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A semigroup approach to the four functions theorem on the discrete hypercube

Laurent Veysseire

May 13, 2017

Abstract

In this paper, we prove a semigroup version of the Ahlswede–Daykin four functions theorem on the discrete hypercube, and of its generalization to $2n$ functions.

Introduction

The Ahlswede–Daykin four function theorem is an inequality from which the more classical FKG inequality[3] can be easily derived. It tells that if four nonnegative functions $\alpha, \beta, \gamma, \delta$ on the discrete hypercube satisfy the Ahlswede–Daykin condition

$$\alpha(x)\beta(y) \leq \gamma(x \vee y)\delta(x \wedge y)$$

then we have $\int \alpha \int \beta \leq \int \gamma \int \delta$, where the integrals are taken with respect to the counting measure on the discrete hypercube.

In this paper, we prove the stability of the Ahlswede–Daykin condition under the heat flow. The stability of this condition under disjoint convolution has been proved in [4].

The Ahlswede–Daykin four function theorem has been generalized to $2n$ functions, independently by Aharoni and Keich in [1] and Rinott and Saks in [6]. We show that the generalized Ahlswede–Daykin condition appearing in this $2n$ functions theorem is also preserved by the heat flow.

The original four functions theorem and $2n$ functions theorem easily follow from the stability of the generalized Ahlswede–Daykin condition under the heat flow, by taking the limit of those inequalities when the time tends to $+\infty$.

1 The four functions theorem

We will denote by $\mathcal{H}_n := \{0, 1\}^{[0, n-1]}$ the discrete hypercube of dimension n . For any two elements x and y of \mathcal{H}_n , we define $x \vee y$ and $x \wedge y$ in the following way:

$$\forall i \in [0, n-1], (x \vee y)_i := \max(x_i, y_i) \quad \text{and} \quad (x \wedge y)_i := \min(x_i, y_i).$$

We consider the following set of 4-tuples of functions on \mathcal{H}_n :

$$A_n := \left\{ (\alpha, \beta, \gamma, \delta) \in \left(\mathbb{R}_+^{\mathcal{H}_n} \right)^4 \mid \forall x, y \in \mathcal{H}_n, \alpha(x)\beta(y) \leq \gamma(x \vee y)\delta(x \wedge y) \right\}.$$

The Ahlswede–Daykin four functions theorem [2] is the following statement.

Theorem 1 *Let $(\alpha, \beta, \gamma, \delta) \in A_n$. Then we have:*

$$\int \alpha \int \beta \leq \int \gamma \int \delta,$$

where $\int f = \sum_{x \in \mathcal{H}_n} f(x)$ for any real function f on \mathcal{H}_n .

Remark 2 *Theorem 1 also holds if we integrate with respect to a measure whose density ρ with respect to the counting measure has the following form:*

$$\rho(x) = \prod_{i \in [0, n-1]} (\mathbb{1}_{x_i=1} p_i + \mathbb{1}_{x_i=0} q_i),$$

where the p_i 's and q_i 's are nonnegative numbers. This kind of measure on \mathcal{H}_n is the product of n measures on the two point space.

Indeed, such ρ satisfy

$$\forall x, y \in \mathcal{H}_n, \rho(x)\rho(y) = \rho(x \vee y)\rho(x \wedge y).$$

Thus, $(\alpha, \beta, \gamma, \delta) \in A_n$ implies $(\alpha\rho, \beta\rho, \gamma\rho, \delta\rho) \in A_n$ too, so we can apply Theorem 1 to this latter one 4-tuple of functions.

Now we introduce a Markov semigroup on \mathcal{H}_n .

For any element x of \mathcal{H}_n , and any $i \in [0, n-1]$, we denote by \tilde{x}_i the element of \mathcal{H}_n defined by

$$(\tilde{x}_i)_j := \begin{cases} x_j & \text{if } j \neq i \\ 1 - x_i & \text{if } i = j. \end{cases}$$

For any $i \in [0, n-1]$, we denote by $L_{(i)}$ the following operator acting on real-valued functions on \mathcal{H}_n :

$$L_{(i)}f(x) := f(\tilde{x}_i) - f(x).$$

We define the Markov semigroup $(P^t)_{t \in \mathbb{R}_+}$ by $P^t = e^{tL}$, where the generator L of the semigroup takes the following form:

$$L = \sum_{i \in \llbracket 0, n-1 \rrbracket} \lambda_i L_{(i)},$$

with $\lambda_i > 0$ the jumping rate in the direction i .

Now, we can state our semigroup version of the four functions theorem.

Theorem 3 *Let $(\alpha, \beta, \gamma, \delta) \in A_n$, then we have*

$$\forall t \geq 0, (P^t \alpha, P^t \beta, P^t \gamma, P^t \delta) \in A_n.$$

Remark 4 *Since the uniform measure on \mathcal{H}_n is the unique reversible measure of (P^t) , taking the limit when t tends to $+\infty$ gives back Theorem 1.*

Corollary 5 *Let f be a log-supermodular function on \mathcal{H}_n and g be a nonincreasing log-modular function on \mathcal{H}_n , then $f * g$ is log-supermodular, where $*$ is the usual convolution operator on $\mathcal{H}_n \simeq (\mathbb{Z}/2\mathbb{Z})^n$.*

Proof: Log-supermodular functions on \mathcal{H}_n are nonnegative functions f that satisfy $f(x)f(y) \leq f(x \vee y)f(x \wedge y)$ for all $x, y \in \mathcal{H}_n$, which is equivalent to $(f, f, f, f) \in A_n$. Log-modular functions on \mathcal{H}_n are nonnegative functions g that satisfy $g(x)g(y) = g(x \vee y)g(x \wedge y)$ for all $x, y \in \mathcal{H}_n$. Log-modular functions can be written as a product of functions: $g(x) = \prod_{i=0}^{n-1} g_i(x_i)$ where the g_i 's are nonnegative functions of $\{0, 1\}$.

