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# Existence of Periodic solutions of integro-differential delay equations with two variable in vector-valued function space

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## ABSTRACT

The aim of this work is to study the existence of a periodic solutions of neutral integro-differential partial functional differential equations with two variable:  $\frac{\partial}{\partial x} \frac{\partial}{\partial y} (\mathbf{u}(\mathbf{x}, \mathbf{y}) - \mathbf{B}\mathbf{u}(\mathbf{x} - \mathbf{r}, \mathbf{y} - \mathbf{r})) + \mathbf{A}\mathbf{u}(\mathbf{x}, \mathbf{y}) = \int_{-\infty}^{\mathbf{x}} \int_{-\infty}^{\mathbf{y}} \mathbf{a}(\mathbf{x} - \mathbf{s}, \mathbf{y} - \xi) \mathbf{u}(\mathbf{s}, \xi) d\xi d\mathbf{s} + \mathbf{f}(\mathbf{x}, \mathbf{y})$ ,  $\mathbf{x}, \mathbf{y} \in [0, 2\pi]$ . Our approach is based on the  $L^p$ -multipliers of linear operators.

**Keywords:** neutral integro-differential equations,  $L^p$ -multipliers, vector-valued function space.

## 1 Introduction

Motivated by the fact that neutral functional differential equations (abbreviated, NFDE) with finite delay arise in many areas of applied mathematics, this type of equations has received much attention in recent years. In particular, the problem of existence of periodic solutions, has been considered by several authors. We refer the readers to papers [\[1\]](#), [\[2\]](#), [\[3\]](#), [\[5\]](#), [\[6\]](#), [\[7\]](#), [\[9\]](#), [\[13\]](#), [\[15\]](#), [\[19\]](#) and the references listed therein for information on this subject.

In this work, we study the existence of periodic solutions for the following neutral partial functional differential equations of the following form

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$$\frac{\partial}{\partial \mathbf{x}} \frac{\partial}{\partial \mathbf{y}} (\mathbf{u}(\mathbf{x}, \mathbf{y}) - \mathbf{B}\mathbf{u}(\mathbf{x} - \mathbf{r}, \mathbf{y} - \mathbf{r})) + \mathbf{A}\mathbf{u}(\mathbf{x}, \mathbf{y}) = \int_{-\infty}^{\mathbf{x}} \int_{-\infty}^{\mathbf{y}} \mathbf{a}(\mathbf{x} - \mathbf{s}, \mathbf{y} - \xi) \mathbf{u}(\mathbf{s}, \xi) d\xi d\mathbf{s} + \mathbf{f}(\mathbf{x}, \mathbf{y}). \quad (1.1)$$

Where  $(A, D(A))$  and  $(B, D(B))$  are a closed linear operators on a Banach space  $X$  such that  $D(A) \subset D(B)$ ,  $a \in L^1(\mathbb{R}_+ \times \mathbb{R}_+)$  and  $f : \mathbb{R} \times \mathbb{R} \rightarrow X$  is a locally  $p$ -integrable and  $2\pi$ -periodic function for  $1 \leq p < \infty$ .

In [4], Bahloul established results on the existence of periodic solutions of Eq. ((1.1)) when  $a = 0$ , namely, for the following partial functional differential equation

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} u(x, y) + Au(x, y) = f(x, y)$$

where  $(A, D(A))$  is a linear operator.

In [3], Bahloul et al established results on the existence for some degenerate differential equation. Resently in [5], Bahloul et al established the existence of periodic solution for the some functional differential equations in UMD space.

$$\frac{d}{dt}(x(t) - L(x_t)) = A(x(t) - L(x_t)) + G(x_t) + f(t) \quad \text{for } t \in \mathbb{R},$$

In [6], Sylvain Koumla, Khalil Ezzinbi and Rachid Bahloul, studied the existence of mild solutions for some partial functional integrodifferential equations with finite delay in Frechet spaces described in the following form:

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x_t) + h(t, x_t) \quad \text{for } t \geq 0$$

where  $A : D(A) \rightarrow X$  is the infinitesimal generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $X$ ,  $B(t)$  is a closed linear operator with domain  $D(B)$  such that  $D(A) \subset D(B)$ .

