, see, for example,

[1, 4]. Also numerical solutions by means of the matrix geometric method [35] are available, see [33, 36] for details.

In performance analysis, one is not only interested in evaluating the system for specific set of parameters but also in the sensitivity of the performance with respect to these parameters. For example, in a queueing model with breakdowns, the breakdown probability is a parameter of key interest and, in this paper, we will analyze the dependence of  $\pi_\theta$  on  $\theta$ , which is significantly more challenging than evaluating  $\pi_\theta$  for a fixed  $\theta$ . Most of the time, it is assumed that these stochastic models are solved for fixed parameters values. However, the parameters of the model are determined through insufficient statistical data (a limited number of observations), leading to uncertainty in the assessment of their values. This parametric uncertainty, induced from the incomplete information of the parameter, is called epistemic uncertainty [29, 32, 54].

In order to estimate the uncertainty of the parameters in the performance measures of the model, two complementary approaches may be used: uncertainty analysis and sensitivity analysis. Uncertainty analysis consists in modelling input parameters as random variables, see, for example, [6, 19, 20]. Then, the sensitivity analysis aims at determining the relative contribution of individual parameters. Several approaches for the propagation of uncertainty have been developed, including interval arithmetic [34, 40], Taylor series expansion [12, 17, 37, 43, 49], moments [17, 31, 43], Monte Carlo analysis [17, 31, 43]. Overviews of these approaches are available in several reviews [55, 18, 21, 24, 25, 41].

Since the seminal work of Ghanem and Spanos in the 90's [14] and later generalized by Xiu and Karniadakis [56], Polynomial Chaos has spread over a broad scientific community, see [42], [8]. Then, due to the specific form of these polynomials, it quickly became apparent that this representation was suitable to compute Sobol' indices [46]. To the best of our knowledge, this is the first attempt to use chaos expansion to investigate the propagation of the epistemic uncertainty through the performance measure of the queueing systems.

The main objective of this work is to develop a numerical procedure to investigate the sensitivity and the propagation of epistemic uncertainty in the input parameters of a queueing systems with unreliable server. As a typical example, a customer arrives at the queueing system according to a Poisson process with rate  $\lambda$ , which we consider as an input parameter of the model. We assume that this parameter is not precisely known and therefore it can be model as a random variable of known probability density function. Clearly, the output measure is a function of this input random variable. The epistemic uncertainty is propagated through a functional relationship of the type  $Y(\omega) = g(\lambda(\omega))$  that links the input parameter to the output measures. In this work, special attention is given to the output stationary distribution. From a statistical point of view the quantity of interest may be the moments of the functional stationary distribution. Such quantities can easily be computed from the coefficient of the polynomial Chaos representation of  $g(\lambda(\omega))$ .

Over the past few years, several techniques have emerged to compute the coefficients of polynomial Chaos. They can be classified in two main families: intrusives methods and non-intrusive methods [38]. In our context,  $Y(\omega)$  can be explicitly expressed as a functional relationship and therefore the two approaches are of equal complexity. In this paper, projection method based on the Gauss quadrature rules are used to obtain the chaos coefficients. Then statistical moments and Sobol' sensitivity indices can easily be computed from polynomials chaos expansion.

The paper is organized as follows. The next section, is devoted to restate some basic notions about orthogonal polynomial and Chaos expansion. Then, the Sobol' indices which are an important tool in sensitivity analysis are introduced. Epistemic uncertainty in input parameters of two queueing systems with unreliable server is considered in Section 4. In the first queue, a single input parameter is considered as random. On the other hand, the second queueing system considers four input

random variables and a sensitivity analysis is performed. For the two queues, numerical results for the computation of statistical moments are obtained using both chaos expansion and Monte-Carlo simulation. Finally, Section 5 draws some conclusions.

## 2. POLYNOMIAL CHAOS EXPANSION

Polynomial Chaos expansion shares many elements with power series involving orthogonal polynomials, which are commonly used in the spectral community. This is why the following subsection is devoted to restate some results concerning orthogonal polynomials.

### 2.1. Orthogonal Polynomials: univariate case.

Let  $V$  be the real vector space of all polynomials in one variable with real coefficients and let be the positive inner product on  $V$  defined as

$$\langle u, v \rangle = \int_I u(x)v(x)f(x)dx \quad \forall u, v \in V$$

where  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}^+$  is a nonnegative integrable function of  $x$  hereinafter referred to as a ‘weight function’. The set of polynomials  $\{\Psi_n\}_{n \geq 0}$  are said to be orthogonal with respect to the weight function  $f(x)$ , if  $\Psi_n$  is a polynomials of degree  $n$  and

$$\langle \Psi_n, \Psi_m \rangle = \int_I \Psi_n(x)\Psi_m(x)f(x)dx = h_n^2 \delta_{n,m}, \quad n, m \in \mathbb{N}, \quad (2.1)$$

where  $\delta_{n,m}$  is the Kronecker’s delta function, and  $h_n$  are non-zero constants. We recall that, for orthogonal polynomials of degree  $d = 0$ ,  $\Psi_0$  is always equal to one ( $\Psi_0 = 1$ ). Furthermore, the system (2.1) is called orthonormal if  $h_n = 1$ .

The most general way to build such polynomials rely on the three-term recurrence relation

$$\begin{cases} \Psi_{n+1}(x) = (x - a_n)\Psi_n(x) - b_n\Psi_{n-1}(x) \\ \Psi_0(x) = 1, \quad \Psi_{-1}(x) = 0 \end{cases} \quad (2.2)$$

with

$$\begin{cases} a_n = \frac{\langle x\Psi_n, \Psi_n \rangle}{\langle \Psi_n, \Psi_n \rangle}, \quad n \in \mathbb{N} \\ b_n = \frac{\langle \Psi_n, \Psi_n \rangle}{\langle \Psi_{n-1}, \Psi_{n-1} \rangle}, \quad n \in \mathbb{N}^* \end{cases} \quad (2.3)$$

(see Gautschi [13] for more details). Furthermore, according to Favard’s theorem, for a given weight function  $f$ , corresponds a unique set of coefficients  $(a_n, b_n)_{n \in \mathbb{N}}$ . From a numerical point of view, the integrals appearing into (2.3) can be evaluated using Fejér quadrature rule, see [39], for example. Then, taking an orthogonal polynomial basis of degree  $n$ ,  $\{\Psi_0, \Psi_1, \dots, \Psi_n\}$ , any sufficiently regular function  $u : I \subset \mathbb{R} \rightarrow \mathbb{R}$  may be represented by its projection  $\Pi_n u$  on such basis, i.e.

$$\Pi_n u(x) = \sum_{i=0}^n \hat{u}_i \Psi_i(x), \quad (2.4)$$

where the coefficients of the projection can be computed by evaluating  $\hat{u}_i = \frac{\langle u, \Psi_i \rangle}{\langle \Psi_i, \Psi_i \rangle}$ . Depending on the regularity of the function  $u$  and the choice of the polynomial basis, upper bounds of the truncation error  $\|u - \Pi_n u\|$  can be derived [16]. In particular, when  $u$  is  $C^\infty$ , one can observe the so-called spectral convergence (the truncation error decays exponentially fast with respect to  $n$ ).

The construction of multivariate orthogonal polynomial basis rely on univariate polynomial basis as will be described in the following section.

