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# Weak convergence of posteriors conditional on maximum pseudo-likelihood estimates and implications in ABC

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## Abstract

The weak convergence of posterior distributions conditional on maximum pseudo-likelihood estimates (MPLE) is studied and exploited to justify the use of MPLE as summary statistics in approximate Bayesian computation (ABC). Our study could be generalized by replacing the pseudo-likelihood by other estimating functions (e.g. quasi-likelihoods and contrasts).

*Keywords:* Approximate Bayesian computation, Bernstein - von Mises theorem, Weak convergence.

*2010 MSC:* 62F12, 62Fxx, 65C60

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## 1. Introduction

Approximate Bayesian computation (ABC) has been developed to make Bayesian inference with models that can be used to generate data sets but whose probability distribution of state variables is intractable (Marin et al., 2012). The intractability of this distribution makes impossible the application of the exact Bayesian approach, even by using numerical algorithms. ABC provides a sample from the parameter space that is approximately distributed under a posterior distribution of parameters conditional on sum-

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9   mary statistics. In general, this posterior does not coincide with the poste-  
10   rior distribution of parameters conditional on full raw data. Here, we are  
11   interested in the specific case where (some of) the summary statistics are  
12   point estimates of parameters (PEP), as in Drovandi et al. (2011), Fearn-  
13   head and Prangle (2012), Gleim and Pigorsch (2013) and Mengersen et al.  
14   (2013).

15       In the classical Bayesian framework, posteriors conditional on PEP can  
16   be viewed as specific cases of posteriors conditional on partial information  
17   (Doksum and Lo, 1990; Soubeyrand et al., 2009). Here, we provide new  
18   results of weak convergence when PEP are either maximum likelihood esti-  
19   mates (MLE) or pseudo-maximum likelihood estimates (MPLE). The case  
20   where PEP are MPLE is of specific interest because, in ABC, it may be  
21   possible to compute MPLE via simplifications of the dependence structure  
22   in the model, and to use MPLE as summary statistics.

23       The results of weak convergence that are provided in this note can be  
24   viewed as new extensions of the Bernstein – von Mises (BvM) theorem.  
25   For parametric models from which independent observations are made, the  
26   BvM theorem (i) states conditions under which the posterior distribution  
27   is asymptotically normal and (ii) subsequently leads to the efficiency of  
28   Bayesian point estimators and to the convergence of Bayesian confidence  
29   sets to frequentist limit confidence sets (Walker, 1969; Freedman, 1999).  
30   Thus, the BvM theorem can be viewed as a frequentist justification of pos-  
31   terior distributions for the estimation of parameters. Numerous extensions  
32   of the BvM theorem have been proposed, for instance, when the model is  
33   semiparametric or nonparametric (Bickel and Kleijn, 2012; Bontemps, 2011;  
34   Castillo, 2012a,b; Castillo and Nickl, 2013; Rivoirard and Rousseau, 2012),  
35   when observations are dependent (Borwanker et al., 1971; Tamaki, 2008),

36 when the model is misspecified (Kleijn and van der Vaart, 2012), and when  
37 the model is nonregular (e.g. when the true value of the parameter is on the  
38 boundary of the parameter space; Bochkina and Green, 2014).

39 Here, we extend the BvM theorem (i) when raw observations are replaced  
40 by the MLE (Lemma 1) or an MPLE (Lemma 2), and (ii) when the posterior  
41 conditional on an MPLE is approximated via ABC (Theorem 1). Using a  
42 posterior distribution (approximate or not) conditional on an MPLE, that  
43 was built by ignoring some dependences in the model, can be viewed as  
44 using a misspecified model like in Kleijn and van der Vaart (2012).

45 The BvM extensions obtained in the classical Bayesian framework (Point  
46 (i) in the paragraph above) are viewed as stepping stones that lead to the  
47 BvM extension obtained in the ABC framework (Point (ii)). Advancing  
48 theory in ABC has generally no direct practical implications because as-  
49 sumptions that may be required to prove theorems cannot be checked for a  
50 real-life implicit stochastic model whose distribution theory is intractable.  
51 However, showing an analytic result for a large class of theoretically tractable  
52 models may lead to conjecture that the result holds for some stochastic im-  
53 plicit models. Specifically, the work presented here allows us to conjecture  
54 that (i) an ABC-posterior distribution conditional on an MPLE is asymp-  
55 totically normal and centered around the MPLE, and (ii) resulting point  
56 estimates and confidence sets converge to their frequentist analogues.

## 57 **2. Main notations**

58 Observed data  $\mathcal{D} \in \mathbb{D}$  are assumed to be generated under the stochas-  
59 tic model  $\mathcal{M}_\theta$  parametrized by  $\theta \in \Theta$  with prior density  $\pi$ . The data  
60 space  $\mathbb{D}$  and the parameter space  $\Theta \subset \mathbb{R}^q$  ( $q \in \mathbb{N}^*$ ) are both included in

61 multidimensional sets of real vectors. The probability distribution func-  
 62 tions (p.d.f.) of the model and the prior are defined with respect to the  
 63 Lebesgue measure. Let  $p(\mathcal{D} \mid \theta)$  denote the likelihood of the model and  
 64  $p(\theta \mid \mathcal{D}) = p(\mathcal{D} \mid \theta)\pi(\theta)/p(\mathcal{D})$  the full sample posterior of the parameter  
 65 vector  $\theta$ . The vector  $\hat{\theta}_{ML} \in \Theta$  is the maximum likelihood estimate (MLE)  
 66 of  $\theta$ :  $\hat{\theta}_{ML} = \operatorname{argmax}_{\theta \in \Theta} p(\mathcal{D} \mid \theta)$ . The posterior of parameters conditional  
 67 on the MLE is  $p(\theta \mid \hat{\theta}_{ML}) = p(\hat{\theta}_{ML} \mid \theta)\pi(\theta)/p(\hat{\theta}_{ML})$ , where  $p(\hat{\theta}_{ML} \mid \theta)$  is  
 68 the p.d.f. of the MLE given  $\theta$ . Besides, we are interested in models whose  
 69 likelihoods are not tractable because of the dependence structure in the  
 70 data, but for which we can build tractable pseudo-likelihoods, say  $\tilde{p}(\mathcal{D} \mid \theta)$ .  
 71 A pseudo-likelihood is generally built by ignoring some of the dependen-  
 72 cies in the data (Gaetan and Guyon, 2008; Gourieroux et al., 1983). The  
 73 vector  $\hat{\theta}_{MPL} \in \Theta$  is a maximum pseudo-likelihood estimate (MPLE) of  $\theta$ :  
 74  $\hat{\theta}_{MPL} = \operatorname{argmax}_{\theta \in \Theta} \tilde{p}(\mathcal{D} \mid \theta)$ . The posterior of parameters conditional on the  
 75 MPLE is  $p(\theta \mid \hat{\theta}_{MPL}) = p(\hat{\theta}_{MPL} \mid \theta)\pi(\theta)/p(\hat{\theta}_{MPL})$ , where  $p(\hat{\theta}_{MPL} \mid \theta)$  is the  
 76 p.d.f. of the MPLE given  $\theta$ .