In [14], Lizama gave necessary and sufficient conditions for the existence of periodic solutions of the following equation.

$$\frac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t) \quad \text{for } t \in \mathbb{R},$$

where  $A$  is a closed linear operator on an UMD-space  $Y$ . We note that, starting with the work [14], the problem of characterization of maximal regularity for avolution equations with periodic initial conditions have been studied intensively in the last years. See e.g [8], [12] and [16], see also [10], [11], [17] and [18] and references therien.

the main novelty in this paper relies in the presence of two variable. This work can be considered as a progress on the treatment of such kind of problems.

This work is organized as follows : After a creterion in the second section, we are able to characterize in section 3 the existence and uniqueness of strong  $L^p$ -solution for the problem ((1.1)) solely in terms of a property of  $L^p$ -multiplier for the sequence of operators  $-kz(-kzI + kzB_{kz} + A - L_{k,z})^{-1}$ . We optain that the following assertion are equivalent in general Banach spaces:

- (1)  $(-kzI + kzB_{kz} + A - \tilde{a}(k, z))$  is invertible and  $\{-kz(-kzI + kzB_{kz} + A - \tilde{a}(k, z))^{-1}, (k, z) \in \mathbb{Z} \times \mathbb{Z}\}$  is an  $L^p$ -muliplier.

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(2) For every  $f \in L^p(\mathbb{T} \times \mathbb{T}; X)$  there exist a unique function  $u \in H^{1,p}(\mathbb{T} \times \mathbb{T}; X)$  such that  $u \in D(A)$  and equation ((1.1)) holds for a.e  $(x, y) \in [0, 2\pi] \times [0, 2\pi]$ .

In section 4, we give the main abstract result ( theorem ((4.1))) of this work.

## 2 A creterion for periodic solutions

Let  $X$  be a Banach Space. Firstly, we denote By  $\mathbb{T}$  the group defined as the quotient  $\mathbb{R}/2\pi\mathbb{Z}$ . There is an identification between functions on  $\mathbb{T}$  and  $2\pi$ -periodic functions on  $\mathbb{R}$ . We consider the interval  $[0; 2\pi)$  as a model for  $\mathbb{T}$ . Given  $1 \leq p < \infty$ , we denote by  $L^p(\mathbb{T} \times \mathbb{T}; X)$  the space of  $2\pi$ -periodic locally  $p$ -integrable functions from  $\mathbb{R} \times \mathbb{R}$  into  $X$ , with the norm:

$$\|f\|_p := (\int_0^{2\pi} \int_0^{2\pi} \|f(x, y)\|^p dx dy)^{1/p}$$

For  $f \in L^p(\mathbb{T} \times \mathbb{T}; X)$ , we denote by  $\hat{f}(k, z)$ ,  $(k, z) \in \mathbb{Z}$  the  $(k, z)$ -th Fourier coefficient of  $f$  that is defined by:

$$\hat{f}(k, z) = (\frac{1}{2\pi})^2 \int_0^{2\pi} \int_0^{2\pi} e^{-i(kx+zt)} f(x, t) dx dt.$$

for  $(k, z) \in \mathbb{Z} \times \mathbb{Z}$  and  $(x, y) \in \mathbb{R} \times \mathbb{R}$ .