One can rewrite it $g(x) = C \prod_{i=0}^{n-1} p_i^{x_i} (1 - p_i)^{1-x_i}$ where C is a nonnegative constant and $p_i = \frac{g_i(1)}{g_i(0) + g_i(1)}$ (the p_i 's are well defined except in the trivial case where $g = 0$). If g is nonincreasing, then all the p_i 's are smaller than $\frac{1}{2}$. If $\forall i, p_i < \frac{1}{2}$, we set $\lambda_i = -\frac{1}{2} \ln(1 - 2p_i)$, and we have, for P^t the Markov semigroup generated by $L = \sum_{i \in \llbracket 0, n-1 \rrbracket} \lambda_i L_{(i)}$,

$$f * g = CP^1 f,$$

so according to Theorem 3, $(P^1 f, P^1 f, P^1 f, P^1 f) \in A_n$, thus $P^1 f$ is log-supermodular and so is $f * g$.

The case where some of the p_i 's are equal to $\frac{1}{2}$ can be deduced from above by a convergence argument. \square

The fact that P^t is a Markov semigroup implies that for any nonnegative function f , $P^t f$ is a nonnegative function too. So to prove Theorem 3, it only remains to check that

$$\forall x, y \in \mathcal{H}_n, P^t \alpha(x) P^t \beta(y) \leq P^t \gamma(x \vee y) P^t \delta(x \wedge y).$$

We first prove the theorem in the case $n = 1$.

In order to simplify the notations, and to spare parentheses and the ink needed for printing them, we will write α_x instead of $\alpha(x)$, and likewise for the other functions. We also define the following quantities, which are nonnegative if $(\alpha, \beta, \gamma, \delta) \in A_1$:

$$\begin{aligned}\Delta_{00} &:= \gamma_0\delta_0 - \alpha_0\beta_0 & \Delta_{01} &:= \gamma_1\delta_0 - \alpha_0\beta_1 \\ \Delta_{10} &:= \gamma_1\delta_0 - \alpha_1\beta_0 & \Delta_{11} &:= \gamma_1\delta_1 - \alpha_1\beta_1.\end{aligned}$$

The quantities $\Delta_{00}(t)$, $\Delta_{01}(t)$, $\Delta_{10}(t)$ and $\Delta_{11}(t)$ are defined on a similar way, by replacing the functions α , β , γ and δ with $P^t\alpha$, $P^t\beta$, $P^t\gamma$ and $P^t\delta$.

The following lemma is a key argument.

Lemma 6 *If $(\alpha, \beta, \gamma, \delta) \in A_1$, then we have*

$$\alpha_0\beta_1 + \alpha_1\beta_0 \leq \gamma_0\delta_1 + \gamma_1\delta_0.$$

Proof of Lemma 6: If $\gamma_1\delta_0 = 0$, then we have $\alpha_0\beta_1 = \alpha_1\beta_0 = 0$ because Δ_{01} and Δ_{10} are nonnegative (α and β are nonnegative too), so we have to prove $\gamma_0\delta_1 \geq 0$, which is true.

If $\gamma_1\delta_0 > 0$, then we can write $\gamma_0 \geq \frac{\alpha_0\beta_0}{\delta_0}$ because $\Delta_{00} \geq 0$, and $\delta_1 \geq \frac{\alpha_1\beta_1}{\gamma_1}$ because $\Delta_{11} \geq 0$. So we get:

$$\gamma_0\delta_1 + \gamma_1\delta_0 - \alpha_0\beta_1 - \alpha_1\beta_0 \geq \frac{\alpha_0\alpha_1\beta_0\beta_1}{\gamma_1\delta_0} + \gamma_1\delta_0 - \alpha_0\beta_1 - \alpha_1\beta_0 = \frac{\Delta_{01}\Delta_{10}}{\gamma_1\delta_0} \geq 0. \square$$

Now we prove Theorem 3 by two different ways.

First proof of Theorem 3: In the case $n = 1$, we have

$$P^t = e^{tL} = e^{\lambda_0 t L_{(0)}} = I + \frac{1}{2}(1 - e^{-2\lambda_0 t})L_{(0)}.$$

So for a fixed $t \geq 0$, we set $p = \frac{1 - e^{-2\lambda_0 t}}{2} \in [0, \frac{1}{2})$, and we have

$$P^t f(0) = (1 - p)f_0 + pf_1, \text{ and } P^t f(1) = pf_0 + (1 - p)f_1.$$

Then we have

$$\begin{aligned}\Delta_{00}(t) &= ((1 - p)\gamma_0 + p\gamma_1)((1 - p)\delta_0 + p\delta_1) - ((1 - p)\alpha_0 + p\alpha_1)((1 - p)\beta_0 + p\beta_1) \\ &= (1 - p)^2\Delta_{00} + p^2\Delta_{11} + p(1 - p)(\gamma_0\delta_1 + \gamma_1\delta_0 - \alpha_0\beta_1 - \alpha_1\beta_0) \geq 0,\end{aligned}$$

where we used Lemma 6 to get the nonnegativity of the last term. We have $\Delta_{11}(t) \geq 0$ with the same proof, by swapping p and $1 - p$. We also have

$$\begin{aligned}\Delta_{01}(t) &= (p\gamma_0 + (1 - p)\gamma_1)((1 - p)\delta_0 + p\delta_1) - ((1 - p)\alpha_0 + p\alpha_1)(p\beta_0 + (1 - p)\beta_1) \\ &= p(1 - p)(\Delta_{00} + \Delta_{11}) + p^2(\gamma_0\delta_1 + \gamma_1\delta_0 - \alpha_0\beta_1 - \alpha_1\beta_0) + (1 - 2p)\Delta_{01} \geq 0,\end{aligned}$$

where we used Lemma 6 to lower bound the second term. The proof of the nonnegativity of $\Delta_{10}(t)$ is similar. So we are done for the case $n = 1$.