## 2.2. Orthogonal Polynomials: multivariate case.

The setting is now the real vector space  $V^d$  of all polynomials in  $d$  variables with real coefficients, together with the positive inner product on  $V^d$  defined as

$$\langle u, v \rangle = \int_I u(\mathbf{x})v(\mathbf{x})f(\mathbf{x})d\mathbf{x} \quad \forall u, v \in V^d, \quad (2.5)$$

where, now,  $f : I \subset \mathbb{R}^d \rightarrow \mathbb{R}^+$  is a nonnegative integrable function of  $\mathbf{x} = (x_1, x_2, \dots, x_d)$ . We further assume that the weight function  $f$  can be written as the product of univariate weight functions, i.e.  $f(x_1, x_2, \dots, x_d) = f_1(x_1)f_2(x_2) \dots f_d(x_d)$ , where  $f_i : I_i \subset \mathbb{R} \rightarrow \mathbb{R}^+$  are nonnegative integrable function. It would be tempting to construct the elements of the multivariate polynomial basis in a similar fashion i.e. as a tensor product of monivariate elements. However, such construction leads to uncomplete basis (all monomials up to a certain degrees are not represented). To ensure that  $\{\Psi_0(\mathbf{x}), \Psi_1(\mathbf{x}), \dots, \Psi_{P_d^p}(\mathbf{x})\}$  is a multivariate polynomial basis of degree  $d$  with  $(P_d^p + 1)$  elements, the procedure described in the sequel is usually adopted. We first define  $\Psi_{\alpha_1 \dots \alpha_d}(\mathbf{x})$  as the tensor product of the elements of a univariate polynomial basis, of degree  $n$  i.e.

$$\Psi_{\alpha_1 \dots \alpha_d}(\mathbf{x}) = \prod_{i=1}^d \Psi_{\alpha_i}(x_i), \quad (2.6)$$

where  $\alpha_i \in \{0, 1, \dots, n\}$ . However, not all the elements of the form (2.6) are retained when constructing the multivariate polynomial basis (that would lead to an uncomplete basis of degree  $nd$  with  $(n+1)^d$  elements). Instead, for a given degree  $p$ , only the elements that satisfy  $\sum_{i=0}^d \alpha_i \leq p$  in (2.6) are kept and a one to one correspondence between the multi-index  $(\alpha_0, \dots, \alpha_d)$  and the  $i^{\text{th}}$  element  $\Psi_i(\mathbf{x})$  of the multivariate basis is set. Proceeding that way, it can be shown that the number of elements of a complete multivariate polynomial basis of degree  $p$  is

$$P_d^p + 1 = \binom{p+d}{d} = \frac{(p+d)!}{d!p!}. \quad (2.7)$$

Similarly to (2.4), any function  $u : \mathbb{R}^d \rightarrow \mathbb{R}$  may be represented by its projection  $\Pi_p u$  on such basis, i.e.

$$\Pi_p u(\mathbf{x}) = \sum_{i=0}^{P_d^p} \hat{u}_i \Psi_i(\mathbf{x}). \quad (2.8)$$

Having set the basic framework of orthogonal polynomials, we now look into the way they can be efficiently used in the field of probability, where they are usually refered as Polynomial Chaos (PC).

## 2.3. Polynomial Chaos expansion.

Denote by  $(\Omega, \mathcal{A}, \mathcal{P})$  the probability space, where as usual  $\Omega$  is the set of all possible outcomes,  $\mathcal{A}$  is a  $\sigma$ -algebra over  $\Omega$ , and  $\mathcal{P}$  is a function  $\mathcal{A} \rightarrow [0, 1]$  that gives a probability measure on  $\mathcal{A}$ . Consider an  $\mathbb{R}^d$ -valued independent random vector  $\mathbf{X} = (X_1, \dots, X_d)$  that describes input uncertainties. The probability law of  $\mathbf{X}$  may be defined by the probability density function

$$f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^d f_i(x_i), \quad (2.9)$$

where  $f_i(x_i)$  is the marginal probability density of  $X_i$  defined on  $(\Omega_i, \mathcal{A}_i, \mathcal{P}_i)$ . Let us now denote  $L_2(\Omega_i, \mathcal{A}_i, \mathcal{P}_i)$  the space of real random variables with finite second order moments, i.e. such that

$$E(X_i^2) = \int x_i^2 f_i(x_i) dx_i < \infty, \quad (2.10)$$

where  $E$  stands for the mathematical expectation.  $L_2(\Omega_i, \mathcal{A}_i, \mathcal{P}_i)$  is a Hilbert space which can be provided with a set of complete orthogonal basis  $\{\Psi_j^i(x)\}_{j \geq 0}$  that are consistent with the density of  $X_i$ . For example, Legendre polynomials are associated with uniform distributions whereas Hermite polynomials are associated with Gaussian distributions. Similarly,  $L_2(\Omega, \mathcal{A}, \mathcal{P})$  is provided with a set of complete multivariate orthogonal basis  $\{\Psi_j(\mathbf{x})\}_{j \geq 0}$  which, in turn, is consistent with the density of  $\mathbf{X}$ .

Let  $\mathbf{Y} = (Y_1, \dots, Y_\ell) : \Omega \rightarrow \mathbb{R}^\ell$  such that  $Y_i \in L_2(\Omega, \mathcal{A}, \mathcal{P})$  for  $i = 1, \dots, \ell$  be a mathematical model. For the sake of simplicity, we will consider only one component of this model, denoted by  $Y$ , since the same procedure apply identically to all the other components. Since  $Y$  is assumed to belong to  $L_2(\Omega, \mathcal{A}, \mathcal{P})$ , it can be represented as [14, 56]

$$\begin{aligned} Y(X_1, \dots, X_d) &= z_0 \Psi_0 + \sum_{i_1=1}^{\infty} z_{i_1} \Psi_1(X_{i_1}) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} z_{i_1 i_2} \Psi_2(X_{i_1}, X_{i_2}) + \\ &+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} z_{i_1 i_2 i_3} \Psi_3(X_{i_1}, X_{i_2}, X_{i_3}) + \dots, \end{aligned} \quad (2.11)$$

which, after some rearranging, can be rewritten in a more convenient way as

$$Y(\mathbf{X}) = \sum_{j=0}^{\infty} y_j \Psi_j(\mathbf{X}) \quad (2.12)$$

Then, similarly to equation (2.8), this serie is truncated by keeping terms up to a degree  $p$

$$Y(\mathbf{X}) \approx \Pi_p Y(\mathbf{X}) = \sum_{j=0}^{P_d^p} y_j \Psi_j(\mathbf{X}). \quad (2.13)$$

### Example 2.1.

Let  $X = (X_{i_1}, X_{i_2})$  be a random normal vector. The functional approximation of the response  $Y = Y(\mathbf{X})$  may be approximated with Hermite-Chaos expansions according to (2.13). If we fix the degree of the Chaos expansions to be  $d = 2$ , we obtain the series expansion with  $P_2^2 = 6$  terms as follows:

$$\begin{aligned} Y &\approx \sum_{i=0}^5 y_i \Psi_i(X_1, X_2) \\ &= y_0 \Psi_{00} + y_1 \Psi_{10} + y_2 \Psi_{01} + y_3 \Psi_{11} + y_4 \Psi_{20} + y_5 \Psi_{02}, \end{aligned} \quad (2.14)$$

where  $\Psi_{i_1 i_2}(X_1, X_2) = \Psi_{i_1}(X_1) \Psi_{i_2}(X_2)$  is the product of univariate Hermite polynomials of degree  $i_1$  and  $i_2$  satisfying  $i_1 + i_2 \leq 2$ . Such polynomials can easily be computed according to the recurrence formula (2.2) together with (2.3) with  $a_n = 0$  and  $b_n = n$ . Should we have used Legendre polynomials instead of Hermite's ones, they would be built by taking  $a_n = 0$  and  $b_n = \frac{n^2}{(4n^2-1)}$ . Table 1 shows the bivariate Hermite polynomial basis constructed from univariate basis  $\{1, X_1, X_1^2 - 1\}$  and  $\{1, X_2, X_2^2 - 1\}$  and the link between the main index  $j$  and the multi-index  $(i_1, i_2)$ . For completeness, the same Table also provides bivariate Legendre polynomial basis constructed from univariate basis  $\{1, X_1, X_1^2 - 1/3\}$  and  $\{1, X_2, X_2^2 - 1/3\}$ .

