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Existence and uniqueness of functional differential equations with n delay

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Abstract

In this paper we give a necessary and sufficient conditions for the existence and uniqueness of periodic solutions of functional differential equations with n delay $\frac{d}{dt}x(t) = Ax(t) + \sum_{j=1}^n Bx(t - r_j) + f(t)$. The conditions are obtained in terms of R-boundedness of operator valued Fourier multipliers.

Mathematics Subject Classification: xxxxxx

Keywords: functional differential equations with n delay, R-bounded.

1 Introduction

Let A and B be two closed linear operators defined on a Banach space X with domains $D(A)$ and $D(B)$, respectively such that $D(A) \subset D(B)$. In this paper we show existence and uniqueness of solutions for the following differential equation with n delay

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + \sum_{j=1}^n Bx(t - r_j) + f(t) \\ x(0) = x(2\pi). \end{cases} \quad (1)$$

where $f \in L^p([-r_{2\pi}, 0], X)$ for some $1 \leq p < \infty$, $r_{2\pi} = 2\pi N$ ($N \in \mathbb{N}$) and we suppose B is bounded. The theory of operator-valued Fourier multipliers has attracted the attention of many papers in recent years. For example, this theory was used in [1] to obtain results about equations $\frac{dx(t)}{dt} = Ax(t) + f(t)$, and in [11] to obtain results about delay equation $\frac{dx(t)}{dt} = Ax(t) + F(x_t) + f(t)$. In [6], S.Bu studied L^p -Maximal Regularity of Degenerate delay Equations with Periodic Conditions. We note that in the special case when $B = 0$, maximal regularity of Eq. (1) has been studied by Arendt and Bu in L^p -spaces case and Besov spaces case [[1], [2]], Bu and Kim in TriebelLizorkin spaces case [8]. The corresponding integro-differential equations were treated by Keyantuo

and Lizama [[17], [18]], Bu and Fang [7]. In this paper, we characterize the existence and uniqueness for the n delay equation (1) under the condition that X is a UMD space. Here the operator A is not necessarily the generator of a C_0 -semigroup. We use the operator valued multiplier Fourier method. The organisation of this work is as follows : In section 2, we present preliminary results on UMD spaces and L^p -multiplier. In section 3, we study the existence of periodic strong solution for Eq.(1) with finite delay. In section 4, we give the main abstract result (theorem [4.2]) of this work.

1) for every $f \in L^p(\mathbb{T}; X); 1 < p < \infty$, there exists a unique 2π -periodic strong L^p -solution of Eq. (1).

2) $(ikI - A - \sum_{j=1}^n B_{j,k})$ has bounded invertible for all $k \in \mathbb{Z}$ and $\{ik(ikI - A - \sum_{j=1}^n B_{j,k})^{-1}\}_{k \in \mathbb{Z}}$ is R-bounded.

2 Preliminary Notes

Let X be a Banach Space. Firstly, we denote By \mathbb{T} the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. There is an identification between functions on \mathbb{T} and 2π -periodic functions on \mathbb{R} . We consider the interval $[0, 2\pi)$ as a model for \mathbb{T} .

Definition 2.1. *A Banach space X is said to be UMD space if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for all $1 < p < \infty$.*

Example 2.2. : [9]

1. Any Hilbert space is an UMD space.
2. $L^p(0,1)$ are UMD spaces for every $1 < p < \infty$.
3. Any closed subspace of a UMD space is a UMD space.

Definition 2.3. [1]

A family of operators $T = (T_j)_{j \in \mathbb{N}^*} \subset B(X, Y)$ is called R-bounded (**Rademacher bounded or randomized bounded**), if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $n \in \mathbb{N}, T_j \in T, x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$ -valued random variables r_j on a probability space (Ω, M, μ) the inequality

$$\left\| \sum_{j=1}^n r_j T_j x_j \right\|_{L^p(0,1;Y)} \leq C \left\| \sum_{j=1}^n r_j x_j \right\|_{L^p(0,1;X)}$$

is valid. The smallest C is called R-bounded of $(T_j)_{j \in \mathbb{N}^*}$ and it is denoted by $R_p(T)$.

Definition 2.4. [11]

For $1 \leq p < \infty$, a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset \mathbf{B}(X, Y)$ is said to be an L^p -multiplier if for each $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

Proposition 2.5. [1, Proposition 1.11] Let X be a Banach space and $\{M_k\}_{k \in \mathbb{Z}}$ be an L^p -multiplier, where $1 \leq p < \infty$. Then the set $\{M_k\}_{k \in \mathbb{Z}}$ is R -bounded.