### 77 3. Posterior conditional on the MLE

78 The full sample posterior  $p(\theta \mid \mathcal{D})$  and the posterior conditional on the  
 79 MLE  $p(\theta \mid \hat{\theta}_{ML})$  exactly coincide in specific cases (e.g. when the MLE are  
 80 sufficient statistics), but do not coincide in general. Our aim, in this section,  
 81 is to provide an asymptotically equivalent distribution for  $p(\theta \mid \hat{\theta}_{ML})$ .

Bernstein – von Mises (BvM) theorems provide, for various statistical  
 models, the asymptotic behavior of posteriors distributions. For example,  
 following Walker (1969) and Lindley (1965, p. 130), we consider a set  $\mathcal{D} =$   
 $(D_1, \dots, D_n)$  of  $n$  i.i.d. variables drawn from a parametric distribution with

density  $f(\cdot | \theta)$  with respect to a  $\sigma$ -finite measure on the real line, where  $\theta$  is in  $\Theta \subset \mathbb{R}^q$ . Under this setting and additional regularity conditions, the BvM theorem establishes the asymptotic normality of the full sample posterior (Walker, 1969, Theorem 2 and conclusion): the full sample posterior density of  $\theta$  is, for large  $n$ , equivalent to the normal density with mean vector equal to the MLE  $\hat{\theta}_{ML}$  and covariance matrix equal to  $\Omega_n(\hat{\theta}_{ML})^{-1}$ :

$$p(\theta | \mathcal{D}) \underset{n \rightarrow \infty}{\sim} \phi_{\hat{\theta}_{ML}, \Omega_n(\hat{\theta}_{ML})^{-1}}(\theta),$$

82 where  $\phi_{\mu, \Sigma}$  denotes the density of the normal distribution with mean vector  
 83  $\mu$  and covariance matrix  $\Sigma$ , and  $\Omega_n(\alpha)$  is the  $q \times q$  matrix with element  $(i, j)$   
 84 equal to  $(-\partial^2 \log p(\mathcal{D} | \theta) / \partial \theta_i \partial \theta_j)_{\theta=\alpha}$ .

85 To provide an asymptotically equivalent distribution for  $p(\theta | \hat{\theta}_{ML})$  as  
 86 in BvM theorems, we assume in Lemma 1 (see below) that the MLE is  
 87 asymptotically normal and consistent. For example, consider the same sta-  
 88 tistical model than above and assume that assumptions made in Lehmann  
 89 and Casella (1998, Theorem 5.1 of the MLE asymptotic normality, p. 463)  
 90 are satisfied. In particular, assume that data were generated with parameter  
 91 vector  $\theta$ . Then, the density of  $\hat{\theta}_{ML}$  is, for large  $n$  and given  $\theta$ , equivalent to  
 92 the normal density with mean vector equal to the true parameter vector  $\theta$   
 93 and covariance matrix equal to  $n^{-1}I(\theta)^{-1}$ :

$$p(\hat{\theta}_{ML} | \theta) \underset{n \rightarrow \infty}{\sim} \phi_{\theta, n^{-1}I(\theta)^{-1}}(\hat{\theta}_{ML}). \quad (1)$$

94 where  $I(\theta)$  denotes the  $q \times q$  Fisher information matrix.

**Lemma 1** (Asymptotic normality of the posterior conditional on the MLE).  
*Consider the modeling setting described in Section 2 and suppose that the MLE satisfies Equation (1) with non-singular matrix  $I(\theta)$ . Assume in addition that the prior  $\pi$  is a positive and Lipschitz function over  $\Theta$ , that*

$\theta \mapsto |I(\theta)|$  (determinant of  $I(\theta)$ ) and  $\theta \mapsto x'I(\theta)x$  (for all  $x \in \mathbb{R}^q$ ,  $x'$  being the transpose of  $x$ ) are Lipschitz functions over  $\Theta$ , and that the constant which arises in the Lipschitz condition for  $\theta \mapsto x'I(\theta)x$  and which is a function of  $x$ , is also a Lipschitz function over  $\mathbb{R}^q$ . Then, when  $n \rightarrow \infty$ , the posterior density  $p(\theta \mid \hat{\theta}_{ML})$  conditional on the MLE is asymptotically equivalent to the density of the normal distribution with mean vector  $\hat{\theta}_{ML}$  and covariance matrix  $n^{-1}I(\hat{\theta}_{ML})^{-1}$  over a subset  $B_n$  of  $\Theta$  whose measure with respect to this normal density is asymptotically one in probability:

$$\begin{aligned}
 p(\theta \mid \hat{\theta}_{ML}) &\underset{n \rightarrow \infty}{\sim} \phi_{\hat{\theta}_{ML}, n^{-1}I(\hat{\theta}_{ML})^{-1}}(\theta), \quad \forall \theta \in B_n \\
 \lim_{n \rightarrow \infty} \int_{B_n} \phi_{\hat{\theta}_{ML}, n^{-1}I(\hat{\theta}_{ML})^{-1}}(\theta) d\theta &= 1.
 \end{aligned}$$

95 Thus, over the subset  $B_n$  which asymptotically contains all the mass  
 96 of the normal density  $\phi_{\hat{\theta}_{ML}, n^{-1}I(\hat{\theta}_{ML})^{-1}}(\cdot)$ , the posterior conditional on the  
 97 MLE is asymptotically equivalent to this normal distribution.