$j$	$(i_1, i_2)$	$\Psi_{i_1 i_2}$ (Hermite)	$\Psi_{i_1 i_2}$ (Legendre)
0	(0, 0)	1	1
1	(1, 0)	$X_1$	$X_1$
2	(0, 1)	$X_2$	$X_2$
3	(1, 1)	$X_1 X_2$	$X_1 X_2$
4	(2, 0)	$X_1^2 - 1$	$X_1^2 - 1/3$
5	(0, 2)	$X_2^2 - 1$	$X_2^2 - 1/3$

TABLE 1. Bivariate Hermite and Legendre Polynomial Basis

#### 2.4. Computing PC coefficients.

The way to compute the PC coefficients  $\{y_j\}_{0 \leq j \leq P_d^p}$  appearing into equation (2.13) can be casted into two different families: projection methods and regression methods [38]. Here we have used the first one which consists in premultiplying (2.13) by  $\Psi_j(\mathbf{X})$  and by taking the expectation of the resulting product. Using the orthogonality of the PC basis, most of the terms cancel and we end up with

$$y_j = E(Y(\mathbf{X})\Psi_j(\mathbf{X})) = \int_{I \subset \mathbb{R}^d} Y(\mathbf{x})\Psi_j(\mathbf{x})f(\mathbf{x})d\mathbf{x} \quad \text{for } j = 0, 1, \dots, P_d^p. \quad (2.15)$$

The above integral can be evaluated through different techniques: going from rough Monte-Carlo sampling simulation to Gaussian quadrature rule or sparse quadrature rules when the dimension  $d$  of the input random is high. Here we evaluate such integrals using Gaussian quadrature rules which take the form

$$\int_{I \subset \mathbb{R}^d} Y(\mathbf{x})f(\mathbf{x})d\mathbf{x} \approx \sum_{i_1=1}^{N_{g_1}} \sum_{i_2=1}^{N_{g_2}} \dots \sum_{i_d=1}^{N_{g_d}} \omega_{i_1} \omega_{i_2} \dots \omega_{i_d} Y(\tilde{x}_{i_1}, \tilde{x}_{i_2}, \dots, \tilde{x}_{i_d}), \quad (2.16)$$

where  $\{\omega_{i_k}\}_{1 \leq i_k \leq N_{g_k}}$  are the quadrature weight and  $\{\tilde{x}_{i_k}\}_{1 \leq i_k \leq N_{g_k}}$  are the quadrature points. The Gaussian quadrature rules are such that the evaluation of the integral is exact if  $Y$  is a multivariate polynomial containing monomials  $x_{i_k}$  of maximum degree  $2N_{g_k} - 1$ . From equations (2.15) and (2.16), we see that it suffices to evaluate the response  $Y$  at  $N_{g_1} \times N_{g_2} \dots \times N_{g_d}$  deterministic quadrature points in order to compute the PC coefficients. Furthermore since  $\Psi_0 = 1$ , setting  $j = 0$  in equation (2.15) leads to

$$y_0 = E(Y(\mathbf{X})), \quad (2.17)$$

i.e. the first coefficient of the PC expansion is the expectation of the random response of the system. Similarly, by considering the approximation of  $Y^2$

$$Y(\mathbf{X})^2 \approx \sum_{i=0}^{P_d^p} \sum_{j=0}^{P_d^p} y_i y_j \Psi_i(\mathbf{X}) \Psi_j(\mathbf{X}), \quad (2.18)$$

and taking the expectation on each side, the orthogonality of the PC basis leads to

$$E(Y(\mathbf{X})^2) = \sum_{i=0}^{P_d^p} y_i^2, \quad (2.19)$$

from which the variance of the random response of the system can easily be deduced. Furthermore, the PC decomposition provides a convenient way of computing Sobol' indices, as explained in the next section.

### 3. SENSITIVITY ANALYSIS

The purpose of sensitivity analysis, is to investigate the influence of each input parameter and their possible interactions onto the output measures. They can be casted into two main families: local analysis based on a local perturbation around an average value and global analysis that consider input parameters as random variables and decompose the output variance into several components. The Sobol' indices belong this last type of family.

#### 3.1. Sobol' indices.

As in the previous section, we consider the mathematical model  $Y = Y(\mathbf{X})$ , where the input parameter  $\mathbf{X} = (X_1, \dots, X_d)$  are  $d$  independant random variables belonging to  $L_2(\Omega_i, \mathcal{A}_i, \mathcal{P}_i)$  for  $i = 1, \dots, d$  and similarly,  $Y$  is assumed to belong to  $L_2(\Omega, \mathcal{A}, \mathcal{P})$ . In 1993, Sobol [45] proposed an indicator of the influence of the input parameter  $X_i$  defined by

$$S_i = \frac{V(E(Y/X_i))}{V(Y)} = \frac{V_i}{V}, \quad (3.1)$$

commonly termed 'first order Sobol' indices'.  $V_i = V(E(Y/X_i))$  is the conditionnal variance of  $Y$  with respect to  $X_i$  and  $V = V(Y)$  is the total variance of  $Y$ . Similarly, sensitivity indices of higher order can be defined by first introducing the following decomposition of the total variance, which is valid for any output  $Y$  belonging to  $L_2(\Omega, \mathcal{A}, \mathcal{P})$

$$V = \sum_{i=1}^d V_i + \sum_{1 \leq i_1 < i_2 \leq d} V_{i_1 i_2} + \sum_{1 \leq i_1 < i_2 < i_3 \leq d} V_{i_1 i_2 i_3} + \dots + V_{i_1 \dots i_d}, \quad (3.2)$$

where

$$\begin{aligned} V_{i_1} &= V(E(Y/X_{i_1})) \\ V_{i_1 i_2} &= V(E(Y/X_{i_1}, X_{i_2})) - V_{i_1} - V_{i_2} \\ V_{i_1 i_2 i_3} &= V(E(Y/X_{i_1}, X_{i_2}, X_{i_3})) - V_{i_1 i_2} - V_{i_1 i_3} - V_{i_2 i_3} - V_{i_1} - V_{i_2} - V_{i_3} \\ &\dots \\ V_{i_1 \dots i_d} &= V - \sum_{i=1}^d V_i - \sum_{1 \leq i_1 < i_2 \leq d} V_{i_1 i_2} - \dots - \sum_{1 \leq i_1 < i_2 \dots < i_{d-1} \leq d} V_{i_1 \dots i_{d-1}} \end{aligned} \quad (3.3)$$

Then, Sobol' indices of order  $k$  are given by

$$S_{i_1 \dots i_k} = \frac{V_{i_1 \dots i_k}}{V}. \quad (3.4)$$

Although Sobol' indices could be computed by estimating integrals appearing into equations (3.3), such a procedure would be both computationally expensive and hardly tractable. Instead, computing those indices from the PC representation of the random output  $Y$  turns out to be an efficient alternative and has been the method of choice for many years since the seminal work of Sudret [46]. For that, we define by  $I_{i_1, i_2, \dots, i_s}$  ( $s \leq d$ ) the set of  $d$ -dimensional vectors  $\alpha = (\alpha_1, \dots, \alpha_d)$  with  $\alpha \neq \mathbf{0}$  and  $0 \leq \alpha_1 \leq \dots \leq \alpha_d \leq d$  that selects elements of the PC basis  $\Psi_{\alpha_1 \dots \alpha_d}$  defined by (2.6) containing solely the variables  $X_{i_1}, X_{i_2}, \dots, X_{i_s}$ .