*Remark 2.1.* [4]

For all  $(k, z) \in \mathbb{Z} \times \mathbb{Z}$  and  $(x, y) \in \mathbb{T} \times \mathbb{T}$ , we have

$$(1) \quad \hat{f}(0, 0) = (\frac{1}{2\pi})^2 \int_0^{2\pi} \int_0^{2\pi} f(x, y) dx dy.$$

$$(2) \quad \hat{f}(k, 0) = (\frac{1}{2\pi})^2 \int_0^{2\pi} \int_0^{2\pi} e^{-ikx} f(x, y) dx dy.$$

$$(3) \quad \hat{f}(0, z) = (\frac{1}{2\pi})^2 \int_0^{2\pi} \int_0^{2\pi} e^{-izt} f(x, y) dx dy.$$

$$(4) \quad \hat{f}(k, y) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} f(x, y) dx.$$

$$(5) \quad \hat{f}(x, z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-izy} f(x, y) dy.$$

**Notation 2.1.** : Let  $(k, z) \in \mathbb{Z} \times \mathbb{Z}$  and  $1 \leq p < \infty$ . Denote by

$$H^{1,p}(\mathbb{T} \times \mathbb{T}; X) = \left\{ u \in L^p(\mathbb{T} \times \mathbb{T}; X) : \exists v \in L^p(\mathbb{T} \times \mathbb{T}; X), \hat{v}(k, z) = -kz\hat{u}(k, z) \right\}. \text{ and } B_{kz} = e^{-(ikx+izy)} B$$

*Remark 2.2.* [9]

Let  $f \in L^1(\mathbb{T}; X)$ . If  $g(t) = \int_0^t f(s) ds$  and  $k \in \mathbb{Z}, k \neq 0$ , then

$$\hat{g}(k) = \frac{i}{k} \hat{f}(0) - \frac{i}{k} \hat{f}(k)$$

**Lemma 2.2.** [4]

Let  $f \in L^1(\mathbb{T} \times \mathbb{T}; X)$ .

If  $w(x, y) = \int_0^x \int_0^y f(s, \xi) d\xi ds$  and  $(k, z) \in \mathbb{Z} \times \mathbb{Z} - \{(0, 0)\}$  then

$$\widehat{w}(k, z) = -\frac{1}{kz}\widehat{f}(0, 0) + \frac{1}{kz}\widehat{f}(0, z) + \frac{1}{kz}\widehat{f}(k, 0) - \frac{1}{kz}\widehat{f}(k, z)$$

**Lemma 2.3.** [4] **((2.2))**

Let  $1 \leq p < \infty$  and let  $f, g \in L^1(\mathbb{T} \times \mathbb{T}; X)$ . The following are equivalent:

(1)  $\widehat{v}(0, 0) = \widehat{v}(k, 0) = \widehat{v}(0, z) = 0$  and there exists  $a \in X$  such that

$$u(x, y) = a + \int_0^x \int_0^y v(s, \xi) d\xi ds, \quad (x, y) \in \mathbb{T} \times \mathbb{T}$$

(2)  $\widehat{v}(k, z) = -kz\widehat{u}(k, z)$ ,  $((k, z) \in \mathbb{Z} \times \mathbb{Z})$

*Remark 2.3.* By Lemma **((2.3))** we have  $u \in H^{1,p}(\mathbb{T} \times \mathbb{T}; X) \Leftrightarrow$  there exists  $v \in L^p(\mathbb{T} \times \mathbb{T}; X)$  such that  $\widehat{v}(0, 0) = \widehat{v}(k, 0) = \widehat{v}(0, z) = 0$  and there exists  $a \in X$  such that  $u(x, y) = a + \int_0^x \int_0^y v(s, \xi) d\xi ds$ ,  $(x, y) \in \mathbb{T} \times \mathbb{T}$

Now we can formulate the following multiplier definition.

**Definition 2.1.** For  $1 \leq p < \infty$ , a sequence  $\{M_{k,z}\}_{(k,z) \in \mathbb{Z} \times \mathbb{Z}}$  is said to be an  $L^p$ -multiplier (or  $(L^p, L^p)$ -multiplier) if for each  $f \in L^p(\mathbb{T} \times \mathbb{T}, X)$ , there exists  $u \in L^p(\mathbb{T} \times \mathbb{T}, Y)$  such that  $\widehat{g}(k, z) = M_{k,z}\widehat{f}(k, z)$  for all  $(k, z) \in \mathbb{Z} \times \mathbb{Z}$ .