Now we assume $n \geq 2$. We can remark that the $L_{(i)}$ commute with each other, so we have $P^t = \prod_{i \in [0, n-1]} e^{\lambda_i t L_{(i)}}$ where the order of the terms in the product does not matter, because they commute with each other. So to prove Theorem 3, it suffices to prove that for each $i \in [0, n-1]$, the operator $e^{\lambda_i t L_{(i)}}$ stabilizes A_n .

Let $i \in [0, n-1]$. For any $x \in \mathcal{H}_n$, we set \bar{x}_0 the element of \mathcal{H}_n whose coordinates are the same than the ones of x except the i^{th} which is replaced with 0, and we define $\bar{x}_1 = \bar{x}_{0i}$ likewise. Let x and y be two fixed points of \mathcal{H}_n . We define the four functions α', β', γ' and δ' on \mathcal{H}_1 on the following way:

$$\begin{aligned} \alpha'_u &:= \alpha(\bar{x}_u) & \beta'_u &:= \beta(\bar{y}_u) \\ \gamma'_u &:= \gamma(\bar{x} \vee \bar{y}_u) & \delta'_u &:= \delta(\bar{x} \wedge \bar{y}_u) \end{aligned}$$

The fact that $(\alpha', \beta', \gamma', \delta') \in A_1$ trivially follows from $(\alpha, \beta, \gamma, \delta) \in A_n$. We also have

$$\begin{aligned} e^{t\lambda_i L_{(i)}} \alpha(x) &= e^{t\lambda_i L_{(0)}} \alpha'(x_i) & e^{t\lambda_i L_{(i)}} \beta(y) &= e^{t\lambda_i L_{(0)}} \beta'(y_i) \\ e^{t\lambda_i L_{(i)}} \gamma(x \vee y) &= e^{t\lambda_i L_{(0)}} \gamma'(x_i \vee y_i) & e^{t\lambda_i L_{(i)}} \delta(x \wedge y) &= e^{t\lambda_i L_{(0)}} \delta'(x_i \wedge y_i) \end{aligned}$$

So, applying Theorem 3 in the one dimensional case to $(\alpha', \beta', \gamma', \delta')$ gives us

$$e^{t\lambda_i L_{(i)}} \alpha(x) e^{t\lambda_i L_{(i)}} \beta(y) \leq e^{t\lambda_i L_{(i)}} \gamma(x \vee y) e^{t\lambda_i L_{(i)}} \delta(x \wedge y).$$

This inequality being true for every $x, y \in \mathcal{H}_n$, we have

$$(e^{t\lambda_i L_{(i)}} \alpha, e^{t\lambda_i L_{(i)}} \beta, e^{t\lambda_i L_{(i)}} \gamma, e^{t\lambda_i L_{(i)}} \delta) \in A_n.$$

Thus $e^{t\lambda_i L_{(i)}}$ stabilizes A_n and so does P^t . \square

We also give a second proof of Theorem 3, which should be more likely generalizable to semigroups with more complicated generators than the first one.

Second proof of Theorem 3: Let A_n^* be the subset of A_n in which we have the strict inequalities

$$\alpha(x)\beta(y) < \gamma(x \vee y)\delta(x \wedge y).$$

To prove that P^t stabilizes A_n , it suffices to prove that it stabilizes A_n^* , because P^t is continuous and A_n is the closure of A_n^* . Let $(\alpha, \beta, \gamma, \delta) \in A_n^*$. Assume that at some time t , we have $(P^t \alpha, P^t \beta, P^t \gamma, P^t \delta) \notin A_n^*$. Because of the continuity in t of P^t , there exists a first time $T > 0$ for which $(P^T \alpha, P^T \beta, P^T \gamma, P^T \delta) \notin A_n^*$. So we have $\forall t <$

$T, (P^t\alpha, P^t\beta, P^t\gamma, P^t\delta) \in A_n^*$ and there exists some x and y in \mathcal{H}_n such that $\Delta_{xy}(T) = 0$, with the notation

$$\Delta_{xy}(t) := P^t\gamma(x \vee y)P^t\delta(x \wedge y) - P^t\alpha(x)P^t\beta(y).$$

We have for every $t \geq 0$

$$\frac{d\Delta_{xy}(t)}{dt} = LP^t\gamma(x \vee y)P^t\delta(x \wedge y) + P^t\gamma(x \vee y)LP^t\delta(x \wedge y) - LP^t\alpha(x)P^t\beta(y) - P^t\alpha(x)LP^t\beta(y).$$

In the case $n = 1$, we have

$$\begin{aligned} \left. \frac{d\Delta_{00}(t)}{dt} \right|_{t=0} &= \lambda_0 [(\gamma_1 - \gamma_0)\delta_0 + \gamma_0(\delta_1 - \delta_0) - (\alpha_1 - \alpha_0)\beta_0 - \alpha_0(\beta_1 - \beta_0)] \\ &= \lambda_0 [-2\Delta_{00} + \gamma_1\delta_0 + \gamma_0\delta_1 - \alpha_1\beta_0 - \alpha_0\beta_1] \geq -2\lambda_0\Delta_{00}, \end{aligned}$$

where we used Lemma 6 for the last inequality. Doing the same computation, we get $\left. \frac{d\Delta_{11}(t)}{dt} \right|_{t=0} \geq -2\lambda_0\Delta_{11}$. We have

$$\begin{aligned} \left. \frac{d\Delta_{01}(t)}{dt} \right|_{t=0} &= \lambda_0 [(\gamma_0 - \gamma_1)\delta_0 + \gamma_1(\delta_1 - \delta_0) - (\alpha_1 - \alpha_0)\beta_1 - \alpha_0(\beta_0 - \beta_1)] \\ &= \lambda_0 [-2\Delta_{01} + \Delta_{00} + \Delta_{11}] \geq -2\lambda_0\Delta_{01}. \end{aligned}$$

And likewise, $\left. \frac{d\Delta_{10}(t)}{dt} \right|_{t=0} \geq -2\lambda_0\Delta_{10}$.