This way, the multi-indices defined by  $I_i$  will select elements of the PC basis depending only on the variable  $X_i$ . Similarly,  $I_{i,j}$  will select elements of the PC basis that depend on  $X_i$  and  $X_j$ , to the exclusion of any other variable. With this notation, Sobol's indices of order  $k$  are simply given as a function of the PC coefficients as follows:

$$S_{i_1 \dots i_k} = \frac{1}{V} \sum_{(\alpha_1, \dots, \alpha_d) \in I_{i_1, \dots, i_k}} y_{\alpha_1, \dots, \alpha_d}^2. \quad (3.5)$$



For a problem with  $d$  input random parameters, it can be shown that  $(2^d - 1)$  Sobol' indices can be computed for each output random quantity of interest. When the number of input r.v. is large, the number of Sobol' indices grows exponentially and it becomes difficult to draw information from these statistics. This is why, in 1996, Homma and Saltelli [22] introduced the total sensitivity indices  $S_{T_i}$  ( $i = 1, \dots, d$ ) which measures the total effect of the  $i^{th}$  random input parameter. It is defined as the sum of all sensitivity indices  $S_{i_1, \dots, i_k}$  ( $k = 1, \dots, d$ ) for which, one of the indices  $i_1, i_2, \dots, i_k$  is equal to  $i$ , i.e.

$$S_{T_i} = \sum_{\substack{k=1 \\ (i_1, \dots, i_k) \in J_i^k}}^d S_{i_1, \dots, i_k},$$

where  $J_i^k$  is the set of  $k$ -dimensional vectors  $(i_1, \dots, i_k)$  with  $1 \leq i_1 < \dots < i_k \leq d$ , such that one of its components is equal to  $i$ . We can also compute these indices from the PC coefficients as follows:

$$S_{T_i} = 1 - \frac{1}{V} \sum_{(\alpha_1, \dots, \alpha_d) \in \bar{I}_i} y_{\alpha_1, \dots, \alpha_d}^2,$$

where  $\bar{I}_i$  is the complementary set of  $I_i$ .

**Example 3.1.**

The computation of the sensitivity indices for the example 2.1 where,  $p = 2$  and  $d = 2$  gives

$$\begin{aligned} S_1 &= \frac{1}{V} (y_{1,0}^2 + y_{2,0}^2), \\ S_2 &= \frac{1}{V} (y_{0,1}^2 + y_{0,2}^2), \\ S_{1,2} &= \frac{1}{V} y_{1,1}^2, \end{aligned} \tag{3.6}$$

with

$$V = y_{1,0}^2 + y_{0,1}^2 + y_{1,1}^2 + y_{2,0}^2 + y_{0,2}^2 \tag{3.7}$$

In accordance with (3.2), we see that the variance  $V$  of the output can be divided into three parts:  $V = S_1 + S_2 + S_{1,2}$ . The first part  $S_1$  represents the influence of the first input random variable alone; the second  $S_2$  represents the effect of the second one alone whereas  $S_{1,2}$  accounts for the combined effect of the two input random variables.

Similarly, according to the definition of the total sensitivity indices, we have

$$S_{T_1} = S_1 + S_{1,2},$$

and

$$S_{T_2} = S_2 + S_{1,2}.$$

#### 4. UNCERTAINTY ANALYSIS IN UNRELIABLE QUEUEING MODELS

In this section, we consider the PC expansion for propagating the uncertainty in performance measures of queueing models with breakdowns and repairs, due to epistemic uncertainties in the model input parameters. Section 4.1 studies the M/G/1/N queue with breakdowns, when the perturbation of a single parameter is introduced. Then, section 4.2 addresses the case of the M/M/1/N queue with breakdowns, for multiple random input parameters.

#### 4.1. The M/G/1/N queue with breakdowns and repairs.

Consider a finite capacity M/G/1/N queue with server subject to breakdowns and repairs. Assume that the customers arrive at the system according to a Poisson stream with rate  $\lambda$ , the service times are general and independent identically distributed with mean  $1/\mu$  and we denote the service time distribution by  $F(x)$ . There can be at most  $N$  customers in the queue (including the one being served), and customers attempting to enter the queue when there are already  $N$  customers present are lost. The service discipline is assumed to be FCFS. At the beginning of each service, there is a probability  $\theta$  that the server breaks down (and the customer is sent back to the queue) and enters a repair state. The time of repair is exponentially distributed with rate  $r$ . The only instant of time when a server can break down is right at the beginning of a service. This system is modeled by a Markov chain  $X = \{X_n : n \in \mathbb{N}\}$ , embedded at service completions and completion of a repair and has state space  $S = \{0, 1, \dots, N-1\}$ . Let  $\Xi = (a_{i,j})_{i,j \in S}$  be the transition probability matrix of the Markov chain  $X$ . The matrix  $\Xi$  has the following form:

$$\Xi = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-2} & 1 - \sum_{k=0}^{N-2} a_k \\ a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-2} & 1 - \sum_{k=0}^{N-2} a_k \\ 0 & a_0 & a_1 & a_2 & \cdots & a_{N-3} & 1 - \sum_{k=0}^{N-3} a_k \\ 0 & 0 & a_0 & a_1 & \cdots & a_{N-4} & 1 - \sum_{k=0}^{N-4} a_k \\ & & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_0 & 1 - a_0 \end{pmatrix}, \quad (4.1)$$

where

$$a_k = \theta \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} dF(x) + (1 - \theta) \frac{r}{r + \lambda} \left( \frac{\lambda}{\lambda + r} \right)^k, \quad k = 0, \dots, N-2. \quad (4.2)$$

Note that the Markov chain  $X$  is unichain and therefore the stationary distribution exists. Let  $\pi$  denote the stationary distribution of the queue-length process embedded at service completions and completion of a repair in the M/G/1/N queue with breakdowns and repairs (see [2] for details). In the sequel, we consider  $\pi$  as a function of the breakdown probability  $\theta$ , denoted by  $\pi(\theta)$ . Assume that the probability  $\theta$  is estimated from insufficient statistical data, and hence has uncertainty associated with it. In the next section, we will discuss a functional approach based on PC expansion for computing the epistemic uncertainty in stationary distribution  $\pi(\theta)$ , due to epistemic uncertainties in  $\theta$ .

##### 4.1.1. Epistemic uncertainty in queueing system.

In this model, we will assume that the uncertainty only affects one parameter. To analyze the propagation of epistemic uncertainty, the mean and the variance of the responses will be computed using the PC expansion.

Let us consider  $\boldsymbol{\pi}(\theta) = (\pi_0(\theta), \dots, \pi_{N-1}(\theta))$  be the random response or the model output of the queueing system. We consider the following perturbation model on the uncertain parameter:

$$\theta(\omega) = \hat{\theta} + \sigma_\theta \varepsilon(\omega), \quad (4.3)$$

where  $\theta(\omega)$  is a random variable with uniform density on the interval  $[0, 1]$ . The parameters  $\hat{\theta}$  and  $\sigma_\theta$  represent the mean and the standard deviation of  $\theta$  and they can be estimated by statistical methods.  $\varepsilon(\omega)$  can be considered as a random noise inflicted on  $\theta$  and it modelizes epistemic uncertainty. We assume that  $\varepsilon(\omega)$  is a random variable with uniform distribution on  $[-1, 1]$ . Then,

the approximation of any component  $\pi_\ell(\theta)$  of the stationary distribution by the PC expansions writes

$$\pi_\ell(\theta) \approx \sum_{i=0}^{P_d^p} y_i \Psi_i(\theta),$$

where  $\{\Psi_i(\theta)\}_{0 \leq i \leq P_d^p}$  form an orthonormal Legendre polynomial basis, and  $y_i$  are the coefficients of the approximating series expansion, see (2.13).