Theorem 2.6. (Marcinkiewicz operator-valued multiplier Theorem). Let X, Y be UMD spaces and $\{M_k\}_{k \in \mathbb{Z}} \subset B(X, Y)$. If the sets $\{M_k\}_{k \in \mathbb{Z}}$ and $\{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}}$ are R -bounded, then $\{M_k\}_{k \in \mathbb{Z}}$ is an L^p -multiplier for $1 < p < \infty$.

We observe that the condition of R -boundedness for $(M_k)_{k \in \mathbb{Z}}$ is necessary.

Remark 2.7. [13]

Let $f \in L^1(\mathbb{T}; X)$. If $g(t) = \int_0^t f(s)ds$ and $k \in \mathbb{Z}, k \neq 0$, then

$$\hat{g}(k) = \frac{i}{k} \hat{f}(0) - \frac{i}{k} \hat{f}(k)$$

3 A criterion for periodic solutions

Notation 3.1. Let $k \in \mathbb{Z}$. Denote by $B_{j,k} := e^{-ikr_j} B$, $\Delta_k = (ikI - A - \sum_{j=1}^n B_{j,k})$ and $\sigma_{\mathbb{Z}}(\Delta) := \{k \in \mathbb{Z} : \Delta_k \text{ is not bijective}\}$
 $H^{1,p}(\mathbb{T}; X) = \{u \in L^p(\mathbb{T}, X) : \exists v \in L^p(\mathbb{T}, X), \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}$

Definition 3.2. Let $f \in L^p(\mathbb{T}; X)$. A function $x \in H^{1,p}(\mathbb{T}; X)$ is said to be a 2π -periodic strong L^p -solution of Eq. (1) if $x(t) \in D(A)$ for all $t \geq 0$ and Eq. (1) holds almost every where.

Lemma 3.3. [1, Lemme 2.1] Let $1 \leq p < \infty$ and $u, v \in L^p(\mathbb{T}; X)$. Then the following assertions are equivalent:

- (i) $\int_0^{2\pi} v(s)ds = 0$ and there exists $x \in X$ such that $u(t) = x + \int_0^t v(s)ds$.
- (ii) $\hat{v}(k) = ik\hat{u}(k)$ for any $k \in \mathbb{Z}$.

Definition 3.4. For $1 \leq p < \infty$, we say that a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset \mathbf{B}(X, Y)$ is an $(L^p, H^{1,p})$ -multiplier, if for each $f \in L^p(\mathbb{T}, X)$ there exists $u \in H^{1,p}(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

Lemma 3.5. Let $1 \leq p < \infty$ and $(M_k)_{k \in \mathbb{Z}} \subset \mathbf{B}(X)$ ($\mathbf{B}(X)$ is the set of all bounded linear operators from X to X). Then the following assertions are equivalent:

- (i) $(M_k)_{k \in \mathbb{Z}}$ is an $(L^p, H^{1,p})$ -multiplier.
- (ii) $(ikM_k)_{k \in \mathbb{Z}}$ is an (L^p, L^p) -multiplier.

Proposition 3.6. Let A be a closed linear operator defined on an UMD space X . Suppose that $\sigma_{\mathbb{Z}}(\Delta) = \emptyset$. Then the following assertions are equivalent :

- (i) $\{ik(ikI - A - \sum_{j=1}^n B_{j,k})^{-1}\}_{k \in \mathbb{Z}}$ is an L^p -multiplier for $1 < p < \infty$
- (ii) $\{ik(ikI - A - \sum_{j=1}^n B_{j,k})^{-1}\}_{k \in \mathbb{Z}}$ is R -bounded.

Proof. By [1, Proposition 1.11] it follows that (i) implies (ii). Conversely, define $M_k = ik(C_k - A)^{-1}$, where $C_k = ikI - \sum_{j=1}^n B_{j,k}$. By Theorem 2.6 is sufficient to prove that the set $(k(M_{k+1} - M_k))_{k \in \mathbb{Z}}$ is R-bounded. We claim first that the set $(\sum_{j=1}^n B_{j,k})_{k \in \mathbb{Z}}$ is R-bounded. since given $x_j \in D(A)$ we have :

$$\begin{aligned} \left\| \sum_{l=1}^m r_l \left(\sum_{j=1}^n B_{j,l} \right) x_l \right\|_{L^p(0,1;X)}^p &= \int_0^1 \left\| \sum_{l=1}^m r_l(t) B \left(\sum_{j=1}^n e^{-ilr_j} x_l \right) \right\|_X^p dt \\ &= \int_0^1 \left\| B \left(\sum_{l=1}^m r_l(t) \sum_{j=1}^n e^{-ilr_j} x_l \right) \right\|_X^p dt \\ &\leq \|B\|^p \int_0^1 \left\| \sum_{l=1}^m r_l(t) \sum_{j=1}^n e^{-ilr_j} x_l \right\|_X^p dt \end{aligned}$$

