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Insensitizing controls for a semilinear parabolic equation: a numerical approach

Franck Boyer ^{*} Víctor Hernández-Santamaría ^{†‡} Luz de Teresa [§]

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Abstract

In this paper, we study the insensitizing control problem in the discrete setting of finite-differences. We prove the existence of a control that insensitizes the norm of the observed solution of a 1-D semi discrete parabolic equation. We derive a (relaxed) observability estimate that yields a controllability result for the cascade system arising in the insensitizing control formulation. Moreover, we deal with the problem of computing numerical approximations of insensitizing controls for the heat equation by using the Hilbert Uniqueness Method (HUM). We present various numerical illustrations.

Keywords: Insensitizing controls, semi discrete Carleman estimates, observability, controllability, HUM.
MSC2010: 35K15; 65M06; 93C20

1 Introduction

1.1 The insensitizing control problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded and open set with boundary $\partial\Omega \in \mathcal{C}^2$. Let $T > 0$ and ω be an open and non empty subset of Ω . We consider the following parabolic equation

$$\begin{cases} \partial_t y - \Delta y + f(y) = \mathbf{1}_\omega v + \xi & \text{in } Q = \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ y(0) = y_0 + \tau w_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where f is a globally Lipschitz-continuous function, ξ and y_0 are given in $L^2(Q)$ and $L^2(\Omega)$, respectively. In (1.1), $y = y(x, t)$ is the state and $v = v(x, t)$ is a control function supported in ω . We may mention the dependence of y on the data by writing $y[y_0, \xi, v, w_0, \tau]$ if necessary.

The data of equation (1.1) are incomplete in the following sense:

- $w_0 \in L^2(\Omega)$ is unknown and $|w_0|_{L^2(\Omega)} = 1$,
- $\tau \in \mathbb{R}$ is unknown and small enough.

Let Ψ be a differentiable functional defined on the set of solutions to (1.1). We say that the control v insensitizes $\Psi(y)$ for the initial data y_0 and the source term ξ if

$$\left. \frac{\partial \Psi(y[y_0, \xi, v, w_0, \tau])}{\partial \tau} \right|_{\tau=0} = 0, \quad \forall w_0 \in L^2(\Omega). \quad (1.2)$$

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When (1.2) holds the functional Ψ is locally insensitive to the perturbations of the initial data. There are several possible choices of Ψ depending on the considered applications. In this paper, we will only consider the most standard choice of Ψ which is to consider the square of the L^2 -norm of the state y in some observation subset $\mathcal{O} \subset \Omega$, namely,

$$\Psi(y) := \frac{1}{2} \int_0^T \int_{\mathcal{O}} y^2 dx dt. \quad (1.3)$$

It is by now well known that, for this particular functional, the insensitivity condition (1.2) is equivalent to a null-control problem for a coupled system of parabolic PDEs. This equivalence is given in the following result.

Proposition 1.1 *Let us consider the following cascade system of semilinear parabolic equations*

$$\begin{cases} \partial_t y - \Delta y + f(y) = \mathbf{1}_\omega v + \xi & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \quad (1.4)$$

$$\begin{cases} -\partial_t q - \Delta q + f'(y)q = \mathbf{1}_{\mathcal{O}} y & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T) = 0 & \text{in } \Omega. \end{cases} \quad (1.5)$$

Then, a control v satisfies the insensitivity condition (1.2) for the functional (1.3) and the problem (1.1), if and only if the associated solution of (1.4)-(1.5) satisfies

$$q(0) = 0. \quad (1.6)$$

Observe that (1.6) is precisely a null controllability property for the cascade system (1.4)-(1.5). However, this situation is more complex than a standard control problem. In fact, two main difficulties arise. On the one hand, the control v acts indirectly on the equation satisfied by q by means of the localized coupling term $\mathbf{1}_{\mathcal{O}} y$. On the other hand, note that (1.4) is forward in time while (1.5) is backward in time. The irreversibility of the heat equation imposes additional difficulties that do not appear in more classical cascade systems in which both equations evolve along the same direction of time (see [17]).