#### 4.1.2. Numerical results.

In this section, the parameters  $N$ ,  $\lambda$  and  $r$  are deterministic and their values are set to  $N = 7$ ,  $\lambda = 1$ ,  $r = 0.4$ , respectively. The only input random parameter is  $\theta$  and it is given by

$$\theta = 0.5 + 0.28 \varepsilon, \quad \varepsilon \sim \mathcal{U}(-1, 1).$$

For numerical computations, we propose two types of distribution for the service time, namely, Erlang ( $E_2$ ) and Hyperexponential of second order ( $H_2$ ). Those two distributions are described in the following subsections.

#### 4.1.3. Epistemic uncertainty in the M/E<sub>2</sub>/1 queue with server breakdowns and repairs.

In (4.2), the density  $F'(x) = f(x)$  of the service time is assumed to be Erlang of second order, i.e.

$$f(x) = \frac{\mu_1 \mu_2}{\mu_2 - \mu_1} (e^{-\mu_1 x} - e^{-\mu_2 x}) \mathbf{1}_{[0, +\infty)}(x),$$

where the rates  $\mu_1 = 4$  and  $\mu_2 = 2$ .

We determine the stationary distribution vector with respect to the random variable  $\theta(\omega)$  by solving the system  $\pi_\theta \Xi_\theta = \pi_\theta$  and  $\sum \pi_\theta = 1$ , where  $\Xi_\theta$  denotes the transition probability given by (4.1). Therefore, the random outputs of interest are the stationary distributions  $\pi_i$ ,  $i \in \{0, \dots, 6\}$ . Their projection is performed on a univariate PC basis of degree  $n = 4$  and the coefficients are computed using a 6-points Gaussian quadrature rule as explained in section 2.4. Numerical results for the expectations and the variances using a Monte-Carlo simulations of a sample size  $N_{MC} = 1000$  and  $N_{MC} = 100000$  are given in Table 2 and Table 3, respectively. Results are compared with those of the PC expansion. We note that the Monte Carlo simulations (MC in short) converge to the PC results; however the later are obtained at a fraction of the cost of the MC simulations. Similarly moments of higher order (skewness and kurtosis) are given in Tables 4 and 5, and the same conclusions can be drawn.

$E(\pi_i)$	PC	MC ( $N_{MC} = 1000$ )	MC ( $N_{MC} = 100000$ )
$\pi_0$	0.0237	0.0230	0.0237
$\pi_1$	0.0413	0.0422	0.0413
$\pi_2$	0.0660	0.0667	0.0660
$\pi_3$	0.1018	0.1026	0.1018
$\pi_4$	0.1557	0.1563	0.1558
$\pi_5$	0.2393	0.2401	0.2392
$\pi_6$	0.3722	0.3719	0.3722

TABLE 2. Expected value of the steady-state vector in M/E<sub>2</sub>/1/7 queue

Figure 1 show the density of the output stationary distributions for a random uniform input of the form (4.3). The shape of the densities show that they are far from being uniform. Furthermore, the plots are coherent with Tables 4 and 5. Indeed, a positive skewness reflects a random variable with higher probabilities in the left part of its support, and vice versa.

$V(\pi_i)$	PC $\times 10^{-4}$	MC $\times 10^{-4}$ ( $N_{MC} = 1000$ )	MC $\times 10^{-4}$ ( $N_{MC} = 100000$ )
$\pi_0$	1.7753	1.6686	1.7809
$\pi_1$	3.6413	3.6789	3.6423
$\pi_2$	5.2349	5.1333	5.2175
$\pi_3$	5.0800	4.9922	5.0508
$\pi_4$	1.8737	1.8585	1.8732
$\pi_5$	2.1440	2.0101	2.1534
$\pi_6$	58.7699	61.0693	58.7199

TABLE 3. Variance of the steady-state vector in M/E<sub>2</sub>/1/7 queue

Skew( $\pi_i$ )	PC	MC ( $N_{MC} = 1000$ )	MC ( $N_{MC} = 100000$ )
$\pi_0$	0.5053	0.5429	0.5064
$\pi_1$	0.3565	0.2911	0.3550
$\pi_2$	0.1730	0.1435	0.1701
$\pi_3$	-0.0722	-0.0927	-0.0730
$\pi_4$	-0.5848	-0.6616	-0.5922
$\pi_5$	-0.7140	-0.7968	-0.7036
$\pi_6$	0.0543	0.0692	0.0540

TABLE 4. Skewness coefficient of the steady-state vector in M/E<sub>2</sub>/1/7 queue

Kurt( $\pi_i$ )	PC	MC ( $N_{MC} = 1000$ )	MC ( $N_{MC} = 100000$ )
$\pi_0$	2.0384	2.1425	2.0341
$\pi_1$	1.8968	1.9122	1.8953
$\pi_2$	1.7864	1.7682	1.7900
$\pi_3$	1.7463	1.7513	1.7495
$\pi_4$	2.0301	2.0290	2.0385
$\pi_5$	2.1921	2.3603	2.1972
$\pi_6$	1.7759	1.8007	1.7724

TABLE 5. Kurtosis coefficient of the steady-state vector in M/E<sub>2</sub>/1/7 queue

#### 4.1.4. Epistemic uncertainty in the M/H<sub>2</sub>/1 queue with server breakdowns and repairs.

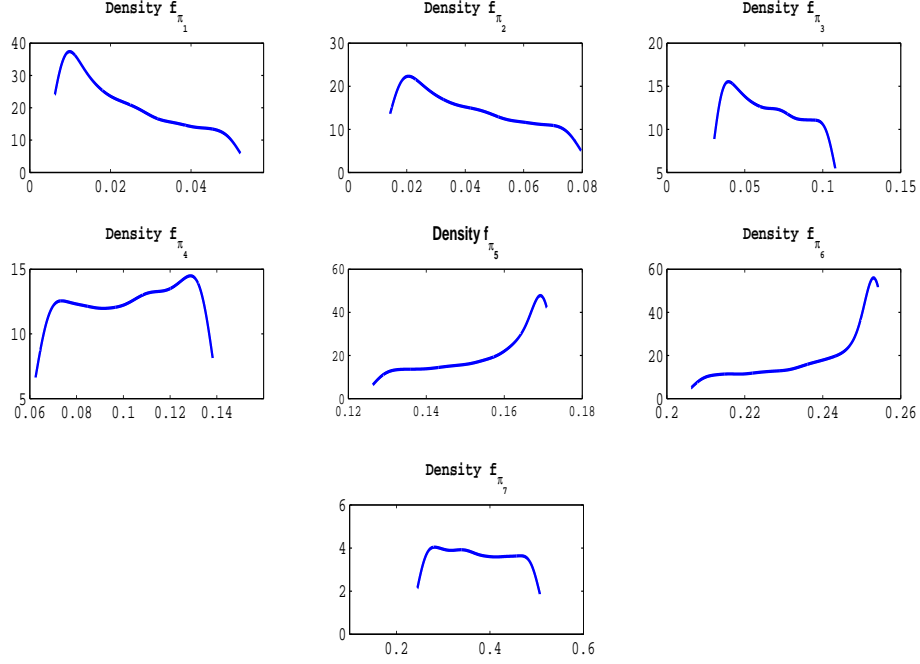
In this section, the density  $F'(x) = f(x)$  of the service time is now assumed to be Hyperexponential of second order i.e.

$$f(x) = (\gamma\mu_1 e^{-\mu_1 x} + (1 - \gamma)\mu_2 e^{-\mu_2 x})\mathbf{1}_{[0, \infty)}(x),$$

where the rates are  $\mu_1 = 3/2$ ,  $\mu_2 = 3$ ; and  $\gamma = 0.3$ . All the other parameters are set as in the previous section, including the random input. Similarly, Tables 6, 7, 8 and 9 show the moments of the stationary distributions when the Hyperexponential law is used instead of the Erlang one for the service time. Here also, we can observe the good convergence properties of the PC, showing the robustness of the method independently of the law of the service time.