98 From a frequentist point of view, the BvM theorem (which concerns the  
 99 full sample posterior  $p(\theta \mid \mathcal{D})$ ) is a justification of the Bayesian approach  
 100 for parameter estimation since the Bayesian confidence sets asymptotically  
 101 coincide with the frequentist limit confidence sets (Freedman, 1999). Lemma  
 102 1 shows a similar result for the posterior conditional on the MLE  $p(\theta \mid$   
 103  $\hat{\theta}_{ML})$ . Thus, Lemma 1 can also be viewed as a justification of the use  
 104 of the posterior conditional on asymptotically normal MLE for parameter  
 105 estimation. Note that results similar to the one provided by Lemma 1 have  
 106 already been obtained for the estimation of an univariate location parameter;  
 107 see Doksum and Lo (1990) and references therein.

108 Regarding assumptions in Lemma 1, the asymptotic normality of the  
 109 MLE (Equation (1)) requires classical but strong assumptions (even for the  
 110 simple i.i.d. case). However, the asymptotic normality of the MLE has been

111 obtained for numerous modeling and sampling settings, even in non-i.i.d.  
 112 cases. Lemma 1 is also based on a series of Lipschitz assumptions concerning  
 113 the prior  $\pi$  and the Fisher information matrix  $I(\theta)$ . These assumptions are  
 114 satisfied for classical distributions (e.g. when  $\pi$  is uniform on a bounded  
 115 domain and when  $(D_1, \dots, D_n)$  are independent normal variables with mean  
 116  $\mu$  and variance 1).

*Proof.* Under the assumptions of the theorem of the MLE asymptotic normality (i.e. under Equation (1)),  $p(\hat{\theta}_{ML} | \theta) = \eta_n(\theta) + \varepsilon_n(\theta)$ , where  $\eta_n(\theta) = \phi_{\theta, n^{-1}I(\theta)^{-1}}(\hat{\theta}_{ML})$  and  $\varepsilon_n(\theta) \underset{n \rightarrow \infty}{=} o(\eta_n(\theta))$ . Therefore,

$$p(\theta | \hat{\theta}_{ML}) \underset{n \rightarrow \infty}{=} \frac{\{\eta_n(\theta) + \varepsilon_n(\theta)\}\pi(\theta)}{\int_{\Theta} \eta_n(\alpha)\pi(\alpha)d\alpha + \int_{\Theta} \varepsilon_n(\alpha)\pi(\alpha)d\alpha} = \frac{\eta_n(\theta)\pi(\theta)(1 + o(1))}{\int_{\Theta} \eta_n\pi + \int_{\Theta} \varepsilon_n\pi}.$$

117 The densities  $\pi$  and  $\eta_n$  being positive over  $\Theta$  and the density  $\eta_n$  converg-  
 118 ing to the Dirac distribution at the true parameter vector  $\theta_0$  when  $n \rightarrow \infty$ ,  
 119  $\int_{\Theta} \eta_n\pi$  is positive and its limit, namely  $\pi(\theta_0)$ , is also positive. Besides,  
 120  $\theta \mapsto p(\hat{\theta}_{ML} | \theta)$  being asymptotically equivalent to  $\eta_n$ , it also converges  
 121 to the Dirac distribution at the true parameter vector  $\theta_0$  when  $n \rightarrow \infty$ ,  
 122 and  $\int_{\Theta} p(\hat{\theta}_{ML} | \theta)\pi(\theta)d\theta \rightarrow \pi(\theta_0) > 0$ . Therefore,  $\int_{\Theta} \varepsilon_n\pi = \int_{\Theta} p(\hat{\theta}_{ML} |$   
 123  $\theta)\pi(\theta)d\theta - \int_{\Theta} \eta_n\pi$  converges to 0,  $\int_{\Theta} \varepsilon_n\pi = o(\int_{\Theta} \eta_n\pi)$  and

$$p(\theta | \hat{\theta}_{ML}) \underset{n \rightarrow \infty}{=} \frac{\pi(\theta)}{\pi(\theta_0)}\eta_n(\theta)(1 + o(1)). \quad (2)$$

124 Let  $B_n$  be the ball of center  $\theta_0$  and radius  $r_n$  such that  $r_n \rightarrow 0$  and  
 125  $r_n\sqrt{n} \rightarrow \infty$  ( $r_n$  converges to zero at a lower rate than  $1/\sqrt{n}$ ). Since  $\pi$  is  
 126 Lipschitzian (i.e.  $\exists A_1 < \infty, \forall \theta_1, \theta_2 \in \Theta, |\pi(\theta_1) - \pi(\theta_2)| \leq A_1\|\theta_1 - \theta_2\|$ ),

$$\pi(\theta) \underset{n \rightarrow \infty}{=} \pi(\theta_0)(1 + o(1)), \quad \forall \theta \in B_n. \quad (3)$$

Let us now derive an equivalent function for  $\eta_n$  which can be written:

$$\eta_n(\theta) = \phi_{\theta, n^{-1}I(\theta)^{-1}}(\hat{\theta}_{ML}) = \frac{\sqrt{n}|I(\theta)|^{1/2}}{(2\pi)^{d/2}} \exp\left(-\frac{n}{2}(\hat{\theta}_{ML} - \theta)'I(\theta)(\hat{\theta}_{ML} - \theta)\right).$$



Using the Lipschitz condition on  $\theta \mapsto I(\theta)$  (i.e.  $\exists A_2 < \infty, \forall \theta_1, \theta_2 \in \Theta, |I(\theta_1) - I(\theta_2)| \leq A_2 \|\theta_1 - \theta_2\|$ ), one can state that for all  $\theta \in B_n$ ,  $|I(\theta)|^{1/2} \underset{n \rightarrow \infty}{=} |I(\hat{\theta}_{ML})|^{1/2}(1 + o(1))$ . Besides, the two Lipschitz conditions concerning  $\theta \mapsto x'I(\theta)x$  can be written as follows:

$$\begin{aligned} \forall x \in \mathbb{R}^q, \exists A_3(x) < \infty, \forall \theta_1, \theta_2 \in \Theta, |x'I(\theta_1)x - x'I(\theta_2)x| &\leq A_3(x) \|\theta_1 - \theta_2\| \\ \exists A_4 < \infty, \forall x_1, x_2 \in \mathbb{R}^q, |A_3(x_1) - A_3(x_2)| &\leq A_4 \|x_1 - x_2\|. \end{aligned}$$

