Existence and uniqueness of functional differential equations with n delay

Bahloul Rachid

Existence and uniqueness of functional differential equations with n delay

Bahloul Rachid
Faculty of Sciences and Technology, Fez, Morocco
bahloulr33@hotmail.com

Abstract

In this paper we give a necessary and suffcient conditions for the existence and uniqueness of periodic solutions of functional differential equations with n delay $\frac{d}{dt}x(t) = Ax(t) + \sum_{j=1}^{n} Bx(t - r_j) + f(t)$. The conditions are obtained in terms of R-boundedness of operator valued Fourier multipliers.

Mathematics Subject Classification: xxxxx

Keywords: functional differential equations with n delay, R-bounded.

1 Introduction

Let A and B be two closed linear operators defined on a Banach space X with domains D(A) and D(B), respectively such that $D(A) \subset D(B)$. In this paper we show existence and uniqueness of solutions for the following differential equation with n delay

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + \displaystyle\sum_{j=1}^{n} Bx(t - r_j) + f(t) \\ \\ x(0) = x(2\pi). \end{cases} \tag{1}$$

where $f \in L^p([-r_{2\pi}, 0], X)$ for some $1 \leq p < \infty, r_{2\pi} = 2\pi N$ $(N \in \mathbb{N})$ and we suppose B is bounded. The theory of operator-valued Fourier multipliers has attracted the attention of many papers in recent years. For example, this theory was used in [1] to obtain results about equations $\frac{dx(t)}{dt} = Ax(t) + f(t)$, and in [11] to obtain results about delay equation $\frac{dx(t)}{dt} = Ax(t) + F(x_t) + f(t)$. In [6], S.Bu studied L^p-Maximal Regularity of Degenerate delay Equations with Periodic Conditions. We note that in the special case when $B = 0$, maximal regularity of Eq. (1) has been studied by Arendt and Bu in L^p-spaces case and Besov spaces case [[1], [2]], Bu and Kim in TriebelLizorkin spaces case [8]. The corresponding integro-differential equations were treated by Keyantuo

and Lizama [[17], [18]], Bu and Fang [7]. In this paper, we characterize the existence and uniqueness for the n delay equation (1) under the condition that X is a UMD space. Here the operator A is not necessarily the generator of a C_0-semigroup. We use the operator valued multiplier Fourier method. The organisation of this work is as follows : In section 2, we present preliminary results on UMD spaces and L^P-multiplier. In section 3, we study the existence of periodic strong solution for Eq.(1) with finite delay. In section 4, we give the main abstract result (theorem [4.2]) of this work.
1) for every $f \in L^p(\mathbb{T}; X); 1 < p < \infty$, there exists a unique 2π-periodic strong L^p-solution of Eq. (1).
2) $(ikI - A - \sum_{j=1}^{n} B_{j,k})$ has bounded invertible for all $k \in \mathbb{Z}$ and $\{ik(ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

2 Preliminary Notes

Let X be a Banach Space. Firstly, we denote By $\mathbb{T}$ the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. There is an identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$. We consider the interval $[0, 2\pi)$ as a model for $\mathbb{T}$.

Definition 2.1. *A Banach space X is said to be UMD space if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for all $1 < p < \infty$.*

Example 2.2. *: [9]*
1.Any Hilbert space is an UMD space.
2. $L^p(0.1)$ are UMD spaces for every $1 < p < \infty$.
3. Any closed subspace of a UMD space is a UMD space.

Definition 2.3. *[1]*
*A family of operators $T = (T_j)_{j\in\mathbb{N}^*} \subset B(X, Y)$ is called R-bounded (**Rademacher bounded or randomized bounded**), if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $n \in N, T_j \in T, x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables r_j on a probability space (Ω, M, μ) the inequality*

$$\left\| \sum_{j=1}^{n} r_j T_j x_j \right\|_{L^p(0,1;Y)} \leq C \left\| \sum_{j=1}^{n} r_j x_j \right\|_{L^p(0,1;X)}$$

is valid. The smallest C is called R-bounded of $(T_j)_{j\in\mathbb{N}^*}$ and it is denoted by $R_p(T)$.