By (Lemma 1.7, [1]) we obtain that

$$\left\| \sum_{l=1}^m r_l \left(\sum_{j=1}^n B_{j,l} \right) x_l \right\|_{L^p(0,1;X)}^p \leq 2n^p \|B\|^p \int_0^1 \left\| \sum_{l=1}^m r_l(t) x_l \right\|_X^p dt$$

We conclude that

$$\left\| \sum_{l=1}^m r_l \left(\sum_{j=1}^n B_{j,l} \right) x_l \right\|_{L^p(0,1;X)} \leq 2^{1/p} n \|B\|.$$

and the claim is proved. Next. We note the following identities

$$\begin{aligned} k[M_{k+1} - M_k] &= k[i(k+1)(C_{k+1} - AD)^{-1} - ik(C_k - AD)^{-1}] \\ &= k(C_{k+1} - AD)^{-1}[i(k+1)(C_k - AD) - ik(C_{k+1} - AD)](C_k - AD)^{-1} \\ &= k(C_{k+1} - AD)^{-1}[ik(C_k - C_{k+1}) + i(C_k - A)](C_k - AD)^{-1} \\ &= k(C_{k+1} - AD)^{-1}[ik(C_k - C_{k+1})(C_k - AD)^{-1} + iI] \\ &= \frac{-ik}{k+1} M_{k+1}(C_k - C_{k+1})M_k + \frac{k}{k+1} M_{k+1}. \end{aligned}$$

We have

$$C_k - C_{k+1} = -iI + \sum_{j=1}^n B e^{-ikr_j} (1 - e^{-ir_j}).$$

Since products and sums of R-bounded sequences is R-bounded [11, Remark 2.2]. Then $\{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}}$ is R-bounded and by theorem 2.6, $\{M_k\}_{k \in \mathbb{Z}}$ is an L^p -multiplier. \square

Theorem 3.7. *Let X be a Banach space. Suppose that for every $f \in L^p(\mathbb{T}; X)$ there exists a unique strong solution of Eq. (1) for $1 \leq p < \infty$. Then*

1. for every $k \in \mathbb{Z}$ the operator $\Delta_k = (ikI - A - \sum_{j=1}^n B_{j,k})$ has bounded inverse
2. $\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is R -bounded.

Before to give the proof of Theorem (3.7), we need the following Lemma.

Lemma 3.8. *if $(ikI - A - \sum_{j=1}^n B_{j,k}(x)) = 0$ for all $k \in \mathbb{Z}$, then $u(t) = e^{ikt}x$ is a 2π -periodic strong L^p -solution of the following equation (1) corresponding to the function $f = 0$.*

Proof. $(ikI - A - \sum_{j=1}^n B_{j,k}(x)) = 0 \Rightarrow ikx = Ax + \sum_{j=1}^n B_{j,k}x$.

We have $u(t) = e^{ikt}x$ then

$$u'(t) = ik e^{ikt}x = e^{ikt}(ikx) = e^{ikt}[Ax + \sum_{j=1}^n B_{j,k}x] = Au(t) + \sum_{j=1}^n Bu(t - r_j).$$

Proof of Theorem 3.7 1) Let $k \in \mathbb{Z}$ and $y \in X$. Then for $f(t) = e^{ikt}y$, there exists $x \in H^{1,p}(\mathbb{T}; X)$ such that:

$$\frac{d}{dt}x(t) = Ax(t) + \sum_{j=1}^n Bx(t - r_j) + f(t)$$

Taking Fourier transform, by Lemma 3.3 we have :

$$\hat{x}'(k) = ik\hat{x}(k) = A\hat{x}(k) + \sum_{j=1}^n B_{j,k}\hat{x}(k) + \hat{f}(k).$$

Then we obtain : $(ikI - A - \sum_{j=1}^n B_{j,k})\hat{x}(k) = \hat{f}(k) = y \Rightarrow (ikI - A - \sum_{j=1}^n B_{j,k})$ is surjective.