127 These conditions imply that over  $B_n$ ,  $|(\hat{\theta}_{ML} - \theta)'I(\theta)(\hat{\theta}_{ML} - \theta) - (\hat{\theta}_{ML} -$   
 128  $\theta)'I(\hat{\theta}_{ML})(\hat{\theta}_{ML} - \theta)|$  is bounded from above by  $2r_n(A_3(0) + 2r_n A_4)$ . There-  
 129 fore,  $(\hat{\theta}_{ML} - \theta)'I(\theta)(\hat{\theta}_{ML} - \theta) \underset{n \rightarrow \infty}{=} (\hat{\theta}_{ML} - \theta)'I(\hat{\theta}_{ML})(\hat{\theta}_{ML} - \theta)(1 + o(1))$  and

$$\eta_n(\theta) \underset{n \rightarrow \infty}{=} \phi_{\hat{\theta}_{ML}, n^{-1}I(\hat{\theta}_{ML})^{-1}}(\theta)(1 + o(1)), \quad \forall \theta \in B_n. \quad (4)$$

Using Equations (2-4), we obtain the first equation of Lemma 1:

$$p(\theta | \hat{\theta}_{ML}) \underset{n \rightarrow \infty}{=} \phi_{\hat{\theta}_{ML}, n^{-1}I(\hat{\theta}_{ML})^{-1}}(\theta)(1 + o(1)), \quad \forall \theta \in B_n.$$

130 The 2nd equation of Lemma 1 is shown below. Let  $p \in (0, 1)$ , and con-  
 131 sider  $R_{n,p}$  the region consisting of the vectors  $\theta \in \Theta$  satisfying  $n(\hat{\theta}_{ML} -$   
 132  $\theta)'I(\hat{\theta}_{ML})(\hat{\theta}_{ML} - \theta) \leq \chi_d^2(p)$ , where  $\chi_d^2(p)$  is the quantile of order  $p$  of  
 133 the chi-square distribution of order  $d$  (i.e. the dimension of  $\Theta$ ). Using  
 134 the link between the normal and chi-square distributions,  $R_{n,p}$  satisfies:  
 135  $\int_{R_{n,p}} \phi_{\hat{\theta}_{ML}, n^{-1}I(\hat{\theta}_{ML})^{-1}}(\theta) d\theta = p$ . Moreover, from a property of the Rayleigh  
 136 quotient,  $(\hat{\theta}_{ML} - \theta)'I(\hat{\theta}_{ML})(\hat{\theta}_{ML} - \theta) \geq \lambda_{\min}\{I(\hat{\theta}_{ML})\} \|\hat{\theta}_{ML} - \theta\|^2$ , where  
 137  $\lambda_{\min}\{I(\hat{\theta}_{ML})\}$  is the minimum eigenvalue of  $I(\hat{\theta}_{ML})$ . Therefore,

$$\forall \theta \in R_{n,p}, \quad \|\hat{\theta}_{ML} - \theta\| \leq \lambda_{\min}\{I(\hat{\theta}_{ML})\}^{-1} \sqrt{\chi_d^2(p)/n}. \quad (5)$$

From the MLE asymptotic normality,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( (\hat{\theta}_{ML} - \theta_0)'I(\theta_0)(\hat{\theta}_{ML} - \theta_0) \leq \sqrt{\chi_d^2(p)/n} \right) = p.$$

138 Moreover,  $(\hat{\theta}_{ML} - \theta_0)'I(\theta_0)(\hat{\theta}_{ML} - \theta_0) \geq \lambda_{\min}\{I(\theta_0)\}\|\hat{\theta}_{ML} - \theta_0\|$ , where  
 139  $\lambda_{\min}\{I(\theta_0)\}$  is the minimum eigenvalue of  $I(\theta_0)$ . Therefore,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \|\hat{\theta}_{ML} - \theta_0\| \leq \lambda_{\min}\{I(\theta_0)\}^{-1} \sqrt{\chi_d^2(p)/n} \right) \geq p. \quad (6)$$

Since  $\|\theta - \theta_0\| \leq \|\hat{\theta}_{ML} - \theta\| + \|\hat{\theta}_{ML} - \theta_0\|$ , one obtains using Eq. (5) and (6),

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \forall \theta \in R_{n,p}, \|\theta - \theta_0\| \leq \left[ \lambda_{\min}\{I(\hat{\theta}_{ML})\}^{-1} + \lambda_{\min}\{I(\theta_0)\}^{-1} \right] \sqrt{\chi_d^2(p)/n} \right) \geq p.$$

Since  $r_n$  goes to zero more slowly than  $1/\sqrt{n}$ , the previous inequality yields:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P} (\forall \theta \in R_{n,p}, \|\theta - \theta_0\| \leq r_n) &\geq p \\ \lim_{n \rightarrow \infty} \mathbb{P} (R_{n,p} \subset B_n) &\geq p \\ \lim_{n \rightarrow \infty} \mathbb{P} \left( \int_{B_n} \phi_{\hat{\theta}_{ML}, n^{-1}I(\hat{\theta}_{ML})^{-1}}(\theta) d\theta \geq \int_{R_{n,p}} \phi_{\hat{\theta}_{ML}, n^{-1}I(\hat{\theta}_{ML})^{-1}}(\theta) d\theta \right) &\geq p \\ \lim_{n \rightarrow \infty} \mathbb{P} \left( \int_{B_n} \phi_{\hat{\theta}_{ML}, n^{-1}I(\hat{\theta}_{ML})^{-1}}(\theta) d\theta \geq p \right) &\geq p. \end{aligned}$$

The last inequality obtained for any  $p \in (0, 1)$  implies that  $\int_{B_n} \phi_{\hat{\theta}_{ML}, n^{-1}I(\hat{\theta}_{ML})^{-1}}(\theta) d\theta$  converges to one in probability when  $n \rightarrow \infty$ , i.e.

$$\lim_{n \rightarrow \infty} \int_{B_n} \phi_{\hat{\theta}_{ML}, n^{-1}I(\hat{\theta}_{ML})^{-1}}(\theta) d\theta \stackrel{\mathbb{P}}{=} 1.$$

140

□

#### 141 4. Posterior conditional on an MPLE

142 Here, we propose a lemma analogous to Lemma 1 but concerning an  
 143 MPLE (maximum pseudo-likelihood estimate) instead of the MLE.