Definition 2.4. *[11]*
For $1 \leq p < \infty$, a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathbf{B}(X, Y)$ is said to be an L^p-multiplier if for each $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

Proposition 2.5. *[1, Proposition 1.11] Let X be a Banach space and $\{M_k\}_{k\in\mathbb{Z}}$ be an L^p-multiplier, where $1 \leq p < \infty$. Then the set $\{M_k\}_{k\in\mathbb{Z}}$ is R-bounded.*

Theorem 2.6. ***(Marcinkiewicz operator-valud multiplier Theorem).*** *Let X, Y be UMD spaces and $\{M_k\}_{k\in\mathbb{Z}} \subset B(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ are*
R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.

We observe that the condition of R-boundedness for $(M_k)_{k\in\mathbb{Z}}$ is necessary.

Remark 2.7. *[13]*
Let $f \in L^1(\mathbb{T};X)$. If $g(t) = \int_0^t f(s)ds$ and $k \in \mathbb{Z}, k \neq 0$, then

$$\hat{g}(k) = \tfrac{i}{k}\hat{f}(0) - \tfrac{i}{k}\hat{f}(k)$$

3 A criterion for periodic solutions

Notation 3.1. *Let $k \in \mathbb{Z}$. Denote by $B_{j,k} := e^{-ikr_j}B$,*
$\Delta_k = (ikI - A - \sum_{j=1}^n B_{j,k})$ and $\sigma_{\mathbb{Z}}(\Delta) := \{k \in \mathbb{Z} : \Delta_k \text{is not bijective}\}$
$H^{1,p}(\mathbb{T};X) = \{u \in L^p(\mathbb{T},X) : \exists v \in L^p(\mathbb{T},X), \hat{v}(k) = ik\hat{u}(k) \text{for all } k \in \mathbb{Z}\}$

Definition 3.2. *Let $f \in L^p(\mathbb{T};X)$. A function $x \in H^{1,p}(\mathbb{T};X)$ is said to be a 2π-periodic strong L^p-solution of Eq. (1) if $x(t) \in D(A)$ for all $t \geq 0$ and Eq. (1) holds almost every where.*

Lemma 3.3. *[1, Lemme 2.1] Let $1 \leq p < \infty$ and $u, v \in L^p(\mathbb{T};X)$. Then the following assertions are equivalent:*
(i) $\int_0^{2\pi} v(s)ds = 0$ and there exists $x \in X$ such that $u(t) = x + \int_0^t v(s)ds$.
(ii) $\hat{v}(k) = ik\hat{u}(k)$ for any $k \in \mathbb{Z}$.

Definition 3.4. *For $1 \leq p < \infty$, we say that a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathbf{B}(X,Y)$ is an $(L^p, H^{1,p})$-multiplier, if for each $f \in L^p(\mathbb{T},X)$ there exists $u \in H^{1,p}(\mathbb{T},Y)$ such that $\hat{u}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$.*

Lemma 3.5. *Let $1 \leq p < \infty$ and $(M_k)_{k\in\mathbb{Z}} \subset \mathbf{B}(X)$ ($\mathbf{B}(X)$ is the set of all bounded linear operators from X to X). Then the following assertions are equivalent:*
(i) $(M_k)_{k\in\mathbb{Z}}$ is an $(L^p, H^{1,p})$-multiplier.
(ii) $(ikM_k)_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

Proposition 3.6. *Let A be a closed linear operator defined on an UMD space X. Suppose that $\sigma_{\mathbb{Z}}(\Delta) = \phi$ .Then the following assertions are equivalent :*
(i) $\{ik(ikI - A - \sum_{j=1}^n B_{j,k})^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$
(ii) $\{ik(ikI - A - \sum_{j=1}^n B_{j,k})^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

Proof. By [1, Proposition 1.11] it follows that (i) implies (ii). Conversely, define $M_k = ik(C_k - A)^{-1}$, where $C_k = ikI - \sum_{j=1}^{n} B_{j,k}$, By Theorem 2.6 is sufficient to prove that the set $(k(M_{k+1} - M_k))_{k\in\mathbb{Z}}$ is R-bounded. We claim first that the set $(\sum_{j=1}^{n} B_{j,k})_{k\in\mathbb{Z}}$ is R-bounded.
since given $x_j \in D(A)$ we have :

$$\begin{aligned}\left\|\sum_{l=1}^{m} r_l(\sum_{j=1}^{n} B_{j,l})x_l\right\|^p_{L^p(0,1;X)} &= \int_0^1 \left\|\sum_{l=1}^{m} r_l(t)B(\sum_{j=1}^{n} e^{-ilr_j}x_l)\right\|^p_X dt \\ &= \int_0^1 \left\|B(\sum_{l=1}^{m} r_l(t)\sum_{j=1}^{n} e^{-ilr_j}x_l)\right\|^p_X dt \\ &\leq ||B||^p \int_0^1 \left\|\sum_{l=1}^{m} r_l(t)\sum_{j=1}^{n} e^{-ilr_j}x_l\right\|^p_X dt\end{aligned}$$