Now in the case when $n \geq 2$, using the same $(\alpha', \beta', \gamma', \delta')$ as in the first proof, and the differential inequations obtained in the case $n = 1$, we get for every $x, y \in \mathcal{H}_n$ and every $i \in [0, n-1]$,

$$L_{(i)}\gamma(x \vee y)\delta(x \wedge y) + \gamma(x \vee y)L_{(i)}\delta(x \wedge y) - L_{(i)}\alpha(x)\beta(y) - \alpha(x)L_{(i)}\beta(y) \geq -2\Delta_{xy}.$$

Thus we have

$$\begin{aligned} \left. \frac{d\Delta_{xy}}{dt} \right|_{t=0} &= \sum_{i \in [0, n-1]} \lambda_i [L_{(i)}\gamma(x \vee y)\delta(x \wedge y) + \gamma(x \vee y)L_{(i)}\delta(x \wedge y) - L_{(i)}\alpha(x)\beta(y) - \alpha(x)L_{(i)}\beta(y)] \\ &\geq -2 \sum_{i \in [0, n-1]} \lambda_i \Delta_{xy}. \end{aligned}$$

So, setting $\lambda := \sum_{i \in [0, n-1]} \lambda_i$, and replacing $(\alpha, \beta, \gamma, \delta)$ with $(P^t\alpha, P^t\beta, P^t\gamma, P^t\delta)$, we get for any $0 \leq t \leq T$

$$\frac{d\Delta_{xy}(t)}{dt} \geq -2\lambda\Delta_{xy}(t).$$

Integrating this inequality between 0 and T yields

$$\Delta_{xy}(T) \geq \Delta_{xy}(0)e^{-2\lambda T} > 0.$$

But there must exist x and y such that $\Delta_{xy}(T) = 0$, so our assumption that P^t does not stabilize A_n^* was wrong. \square

2 generalization to $2n$ functions

Theorem 1 has been generalized to $2n$ functions, independently by Aharoni and Keich [1], and Rinott and Saks [6]. The proof we present here is widely inspired from the one of [1].

Definition 7 Let $0 < n \in \mathbb{N}$, and $x = (x_1, \dots, x_n) \in (\mathcal{H}_m)^n$. Then we set, for $1 \leq i \leq n$:

$$\varphi_i(x) := \bigvee_{\substack{S \subset [1, n] \\ |S|=i}} \left(\bigwedge_{j \in S} x_j \right).$$

A other way to say that is to set $\varphi_i(x)_j$, the j^{th} coordinate of $\varphi_i(x)$, to be 1 if at least i x_k among the n have their j^{th} coordinate equal to 1, and to set $\varphi_i(x)_j = 0$ otherwise.

We also define the following set of $2n$ -tuples of functions:

$$B_m^n := \left\{ (f_1, \dots, f_n, g_1, \dots, g_n) \in (\mathbb{R}_+^{\mathcal{H}_m})^{2n} \mid \forall x \in (\mathcal{H}_m)^n, \prod_{i=1}^n f_i(x_i) \leq \prod_{i=1}^n g_i(\varphi_i(x)) \right\}.$$

Then the $2n$ functions theorem is the following one:

Theorem 8 Let $(f_1, \dots, f_n, g_1, \dots, g_n) \in B_m^n$. Then we have:

$$\prod \int f_i \leq \prod \int g_i.$$

We will show the following semigroup version of this theorem:

Theorem 9 Let $(f_1, \dots, f_n, g_1, \dots, g_n) \in B_m^n$, and $P^t = e^{tL}$ be the Markov semigroup on the state space \mathcal{H}_m , generated by $L = \sum_{j \in [0, m-1]} \lambda_j L_{(j)}$. Then we have, for all $t \geq 0$

$$(P^t f_1, \dots, P^t f_n, P^t g_1, \dots, P^t g_n) \in B_m^n$$

Proof : Like for Theorem 3, we first prove Theorem 9 in the one dimensional case ($m = 1$).

Step 1: We first reduce the problem to simple cases.

We may assume that for each $i \in [1, n]$, $f_i(0)$, $f_i(1)$, $g_i(0)$ and $g_i(1)$ are positive. Indeed, for $\varepsilon > 0$, we may set $f_i^\varepsilon(x) = \max(\varepsilon, f_i(x))$ and $g_i^\varepsilon(x) = \max(\varepsilon, g_i(x))$, where $\varepsilon' = \inf_{1 \leq k \leq n} \left(\frac{\varepsilon^n}{M^{n-k}} \right)^{\frac{1}{k}}$ with $M = \sup_{i,x} f_i(x)$.

Let us check that $(f_1^\varepsilon, \dots, f_n^\varepsilon, g_1^\varepsilon, \dots, g_n^\varepsilon) \in B_m^n$. Let $x \in \{0, 1\}^n$. If for every $1 \leq i \leq n$, we have $f_i(x_i) \geq \varepsilon'$, then $\prod f_i^\varepsilon(x_i) = \prod f_i(x_i) \leq$

$\prod g_i(\varphi_i(x)) \leq \prod g_i^\varepsilon(\varphi_i(x))$. Otherwise, there exist k indexes i satisfying $f_i(x_i) < \varepsilon'$, where $1 \leq k \leq n$, and we have $\prod f_i^\varepsilon(x_i) \leq \varepsilon'^k M^{n-k} \leq \varepsilon^n \leq \prod g_i^\varepsilon(\varphi_i(x))$.

Furthermore, we have $\varepsilon' \leq \varepsilon$ (just take $k = n$), so f_i^ε tends to f_i and g_i^ε tends to g_i when ε tends to 0. Thus if Theorem 9 is true for f_i^ε and g_i^ε , it also holds for f_i and g_i by making ε tend to 0.