**Definition 2.2.** : For  $1 \leq p < \infty$ , a sequence  $\{M_{k,z}\}_{(k,z) \in \mathbb{Z} \times \mathbb{Z}}$  is said to be an  $(L^p, H^{1,p})$ -multiplier if for each  $f \in L^p(\mathbb{T} \times \mathbb{T}, X)$ , there exists  $u \in H^{1,p}(\mathbb{T} \times \mathbb{T}, Y)$  such that  $\widehat{g}(k, z) = M_{k,z}\widehat{f}(k, z)$  for all  $(k, z) \in \mathbb{Z} \times \mathbb{Z}$ .

We recall the following results

**Lemma 2.4.** [4] Let  $f, g \in L^p([0, 2\pi] \times [0, 2\pi]; X)$ , where  $1 \leq p < \infty$ . Then the following are equivalent.

(i)  $f(x, y) \in D(A)$  and  $Af(x, y) = g(x, y)$ ,

(ii)  $\widehat{f}(k, z) \in D(A)$  and  $A\widehat{f}(k, z) = \widehat{g}(k, z)$  for all  $(k, z) \in \mathbb{Z} \times \mathbb{Z}$

Next we give some preliminaries. Given  $a \in L^1(\mathbb{R}_+ \times \mathbb{R}_+)$  and  $u : [0, 2\pi] \times [0, 2\pi] \rightarrow X$  (extended by periodicity to  $\mathbb{R}$ ), we define

$$F(x, y) = \int_{-\infty}^x \int_{-\infty}^y a(x-s, y-\xi)u(s, \xi) ds d\xi \quad (2.1)$$

Let  $\tilde{a}(\lambda, \mu) = \int_0^{+\infty} \int_0^{+\infty} e^{-(\lambda x + \mu y)} a(x, y) dx dy$ ,  $\lambda, \mu \in \mathbb{C}$ , be the Laplace transform of  $a$ . An easy computation shows that:

$$\widehat{F}(k, z) = \tilde{a}(ik, iz)\widehat{u}(k, z), \quad \text{for } k, z \in \mathbb{Z} \quad (2.2)$$

*Remark 2.4.* For all  $(\lambda, \mu) \in \mathbb{C} \times \mathbb{C}$  and  $(x, t) \in \mathbb{R} \times \mathbb{R}$ , we have

(1)  $\tilde{f}(0, 0) = \int_0^{+\infty} \int_0^{+\infty} f(x, y) dx dy$ .

(2)  $\tilde{f}(\lambda, 0) = \int_0^{+\infty} \int_0^{+\infty} e^{-i\lambda x} f(x, y) dx dy$ .

(3)  $\tilde{f}(0, \mu) = \int_0^{+\infty} \int_0^{+\infty} e^{-i\mu y} f(x, y) dx dy$ .

(4)  $\tilde{f}(\lambda, y) = \int_0^{+\infty} e^{-i\lambda x} f(x, y) dx$ .

(5)  $\tilde{f}(x, \mu) = \int_0^{+\infty} e^{-i\mu y} f(x, y) dy$ .

### 3 A characterization of strong $L^p$ -solution

**Definition 3.1.** : Let  $f \in L^p(\mathbb{T} \times \mathbb{T}; X)$ . A function  $u \in H^{1,p}(\mathbb{T} \times \mathbb{T}; X)$  is said to be a  $2\pi$ -periodic strong  $L^p$ -solution of Eq. ((1.1)) if  $u \in D(A)$ ,  $u(0, y) = u(2\pi, y)$ ,  $u(x, 0) = u(x, 2\pi)$  for all  $(x, y) \in \mathbb{T} \times \mathbb{T}$  and Eq.((1.1)) holds almost every where.