**Lemma 2** (Asymptotic normality of the posterior conditional on an MPLE).

*Consider the modeling setting provided in Section 2. Assume that, given the vector  $\theta$  under which the data  $\mathcal{D}$  were generated, the p.d.f. of the MPLE*

$\hat{\theta}_{MPL}$  is equivalent to the normal density with mean vector  $\theta$  and covariance matrix  $g(n)^{-1}J(\theta)^{-1}$ :

$$p(\hat{\theta}_{MPL} | \theta) \underset{n \rightarrow \infty}{\sim} \phi_{\theta, g(n)^{-1}J(\theta)^{-1}}(\hat{\theta}_{MPL}),$$

where  $g$  is a positive increasing function such that  $g(n) \rightarrow \infty$  and  $J(\theta)$  is a positive-definite matrix. Assume in addition that the prior  $\pi$  is a positive and Lipschitz function over  $\Theta$ , that  $\theta \mapsto |J(\theta)|$  and  $\theta \mapsto x'J(\theta)x$  (for all  $x \in \mathbb{R}^q$ ,  $x'$  being the transpose of  $x$ ) are Lipschitz functions over  $\Theta$ , and that the constant which arises in the Lipschitz condition for  $\theta \mapsto x'J(\theta)x$  and which is a function of  $x$ , is also a Lipschitz function over  $\mathbb{R}^q$ . Then, when  $n \rightarrow \infty$ , the posterior density  $p(\theta | \hat{\theta}_{MPL})$  conditional on the MPLE is asymptotically equivalent to the density of the normal distribution with mean vector  $\hat{\theta}_{MPL}$  and covariance matrix  $g(n)^{-1}J(\hat{\theta}_{MPL})^{-1}$  over a subset  $B_n$  of  $\Theta$  whose measure with respect to this normal density is asymptotically one:

$$p(\theta | \hat{\theta}_{MPL}) \underset{n \rightarrow \infty}{\sim} \phi_{\hat{\theta}_{MPL}, g(n)^{-1}J(\hat{\theta}_{MPL})^{-1}}(\theta), \quad \forall \theta \in B_n$$

$$\lim_{n \rightarrow \infty} \int_{B_n} \phi_{\hat{\theta}_{MPL}, g(n)^{-1}J(\hat{\theta}_{MPL})^{-1}}(\theta) d\theta = 1.$$

144 Lemma 2 justifies the use of the posterior conditional on the MPLE for  
 145 parameter estimation because the Bayesian confidence sets that are pro-  
 146 vided by this posterior asymptotically coincide with the frequentist limit  
 147 confidence sets obtained by maximizing the pseudo-likelihood.

148 The asymptotic normality of the MPLE required in Lemma 2 has been  
 149 obtained for various models, especially random Markov fields and spatial  
 150 point processes; see Gaetan and Guyon (2008, chap. 5), Gouieroux et al.  
 151 (1983), Møller and Waagepetersen (2004, chap. 9) and references therein.  
 152 It has to be noted that information is lost when MPLE are used rather than  
 153 MLE and, consequently, that estimation accuracy is decreased (e.g. this

154 has been shown for simple Markovian models using asymptotic estimation  
 155 variances (Gaetan and Guyon, 2008, chap. 5)).

156 *Proof.* Follow the proof of Lemma 1 by assuming that the radius  $r_n$  of the  
 157 ball  $B_n$  satisfies  $r_n\sqrt{g(n)} \rightarrow \infty$  instead of  $r_n\sqrt{n} \rightarrow \infty$ .

158 □

## 159 5. Approximate posterior conditional on an MPLE

160 Here, we derive implications of Lemma 2 in the framework of approxi-  
 161 mate Bayesian computation (ABC) when (some of) the summary statistics  
 162 are MPLE (see Theorem 1 and Corollary 1). We consider the (simple) ABC-  
 163 rejection algorithm based on independent simulations, on a set of summary  
 164 statistics and on a tolerance threshold (Pritchard et al., 1999):

165 *ABC-rejection.* Perform the next 3 steps for  $i$  in  $\{1, \dots, I\}$ , independently:

- 166 • Generate  $\theta_i$  from  $\pi$  and simulate  $\mathcal{D}_i$  from  $\mathcal{M}_{\theta_i}$ ;
- 167 • Compute summary statistics  $S_i = s(\mathcal{D}_i)$ , where  $s$  is a function from  $\mathbb{D}$   
 168 to the space  $\mathbb{S}$  of statistics;
- 169 • Accept  $\theta_i$  if  $d(S_i, S) \leq \epsilon$ , where  $d$  is a distance over  $\mathbb{S}$  and  $\epsilon$  is a  
 170 tolerance threshold for the distance between the observed statistics  
 171  $S = s(\mathcal{D})$  and the simulated ones  $S_i$ .

The set of accepted parameters, say  $\Theta_{\epsilon, I} = \{\theta_i : d(S_i, S) \leq \epsilon, i = 1, \dots, I\}$ ,  
 forms a sample from the following posterior:

$$p_\epsilon(\theta \mid S) = \frac{\left( \int_{\mathcal{B}_d(S, \epsilon)} h(s \mid \theta) ds \right) \pi(\theta)}{\int_{\Theta} \left( \int_{\mathcal{B}_d(S, \epsilon)} h(s \mid \alpha) ds \right) \pi(\alpha) d\alpha},$$

172 where the ball  $\mathcal{B}_d(S, \epsilon)$  in the  $d$ -dimensional space  $\mathbb{S}$  is the set of points  
 173 from which the distance to  $S$  is less than  $\epsilon$ , and  $h(S \mid \theta)$  is the conditional

174 probability distribution function of  $S$  given  $\theta$ . When  $\epsilon$  tends to zero and  $d$  is  
175 appropriate,  $p_\epsilon(\theta | S)$  is a good approximation of  $p(\theta | S)$  under regularity  
176 assumptions (see Blum (2010) and Soubeyrand et al. (2013, Appendix A)):  
177  $p_\epsilon(\theta | S)$  and  $p(\theta | S)$  are asymptotically equivalent. However, if  $S$  is not  
178 sufficient, then  $p(\theta | S) \neq p(\theta | \mathcal{D})$  and information is lost by using  $S$  instead  
179 of  $\mathcal{D}$ .