By (Lemma 1.7, [1]) we obtain that

$$\left\|\sum_{l=1}^{m} r_l(\sum_{j=1}^{n} B_{j,l})x_l\right\|^p_{L^p(0,1;X)} \leq 2n^p||B||^p \int_0^1 \left\|\sum_{l=1}^{m} r_l(t)x_l\right\|^p_X dt$$

We conclude that

$$\left\|\sum_{l=1}^{m} r_l(\sum_{j=1}^{n} B_{j,l})x_l\right\|_{L^p(0,1;X)} \leq 2^{1/p}n||B||.$$

and the claim is proved. Next. We note the following identities

$$\begin{aligned}k\left[M_{k+1} - M_k\right] &= k\left[i(k+1)(C_{k+1} - AD)^{-1} - ik(C_k - AD)^{-1}\right] \\ &= k(C_{k+1} - AD)^{-1}[i(k+1)(C_k - AD) - ik(C_{k+1} - AD)](C_k - AD)^{-1} \\ &= k(C_{k+1} - AD)^{-1}[ik(C_k - C_{k+1}) + i(C_k - A)](C_k - AD)^{-1} \\ &= k(C_{k+1} - AD)^{-1}[ik(C_k - C_{k+1})(C_k - AD)^{-1} + iI] \\ &= \frac{-ik}{k+1}M_{k+1}(C_k - C_{k+1})M_k + \frac{k}{k+1}M_{k+1}.\end{aligned}$$

We have

$$C_k - C_{k+1} = -iI + \sum_{j=1}^{n} Be^{-ikr_j}(1 - e^{-ir_j}).$$

Since products and sums of R-bounded sequences is R-bounded [11, Remark 2.2]. Then $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded and by theorem 2.6, $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. □

Theorem 3.7. *Let X be a Banach space. Suppose that for every $f \in L^p(\mathbb{T}; X)$ there exists a unique strong solution of Eq. (1) for $1 \leq p < \infty$. Then*

1. *for every* $k \in \mathbb{Z}$ *the operator* $\Delta_k = (ikI - A - \sum_{j=1}^{n} B_{j,k})$ *has bounded inverse*

2. $\left\{ik\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ *is R-bounded.*

Before to give the proof of Theorem (3.7), we need the following Lemma.

Lemma 3.8. *if* $(ikI - A - \sum_{j=1}^{n} B_{j,k}(x)) = 0$ *for all* $k \in \mathbb{Z}$*, then* $u(t) = e^{ikt}x$ *is a* 2π*-periodic strong* L^p*-solution of the following equation (1) corresponing to the function* $f = 0$*.*

Proof. $(ikI - A - \sum_{j=1}^{n} B_{j,k}(x)) = 0 \Rightarrow ikx = Ax + \sum_{j=1}^{n} B_{j,k}x$.
We have $u(t) = e^{ikt}x$ then

$$u'(t) = ike^{ikt}x = e^{ikt}(ikx) = e^{ikt}[Ax + \sum_{j=1}^{n} B_{j,k}x] = Au(t) + \sum_{j=1}^{n} Bu(t - r_j).$$

Proof of Theorem 3.7 1) Let $k \in \mathbb{Z}$ and $y \in X$. Then for $f(t) = e^{ikt}y$, there exists $x \in H^{1,p}(\mathbb{T}; X)$ such that:

$$\frac{d}{dt}x(t) = Ax(t) + \sum_{j=1}^{n} Bx(t - r_j) + f(t)$$

Taking Fourier transform, by Lemma 3.3 we have :

$$\hat{x'}(k) = ik\hat{x}(k) = A\hat{x}(k) + \sum_{j=1}^{n} B_{j,k}\hat{x}(k) + \hat{f}(k).$$