If $(ikI - A - \sum_{j=1}^n B_{j,k})u = 0$, then by Lemma 3.8 $x(t) = e^{ikt}u$ is a 2π -periodic strong L^p -solution of Eq. (1) corresponding to the function $f = 0$ Hence $x(t) = 0$ and $u = 0$ then $(ikI - A - \sum_{j=1}^n B_{j,k})$ is injective.

2) Let $f \in L^p(\mathbb{T}, X)$. By hypothesis, there exists a unique $x \in H^{1,p}(\mathbb{T}, X)$ such that the Eq. (1) is valid. Taking Fourier transforms, we deduce that $\hat{x}(k) = (ikI - A - \sum_{j=1}^n B_{j,k})^{-1}\hat{f}(k)$ for all $k \in \mathbb{Z}$. Hence

$$ik\hat{x}(k) = ik(ikI - A - \sum_{j=1}^n B_{j,k})^{-1}\hat{f}(k)$$

On the other hand, since $x \in H^{1,p}(\mathbb{T}, X)$, there exists $v \in L^p(\mathbb{T}, X)$ such that $\hat{v}(k) = ik\hat{x}(k) = ik(ikI - A - \sum_{j=1}^n B_{j,k})^{-1}\hat{f}(k)$ i.e $\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is an L^p -multiplier. Then $\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is R -bounded. \square

4 Existence of mild solutions of Eq. (1)

It is well known that in many important applications the operator A is the infinitesimal generator of C_0 -semigroup $(T(t))_{t \geq 0}$ on the space X . Let A be a generator of a C_0 -semigroup $(T(t))_{t \geq 0}$.

Definition 4.1. Assume that A generates a C_0 -semigroup $(T(t))_{t \geq 0}$ on X . A function x is called a mild solution of Eq. (1) if :

$$x(t) = T(t)\varphi + \int_0^t T(t-s) \left(\sum_{j=1}^n Bx(s-r_j) + f(s) \right) ds \text{ for } 0 \leq t \leq 2\pi.$$

Remark 4.2. [14, Remark 4.2]

Let $(T(t))_{t \geq 0}$ be the C_0 -semigroup generated by A . If $g : [0, a] \rightarrow X$ is a continuous function, then $\int_0^t \int_0^s T(t-\xi)g(\xi)d\xi ds \in D(A)$ and

$$A \int_0^t \int_0^s T(t-\xi)g(\xi)d\xi ds = \int_0^t (T(t-s) - I)g(s)ds \text{ for all } 0 \leq t \leq a.$$

Lemma 4.3. [10]

Assume that A generates a C_0 -semigroup $(T(t))_{t \geq 0}$ on X , if x is a mild solution of Eq. (1) then

$$x(t) = \varphi + A \int_0^t x(s)ds + \int_0^t \left(\sum_{j=1}^n Bx(s-r_j) + f(s) \right) ds \text{ for } 0 \leq t \leq 2\pi.$$

Theorem 4.4. Assume that A generates a C_0 -semigroup $(T(t))_{t \geq 0}$ on X and $f \in L^p(\mathbb{T}, X)$ for some $1 \leq p < \infty$, if x is a mild solution of Eq. (1). Then

$$(ikI - A - \sum_{j=1}^n B_{j,k})\hat{x}(k) = \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

Proof. Let x be a mild solution of Eq. (1). Then by Lemma 4.3, we have

$$x(t) = \varphi + A \int_0^t x(s)ds + \int_0^t \left(\sum_{j=1}^n Bx(s-r_j) + f(s) \right) ds$$

For $t = 2\pi$, we have

$$x(2\pi) = \varphi + A \int_0^{2\pi} x(s)ds + \int_0^{2\pi} \left(\sum_{j=1}^n Bx(s-r_j) + f(s) \right) ds;$$

Since: $x(2\pi) = \varphi$, then

$$\begin{aligned}
& A \int_0^{2\pi} x(s) ds + \int_0^{2\pi} \left(\sum_{j=1}^n Bx(s-r_j) + f(s) \right) ds = 0 \\
& \Rightarrow \frac{1}{2\pi} A \int_0^{2\pi} x(s) ds + \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{j=1}^n Bx(s-r_j) + f(s) \right) ds = 0 \\
& \Rightarrow \frac{1}{2\pi} A \int_0^{2\pi} x(s) ds + \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^n Bx(s-r_j) ds + \frac{1}{2\pi} \int_0^{2\pi} f(s) ds = 0 \\
& \Rightarrow \frac{1}{2\pi} A \int_0^{2\pi} e^{-i0s} x(s) ds + \frac{1}{2\pi} \int_0^{2\pi} e^{-i0s} \sum_{j=1}^n Bx(s-r_j) ds + \frac{1}{2\pi} \int_0^{2\pi} e^{-i0s} f(s) ds = 0 \\
& \Rightarrow (0 - A - \sum_{j=1}^n B_{j,0}) \hat{x}(0) = \hat{f}(0),
\end{aligned}$$

which shows that the assertion holds for $k = 0$.