This problem, originally addressed by Lions [23], has been thoroughly studied in different contexts. In [2], the authors relaxed condition (1.2) as follows: given $\varepsilon > 0$, the control v is said to ε -insensitize Ψ if

$$\left| \frac{\partial \Psi(y[y_0, \xi, v, w_0, \tau])}{\partial \tau} \right|_{\tau=0} \leq \varepsilon |w_0|_{L^2(\Omega)}, \quad \forall w_0 \in L^2(\Omega).$$

As in the previous proposition, we can show that the ε -insensitivity property is equivalent to the condition $|q(0)|_{L^2(\Omega)} \leq \varepsilon$ for the solution of (1.5). Hence, this problem corresponds to an approximate controllability problem for the coupled system (1.4)-(1.5), instead of a null-control problem. In this context, the authors proved the existence of such controls in the presence of both unknown initial and boundary data, when $\mathcal{O} \cap \omega \neq \emptyset$. In [25], two main results are given. On the one hand, the author proved that we cannot expect the existence of insensitizing controls for every $y_0 \in L^2(\Omega)$ when $\Omega \setminus \bar{\omega} \neq \emptyset$, even in the linear case where $f = 0$. On the other hand, for $y_0 = 0$ and a suitable hypothesis on the source term ξ , the author proved the existence of insensitizing controls such that (1.2) holds as soon as $\mathcal{O} \cap \omega \neq \emptyset$. The main step of the proof is to consider the linearized system

$$\begin{cases} \partial_t y - \Delta y + ay = \mathbf{1}_\omega v + \xi & \text{in } Q, \\ -\partial_t q - \Delta q + bq = \mathbf{1}_{\mathcal{O}} y & \text{in } Q, \\ y = q = 0 & \text{on } \Sigma, \\ y(0) = 0 & \text{in } \Omega, \\ q(T) = 0 & \text{in } \Omega, \end{cases}$$

with $a, b \in L^\infty(Q)$ and the associated adjoint system

$$\begin{cases} -\partial_t z - \Delta z + az = \mathbf{1}_O p & \text{in } Q, \\ \partial_t p - \Delta p + bp = 0 & \text{in } Q, \\ p = z = 0 & \text{on } \Sigma, \\ p(0) = p_0 & \text{in } \Omega, \\ z(T) = 0 & \text{in } \Omega, \end{cases}$$

for which the following observability inequality is proved, for some $\mathcal{M} > 0$,

$$\iint_Q e^{-\frac{\mathcal{M}}{t}} |z|^2 \leq C_{obs} \|z\|_{L^2(\omega \times (0, T))}^2. \quad (1.7)$$

With this estimate, a controllability result is obtained for the linearized system, and a precise analysis of the dependence of C_{obs} and \mathcal{M} with respect to a and b permits to conclude in the nonlinear case by a fixed point argument. Note that, since z satisfies an equation in which p acts as a source term, one cannot use usual energy estimates for z to obtain, from (1.7), a bound on $\|z(0)\|_{L^2}$ by the observation term $\|z\|_{L^2(\omega \times (0, T))}$. This is the main reason why this analysis is restricted to the case $y_0 = 0$.

This result was generalized in [3] and [4] to nonlinearities with certain superlinear growth and nonlinear terms depending on the state y and its gradient. Regarding the class of initial data y_0 that can be insensitized, the work of de Teresa and Zuazua [26] gives different results of positive and negative nature. More recently, there are many works within the context of insensitizing controls for other functionals rather than (1.3) and equations of different nature. For instance, in [19], the author considers a functional involving the gradient of the state for a linear heat system and in [18] treats the case of the curl of the solution for a Stokes system. In [10] and [20], the authors studied the insensitizing controls of the Navier-Stokes equation and the Boussinesq system.

1.2 Statement of the problem

In this article, we are interested in studying the insensitizing control problem from another perspective. The main goal of this paper is to present numerical methods as well as associated theoretical and numerical results concerning the computation of insensitizing controls for semilinear parabolic problems.

The outline of the paper is as follows. First, we build a semi discrete approximation of the PDE under study and by means of semi discrete Carleman estimates taken from [6] we deduce a ‘‘relaxed’’ observability inequality for the linearized equation, which is uniform with respect to the discretization parameter (see Section 2). This allows us to establish the existence of suitable insensitizing semi discrete controls within this framework for the initial nonlinear problem we are interested in (see Section 3). We then propose in Section 4 a fully discrete version of this approach that will be the heart of our computational code. To perform the actual computation of the controls we will use the penalized Hilbert Uniqueness Method (HUM) approach (as discussed for instance in [5]) and we present numerical results in Section 5.