**Theorem 1** (Asymptotic normality of the ABC-posterior conditional on an MPLLE). *Consider the ABC-rejection algorithm that samples in the posterior  $p_\epsilon(\theta | \hat{\theta}_{MPL})$  of  $\theta$  conditional on the vector of summary statistics  $S = \hat{\theta}_{MPL}$ . Assume that when  $\epsilon \rightarrow 0$ ,  $p_\epsilon(\theta | \hat{\theta}_{MPL})$  converges pointwise to  $p(\theta | \hat{\theta}_{MPL})$ . Then, under assumptions of Lemma 2, when  $n \rightarrow \infty$  and  $\epsilon \rightarrow 0$ , the posterior  $p_\epsilon(\theta | \hat{\theta}_{MPL})$  is asymptotically equivalent to the density of the normal distribution with mean vector  $\hat{\theta}_{MPL}$  and covariance matrix  $g(n)^{-1}J(\hat{\theta}_{MPL})^{-1}$  over a subset  $B_n$  of  $\Theta$  whose measure with respect to this normal density goes to one in probability and that does not depend on  $\epsilon$ :*

$$p_\epsilon(\theta | \hat{\theta}_{MPL}) \underset{n \rightarrow \infty, \epsilon \rightarrow 0}{\sim} \phi_{\hat{\theta}_{MPL}, g(n)^{-1}J(\hat{\theta}_{MPL})^{-1}}(\theta), \quad \forall \theta \in B_n$$

$$\lim_{n \rightarrow \infty} \int_{B_n} \phi_{\hat{\theta}_{MPL}, g(n)^{-1}J(\hat{\theta}_{MPL})^{-1}}(\theta) d\theta \stackrel{\mathbb{P}}{=} 1.$$

180 As explained in the introduction, this result leads us to conjecture that,  
181 for some stochastic implicit models, (i) the ABC-posterior distribution con-  
182 ditional on an MPLLE is asymptotically normal and centered around the  
183 MPLLE, and (ii) resulting point estimates and confidence sets converge to  
184 their frequentist analogues. Corollary 1 in Appendix A provides an anal-  
185 ogous result when the MPLLE is used in conjunction with supplementary  
186 statistics.

*Proof.* This result is directly obtained from Lemma 2 by simply noting that

the subset  $B_n$  does not depend on  $\epsilon$ . Indeed, following the proof of Lemma 1, pointwise in  $\theta$  we have:

$$\begin{aligned} p_\epsilon(\theta | \hat{\theta}_{MPL}) &\underset{\epsilon \rightarrow 0}{=} p(\theta | \hat{\theta}_{MPL})(1 + o_\epsilon(1)) \\ &\underset{n \rightarrow \infty, \epsilon \rightarrow 0}{=} \frac{\pi(\theta)}{\pi(\theta_0)} \eta_n(\theta)(1 + o_n(1))(1 + o_\epsilon(1)). \end{aligned}$$

187 Then,  $B_n$  is used to provide an equivalent of  $\pi(\theta)\pi(\theta_0)^{-1}\eta_n(\theta)$ . This term  
188 does not depend on  $\epsilon$  and, consequently,  $B_n$  has not to be dependent on  $\epsilon$ .

189

□

## 190 6. Example

The simplified example presented here illustrates the application of ABC conditional on an MPLE and a supplementary statistic. The model  $\mathcal{M}_\theta$  under consideration is the following bivariate normal distribution:

$$\mathcal{N}\left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right),$$

parameterized by the mean  $\mu$  and the correlation  $\rho$ ; we set  $\theta = (\mu, \rho)$ . Observed data  $\mathcal{D} = \{(D_k^{(1)}, D_k^{(2)}) : k = 1, \dots, n\}$  are  $n = 100$  vectors drawn under this normal distribution with  $\mu = 0$  and  $\rho = 0.5$ . We use a uniform prior distribution  $\pi$  over the rectangular domain  $(-3, 3) \times (-1, 1)$ . The maximum likelihood estimates of  $\mu$  and  $\rho$  are the empirical mean of  $(D_k^{(1)} + D_k^{(2)})/2$  and the empirical correlation of  $(D_k^{(1)}, D_k^{(2)})$ ,  $k = 1, \dots, n$ . Here, we applied ABC with the two following statistics:

$$S = s(\mathcal{D}) = \begin{pmatrix} \hat{\mu}_{MPL} \\ S_0 \end{pmatrix} = \frac{1}{n} \sum_{i=1}^n \begin{pmatrix} D_k^{(1)} \\ \mathbf{1}\{\text{sign}(D_k^{(1)}) = \text{sign}(D_k^{(2)})\} \end{pmatrix},$$

191 where  $\hat{\mu}_{MPL}$  is an MPLE of  $\mu$  that uses only partial information contained  
192 in the sample (i.e. only the first component of sampled vectors), and  $S_0$  is a

193 supplementary statistic that gives the mean number of vectors in the sample  
 194 whose components  $D_k^{(1)}$  and  $D_k^{(2)}$  have the same sign ( $\mathbf{1}\{\cdot\}$  is the indicator  
 195 function).

196 To assess the convergence of ABC when  $\epsilon$  tends to zero, we applied ABC  
 197 with varying  $\epsilon$ , with  $I = 10^5$  simulations, and with the distance  $d(S_i, S) =$   
 198  $(\hat{\mu}_{MPL,i} - \hat{\mu}_{MPL})^2 + (S_{0,i} - S_0)^2$ , where  $S_i = (\hat{\mu}_{MPL,i}, S_{0,i})$  is the vector of  
 199 statistics computed for the simulation  $i$ . As usual in ABC–rejection, instead  
 200 of fixing  $\epsilon$ , we fixed the sample size  $\tau$  of the posterior sample (i.e. the number  
 201 of accepted parameter vectors); note that  $\epsilon$  decreases when  $\tau$  decreases.  
 202 The sample size  $\tau$  was fixed at values ranging from 10 to 5000. For each  
 203 value of  $\tau$ , we computed the local posterior probability (LPP) around the  
 204 true parameter vector  $\theta = (0, 0.5)$  as the proportion of accepted parameter  
 205 vectors in the small rectangle  $[-0.015, 0.015] \times [0.495, 0.505]$  whose center  
 206 is  $\theta = (0, 0.5)$  and whose sides are 200 times smaller than the sides of the  
 207 parameter space  $(-3, 3) \times (-1, 1)$ . We expect that this LPP increases with  
 208 the efficiency of the inference procedure. The LPP was computed for 50000  
 209 datasets and Figure 1 shows its mean and standard deviation when  $\tau$  varies.  
 210 The mean LPP around the true parameters increases when the sample size  
 211  $\tau$  (and  $\epsilon$ ) tends to zero; meanwhile, the dispersion of the LPP increases.  
 212 This is the signature of the classical bias–variance trade-off.