Now, define

$$v(t) = \int_0^t x(s) ds$$

and

$$g(t) = x(t) - \varphi - \int_0^t \left(\sum_{j=1}^n Bx(s-r_j) + f(s) \right) ds$$

by Remark 2.7 We have:

$$\begin{aligned}
\hat{v}(k) &= \frac{i}{k} \hat{x}(0) - \frac{i}{k} \hat{x}(k) \\
A\hat{v}(k) &= \frac{i}{k} A\hat{x}(0) - \frac{i}{k} A\hat{x}(k)
\end{aligned}$$

and

$$\begin{aligned}
\hat{g}(k) &= \hat{x}(k) - \left[\frac{i}{k} G_0 \hat{x}(0) - \frac{i}{k} G_k \hat{x}(k) \right] - \left[\frac{i}{k} \hat{f}(0) - \frac{i}{k} \hat{f}(k) \right] \\
&= \hat{x}(k) - \frac{i}{k} G_0 \hat{x}(0) + \frac{i}{k} G_k \hat{x}(k) - \frac{i}{k} \hat{f}(0) + \frac{i}{k} \hat{f}(k)
\end{aligned}$$

□

Corollary 4.5. *Assume that A generates a C_0 -semigroup $(T(t))_{t \geq 0}$ on X and let $f \in L^p(\mathbb{T}, X) : 1 \leq p < \infty$ and x be a mild solution of Eq. (1). If $(ikI - A - \sum_{j=1}^n B_{j,k})$ has a bounded inverse. Then $(ikI - A - \sum_{j=1}^n B_{j,k})$ is an L^p -multiplier.*

Proof. Let $f \in L^p(\mathbb{T}, X)$ then from Theorem (4.4) we have:

$\hat{x}(k) = (ikD_k - AD_k - G_k)^{-1} \hat{f}(k)$ for all $f \in L^p(\mathbb{T}; X)$, then $(ikI - A - \sum_{j=1}^n B_{j,k})^{-1}$ is an L^p -multiplier. □

5 Main Result

Our main result in this work is to establish that the converse of theorem (3.7) and corollary (4.5) is true, provided X is an UMD space.

Theorem 5.1. (Fejer Theorem) : Let $f \in L^p(\mathbb{T}, X)$. Then

$$f = \lim_{n \rightarrow +\infty} \sigma_n(f)$$

where $\sigma_n(f) = \frac{1}{n+1} \sum_{m=0}^n \sum_{k=-m}^m e_k \hat{f}(k)$, with $e_k(t) = e^{ikt}$.

Theorem 5.2. Let X be an UMD space and $A : D(A) \subset X \rightarrow X$ be a closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$.

- 1) for every $f \in L^p(\mathbb{T}, X)$ there exists a unique strong L^p -solution of Eq.(1).
- 2) $\sigma_{\mathbb{Z}}(\Delta) = \phi$ and $\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is R -bounded.

Proof. 1 \Rightarrow 2) see Theorem 3.7.

1 \Leftarrow 2) Let $f \in L^p(\mathbb{T}; X)$. Define $\Delta_k = (ikI - A - \sum_{j=1}^n B_{j,k})$,

By Proposition 3.6, the family $\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is an L^p -multiplier it is equivalent to

the family $\{\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is an L^p -multiplier that maps $L^p(\mathbb{T}; X)$ into $H^{1,p}(\mathbb{T}; X)$, namely there exists $x \in H^{1,p}(\mathbb{T}, X)$ such that

$$\hat{x}(k) = \Delta_k^{-1} \hat{f}(k) = (ikI - A - \sum_{j=1}^n B_{j,k})^{-1} \hat{f}(k) \quad (2)$$