In order to simplify the presentation, we will only consider here the 1D case but it is worth mentioning that the techniques and results given below still hold in any dimension as soon as we restrict ourselves to finite difference schemes on Cartesian grids (see [7]).

From now on, we consider the following 1-D semi discrete system

$$\begin{cases} \partial_t y^{\mathfrak{M}} + \mathcal{A}^{\mathfrak{M}} y^{\mathfrak{M}} + f(y^{\mathfrak{M}}) = \mathbf{1}_\omega v^{\mathfrak{M}} + \xi^{\mathfrak{M}} & \text{in } \mathbb{R}^{\mathfrak{M}} \times (0, T), \\ y^{\partial \mathfrak{M}} = 0 & \text{in } (0, T), \\ y^{\mathfrak{M}}(0) = y_0^{\mathfrak{M}} + \tau w_0^{\mathfrak{M}}. \end{cases} \quad (1.8)$$

where f is a C^1 globally Lipschitz-continuous function, with $f(0) = 0$. Here $\mathcal{A}^{\mathfrak{M}}$ is the discrete approximation of $\mathcal{A} := -\partial_x^2$ on a mesh \mathfrak{M} whose step size is denoted by $h^{\mathfrak{M}}$, $\partial \mathfrak{M}$ denotes the boundary cells of the mesh and $\mathbb{R}^{\mathfrak{M}}$ is the space of discrete (in space) functions defined on \mathfrak{M} . These notions will be precisely introduced in the Section 1.3. As described in the introduction, we are interested in proving the existence of uniformly bounded semi discrete controls that insensitize the functional

$$\Psi(y^{\mathfrak{M}}) := \frac{1}{2} \int_0^T \int_O |y^{\mathfrak{M}}|^2 dx dt, \quad (1.9)$$

where $y^{\mathfrak{M}}$ is the solution to (1.8). Following the ideas of the continuous case, it can be proved that the insensitizing control problem for (1.8) is equivalent to finding bounded families of semi discrete controls $(v^{\mathfrak{M}})_{\mathfrak{M}}$ such that the solution $(y^{\mathfrak{M}}, q^{\mathfrak{M}})$ of the coupled problem

$$\begin{cases} \partial_t y^{\mathfrak{M}} + \mathcal{A}^{\mathfrak{M}} y^{\mathfrak{M}} + f(y^{\mathfrak{M}}) = \mathbf{1}_{\omega} v^{\mathfrak{M}} + \xi^{\mathfrak{M}} & \text{in } \mathbb{R}^{\mathfrak{M}} \times (0, T), \\ -\partial_t q^{\mathfrak{M}} + \mathcal{A}^{\mathfrak{M}} q^{\mathfrak{M}} + f'(y^{\mathfrak{M}}) q^{\mathfrak{M}} = \mathbf{1}_{\mathcal{O}} y^{\mathfrak{M}} & \text{in } \mathbb{R}^{\mathfrak{M}} \times (0, T), \\ y^{\partial \mathfrak{M}} = q^{\partial \mathfrak{M}} = 0 & \text{in } (0, T), \\ y^{\mathfrak{M}}(0) = y_0^{\mathfrak{M}}, \quad q^{\mathfrak{M}}(T) = 0, \end{cases} \quad (1.10)$$

satisfies the condition

$$q^{\mathfrak{M}}(0) = 0.$$

To accomplish this, we follow the strategy outlined in [25], but taking into account the particularities associated with the semi discrete nature of the problem. In fact, in a first step, we will study controllability properties of the linearized version of (1.10). Then, a fixed point argument allow us to obtain the controllability result for the nonlinear system.

1.3 Discrete settings and notation

Following [6] and [9], we establish the framework of the discrete setting to clarify the exposition of the results. In particular, the notation introduced on those articles, allows to carry out most of the computations in a very intuitive manner and enable us to emulate as close as possible the continuous insensitizing problem as addressed for instance in [4], [25].