Then we obtain : $(ikI - A - \sum_{j=1}^{n} B_{j,k})\hat{x}(k) = \hat{f}(k) = y \Rightarrow (ikI - A - \sum_{j=1}^{n} B_{j,k})$ is surjective.
If $(ikI - A - \sum_{j=1}^{n} B_{j,k})u = 0$, then by Lemma 3.8 $x(t) = e^{ikt}u$ is a 2π-periodic strong L^p-solution of Eq. (1) corresponing to the function $f = 0$ Hence $x(t) = 0$ and $u = 0$ then $(ikI - A - \sum_{j=1}^{n} B_{j,k})$ is injective.
2) Let $f \in L^p(\mathbb{T}, X)$. By hypothesis, there exists a unique $x \in H^{1;p}(\mathbb{T}, X)$ such that the Eq. (1) is valid. Taking Fourier transforms, we deduce that $\hat{x}(k) = (ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1}\hat{f}(k)$ for all $k \in \mathbb{Z}$. Hence

$$ik\hat{x}(k) = ik(ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1}\hat{f}(k)$$

On the other hand, since $x \in H^{1;p}(\mathbb{T}, X)$, there exists $v \in L^p(\mathbb{T}, X)$ such that $\hat{v}(k) = ik\hat{x}(k) = ik(ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1}\hat{f}(k)$ i.e $\{ik\Delta_k^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. Then $\{ik\Delta_k^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded. □

4 Existence of mild solutions of Eq. (1)

It is well known that in many important applications the operator A is the infinitesimal generator of C_0-semigroup $(T(t))_{t\geq 0}$ on the space X. Let A be a generator of a C_0-semigroup $(T(t))_{t\geq 0}$.

Definition 4.1. *Assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X. A function x is called a mild solution of Eq. (1) if :*

$$x(t) = T(t)\varphi + \int_0^t T(t-s)(\sum_{j=1}^{n} Bx(s-r_j) + f(s))ds \text{ for } 0 \leq t \leq 2\pi.$$

Remark 4.2. *[14, Remark 4.2]*
Let $(T(t))_{t\geq 0}$ be the C_0-semigroup generated by A. If $g : [0,a] \to X$ is a continuous function, then $\int_0^t \int_0^s T(t-\xi)g(\xi)d\xi ds \in D(A)$ and

$$A\int_0^t \int_0^s T(t-\xi)g(\xi)d\xi ds = \int_0^t (T(t-s)-I)g(s)ds \text{ for all } 0 \leq t \leq a.$$

Lemma 4.3. *[10]*
Assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X, if x is a mild solution of Eq. (1) then

$$x(t) = \varphi + A\int_0^t x(s)ds + \int_0^t (\sum_{j=1}^{n} Bx(s-r_j) + f(s))ds \text{ for } 0 \leq t \leq 2\pi.$$

Theorem 4.4. *Assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X and $f \in L^p(\mathbb{T}, X)$ for some $1 \leq p < \infty$, if x is a mild solution of Eq. (1). Then*

$$(ikI - A - \sum_{j=1}^{n} B_{j,k})\hat{x}(k) = \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

Proof. Let x be a mild solution of Eq. (1). Then by Lemma 4.3, we have

$$x(t) = \varphi + A\int_0^t x(s)ds + \int_0^t (\sum_{j=1}^{n} Bx(s-r_j) + f(s))ds$$

For $t = 2\pi$, we have

$$x(2\pi) = \varphi + A\int_0^{2\pi} x(s)ds + \int_0^{2\pi} (\sum_{j=1}^{n} Bx(s-r_j) + f(s))ds;$$

Since: $x(2\pi) = \varphi$, then

$$A\int_0^{2\pi} x(s)ds + \int_0^{2\pi}(\sum_{j=1}^{n} Bx(s-r_j) + f(s))ds = 0$$

$$\Rightarrow \frac{1}{2\pi}A\int_0^{2\pi} x(s)ds + \frac{1}{2\pi}\int_0^{2\pi}(\sum_{j=1}^{n} Bx(s-r_j) + f(s))ds = 0$$

$$\Rightarrow \frac{1}{2\pi}A\int_0^{2\pi} x(s)ds + \frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{n} Bx(s-r_j)ds + \frac{1}{2\pi}\int_0^{2\pi} f(s))ds = 0$$