## 4.2. The M/M/1 queue with server breakdowns and threshold-based recovery policy.

Consider a finite capacity M/M/1 queue with server breakdowns and threshold-based recovery policy. If a server provides service for a customer, the server may experience a breakdown, and the repair begins when the number of customers present in the system exceeds some prespecified

FIGURE 1. Marginal probability density  $f_{\pi_i}$  of the steady-state vector  $\pi(\theta)$  in M/E<sub>2</sub>/1/7 queue

$E(\pi_i)$	PC	MC ( $N_{MC} = 1000$ )	MC ( $N_{MC} = 100000$ )
$\pi_0$	0.0397	0.0407	0.0396
$\pi_1$	0.0602	0.0588	0.0602
$\pi_2$	0.0853	0.0847	0.0853
$\pi_3$	0.1179	0.1179	0.1179
$\pi_4$	0.1619	0.1615	0.1619
$\pi_5$	0.2233	0.2238	0.2233
$\pi_6$	0.3118	0.3103	0.3117

TABLE 6. Expected value of the steady-state vector in M/H<sub>2</sub>/1/7 queue

$V(\pi_i)$	PC $\times 10^{-4}$	MC $\times 10^{-4}$ ( $N_{MC} = 1000$ )	MC $\times 10^{-4}$ ( $N_{MC} = 100000$ )
$\pi_0$	4.7367	4.5451	4.7370
$\pi_1$	6.7973	6.5517	6.8120
$\pi_2$	7.1124	7.1342	7.1237
$\pi_3$	4.7642	4.5830	4.7726
$\pi_4$	0.7665	0.8177	0.7653
$\pi_5$	5.4708	5.2979	5.4699
$\pi_6$	64.0300	62.3300	64.0396

TABLE 7. Variance of the steady-state vector in M/H<sub>2</sub>/1/7 queue

threshold level  $q$  ( $1 \leq q \leq N$ ). The stream customers arrive at the queue according to a Poisson law of parameter  $\lambda$ . Arriving customers form a single waiting line based on the order of their arrivals

Skew( $\pi_i$ )	PC	MC ( $N_{MC} = 1000$ )	MC ( $N_{MC} = 100000$ )
$\pi_0$	0.3481	0.2807	0.3531
$\pi_1$	0.1478	0.2379	0.1474
$\pi_2$	-0.0881	-0.0925	-0.0896
$\pi_3$	-0.4210	-0.4300	-0.4230
$\pi_4$	-1.2389	-1.1806	-1.2381
$\pi_5$	-0.3119	-0.3561	-0.3101
$\pi_6$	0.2751	0.3242	0.2747

TABLE 8. Skewness coefficient of the steady-state vector in M/H<sub>2</sub>/1/7 queue

Kurt( $\pi_i$ )	PC	MC ( $N_{MC} = 1000$ )	MC ( $N_{MC} = 100000$ )
$\pi_0$	1.8643	1.9138	1.8624
$\pi_1$	1.7524	1.7200	1.7568
$\pi_2$	1.7297	1.6881	1.7266
$\pi_3$	1.8885	1.8168	1.8935
$\pi_4$	3.3357	3.7223	3.3412
$\pi_5$	1.7598	1.7821	1.7586
$\pi_6$	1.8392	1.8446	1.8372

TABLE 9. Kurtosis coefficient of the steady-state vector in M/H<sub>2</sub>/1/7 queue

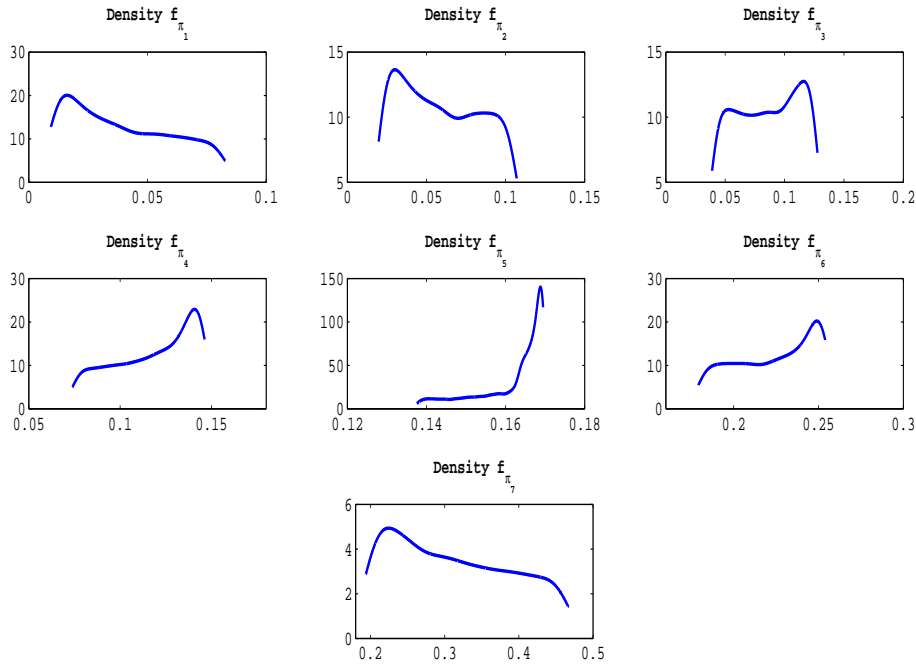


FIGURE 2. Marginal probability density  $f_{\pi_i}$  of of the steady-state vector in M/H<sub>2</sub>/1/7 queue

and the server can serve only one customer at a time. There can be at most  $N$  customers present at the queue (including the one in service), and customers attempting to enter the queue when  $N$

customers are already present are lost. A single server with exponential distributed service times with rate  $\mu$  is considered. We assume that the server can break down only if the system is not empty. The lengths of breakdowns are identically distributed and follow an exponential distribution with rate  $\alpha$ . The duration of reparation of the server is assumed to follow an exponential law with parameter  $\beta$ . After the server being repaired, it switches to working state and continues to provide service until the system becomes empty. The arrival flow customers, service times, breakdown times and repair times are assumed to be mutually independent input random variables. The service discipline is FCFS.

A typical state of this system may be denoted by  $X(t) = \{(N(t), Y(t)); t \geq 0\}$ , where  $N(t)$  is the number of customers in the queue at time  $t$ , and  $Y(t)$  is a random variable representing the server state at time  $t$ . If at time  $t$ , the server is experiencing a breakdown, then  $Y(t) = 1$ ; otherwise, the server is in the working state and  $Y(t) = 0$ . The stochastic process  $X(t)$  is a continuous time Markov chain whose state space  $S = \{(n, 0) : n = 0, \dots, N\} \cup \{(n, 1) : n = 1, \dots, N\}$ . The infinitesimal generator matrix  $Q$  of the continuous time Markov chain  $X(t)$  has the following block-tridiagonal form:

$$Q = \begin{pmatrix} A_0 & B_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ D_1 & B_1 & F_1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & D_2 & B_2 & F_2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & D_3 & B_3 & F_3 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & D_s & M_s & F_s & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & D_{s+1} & M_{s+1} & F_{s+1} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & D_N & E_N \end{pmatrix}, \quad (4.4)$$

where  $B_0 = \lambda, A_0 = -\lambda, D_1 = \begin{pmatrix} \mu \\ 0 \end{pmatrix}, B_i = \begin{pmatrix} -(\lambda + \mu + \alpha) & \alpha \\ 0 & -\lambda \end{pmatrix}, i = 1, \dots, q-1,$

$$F_j = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, j = 1, \dots, N-1, \quad D_k = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}, k = 2, \dots, N$$

$$M_s = \begin{pmatrix} -(\lambda + \alpha + \mu) & \alpha \\ \beta & -(\beta + \lambda) \end{pmatrix}, s = q, \dots, N-1, \quad E_N = \begin{pmatrix} -(\alpha + \mu) & \alpha \\ \beta & -\beta \end{pmatrix}.$$