As mentioned above, we restrict in this paper our analysis to semi discrete systems in one dimension space even though the proposed strategy can be adapted to multi dimensional Cartesian discretizations (see [7]).

Let us set $\Omega = (0, L)$ and consider the elliptic operator $\mathcal{A} = -\partial_x^2$ with homogeneous Dirichlet boundary conditions. We introduce finite differences approximations of the operator \mathcal{A} . Let $0 = x_0 < x_1 < \dots < x_N < x_{N+1} = L$. We refer to this discretization as to the primal mesh $\mathfrak{M} := \{x_i : i = 1, \dots, N\}$. We define $|\mathfrak{M}| := N$ and the boundary points are denoted by $\partial \mathfrak{M} = \{x_0, x_{N+1}\} = \{0, L\}$.

We set $h_{i+\frac{1}{2}} = x_{i+1} - x_i$ and $x_{i+\frac{1}{2}} = (x_{i+1} + x_i)/2$, $i = 0, \dots, N$. The step size is denoted by $h^{\mathfrak{M}} = \max_i h_{i+\frac{1}{2}}$.

We introduce the dual mesh $\overline{\mathfrak{M}} := \{x_{i+\frac{1}{2}} : i = 0, \dots, N\}$ and we set $h_i = (h_{i+\frac{1}{2}} + h_{i-\frac{1}{2}})/2 = x_{i+\frac{i}{2}} - x_{i-\frac{1}{2}}$, $i = 1, \dots, N$.

We denote by $\mathbb{R}^{\mathfrak{M}}$ and $\mathbb{R}^{\overline{\mathfrak{M}}}$ the sets of discrete functions defined on \mathfrak{M} and $\overline{\mathfrak{M}}$, respectively. If $u \in \mathbb{R}^{\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}}$), we denote by u_i (resp. $u_{i+\frac{1}{2}}$) its value corresponding to x_i (resp. $x_{i+\frac{1}{2}}$). For $u \in \mathbb{R}^{\mathfrak{M}}$ we define

$$u^{\mathfrak{M}} = \sum_{i=1}^N \mathbf{1}_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]} u_i \in L^\infty(\Omega).$$

Since no confusion is possible, by abuse of notation, we shall often write u instead of $u^{\mathfrak{M}}$. Additionally, for $u \in \mathbb{R}^{\mathfrak{M}}$ we define

$$\int_{\Omega} u := \int_{\Omega} u^{\mathfrak{M}}(x) dx = \sum_{i=1}^N h_i u_i.$$

For some $u \in \mathbb{R}^{\mathfrak{M}}$, we shall need to associate boundary conditions $u^{\partial \mathfrak{M}} = \{u_0, u_{N+1}\}$. The set of such extended discrete functions is denoted by $\mathbb{R}^{\mathfrak{M} \cup \partial \mathfrak{M}}$. Homogeneous Dirichlet boundary conditions then consist in the choice $u_0 = u_{N+1} = 0$, in short $u^{\partial \mathfrak{M}} = 0$ or even $u|_{\partial \Omega} = 0$.

For $u \in \mathbb{R}^{\overline{\mathfrak{M}}}$ we define

$$u^{\overline{\mathfrak{M}}} = \sum_{i=0}^N \mathbf{1}_{[x_i, x_{i+1}]} u_{i+\frac{1}{2}} \in L^\infty(\Omega).$$

As above, for $u \in \mathbb{R}^{\overline{\mathfrak{M}}}$, we set

$$\int_{\Omega} u := \int_{\Omega} u^{\overline{\mathfrak{M}}}(x) dx = \sum_{i=0}^N h_{i+\frac{1}{2}} u_{i+\frac{1}{2}}.$$

In the same manner, we define the following L^2 -inner product on \mathbb{R}^{m} (resp. $\mathbb{R}^{\overline{\text{m}}}$)

$$(u, v)_{L^2(\Omega)} = \int_{\Omega} u v = \int_{\Omega} u^{\text{m}}(x) v^{\text{m}}(x) dx.$$

$$\left(\text{resp. } (u, v)_{L^2(\Omega)} = \int_{\Omega} u v = \int_{\Omega} u^{\overline{\text{m}}}(x) v^{\overline{\text{m}}}(x) dx. \right)$$

The associated norms will be denoted by $|u|_{L^2(\Omega)}$. We use similar definitions and notations for functions restricted to the domains \mathcal{O} and ω .