We can also assume that $\prod_{i=1}^n f_i(0) = \prod_{i=1}^n g_i(0) = 1$. Indeed, if $\prod_{i=1}^n f_i(0) < \prod_{i=1}^n g_i(0)$, we can set

$$\tilde{g}_1(x) = \begin{cases} g_1(1) & \text{if } x = 1 \\ \frac{\prod_{i=1}^n f_i(0)}{\prod_{i=2}^n g_i(0)} & \text{if } x = 0 \end{cases}$$

Then we have $(f_1, \dots, f_n, \tilde{g}_1, g_2, \dots, g_n) \in B_1^n$, and $g_1 \geq \tilde{g}_1$, so $P^t g_1 \geq P^t \tilde{g}_1$. So if $(P^t f_1, \dots, P^t f_n, P^t \tilde{g}_1, P^t g_2, \dots, P^t g_n)$ belongs to B_1^n , the same is true for $(P^t f_1, \dots, P^t f_n, P^t g_1, \dots, P^t g_n)$. And we can assume that $\prod_{i=1}^n f_i(0) = 1$ by dividing the functions f_i and g_i by $(\prod_{i=1}^n f_i(0))^{\frac{1}{n}}$.

We set $\alpha_i = \ln\left(\frac{f_i(1)}{f_i(0)}\right)$ and $\beta_i = \ln\left(\frac{g_i(1)}{g_i(0)}\right)$. We can assume that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ by changing the order of the f_i 's, because $\varphi_i(\sigma.x) = \varphi(x)$, where $\sigma \in \mathfrak{S}_n$, $x \in \{0, 1\}^n$ and $\sigma.x \in \{0, 1\}^n$ is defined by $(\sigma.x)_i = x_{\sigma(i)}$.

Step 2: We prove Theorem 9 in our simple case thanks to a lemma.

For a subset $S \subset \llbracket 1, n \rrbracket$, and for $0 \leq k_1 \leq k_2 \leq n$, we set:

$$\Lambda_{S, k_1, k_2} := \{C \subset \llbracket 1, n \rrbracket \mid \#C = k_2, \#(C \cap S) \geq k_1\}.$$

The lemma at the core of the proof is the following one:

Lemma 10 *Let $S \subset \llbracket 1, n \rrbracket$, and $0 \leq k_1 \leq k_2 \leq n$. If $(\alpha_i)_{i \in \llbracket 1, n \rrbracket}$ and $(\beta_i)_{i \in \llbracket 1, n \rrbracket}$ are two sequences such that α_i is nonincreasing and for all $1 \leq i \leq n$, $\sum_{j=1}^i \alpha_j \leq \sum_{j=1}^i \beta_j$, then the following inequality holds:*

$$\sum_{C \in \Lambda_{S, k_1, k_2}} \exp\left(\sum_{i \in C} \alpha_i\right) \leq \sum_{C \in \Lambda_{\llbracket 1, \#S \rrbracket, k_1, k_2}} \exp\left(\sum_{i \in C} \beta_i\right).$$

In [1], the same inequality was shown, but only for the sets $\Lambda_k = \{C \subset S, \#C = k\}$.

Let us see how Lemma 10 implies Theorem 9.

In our simple case, we have $\prod_{i=1}^n f_i(x_i) = \exp(\sum_{i|x_i=1} \alpha_i)$ and $\prod_{i=1}^n g_i(x_i) = \exp(\sum_{i|x_i=1} \beta_i)$. For $t > 0$, we have for any $f : \{0, 1\} \mapsto \mathbb{R}$, $P^t f(x) = (1-p)f(x) + pf(1-x)$ with $p = p(t) = \frac{1-e^{-2\lambda_0 t}}{2} < \frac{1}{2}$.

Let $x \in \{0, 1\}^n$, we set $S = \{i | x_i = 1\} \subset [1, n]$. Then we have:

$$\begin{aligned}
\prod_{i=1}^n P^t f_i(x_i) &= \sum_{C \subset [1, n]} \prod_{i=1}^n f_i(\mathbf{1}_C(i)) p^{\sharp(C \Delta S)} (1-p)^{n-\sharp(C \Delta S)} \\
&= \sum_{C \subset [1, n]} \exp\left(\sum_{i \in C} \alpha_i\right) p^{\sharp(C \Delta S)} (1-p)^{n-\sharp(C \Delta S)} \\
&= \sum_{0 \leq k_1 \leq k_2 \leq n} \sum_{\substack{C \subset [1, n] \\ \sharp C = k_2 \\ \sharp(C \cap S) = k_1}} \exp\left(\sum_{i \in C} \alpha_i\right) p^{\sharp S + k_2 - 2k_1} (1-p)^{n-\sharp S - k_2 + 2k_1} \\
&= \sum_{0 \leq k_2 \leq n} \left(\sum_{C \in \Lambda_{S, 0, k_2}} \exp\left(\sum_{i \in C} \alpha_i\right) p^{\sharp S + k_2} (1-p)^{n-\sharp S - k_2} \right. \\
&\quad \left. + \sum_{1 \leq k_1 \leq k_2} \sum_{C \in \Lambda_{S, k_1, k_2}} \exp\left(\sum_{i \in C} \alpha_i\right) p^{\sharp S + k_2 - 2k_1} (1-p)^{n-\sharp S - k_2 + 2k_1} \left(1 - \left(\frac{p}{1-p}\right)^2\right) \right),
\end{aligned}$$

where $C \Delta S = (C \cup S) \setminus (C \cap S)$ is the symmetric difference of the sets C and S , and where we used an Abel transform to get the last equality. We can rewrite this equation

$$\prod_{i=1}^n P^t f_i(x_i) = \sum_{0 \leq k_1 \leq k_2 \leq n} P_{S, k_1, k_2} R_{\sharp S, k_1, k_2},$$

with

$$P_{S, k_1, k_2} := \sum_{C \in \Lambda_{S, k_1, k_2}} \exp\left(\sum_{i \in C} \alpha_i\right)$$

and

$$R_{\sharp S, k_1, k_2} := \begin{cases} p^{\sharp S + k_2} (1-p)^{n-\sharp S - k_2} & \text{if } k_1 = 0 \\ \left(1 - \left(\frac{p}{1-p}\right)^2\right) p^{\sharp S + k_2 - 2k_1} (1-p)^{n-\sharp S - k_2 + 2k_1} & \text{if } k_1 \geq 1. \end{cases}$$