In particular, $x \in L^p(\mathbb{T}; X)$ and there exists $v \in L^p(\mathbb{T}; X)$ such that $\hat{v}(k) = ik\hat{x}(k)$

$$\hat{x}'(k) := \hat{v}(k) = ik\hat{x}(k) \quad (3)$$

By Theorem 5.1 we have for $j \in \{1 \dots n\}$

$$x(t - r_j) = \lim_{l \rightarrow +\infty} \frac{1}{l+1} \sum_{m=0}^l \sum_{k=-m}^m e^{ikt} e^{-ikr_j} \hat{x}(k)$$

Then, since B is bounded linear

$$\sum_{j=1}^n Bx(t - r_j) = \lim_{l \rightarrow +\infty} \frac{1}{l+1} \sum_{m=0}^l \sum_{k=-m}^m e^{ikt} \left(\sum_{j=1}^n B_{j,k} \hat{x}(k) \right)$$

By (2) and (3) we have:

$$\hat{x}'(k) = ik\hat{x}(k) = A\hat{x}(k) + \sum_{j=1}^n B_{j,k} \hat{x}(k) + \hat{f}(k), \text{ for all } k \in \mathbb{Z}$$

Then using that A and B are closed we conclude that $x(t) \in D(A)$ [[1], Lemma 3.1] and from the uniqueness theorem of Fourier coefficients that

$$x'(t) = Ax(t) + \sum_{j=1}^n Bx(t - r_j) + f(t).$$

We have $x \in H^{1,p}(\mathbb{T}, X)$ then by lemma 3.3, $x(0) = x(2\pi)$, then the Eq. (1) has a unique 2π -periodic strong L^p -solution. \square

Theorem 5.3. *Let $1 \leq p < \infty$. Assume that A generates a C_0 -semigroup $(T(t))_{t \geq 0}$ on X . If $\sigma_Z(\Delta) = \emptyset$ and $(ikI - A - \sum_{j=1}^n B_{j,k})^{-1}$ is an L^p -multiplier Then there exists a unique mild solution periodic of Eq. (1).*

Proof. For $f \in L^p(\mathbb{T}; X)$ we define

$$f_l(t) = \frac{1}{l+1} \sum_{m=0}^l \sum_{k=-m}^m e^{ikt} \hat{f}(k)$$

By the Fejér Theorem we can assert that $f_l \rightarrow f$ as $l \rightarrow \infty$ for the norm in $L^p(\mathbb{T}; X)$. We have $(ikI - A - \sum_{j=1}^n B_{j,k})^{-1}$ is an L^p -multiplier then there exists $x \in L^p(\mathbb{T}; X)$ such that $\hat{x}(k) = (ikI - A - \sum_{j=1}^n B_{j,k})^{-1} \hat{f}(k)$ put

$$x_l(t) = \frac{1}{l+1} \sum_{m=0}^l \sum_{k=-m}^m e^{ikt} (ikI - A - \sum_{j=1}^n B_{j,k})^{-1} \hat{f}(k)$$

Using again the Fejér Theorem we obtain that $x_n(t) \rightarrow x(t)$ (as $n \rightarrow \infty$) and $x_n(t)$ is strong L^p -solution of Eq. (1) and $x_n(t)$ verified

$$x_l(t) = T(t)\varphi_l + \int_0^t T(t-s) \left(\sum_{j=1}^n Bx_l(s - r_j) + f_l(s) \right) ds \quad (4)$$

With $t = 2\pi$ we obtain

$$x_l(2\pi) = T(2\pi)\varphi_l + \int_0^{2\pi} T(2\pi - s) \left(\sum_{j=1}^n Bx_l(s - r_j) + f_l(s) \right) ds.$$

from which we infer that the sequence $(\varphi_l)_n$ is convergent to some element φ as $l \rightarrow \infty$ ($\varphi_l = x_l(0) = x_l(2\pi)$). Moreover, φ satisfies the condition

$$\varphi = T(2\pi)\varphi + \int_0^{2\pi} T(2\pi - s) \left(\sum_{j=1}^n Bx(s - r_j) + f(s) \right) ds. \quad (5)$$

Taking the limit as l goes to infinity in (4), we can write

$$x(t) = T(t)\varphi + \int_0^t T(t-s) \left(\sum_{j=1}^n Bx(s-r_j) + f(s) \right) ds := g(t)$$

$g(2\pi) = T(2\pi)y + \int_0^{2\pi} T(2\pi-s) \left(\sum_{j=1}^n Bx(s-r_j) + f(s) \right) ds \stackrel{(5)}{=} \varphi = g(0)$
 Then $x(2\pi) = \varphi \Rightarrow x(2\pi) = x(0)$, we conclude that x is a 2π -periodic mild solution of Eq. (1). \square

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