213 To automatically select the sample size  $\tau$ , we applied the procedure  
 214 proposed by Soubeyrand et al. (2013) where the distance between summary  
 215 statistics is also optimized. In this procedure, the distance is weighted:  
 216  $d(S_i, S; w_1, w_2) = w_1(\hat{\mu}_{MPL,i} - \hat{\mu}_{MPL})^2 + w_2(S_{0,i} - S_0)^2$ , and the triplet  
 217  $(\tau, w_1, w_2)$  is optimized under constraints using an integrated mean square  
 218 error criterion. For one of the 50000 datasets simulated above, Figure 2  
 219 shows ABC–posterior samples obtained when  $d$  is not weighted and  $\tau$  is

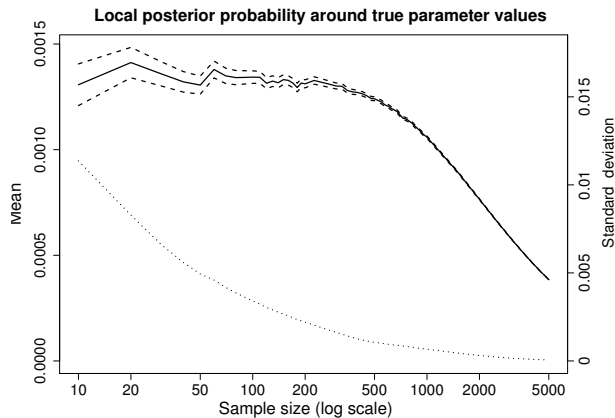


Figure 1: Mean (solid line), pointwise 95%-confidence envelopes (dashed lines) and standard deviation (dotted line) of the local posterior probability around the true parameter vector  $\theta = (0, 0.5)$  as a function of sample size  $\tau$ .

220 fixed at 5000, 1000 and 200, and when  $d$  is weighted and  $(\tau, w_1, w_2)$  is  
 221 optimized. The grey contour line shows the smallest 95%-posterior area  
 222 obtained with the classical Bayesian computation. The first three panels  
 223 illustrate the bias–variance trade-off when  $\tau$  tends to zero. The fourth panel  
 224 illustrates the difference between the classical Bayesian inference conditional  
 225 on all data and the ABC inference conditional on partial information and  
 226 with optimized  $\tau$  (here  $\tau = 585$ ). The relevancy of the optimized sample size  
 227  $\tau = 585$  can be seen by projecting this value on Figure 1: (i) the expected  
 228 LPP around the true parameters is comparable to expected LPP obtained  
 229 with lower  $\tau$ , and (ii) the standard deviation of the LPP is strongly decreased  
 230 compared with standard deviations obtained with lower  $\tau$ .



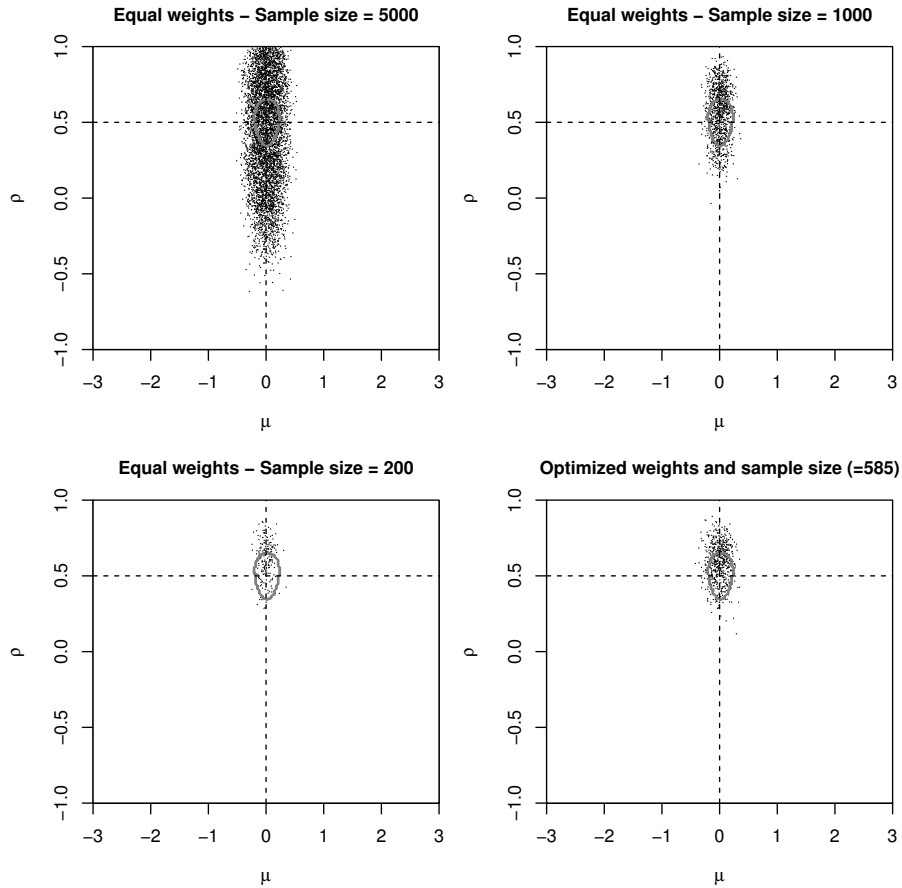


Figure 2: ABC-posterior samples (dots) obtained when  $d$  is not weighted and  $\tau$  is fixed at 5000 (top left), 1000 (top right) and 200 (bottom left), and when  $d$  is weighted and  $(\tau, w_1, w_2)$  is optimized (bottom right). Dashed lines are intersecting at the true value  $(0, 0.5)$  of the parameter vector  $\theta = (\mu, \rho)$ . The grey contour line gives the smallest 95%-posterior area obtained with the classical Bayesian computation.

## 231 7. Discussion

232 We provided a frequentist justification for the use of posterior distribu-  
233 tions conditional on MPLE, both in the classical Bayesian framework and in  
234 the ABC framework. The asymptotic results presented above were obtained  
235 for a large but limited class of models satisfying regularity assumptions. In  
236 real-life studies where ABC is applied, these assumptions cannot be checked  
237 and, consequently, our asymptotic results may not hold. Therefore, it is  
238 crucial (i) to combine MPLE and supplementary summary statistics to tend  
239 to a set of sufficient summary statistics (Joyce and Marjoram, 2008), and  
240 (ii) to apply a method for selecting, weighting or transforming the summary  
241 statistics to avoid to take into account non-relevant statistics. In Section  
242 6, we used the weighted distance between summary statistics proposed by  
243 Soubeyrand et al. (2013) where the weights are optimized with respect to  
244 an integrated mean square error. Other approaches could be applied. For  
245 example, Barnes et al. (2012), Joyce and Marjoram (2008) and Nunes and  
246 Balding (2010) propose a dimension reduction (that can be viewed as a bi-  
247 nary weighting), and Wegmann et al. (2009) proposes a PLS transformation  
248 followed by a binary weighting of the PLS axes (see also Blum et al., 2013;  
249 Fearnhead and Prangle, 2012; Jung and Marjoram, 2011).