For semi discrete functions $u(t)$ in \mathbb{R}^{m} (or $\mathbb{R}^{\overline{\text{m}}}$) for all $t \in (0, T)$, we define the following L^2 -norm

$$\|u\|_{L^2(Q)} = \left(\int_0^T \int_{\Omega} |u(t)|^2 dt \right)^{1/2}.$$

Endowing the space of semi discrete functions $L^2(0, T; \mathbb{R}^{\text{m}})$ (resp. $L^2(0, T; \mathbb{R}^{\overline{\text{m}}})$) with this norm yields a Hilbert space.

Analogously, we shall define the space $L^\infty(0, T; \mathbb{R}^{\text{m}})$ (resp. $L^\infty(0, T; \mathbb{R}^{\overline{\text{m}}})$) by means of the norm

$$\|u\|_{L^\infty(Q)} = \text{ess sup}_{t \in (0, T)} \left(\sup_{i \in \{1, \dots, N\}} |u_i(t)| \right).$$

Similarly, we shall use such norms for spaces of semi discrete functions defined on (or restricted to) the domains $\omega \times (0, T)$ or $\mathcal{O} \times (0, T)$.

In order to manipulate the discrete functions, we define the following translation operators for indices:

$$(\tau^+ u)_{i+\frac{1}{2}} := u_{i+1}, \quad (\tau^- u)_{i+\frac{1}{2}} := u_i, \quad i = 0, \dots, N.$$

A first-order difference operator D_i and an averaging operator A_i are then given by

$$(Du)_{i+\frac{1}{2}} := \frac{1}{h_{i+\frac{1}{2}}} (\tau^+ u - \tau^- u)_{i+\frac{1}{2}},$$

$$(Au)_{i+\frac{1}{2}} = \bar{u}_{i+\frac{1}{2}} := \frac{1}{2} (\tau^+ u + \tau^- u)_{i+\frac{1}{2}}.$$
(1.11)

Both map $\mathbb{R}^{\text{m} \cup \partial \text{m}}$ into $\mathbb{R}^{\overline{\text{m}}}$.

Likewise, we define on the dual mesh translation operators τ^\pm as follows

$$(\tau^+ u)_i := u_{i+\frac{1}{2}}, \quad (\tau^- u)_i := u_{i-\frac{1}{2}}, \quad i = 1, \dots, N.$$

Then, a difference operator \bar{D} and an averaging operator \bar{A} (both mapping $\mathbb{R}^{\overline{\text{m}}}$ into \mathbb{R}^{m}) are given by

$$(\bar{D}u)_i := \frac{1}{h_i} (\tau^+ u - \tau^- u)_i$$

$$(\bar{A}u)_i = \bar{u}_i := \frac{1}{2} (\tau^+ u + \tau^- u)_i$$
(1.12)

Note that there is no need for boundary conditions here.

A continuous function ψ defined on $\bar{\Omega}$ can be sampled on the primal mesh, that is, $\psi^{\text{m}} = \{\psi(x_i) : i = 1, \dots, N\}$, which we identify to

$$\psi^{\text{m}} = \sum_{i=1}^N \mathbf{1}_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]} \psi_i, \quad \psi_i = \psi(x_i), \quad i = 1, \dots, N.$$

We also set

$$\psi^{\partial \text{m}} = \{\psi(x_0), \psi(x_{N+1})\} = \{\psi(0), \psi(L)\},$$

$$\psi^{\text{m} \cup \partial \text{m}} = \{\psi(x_i) : i = 0, \dots, N+1\}.$$

The function ψ can also be sampled on the dual mesh, i.e., $\psi^{\mathfrak{M}} = \{\psi(x_{i+\frac{1}{2}}) : i = 0, \dots, N\}$, which we identify to

$$\psi^{\mathfrak{M}} = \sum_{i=0}^N \mathbf{1}_{[x_i, x_{i+1}]} \psi_{i+\frac{1}{2}}, \quad \psi_{i+\frac{1}{2}} = \psi(x_{i+\frac{1}{2}}), \quad i = 0, \dots, N.$$

In the sequel, we will use the same symbol ψ for both the continuous function and its sampling on the primal or dual mesh. Indeed, from the context, one will be able to deduce the appropriate sampling. For example, with u defined on the primal mesh \mathfrak{M} , in an expression like $\overline{D}(\rho Du)$ where $\rho : \Omega \rightarrow \mathbb{R}$ is a given function, it is clear that the function ρ is sampled on the dual mesh \mathfrak{M} since Du is defined on this mesh and the operator \overline{D} acts on functions defined on this mesh as well.