For $1 \leq i \leq n$, we have $\varphi_i(x) = 1$ if and only if $i \leq \sharp S$. Thus, doing as above, we get:

$$\prod_{i=1}^n P^t g_i(\varphi_i(x)) = \sum_{0 \leq k_1 \leq k_2 \leq n} Q_{\sharp S, k_1, k_2} R_{\sharp S, k_1, k_2}$$

with

$$Q_{\sharp S, k_1, k_2} := \sum_{C \in \Lambda_{[1, \sharp S], k_1, k_2}} \exp\left(\sum_{i \in C} \beta_i\right).$$

From Lemma 10, we get $P_{S, k_1, k_2} \leq Q_{\sharp S, k_1, k_2}$, and since $0 < p < \frac{1}{2}$, we have $R_{\sharp S, k_1, k_2} \geq 0$. Thus we get the desired inequality:

$$\prod_{i=1}^n P^t f_i(x_i) \leq \prod_{i=1}^n P^t g_i(\varphi_i(x)).$$

Step 3: We prove Theorem 9 in the general m -dimensional case. We can repeat the argument of the first proof of Theorem 3. Again, we only have to show that $e^{-\lambda_i t L^{(i)}}$ stabilizes B_m^n . Let $i \in [0, m-1]$ and $x = (x_1, \dots, x_n) \in (\mathcal{H}_m)^n$. Let $(f_1, \dots, f_n, g_1, \dots, g_n) \in B_m^n$. We set, for each $1 \leq j \leq n$, and for $u \in \{0, 1\}$, $f_j(u) := f_j(\bar{x}_{j_u})$ and $\bar{g}_j(u) := g_j(\overline{\varphi_j(x)}_u)$, where for any $y \in \mathcal{H}_m$, \bar{y}_u denotes the element of \mathcal{H}_m whose coordinates are the same that the ones of y , except the i -th, which is set to be u .

We have $(\bar{f}_1, \dots, \bar{f}_n, \bar{g}_1, \dots, \bar{g}_n) \in B_m^1$, and for all $1 \leq j \leq n$,

$$\begin{aligned} e^{\lambda_i t L^{(i)}} f_j(x_j) &= e^{\lambda_i t L^{(0)}} \bar{f}_j(x_{j_i}) \\ e^{\lambda_i t L^{(i)}} g_j(\varphi_j(x)) &= e^{\lambda_i t L^{(0)}} \bar{g}_j(\varphi_j(x)_i). \end{aligned}$$

Then, applying Theorem 9 in dimension 1 to $(\bar{f}_1, \dots, \bar{f}_n, \bar{g}_1, \dots, \bar{g}_n)$ shows that

$$\prod_{j=1}^n e^{\lambda_i t L^{(i)}} f_j(x_j) \leq \prod_{j=1}^n e^{\lambda_i t L^{(i)}} g_j(\varphi_j(x)).$$

Thus $e^{\lambda_i t L^{(i)}}$ stabilizes B_m^n , and so does $P^t = \prod_{i=0}^{m-1} e^{t \lambda_i L^{(i)}}$. \square

Remark 11 *One can also do as in the second proof of Theorem 3 to prove Theorem 9. We can show, using Lemma 10 with $(k_1, k_2) = (\sharp S - 1, \sharp S)$ or $(\sharp S, \sharp S + 1)$, that*

$$\frac{d}{dt} \Delta_x(t) \geq -2 \sum_{i=0}^{m-1} \lambda_i \Delta_x(t),$$

with $x \in \mathcal{H}_m^n$ and

$$\Delta_x(t) := \prod_{i=1}^n P^t g_i(\varphi_i(x)) - \prod_{i=1}^n P^t f_i(x_i).$$

It remains to prove Lemma 10. As in [1], we use a majorization argument. We first recall some basic facts about majorization, which can be found in [5].

Let $N \in \mathbb{N}$ and $x \in \mathbb{R}^N$. Then for $1 \leq i \leq N$, we denote by $x_{[i]}$ the i^{th} greatest coordinate of x . That is, $x_{[i]} = x_{\sigma(i)}$ for some permutation $\sigma \in \mathfrak{S}_N$ satisfying $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \dots \geq x_{\sigma(N)}$.

Definition 12 *Let $x, y \in \mathbb{R}^N$. We say that y majorizes x , and we write it $x \prec y$, if for each $1 \leq i \leq N-1$, we have*

$$\sum_{j=1}^i x_{[j]} \leq \sum_{j=1}^i y_{[j]},$$

and

$$\sum_{j=1}^N x_{[j]} = \sum_{j=1}^N y_{[j]}.$$

We say that y weakly majorizes x , and we write it $x \prec_w y$, if for each $1 \leq i \leq N$, we have

$$\sum_{j=1}^i x_{[j]} \leq \sum_{j=1}^i y_{[j]}.$$

The following proposition gives another definition for majorization.

proposition 13 *Let $x, y \in \mathbb{R}^N$. Then $x \prec_w y$ if and only if for any convex function $f : \mathbb{R} \rightarrow \mathbb{R}$, we have*

$$\sum_{i=1}^N f(x_i) \leq \sum_{i=1}^N f(y_i).$$

We only need the following Lemma, which is a direct consequence of this proposition.

Lemma 14 *Let $x, y \in \mathbb{R}^N$. If $x \prec_w y$, then we have*

$$\sum_{i=1}^N e^{x_i} \leq \sum_{i=1}^N e^{y_i}.$$

Proof of Lemma 14: Let $M \in \mathbb{R}$, we set $x' = (x_1, \dots, x_N, -M) \in \mathbb{R}^{N+1}$ and $y' = (y_1, \dots, y_N, -M - \sum_{i=1}^N (y_i - x_i)) \in \mathbb{R}^{N+1}$. If $x \prec_w y$, then for M large enough, we have $x' \prec_w y'$. Applying Proposition 13 to x' and y' , with $f = \exp$ and making M tend to $+\infty$ gives us the desired inequality. \square

Now we can prove Lemma 10.