$$\Rightarrow \frac{1}{2\pi}A\int_0^{2\pi} e^{-i0s}x(s)ds + \frac{1}{2\pi}\int_0^{2\pi} e^{-i0s}\sum_{j=1}^{n} Bx(s-r_j)ds + \frac{1}{2\pi}\int_0^{2\pi} e^{-i0s}f(s))ds = 0$$

$$\Rightarrow (0 - A - \sum_{j=1}^{n} B_{j,0})\hat{x}(0) = \hat{f}(0),$$

which shows that the assertion holds for $k = 0$.
Now, define

$$v(t) = \int_0^t x(s)ds$$

and

$$g(t) = x(t) - \varphi - \int_0^t (\sum_{j=1}^{n} Bx(s-r_j) + f(s))ds$$

by Remark 2.7 We have:

$$\hat{v}(k) = \frac{i}{k}\hat{x}(0) - \frac{i}{k}\hat{x}(k)$$
$$A\hat{v}(k) = \frac{i}{k}A\hat{x}(0) - \frac{i}{k}A\hat{x}(k)$$

and

$$\begin{aligned}\hat{g}(k) &= \hat{x}(k) - [\frac{i}{k}G_0\hat{x}(0) - \frac{i}{k}G_k\hat{x}(k)] - [\frac{i}{k}\hat{f}(0) - \frac{i}{k}\hat{f}(k)] \\ &= \hat{x}(k) - \frac{i}{k}G_0\hat{x}(0) + \frac{i}{k}G_k\hat{x}(k) - \frac{i}{k}\hat{f}(0) + \frac{i}{k}\hat{f}(k)\end{aligned}$$

□

Corollary 4.5. *Assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X and let $f \in L^p(\mathbb{T}, X) : 1 \leq p < \infty$ and x be a mild solution of Eq. (1). If $(ikI - A - \sum_{j=1}^{n} B_{j,k})$ has a bounded inverse. Then $(ikI - A - \sum_{j=1}^{n} B_{j,k})$ is an L^p-multiplier.*

Proof. Let $f \in L^p(\mathbb{T}, X)$ then from Theorem (4.4) we have:
$\hat{x}(k) = (ikD_k - AD_k - G_k)^{-1}\hat{f}(k)$ for all $f \in L^p(\mathbb{T}; X)$, then
$(ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1}$ is an L^p-multiplier. □

5 Main Result

Our main result in this work is to establish that the converse of theorem (3.7) and corollary (4.5) is true, provided X is an UMD space.

Theorem 5.1. *(Fejer Theorem) : Let $f \in L^p(\mathbb{T}, X)$. Then*

$$f = \lim_{n \to +\infty} \sigma_n(f)$$

where $\sigma_n(f) = \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \hat{f}(k)$, with $e_k(t) = e^{ikt}$.

Theorem 5.2. *Let X be an UMD space and $A : D(A) \subset X \to X$ be a closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$.*
1) for every $f \in L^p(\mathbb{T}, X)$ there exists a unique strong L^p-solution of Eq.(1).
2) $\sigma_{\mathbb{Z}}(\Delta) = \phi$ and $\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is R-bounded.

Proof. $1 \Rightarrow 2$) see Theorem 3.7.
$1 \Leftarrow 2$) Let $f \in L^p(\mathbb{T}; X)$. Define $\Delta_k = (ikI - A - \sum_{j=1}^{n} B_{j,k})$,
By Proposition 3.6, the family $\left\{ik\Delta_k^{-1}\right\}_{k \in \mathbb{Z}}$ is an L^p-multiplier it is equivalent to
the family $\left\{\Delta_k^{-1}\right\}_{k \in \mathbb{Z}}$ is an L^p-multiplier that maps $L^p(\mathbb{T}; X)$ into $H^{1,p}(\mathbb{T}; X)$, namely there exists $x \in H^{1,p}(\mathbb{T}, X)$ such that

$$\hat{x}(k) = \Delta_k^{-1} \hat{f}(k) = (ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1} \hat{f}(k) \tag{2}$$

In particular, $x \in L^p(\mathbb{T}; X)$ and there exists $v \in L^p(\mathbb{T}; X)$ such that $\hat{v}(k) = ik\hat{x}(k)$

$$\widehat{x'}(k) := \hat{v}(k) = ik\hat{x}(k) \tag{3}$$

By Theorem 5.1 we have for $j \in \{1...n\}$

$$x(t - r_j) = \lim_{l \to +\infty} \frac{1}{l+1} \sum_{m=0}^{l} \sum_{k=-m}^{m} e^{ikt} e^{-ikr_j} \hat{x}(k)$$