250 The strong implication of our approach is that an analytic work has to be  
251 made: the dependence structure of the model has to be simplified to write a  
252 tractable pseudo-likelihood and, eventually, to find an analytic expression for  
253 the maximizer. This additional work is however expected to yield relevant  
254 summary statistics directly informing (a subset of) the parameters.

255 In this article, we did not precisely define what is a pseudo-likelihood.  
256 This deliberate choice is justified in the sense that the MPLE could be

257 replaced, in Theorem 1, by any other estimates with similar normal weak  
 258 convergence. Thus, our approach could be generalized by replacing the  
 259 pseudo-likelihood by a quasi-likelihood or a contrast. Another extension  
 260 of Theorem 1 could be derived by conditioning ABC on the score function  
 261 of the pseudo-likelihood instead of the MPLE as proposed by Gleim and  
 262 Pigorsch (2013), Ruli et al. (2013) and Mengersen et al. (2013).

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## 266 **Appendix A. Combining an MPLE and supplementary statistics**

**Corollary 1.** *Consider the posterior  $p_\epsilon(\theta \mid \hat{\theta}_{MPL}, S_0)$  of  $\theta$  conditional on the vector of summary statistics  $S = (\hat{\theta}_{MPL}, S_0)$  in which the ABC-rejection algorithm samples. Assume that when  $\epsilon \rightarrow 0$ ,  $p_\epsilon(\theta \mid \hat{\theta}_{MPL})$  converges pointwise to  $p(\theta \mid \hat{\theta}_{MPL})$ . Assume that there exists a vector  $T_0$  independent of  $\hat{\theta}_{MPL}$  such that  $(\hat{\theta}_{MPL}, T_0)$  brings the same information on  $\theta$  than  $S = (\hat{\theta}_{MPL}, S_0)$ . Then,*

$$p_\epsilon(\theta \mid \hat{\theta}_{MPL}, S_0) \underset{\epsilon \rightarrow 0}{=} \check{p}(\theta \mid T_0)(1 + o_\epsilon(1)),$$

267 where  $\check{p}(\theta \mid T_0)$  is the posterior density of  $\theta$  conditional on the subset of  
 268 statistics  $T_0$  given the modified prior  $p(\theta \mid \hat{\theta}_{MPL}) = p(\hat{\theta}_{MPL} \mid \theta)\pi(\theta)/p(\hat{\theta}_{MPL})$ .

269 *If in addition assumptions of Lemma 2 hold, the modified prior  $p(\theta \mid$   
 270  $\hat{\theta}_{MPL})$  is asymptotically equivalent to the density of the normal distribution  
 271 with mean vector  $\hat{\theta}_{MPL}$  and covariance matrix equal to  $g(n)^{-1}J(\hat{\theta}_{MPL})^{-1}$   
 272 over a subset  $B_n$  of  $\Theta$  whose measure with respect to this normal density  
 273 goes to one in probability and that does not depend on  $\epsilon$ .*

274 Under the first set of assumptions of Corollary 1, we can also write  
 275 that  $p_\epsilon(\theta \mid \hat{\theta}_{MPL}, S_0) \underset{\epsilon \rightarrow 0}{=} \check{p}(\theta \mid \hat{\theta}_{MPL})(1 + o_\epsilon(1))$ , where  $\check{p}(\theta \mid \hat{\theta}_{MPL})$  is  
 276 the posterior density of  $\theta$  conditional on the MPLLE with the modified prior  
 277  $p(\theta \mid T_0) = p(T_0 \mid \theta)\pi(\theta)/p(T_0)$ . Moreover, the existence of  $T_0$  is a strong  
 278 assumption that reduces the applicability of the corollary (a trivial but cer-  
 279 tainly rare example where  $T_0$  exists is obtained when one assumes that  
 280  $(\hat{\theta}_{MPL}, S_0)$  is a normal random vector; Indeed, in this case  $T_0$  can be defined  
 281 as  $T_0 = S_0 - E(S_0 \mid \hat{\theta}_{MPL})$ . Nevertheless, if  $(\hat{\theta}_{MPL}, S_0)$  is asymptotically  
 282 normal, we conjecture that a result similar to Corollary 1 can be obtained.

*Proof.* The first part of the corollary is straightforward:

$$\begin{aligned} p_\epsilon(\theta \mid \hat{\theta}_{MPL}, S_0) &\underset{\epsilon \rightarrow 0}{=} p(\theta \mid \hat{\theta}_{MPL}, S_0)(1 + o_\epsilon(1)) = p(\theta \mid \hat{\theta}_{MPL}, T_0)(1 + o_\epsilon(1)) \\ &= \frac{p(\hat{\theta}_{MPL}, T_0 \mid \theta)\pi(\theta)}{p(\hat{\theta}_{MPL}, T_0)}(1 + o_\epsilon(1)) = \frac{p(T_0 \mid \hat{\theta}_{MPL}, \theta)p(\hat{\theta}_{MPL} \mid \theta)\pi(\theta)}{p(T_0)p(\hat{\theta}_{MPL})}(1 + o_\epsilon(1)) \\ &= \frac{p(T_0 \mid \theta)p(\hat{\theta}_{MPL} \mid \theta)\pi(\theta)}{p(T_0)p(\hat{\theta}_{MPL})}(1 + o_\epsilon(1)) = \frac{p(T_0 \mid \theta)p(\theta \mid \hat{\theta}_{MPL})}{p(T_0)}(1 + o_\epsilon(1)). \end{aligned}$$

283 The term  $p(\theta \mid \hat{\theta}_{MPL})$  can be viewed as a prior that is based on information  
 284 contained in  $\hat{\theta}_{MPL}$  and that is independent from data  $T_0$  used for the final  
 285 inference. The second part of the corollary corresponds to Theorem 1.

286

□

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