The Markov transition diagram of the process is given below

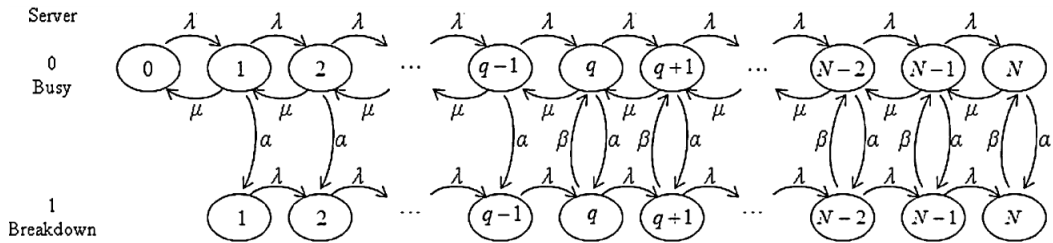


FIGURE 3. Markov transition diagram of the M/M/1/N queue with server breakdowns and threshold-based recovery policy

Denote by  $\pi_{m,n}, m = 0, 1, n \geq 0$  the stationary distribution of the Markov chain  $X(t)$ . We have  $\pi_0 = \pi_{0,0}$ , and  $\pi_n = (\pi_{0,n}, \pi_{1,n}), n = 1, \dots, N$ . The steady-state analysis of the Markov chain  $X(t)$

can be performed via a recursive scheme [58], or by using matrix analytic methods [35]. In the sequel, we follow a different method, and we will provide a functional approach based on the PC expansion for obtaining the stationary distribution  $\pi_{m,n}$  in terms of some parameters, which are obtained under the epistemic uncertainties.

#### 4.2.1. Epistemic uncertainty in queueing system.

In this model, we consider the epistemic uncertainty in all the input parameters of the queueing system,  $\lambda, \mu, \alpha, \beta$ . However, now  $\pi_{m,n}(\boldsymbol{\theta})$  is assumed to be the random output response of the model of the queueing system, where  $\boldsymbol{\theta} = [\theta_1 = \lambda, \theta_2 = \mu, \theta_3 = \alpha, \theta_4 = \beta]$  is the vector that describes the model parameters. We consider an epistemic uncertainty in the input parameter  $\boldsymbol{\theta}$  of known probability density function. Due to the uncertainty in input parameters, the responses  $\pi_{m,n}(\boldsymbol{\theta}(\omega))$  are considered as random variables. The uncertainty analysis of the functional  $\pi_{m,n}$  may be quantified by the computation of its mean and variance and moments of higher order.

Furthermore, we proceed to a sensitivity analysis in order to determine the most sensitive parameters with respect to the performance measures. A convenient way to compute those quantities is to approximate the functional  $\pi_{m,n}(\boldsymbol{\theta}(\omega))$  in the form of the PC expansion. The sensitivity analysis introduced in section 3 help us to point out which parameters generate a higher sensitivity with respect to the stationary distribution. We consider the following perturbation models for all the uncertain parameters:

$$\theta_j(\omega) = \hat{\theta}_j + \sigma_j \varepsilon_j(\omega), \quad j = 1, \dots, 4$$

where  $\theta_j(\omega)$  is random variable,  $\hat{\theta}_j, \sigma_j$ , are respectively, estimated mean and the standard deviation of the random variable  $\theta_j$  obtained by statistical methods.  $\varepsilon_j(\omega)$  is a random noise inflicted on  $\theta_i$  which models epistemic uncertainty. Here,  $\varepsilon_j(\omega)$  is assumed to follow a standard normal distribution. Let  $\boldsymbol{\theta} = [\theta_1(\omega) = \lambda, \theta_2(\omega) = \mu, \theta_3(\omega) = \alpha, \theta_4(\omega) = \beta]$ ,  $\pi_{m,n}(\boldsymbol{\theta}(\omega))$  and  $\boldsymbol{\varepsilon} = [\varepsilon_1, \dots, \varepsilon_4]$  be random fields.

The approximation of the measure  $\pi_{m,n}(\boldsymbol{\theta}(\omega))$  in the form of multivariate PC expansions is given by:

$$\pi_{m,n}(\boldsymbol{\theta}) \approx \sum_{i=0}^{P_4^p} y_i \Psi_i(\boldsymbol{\theta}),$$

where  $\Psi_i(\boldsymbol{\theta})$  is the multivariate orthonormal Hermite polynomial and  $y_i$  are the coefficients of the approximating series expansion, (2.13).

#### 4.2.2. Numerical application.

As an illustrative example, the queueing system described in section 4.2 is studied. The capacity is set to  $N = 5$  and the threshold is equal to  $q = 3$ . In order to study the sensitivity of the queue, we assume that the exact value of all the input parameters,  $\alpha, \beta, \lambda, \mu$ , are not precisely known and we propose the following perturbation on these parameters:

$$\begin{aligned} \alpha &= \hat{\alpha} + \sigma_\alpha \varepsilon_1, \quad \varepsilon_1 \sim \mathcal{N}(0, 1) & \beta &= \hat{\beta} + \sigma_\beta \varepsilon_2, \quad \varepsilon_2 \sim \mathcal{N}(0, 1) \\ \lambda &= \hat{\lambda} + \sigma_\lambda \varepsilon_3, \quad \varepsilon_3 \sim \mathcal{N}(0, 1) & \mu &= \hat{\mu} + \sigma_\mu \varepsilon_4, \quad \varepsilon_4 \sim \mathcal{N}(0, 1) \end{aligned}$$

so  $\alpha, \beta, \lambda, \mu$ , are Gaussian random variables. The mean and the standard deviation of these random variables are set respectively to  $\hat{\mu} = 7.3, \hat{\lambda} = 2, \hat{\alpha} = 3, \hat{\beta} = 4, \sigma_\alpha = 0.04, \sigma_\beta = 0.02, \sigma_\lambda = 0.04$  and  $\sigma_\mu = 0.02$ .

Table 10 gives the first order Sobol' indices for all the components of the stationary distribution vector. From this table, we see that for all but  $\pi_{0,1}$  and  $\pi_{1,3}$ , the most influential input random parameter is  $\lambda$ . Table 11, which gives the second order Sobol' indices, shows that the combined



$\pi_{m,n} \setminus S_{\theta_i}$	$S_{\alpha}$	$S_{\mu}$	$S_{\beta}$	$S_{\lambda}$
$\pi_{0,0}$	0.3799	0.0366	0.0046	0.5789
$\pi_{0,1}$	0.7942	0.0047	0.0096	0.1913
$\pi_{1,1}$	0.0144	0.0038	0.0078	0.9740
$\pi_{0,2}$	0.0632	0.0160	0.0080	0.9126
$\pi_{1,2}$	0.1650	0.0003	0.0109	0.8236
$\pi_{0,3}$	0.0445	0.0473	0.0073	0.9008
$\pi_{1,3}$	0.7812	0.0151	0.0418	0.1619
$\pi_{0,4}$	0.0502	0.0264	0.0006	0.9229
$\pi_{1,4}$	0.3120	0.0124	0.0411	0.6345
$\pi_{0,5}$	0.0528	0.0198	0.0044	0.9229
$\pi_{1,5}$	0.1512	0.0085	0.0396	0.8005

TABLE 10. First order Sobol' indices for stationary distribution vector in M/M/1/5 queue with server breakdowns and threshold-based recovery policy