Then, since B is bounded linear

$$\sum_{j=1}^{n} Bx(t - r_j) = \lim_{l \to +\infty} \frac{1}{l+1} \sum_{m=0}^{l} \sum_{k=-m}^{m} e^{ikt} (\sum_{j=1}^{n} B_{j,k} \hat{x}(k))$$

By (2) and (3) we have:

$$\widehat{x'}(k) = ik\hat{x}(k) = A\hat{x}(k) + \sum_{j=1}^{n} B_{j,k} \hat{x}(k) + \hat{f}(k), \text{ for all } k \in \mathbb{Z}$$

Then using that A and B are closed we conclude that $x(t) \in D(A)$ [[1], Lemma 3.1] and from the uniqueness theorem of Fourier coefficients that

$$x'(t) = Ax(t) + \sum_{j=1}^{n} Bx(t - r_j) + f(t).$$

We have $x \in H^{1,p}(\mathbb{T}, X)$ then by lemma 3.3, $x(0) = x(2\pi)$, then the Eq. (1) has a unique 2π-periodic strong L^p-solution. □

Theorem 5.3. *Let $1 \leq p < \infty$. Assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X. If $\sigma_Z(\Delta) = \emptyset$ and $(ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1}$ is an L^p-multiplier Then there exists a unique mild solution periodic of Eq. (1).*

Proof. For $f \in L^p(\mathbb{T}; X)$ we define

$$f_l(t) = \frac{1}{l+1} \sum_{m=0}^{l} \sum_{k=-m}^{m} e^{ikt} \hat{f}(k)$$

By the Fejér Theorem we can assert that $f_l \to f$ as $l \to \infty$ for the norm in $L^p(\mathbb{T}; X)$. We have $(ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1}$ is an L^p-multiplier then there exists $x \in L^p(\mathbb{T}; X)$ such that $\hat{x}(k) = (ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1} \hat{f}(k)$
put

$$x_l(t) = \frac{1}{l+1} \sum_{m=0}^{l} \sum_{k=-m}^{m} e^{ikt} (ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1} \hat{f}(k)$$

Using again the Fejér Theorem we obtain that $x_n(t) \to x(t)$ (as $n \to \infty$) and $x_n(t)$ is strong L^p-solution of Eq. (1) and $x_n(t)$ verified

$$x_l(t) = T(t)\varphi_l + \int_0^t T(t-s)(\sum_{j=1}^{n} Bx_l(s - r_j) + f_l(s))ds \tag{4}$$

With $t = 2\pi$ we obtain

$$x_l(2\pi) = T(2\pi)\varphi_l + \int_0^{2\pi} T(2\pi - s)(\sum_{j=1}^{n} Bx_l(s - r_j) + f_l(s))ds.$$

from which we infer that the sequence $(\varphi_l)_n$ is convergent to some element φ as $l \to \infty$($\varphi_l = x_l(0) = x_l(2\pi)$). Moreover, φ satisfies the condition

$$\varphi = T(2\pi)\varphi + \int_0^{2\pi} T(2\pi - s)(\sum_{j=1}^{n} Bx(s - r_j) + f(s))ds. \tag{5}$$

Taking the limit as l goes to infinity in (4), we can write

$$x(t) = T(t)\varphi + \int_0^t T(t-s)(\sum_{j=1}^{n} Bx(s-r_j) + f(s))ds := g(t)$$

$g(2\pi) = T(2\pi)y + \int_0^{2\pi} T(2\pi - s)(\sum_{j=1}^{n} Bx(s-r_j) + f(s))ds \overbrace{=}^{(5)} \varphi = g(0)$
Then $x(2\pi) = \varphi \Rightarrow x(2\pi) = x(0)$, we conclude that x is a 2π- periodic mild solution of Eq. (1). □

Acknowledgements. This is a text of acknowledgements.

References

[1] W.Arend and S.Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity, Math.Z. 240, (2002), 311-343.*

[2] W. Arendt, S. Bu, Operator-valued Fourier multipliers on periodic Besov spaces and applications, Proc. Edinburgh Math. Soc. 47, (1), (2004), 15-33.