**Proposition 3.1.** Let  $X$  be a Banach space. Suppose that for every  $f \in L^p(\mathbb{T} \times \mathbb{T}; X)$  there exists a unique strong solution of Eq. ((1.1)) for  $1 \leq p < \infty$ . Then

- (1) for every  $(k, z) \in \mathbb{Z} \times \mathbb{Z}$  the operator  $(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))$  has invertible.
- (2)  $\{-kz(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))^{-1}\}_{(k,z) \in \mathbb{Z} \times \mathbb{Z}}$  is an  $L^p$ -multiplier.

*Proof.* 1. Let  $k, z \in \mathbb{Z}$  and  $w \in X$ . Then for  $f(x, y) = e^{i(kx+zy)}w$ , there exists  $x \in H^{1,p}(\mathbb{T} \times \mathbb{T}; X)$  such that:

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} (u(x, y) - Bu(x-r, y-r)) + Au(x, y) = \int_{-\infty}^x \int_{-\infty}^y a(x-s, y-\xi)u(s, \xi) d\xi ds + f(x, y)$$

Taking Fourier transform. Then we obtain

$$(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))\widehat{u}(k, z) = \widehat{f}(k, z) = w \Rightarrow (-kzI + kzB_{kz} + A - \tilde{a}(ik, iz)) \text{ is surjective.}$$

if  $(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))(v) = 0$ , then,  $u(x, y) = e^{i(kx+zy)}v$  is a  $2\pi$ -periodic strong  $L^p$ -solution of Eq. ((1.1)) corresponding to the function  $f = 0$  Hence  $u(x, y) = 0$  and  $v = 0$  then  $(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))$  is injective.

2) Let  $f \in L^p(\mathbb{T} \times \mathbb{T}; X)$ . By hypothesis, there exists a unique  $u \in H^{1,p}(\mathbb{T} \times \mathbb{T}, X)$  such that the Eq.((1.1)) is valid. Taking Fourier transforms, we deduce that  $\widehat{u}(k, z) = (-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))^{-1}\widehat{f}(k, z)$  for all  $k, z \in \mathbb{Z}$ , then  $\{(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))^{-1}\}_{(k,z) \in \mathbb{Z} \times \mathbb{Z}}$

is an  $L^p$ -multiplier. Hence  $-kz\widehat{u}(k, z) = -kz(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))^{-1}\widehat{f}(k, z)$  for all  $k, z \in \mathbb{Z}$ . On the other hand, since  $u \in H^{1,p}(\mathbb{T} \times \mathbb{T}; X)$ , there exists  $w \in L^p(\mathbb{T} \times \mathbb{T}; X)$  such that  $\widehat{w}(k, z) = -kz\widehat{u}(k, z) = -kz(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))^{-1}\widehat{f}(k, z)$  i.e  $\{-kz(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))^{-1}\}_{(k,z) \in \mathbb{Z} \times \mathbb{Z}}$  is an  $L^p$ -multiplier.  $\square$

### 4 Main result

The next Theorem is the main result of this section.

**Theorem 4.1.** Let  $A$  be a closed linear operator and  $1 \leq p < \infty$ . If

- (1) for every  $(k, z) \in \mathbb{Z} \times \mathbb{Z}$  the operator  $M_{k,z} = (-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))$  has invertible.
- (2)  $\{-kz(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))^{-1}\}_{(k,z) \in \mathbb{Z} \times \mathbb{Z}}$  is an  $L^p$ -multiplier.

Then for every  $f \in L^p(\mathbb{T} \times \mathbb{T}; X)$  there exists a unique  $2\pi$ -periodic strong  $L^p$ -solution of Eq. ((1.1)).