$\pi_{m,n} \setminus S_{\theta_i, \theta_j}$	$S_{\alpha, \mu} \times 10^{-8}$	$S_{\alpha, \beta} \times 10^{-8}$	$S_{\alpha, \lambda} \times 10^{-6}$	$S_{\mu, \beta} \times 10^{-9}$	$S_{\mu, \lambda} \times 10^{-8}$	$S_{\beta, \lambda} \times 10^{-7}$
$\pi_{0,0}$	0.6972	1.3636	0.2051	1.8411	4.9130	1.1427
$\pi_{0,1}$	0.0144	2.8434	0.0127	0.4145	0.0016	0.6002
$\pi_{1,1}$	0.0361	0.0175	0.5309	0.3338	0.0569	1.9217
$\pi_{0,2}$	0.0516	0.7513	0.0183	0.0181	0.0249	0.8644
$\pi_{1,2}$	0.0263	0.0365	0.0143	0.0109	0.0013	7.3456
$\pi_{0,3}$	0.8840	0.0198	0.3924	0.0283	0.4031	0.8201
$\pi_{1,3}$	9.8593	1.6277	0.3311	0.1568	0.0997	0.0147
$\pi_{0,4}$	0.1210	7.1050	4.0803	4.9641	0.0012	8.0038
$\pi_{1,4}$	0.0137	0.0368	0.4395	0.9306	0.6662	8.6512
$\pi_{0,5}$	0.0113	0.2362	0.5077	0.6212	0.0029	0.1801
$\pi_{1,5}$	0.0250	0.0959	0.0187	0.0433	0.0010	0.0427

TABLE 11. Second order Sobol' indices for stationary distribution vector in M/M/1/5 queue with server breakdowns and threshold-based recovery policy

$\pi_{m,n} \setminus S_{\theta_i}^T$	$S_{\alpha}^T$	$S_{\mu}^T$	$S_{\beta}^T$	$S_{\lambda}^T$
$\pi_{0,0}$	0.3799	0.0366	0.0046	0.5789
$\pi_{0,1}$	0.7943	0.0047	0.0096	0.1914
$\pi_{1,1}$	0.0144	0.0038	0.0078	0.9741
$\pi_{0,2}$	0.0634	0.0160	0.0080	0.9128
$\pi_{1,2}$	0.1652	0.0002	0.0109	0.8237
$\pi_{0,3}$	0.0445	0.0473	0.0074	0.9009
$\pi_{1,3}$	0.7812	0.0151	0.0419	0.1619
$\pi_{0,4}$	0.0502	0.0264	0.0005	0.9229
$\pi_{1,4}$	0.3121	0.0124	0.0411	0.6345
$\pi_{0,5}$	0.0528	0.0198	0.0044	0.9230
$\pi_{1,5}$	0.1514	0.0085	0.0396	0.8007

TABLE 12. Total order Sobol' indices for steady-state vector in M/M/1/5 queue with server breakdowns and threshold-based recovery policy

effect of two input random parameter is negligible in comparison with the effect of a single parameter. The same is also true for higher order indices since the total Sobol' indices are almost equal to first order ones, as shown by Table 12. Note that for the components  $\pi_{0,1}$  and  $\pi_{1,3}$ , it is the parameter  $\alpha$  which is most influential.

When considering  $\pi_{0,1}$  and  $\pi_{1,3}$ , we can set  $\alpha$  as being the only random parameters and when considering all the other components of the stationary distribution, we can consider the parameter  $\lambda$  as sole input random variable. This is what we do as a second numerical experiments. Then, Tables 13 to 16 compare the statistical moments of the stationary distribution vector when the four input parameters are random (first column) to the case where only the most influential parameter is random (second column). As expected, the two columns of the tables match rather well.

$E(\pi_{m,n})$	PC (1 r.v.)	PC (4 r.v.)
$\pi_{0,0}$	0.2639	0.2639
$\pi_{0,1}$	0.0723	0.0723
$\pi_{1,1}$	0.1084	0.1084
$\pi_{0,2}$	0.0495	0.0495
$\pi_{1,2}$	0.1827	0.1827
$\pi_{0,3}$	0.0636	0.0636
$\pi_{1,3}$	0.0927	0.0927
$\pi_{0,4}$	0.0428	0.0428
$\pi_{1,4}$	0.0523	0.0523
$\pi_{0,5}$	0.0261	0.0261
$\pi_{1,5}$	0.0457	0.0457

TABLE 13. Expected Value of the stationary distribution in M/M/1/5 queue with server breakdowns and threshold-based recovery policy

$V(\pi_{m,n})$	PC $\times 10^{-6}$ (1 r.v.)	PC $\times 10^{-6}$ (4 r.v.)
$\pi_{0,0}$	14.711	25.338
$\pi_{0,1}$	0.7178	0.8879
$\pi_{1,1}$	2.4845	2.5455
$\pi_{0,2}$	0.4445	0.4905
$\pi_{1,2}$	4.2497	5.0989
$\pi_{0,3}$	0.7925	0.8793
$\pi_{1,3}$	0.6677	0.8450
$\pi_{0,4}$	1.4357	1.5550
$\pi_{1,4}$	0.9757	1.5432
$\pi_{0,5}$	1.2546	1.3592
$\pi_{1,5}$	3.4476	4.3111

TABLE 14. Variance of the stationary distribution in M/M/1/5 queue with server breakdowns and threshold-based recovery policy

## 5. CONCLUSION

In this paper, we have developed a numerical approach based on polynomial chaos expansion, to study the sensitivity and the propagation of the epistemic uncertainty in queueing models that occurs with unreliable servers. To illustrate the applicability of the proposed approach, two

Skew( $\pi_{m,n}$ )	PC (1 r.v.)	PC (4 r.v.)
$\pi_{0,0}$	0.0350	0.0395
$\pi_{0,1}$	0.0443	0.0415
$\pi_{1,1}$	0.0350	0.0371
$\pi_{0,2}$	-0.0722	-0.0411
$\pi_{1,2}$	0.0200	0.0362
$\pi_{0,3}$	-0.0536	-0.0523
$\pi_{1,3}$	-0.0316	-0.0556
$\pi_{0,4}$	-0.0011	-0.0042
$\pi_{1,4}$	-0.0622	-0.0116
$\pi_{0,5}$	0.0488	0.0576
$\pi_{1,5}$	0.0374	0.0668

TABLE 15. Skewness coefficient of the steady-state vector in M/M/1/5 queue with server breakdowns and threshold-based recovery policy

Kurt( $\pi_{m,n}$ )	PC (1 r.v.)	PC (4 r.v.)
$\pi_{0,0}$	3.0016	3.0020
$\pi_{0,1}$	3.0037	3.0046
$\pi_{1,1}$	3.0016	3.0024
$\pi_{0,2}$	3.0071	3.0064
$\pi_{1,2}$	2.9999	3.0023
$\pi_{0,3}$	3.0041	3.0035
$\pi_{1,3}$	3.0017	3.0060
$\pi_{0,4}$	2.9983	2.9979
$\pi_{1,4}$	3.0044	2.9988
$\pi_{0,5}$	3.0013	3.0028
$\pi_{1,5}$	2.9995	3.0052

TABLE 16. Kurtosis coefficient of the steady-state vector in M/M/1/5 queue with server breakdowns and threshold-based recovery policy

models of queueing systems have been investigated. In the first model (M/G/1/N queue with breakdowns and repairs), the epistemic uncertainty only affects one input parameter, whereas in the second model (M/M/1/N queue with server breakdowns and threshold-based recovery policy), it affects four parameters. In the latter case, a sensitivity analysis using Sobol' indices is performed. When considering the stationary distribution as output quantity of interest, it was shown that the parameter  $\lambda$  of the Poisson law modelling the customers arrival at the queue, was the most influential factor. This finding was then confirmed by considering  $\lambda$  as the only random input parameter and by setting the three remaining ones to their average values. In that case, it was shown that the statistical moments of the output measure were relatively insensitive to the other parameters. Finally, comparisons with Monte-Carlo simulations showed the good convergence properties of the chaos expansion.

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