[3] R.Bahloul, M.Bahaj and O.Sidki,Periodic solutions of degenerate equations with finite delay in UMD space, journal of Advances in Dynamical Systems and Application. ISSN 0973-5321, Volume 10, Number 1, (2015) pp. 23-31.

[4] R.Bahloul, Periodic solutions of differential equations with two variable in vector-valued function space, Asian Journal of Mathematics and Computer Research,12(1): 44-53, 2016 ISSN: 2395-4205 (P), ISSN: 2395-4213 (O).

[5] K. Ezzinbi, R.Bahloul et O.Sidki, Periodic Solutions in UMD spaces for some neutral partial function differential equations, Advances in Pure Mathematics, (2016), 6, 713-726, http: dx.doi.org/10.4236/apm.2016.61008

[6] S.Bu, *"L^p-Maximal regularity of degenerate delay equations with Periodic Conditions."* , Banach J.Math.Anal. 8(2014), no. 2, 49-59;

[7] S. Bu and Y. Fang, Maximal regularity for integro-differential equations on periodic Triebel-Lisorkin spaces, Taiwanese J. Math. 12 (2009), no.2, 281292.

[8] S. Bu and J. Kim, Operator-valued Fourier multipliers on peoriodic Triebel spaces, Acta Math. Sinica, English Series 17 (2004), 1525.

[9] J.Bourgain, Some remarks on Banach spaces in which martingale differences sequences are unconditional. Arkiv Math. 21, (1983), 163-168.

[10] Butzer P.L. and Westphal, An access to fractional differentiation via fractional difference quotients, Lecture Notes in Math. 457, 116-145, 1975.

[11] C.Lizama, Fourier multipliers and perodic solutions of delay equatons in Banach spaces, J.Math.Anal.Appl. 324,(2) (2006), 921-933.

[12] C.Lizama and V. Poblete, Periodic Solutions of Fractional Differential Equations With Delay, J.Evol.Equ.11, (2011), 57-70.

[13] Hernan R.Henriquez, Michelle Pierri and Andrea Prokopczyk Periodic Solutions of abstract neutral functional differential equations, J. Math. Ana. Appl. 385, (2012), 608 - 621

[14] M. Girardi, L. Weis, Criteria for R-boundedness of operator families, in : Evolution Equations, in : Lect. Notes Pure Appl. Math., vol. 234, Dekker, New York, 2003, pp. 203221.

[15] Y.Hino, T.Naito,N. Van Minh and J.S.Shin, Almost periodic solution of Differential Equations in Banach Spaces, Taylor and Francis, London,2002.

[16] V. Keyantuo and C. Lizama, Fourier multipliers and integro-differential equations in Banach spaces, J. London Math. Soc. 69 (2004), 737750.

[17] V. Keyantuo and C. Lizama, Maximal regularity for a class of integro-differential equations with infinite delay in Banach spaces, Studia Math.168 (2005), 2550.

[18] V. Keyantuo, C. Lizama, Mild well-posedness of abstract differential equations, in : H. Amann, W. Arendt, M. Hieber, F. Neubrander, S.Nicaise, J. von Below (Eds.), Functional Analysis andEvolution Equations, Birkhuser, Basel, 2008, pp. 371387.

[19] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, 1983.

[20] H. Triebel, Fractals and Spectra. Related to Fourier Analysis and Function Spaces, Monogr. Math., vol. 78, Birkhuser, Basel,1983.

[21] H. Triebel, Theory of Function Spaces, Monogr. Math., vol. 91, Birkhuser, Basel, 1997.

[22] J.Wu, Theory and Applications of Partial Differential Equations, Appl, Math .Sci. 119, Springer-verlag, 1969.

[23] L. Weis, Operator-valued Fourier multiplier theorems and maximal Lp-regularity, Math. Ann. 319 (2001) 735758.

[24] C. Yu, G. Gao. Some results on a class of fractional functional differential equations. Commun.Appl.Nonlinear Anal.,11 (3) (2004) 6775.

[25] X. Zhang. Some results of linear fractional order time-delay system.Appl. Math. Comput. 197 (2008),407411.

[26] F. Zimmermann, On vector-valued Fourier multiplier theorems, Studia Math. 93 (1989) 201222.

Received: Month xx, 20xx