**Lemma 4.2.** [4] Let  $1 \leq p < \infty$  and  $X$  be a Banach space. The following assertions are equivalent

(1)  $\{M_{k,z}\}_{(k,z) \in \mathbb{Z} \times \mathbb{Z}}$  is an  $(L^p, H^p)$ -multiplier.

(2)  $\{-kzM_{k,z}\}_{(k,z) \in \mathbb{Z} \times \mathbb{Z}}$  is an  $L^p$ -multiplier.

*Proof.* 1  $\Rightarrow$  2. Let  $f \in L^p(\mathbb{T} \times \mathbb{T}; X)$ , then there exists  $g \in H^{1,p}(\mathbb{T} \times \mathbb{T}; X)$  ( in particular  $g \in L^p(\mathbb{T} \times \mathbb{T}; X)$ ) such that  $\widehat{g}(k, z) = -kzM_{k,z}\widehat{f}(k, z)$ , then  $\{-kzM_{k,z}\}_{(k,z) \in \mathbb{Z} \times \mathbb{Z}}$  is an  $L^p$ -multiplier.

2  $\Rightarrow$  1. Let  $f \in L^p(\mathbb{T} \times \mathbb{T}; X)$ , by hypothesis  $\exists v \in L^p(\mathbb{T} \times \mathbb{T}; X)$  such that  $\widehat{v}(k, z) = -kzM_{k,z}\widehat{f}(k, z)$ . In particular  $\widehat{v}(0, 0) = \widehat{v}(0, z) = \widehat{v}(k, 0) = 0$ .

Let  $w(x, y) = \int_0^x \int_0^y v(s, \xi) d\xi ds$ , then we have

$$\widehat{w}(x, z) = \int_0^x \left( \frac{i}{z} \widehat{v}(x, 0) - \frac{i}{z} \widehat{v}(x, z) \right) ds, \text{ for all } z \neq 0$$

$$\widehat{w}(k, z) = \frac{i}{k} \left[ \frac{i}{z} \widehat{v}(0, 0) - \frac{i}{z} \widehat{v}(0, z) \right] - \frac{i}{k} \left[ \frac{i}{z} \widehat{v}(k, 0) - \frac{i}{z} \widehat{v}(k, z) \right], \text{ for all } k \neq 0$$

$$\widehat{w}(k, z) = \frac{i}{k} \frac{i}{z} \widehat{v}(k, z) = \frac{-1}{kz} \widehat{v}(k, z) = M_{k,z} \widehat{f}(k, z), \text{ for all } k \neq 0 \text{ and } z \neq 0$$

Let  $u(x, y) = w(x, y) + M_{0,0} \widehat{f}(0, 0) - \widehat{w}(0, 0)$  i.e

$u(x, y) = a + \int_0^x \int_0^y v(s, \xi) d\xi ds$ ,  $a = M_{0,0} \widehat{f}(0, 0) - \widehat{w}(0, 0)$ . Then by Remark ((2.2)) we have  $u \in H^{1,p}(\mathbb{T} \times \mathbb{T}; X)$ .

$$\widehat{u}(k, z) = \widehat{w}(k, z) + 0 = \widehat{w}(k, z) = M_{k,z} \widehat{f}(k, z), \text{ for all } (k, z) \neq (0, 0),$$

Proof of theorem ((4.1)). Let  $f \in L^p(\mathbb{T} \times \mathbb{T}; X)$ . Define  $M_{k,z} = (-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))^{-1}$  by theorem ((4.1)), the family  $\{-kz(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))^{-1}\}_{(k,z) \in \mathbb{Z} \times \mathbb{Z}}$  is an  $L^p$ -multiplier and by Lemma ((4.2)), it is equivalent to the fact that the family  $\{M_{k,z}\}_{(k,z) \in \mathbb{Z} \times \mathbb{Z}}$  is an  $(L^p, H^p)$ -multiplier, i.e, there exists  $u \in H^{1,p}(\mathbb{T} \times \mathbb{T}; X)$  such that

$$\widehat{u}(k, z) = M_{k,z} \widehat{f}(k, z) = (-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))^{-1} \widehat{f}(k, z) \quad (4.1)$$