Proof of Lemma 10: We first show that

$$\sum_{C \in \Lambda_{S, k_1, k_2}} \exp\left(\sum_{i \in C} \alpha_i\right) \leq \sum_{C \in \Lambda_{[[1, \#S]], k_1, k_2}} \exp\left(\sum_{i \in C} \alpha_i\right),$$

so it is sufficient to prove Lemma 10 in the case where $S = [[1, \#S]]$.

Let $S \subset [[1, n]]$. For $n_1, n_2, n_3, n_4 \in [[1, n]]$, we set P_{n_1, n_2, n_3, n_4} the set of every $C \subset [[1, n]]$ satisfying

$$\begin{cases} \#(C \cap S \cap [[1, \#S]]) = n_1 \\ \#(C \cap S \cap [[\#S + 1, n]]) = n_2 \\ \#(C \cap S^C \cap [[1, \#S]]) = n_3 \\ \#(C \cap S^C \cap [[\#S + 1, n]]) = n_4, \end{cases}$$

where $S^C = [1, n] \setminus S$.

Then we have

$$\Lambda_{S, k_1, k_2} = \bigsqcup_{\substack{n_1+n_2 \geq k_1 \\ n_1+n_2+n_3+n_4=k_2}} P_{n_1, n_2, n_3, n_4}$$

and

$$\Lambda_{[1, \#S], k_1, k_2} = \bigsqcup_{\substack{n_1+n_3 \geq k_1 \\ n_1+n_2+n_3+n_4=k_2}} P_{n_1, n_2, n_3, n_4}.$$

Thus, we have

$$\begin{aligned} & \sum_{C \in \Lambda_{[1, \#S], k_1, k_2}} \exp\left(\sum_{i \in C} \alpha_i\right) - \sum_{C \in \Lambda_{S, k_1, k_2}} \exp\left(\sum_{i \in C} \alpha_i\right) = \\ & \sum_{\substack{n_1+n_2 < k_1 \\ n_1+n_3 \geq k_1 \\ n_1+n_2+n_3+n_4=k_2}} \left(\sum_{C \in P_{n_1, n_2, n_3, n_4}} \exp\left(\sum_{i \in C} \alpha_i\right) - \sum_{C \in P_{n_1, n_3, n_2, n_4}} \exp\left(\sum_{i \in C} \alpha_i\right) \right). \end{aligned} \quad (1)$$

Let $n_1, n_2, n_3, n_4 \in \mathbb{N}$ such that $n_1 + n_2 < k_1$, $n_1 + n_3 \geq k_1$ and $n_1 + n_2 + n_3 + n_4 = k_2$. One trivially has $n_3 > n_2$. We construct a bipartite graph whose vertices are the elements of $P_{n_1, n_2, n_3, n_4} \sqcup P_{n_1, n_3, n_2, n_4}$, by putting an edge between $C \in P_{n_1, n_2, n_3, n_4}$ and $C' \in P_{n_1, n_3, n_2, n_4}$ (and we write it $C \sim C'$) if and only if the following conditions are satisfied:

$$\begin{cases} C \cap S \cap [1, \#S] = C' \cap S \cap [1, \#S] \\ C \cap S \cap [\#S + 1, n] \subset C' \cap S \cap [\#S + 1, n] \\ C \cap S^C \cap [1, \#S] \supset C' \cap S^C \cap [1, \#S] \\ C \cap S^C \cap [\#S + 1, n] = C' \cap S^C \cap [\#S + 1, n]. \end{cases}$$

More simply said, we have $C \sim C'$ if and only if C' can be obtained from C by removing $n_3 - n_2$ elements of $C \cap S^C \cap [1, \#S]$ and adding $n_3 - n_2$ elements of $C^C \cap S \cap [\#S + 1, n]$, or symmetrically if and only if C can be obtained from C' by removing $n_3 - n_2$ elements of $C' \cap S \cap [\#S + 1, n]$ and adding $n_3 - n_2$ elements of $C'^C \cap S^C \cap [1, \#S]$.

Since α_i is nonincreasing, the sum of $n_3 - n_2$ α_i 's with $i \leq \#S$ is always greater than or equals to the sum of $n_3 - n_2$ α_i 's with $i > \#S$. Thus, if $C \in P_{n_1, n_2, n_3, n_4}$ and $C' \in P_{n_1, n_3, n_2, n_4}$ are such that $C \sim C'$, then we have $\sum_{i \in C} \alpha_i \geq \sum_{i \in C'} \alpha_i$. Furthermore, any vertex of the graph has exactly $N = \binom{n_3}{n_3 - n_2} \binom{\#S - \#(S \cap [1, \#S]) - n_2}{n_3 - n_2}$ neighbors. Then,

we have:

$$\begin{aligned}
0 &\leq \frac{1}{N} \sum_{\substack{C \in P_{n_1, n_2, n_3, n_4} \\ C' \in P_{n_1, n_3, n_2, n_4} \\ C \sim C'}} \left(\exp\left(\sum_{i \in C} \alpha_i\right) - \exp\left(\sum_{i \in C'} \alpha_i\right) \right) \\
&= \sum_{C \in P_{n_1, n_2, n_3, n_4}} \exp\left(\sum_{i \in C} \alpha_i\right) - \sum_{C' \in P_{n_1, n_3, n_2, n_4}} \exp\left(\sum_{i \in C'} \alpha_i\right).
\end{aligned}$$

Putting this inequality in (1) provides the desired inequality:

$$\sum_{C \in \Lambda_{[[1, \#S]], k_1, k_2}} \exp\left(\sum_{i \in C} \alpha_i\right) - \sum_{C \in \Lambda_{S, k_1, k_2}} \exp\left(\sum_{i \in C} \alpha_i\right) \geq 0.$$

Remark 15 *Using the marriage Lemma, or the fact that the extremal points of the set of bistochastic matrices are permutation matrices ($\frac{1}{N}$ times the adjacency matrix of our graph is trivially a bistochastic matrix), one can show that there exists a bijection f between P_{n_1, n_2, n_3, n_4} and P_{n_1, n_3, n_2, n_4} such that $C \sim f(C)$. Thus there also exists a bijection g from $\Lambda_{[[1, \#S]], k_1, k_2}$ to Λ_{S, k_1, k_2} such that $\sum_{i \in C} \alpha_i \geq \sum_{i \in g(C)} \alpha_i$, and this implies that $X_{\Lambda_{S, k_1, k_2}} \prec_w X_{\Lambda_{[[1, \#S]], k_1, k_2}}$, where the notation X_Λ is explained just below.*

For $\Lambda \subset \mathcal{P}([1, n])$, we set X_Λ and Y_Λ the vectors of \mathbb{R}^Λ whose coordinates are $(X_\Lambda)_C = \sum_{i \in C} \alpha_i$ and $(Y_\Lambda)_C = \sum_{i \in C} \beta_i$, for $C \in \Lambda$.

Let us prove that $X_{\Lambda_{[[1, \#S]], k_1, k_2}} \prec_w Y_{\Lambda_{[[1, \#S]], k_1, k_2}}$. We enumerate the elements $(C_i)_{i=1 \dots \#\Lambda_{[[1, \#S]], k_1, k_2}}$ of $\Lambda_{[[1, \#S]], k_1, k_2}$ in such a way that $(\sum_{j \in C_i} \alpha_j, \sum_{j \in C_i} 2^{-j})$ is a decreasing sequence (indexed by i), for the lexicographic order, which means that for $1 \leq i_1 < i_2 \leq \#\Lambda_{[[1, \#S]], k_1, k_2}$, we have $\sum_{j \in C_{i_1}} \alpha_j \geq \sum_{j \in C_{i_2}} \alpha_j$, and in case of equality, we have furthermore $\sum_{j \in C_{i_1}} 2^{-j} > \sum_{j \in C_{i_2}} 2^{-j}$ (well note that $C \subset [1, n] \mapsto \sum_{j \in C} 2^{-j}$ is injective).

Let $1 \leq k \leq \#\Lambda_{[[1, \#S]], k_1, k_2}$. For $1 \leq j \leq n$, we set $a_k(j) = \#\{i \in [1, k], j \in C_i\}$, and we set $a_k(n+1) = 0$.

We show that $a_k(j)$ is nonincreasing in j . Let $1 \leq j_1 < j_2 \leq n$, then $a_k(j_1) - a_k(j_2) = \#\{i \in [1, k], j_1 \in C_i, j_2 \notin C_i\} - \#\{i \in [1, k], j_2 \in C_i, j_1 \notin C_i\}$. Let $i \in [1, k]$ be such that $j_2 \in C_i$ and $j_1 \notin C_i$. We set $C'_i = (C_i \cup \{j_1\}) \setminus \{j_2\}$. We have $\#C'_i = \#C_i = k_2$ and $\#(C'_i) \cap [1, \#S] \geq \#(C_i) \cap [1, \#S] \geq k_1$, so $C'_i \in \Lambda_{[[1, \#S]], k_1, k_2}$. We also have $\sum_{j \in C'_i} \alpha_j - \sum_{j \in C_i} \alpha_j = \alpha_{j_1} - \alpha_{j_2} \geq 0$ and $\sum_{j \in C'_i} 2^{-j} - \sum_{j \in C_i} 2^{-j} = 2^{-j_1} - 2^{-j_2} > 0$, so there exists $i' < i$ such that $C'_i = C_{i'}$. Thus $i \mapsto i'$ is an injective function from $\{i \in [1, k], j_2 \in C_i, j_1 \notin C_i\}$ to $\{i \in [1, k], j_1 \in C_i, j_2 \notin C_i\}$. Then $\#\{i \in [1, k], j_1 \in C_i, j_2 \notin C_i\} \geq \#\{i \in [1, k], j_2 \in C_i, j_1 \notin C_i\}$, and $a_k(j_1) - a_k(j_2) \geq 0$.

So we have

$$\begin{aligned}
\sum_{i=1}^k (X_{\Lambda_{[[1, \#S]], k_1, k_2}})_{[i]} &= \sum_{i=1}^k \sum_{j \in C_i} \alpha_j = \sum_{j=1}^n a_k(j) \alpha_j \\
&= \sum_{j=1}^n \left((a_k(j) - a_k(j+1)) \sum_{l=1}^j \alpha_l \right) \quad (\text{Abel transformation}) \\
&\leq \sum_{j=1}^n \left((a_k(j) - a_k(j+1)) \sum_{l=1}^j \beta_j \right) = \sum_{j=1}^n a_k(j) \beta_j = \sum_{i=1}^k \sum_{j \in C_i} \beta_j \\
&= \sum_{i=1}^k (Y_{\Lambda_{[[1, \#S]], k_1, k_2}})_{C_i} \leq \sum_{i=1}^k (Y_{\Lambda_{[[1, \#S]], k_1, k_2}})_{[i]}
\end{aligned}$$

where the last inequality occurs because the sum of any k coordinates of the vector $Y_{\Lambda_{[[1, \#S]], k_1, k_2}}$ is always smaller than the sum of the k greatest ones.

Thus we have $X_{\Lambda_{[[1, \#S]], k_1, k_2}} \prec_w Y_{\Lambda_{[[1, \#S]], k_1, k_2}}$ as announced. Then we apply Lemma 14 and get

$$\sum_{C \in \Lambda_{[[1, \#S]], k_1, k_2}} \exp\left(\sum_{i \in C} \alpha_i\right) \leq \sum_{C \in \Lambda_{[[1, \#S]], k_1, k_2}} \exp\left(\sum_{i \in C} \beta_i\right).$$

So Lemma 10 is proved. \square

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