In particular,  $u \in L^p(\mathbb{T} \times \mathbb{T}; X)$  and there exists  $v \in L^p(\mathbb{T} \times \mathbb{T}; X)$  such that

$$\widehat{v}(k, z) = -kz \widehat{u}(k, z) \quad (4.2)$$

Moreover,  $\widehat{\frac{\partial}{\partial x} \frac{\partial}{\partial t} u}(k, z) = -kz \widehat{v}(k, z)$ .

By Fejer's theorem one has in  $L^p(\mathbb{T} \times \mathbb{T}; X)$

$$\widehat{u}(k, z) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sigma_{n,m}(u)(x, y)$$

such that  $\sigma_{n,m}(u)(x, y) = \frac{1}{n+1} \frac{1}{m+1} \sum_{p=0}^n \sum_{k=-p}^p \sum_{l=0}^m \sum_{z=-l}^l e^{i(kx+zt)} \widehat{u}(k, z)$ .

By ((4.1)) and ((4.2)) we have

$$-kz \widehat{u}(k, z) + kzB_{kz} \widehat{u}(k, z) + A \widehat{u}(k, z) = \tilde{a}(ik, iz) \widehat{f}(k, z) + \widehat{f}(k, z)$$

i.e

$$-kz \widehat{u}(k, z) - (-kz)B_{kz} \widehat{u}(k, z) + A \widehat{u}(k, z) = \widehat{F}(k, z) + \widehat{f}(k, z)$$

For all  $(k, z) \in \mathbb{Z} \times \mathbb{Z}$ . Then using that A and B are closed, we conclude that  $u \in D(A)$  and

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} (u(x, y) - Bu(x-r, y-r)) + Au(x, y) = \int_{-\infty}^x \int_{-\infty}^y a(x-s, y-\xi) u(s, \xi) d\xi ds + f(x, y)$$

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Suppose (1) has two strong  $L^p$ -solution  $u_1$  and  $u_2$ , then  $u = u_1 - u_2$  is a solution of Eq. **((1.1))** corresponding to the function  $f = 0$ . Taking Fourier transform on both sides of this equality we get  $(-kzI + kzB_{kz} + A - \tilde{a}(ik, iz))\widehat{u}(k, z)$ , which implies that  $\widehat{u}(k, z) = 0$  for all  $(k, z) \in \mathbb{Z} \times \mathbb{Z}$ , consequently,  $u = 0$ . Then  $u_1 = u_2$ .  $\square$

## 5 Conclusion

For every function  $f \in L^p(\mathbb{T} \times \mathbb{T}; X)$ , this paper deals with the existence of periodic solutions of neutral integro-differential equation with two variable  $\frac{\partial}{\partial x} \frac{\partial}{\partial y} (\mathbf{u}(\mathbf{x}, \mathbf{y}) - \mathbf{B}\mathbf{u}(\mathbf{x} - \mathbf{r}, \mathbf{y} - \mathbf{r})) + \mathbf{A}\mathbf{u}(\mathbf{x}, \mathbf{y}) = \int_{-\infty}^{\mathbf{x}} \int_{-\infty}^{\mathbf{y}} \mathbf{a}(\mathbf{x} - \mathbf{s}, \mathbf{y} - \xi)\mathbf{u}(\mathbf{s}, \xi)\mathbf{d}\xi\mathbf{d}\mathbf{s} + \mathbf{f}(\mathbf{x}, \mathbf{y})$ . Using the method of Fourier transform of equation **((1.1))** and  $L^p$ -multipliers of linear operators. the results obtained in this paper generalizes the result of Bahloul [4].

## Competing Interests

The author declare that no competing interests exist.

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