Remark 1.2 *In the sequel, we shall only use uniform meshes to simplify the notation. In this case, $h_i = h^{\mathfrak{M}}$ and $h_{i+\frac{1}{2}} = h^{\mathfrak{M}}, \forall i$. Thus, we can write $x_i = ih^{\mathfrak{M}}$ and $x_{i+\frac{1}{2}} = (i + \frac{1}{2})h^{\mathfrak{M}}$. However, the analysis for more general (still somehow regular) meshes is possible, see [7] for a detailed discussion.*

Hereinafter, in order to ease the reading of the computations, we will omit the superscript \mathfrak{M} to refer to discrete variables, and the mesh step $h^{\mathfrak{M}}$ will simply be denoted by h .

With the notation we have introduced, a suitable finite-difference approximation of the elliptic operator $\mathcal{A}y = -\partial_x^2 y$ with homogeneous Dirichlet boundary conditions is $\mathcal{A}^{\mathfrak{M}}y = -\overline{D}(Dy)$ for $y \in \mathbb{R}^{\mathfrak{M} \cup \partial\mathfrak{M}}$ satisfying $y^{\partial\mathfrak{M}} = 0$, so that

$$(\mathcal{A}^{\mathfrak{M}}y)_i = -\frac{1}{h^2} (y_{i+1} - 2y_i + y_{i-1}), \quad i = 1, \dots, N.$$

Note that all our results below can be extended to the case of a non-constant diffusion coefficient $x \mapsto \gamma(x)$ by considering the operator $\mathcal{A}^{\mathfrak{M}} = -\overline{D}(\gamma D \cdot)$. In order to concentrate on the particular difficulties related to the coupling between the forward and backward semi discrete parabolic equations in our problem, we will not consider this generalization in the sequel.

We shall need the following uniform discrete Poincaré inequality (which is valid even for non uniform meshes)

$$|y|_{L^2(\Omega)}^2 \leq C(\mathcal{A}^{\mathfrak{M}}y, y)_{L^2(\Omega)}, \quad \forall y \in \mathbb{R}^{\mathfrak{M} \cup \partial\mathfrak{M}}, \quad y = 0 \text{ on } \partial\mathfrak{M}, \quad (1.13)$$

the right-hand side being the square of the discrete H_0^1 -norm of y in the present framework.

We finally introduce the time-dependent weight $e_{\mathcal{M}}(t) = \exp(\mathcal{M}t^{-1})$ and define the Hilbert space

$$L^2(e_{\mathcal{M}}) = \left\{ \xi \in L^2(0, T; \mathbb{R}^{\mathfrak{M}}) : \iint_Q e_{\mathcal{M}} |\xi|^2 < \infty \right\}, \quad (1.14)$$

endowed with its natural norm.

Remark 1.3 *Any $\xi \in L^2(0, T; \mathbb{R}^{\mathfrak{M}})$ compactly supported in $(0, T] \times \overline{\Omega}$ necessarily belongs to $L^2(e_{\mathcal{M}})$ for any value of \mathcal{M} .*

1.4 Statement of the main results

Using a series of tools developed in [6, 7, 9], we are able to prove (see Theorem 2.1) an observability inequality of the form

$$\iint_Q e^{-\frac{\mathcal{M}}{t}} |z|^2 \leq C_{obs} \left(\|z\|_{L^2(\omega \times (0, T))}^2 + e^{-\frac{C}{h}} |p_0|_{L^2(\Omega)}^2 \right), \quad (1.15)$$

valid for every solution of the adjoint linear system

$$\begin{cases} -\partial_t z + \mathcal{A}^{\mathfrak{M}}z + az = \mathbf{1}_{\mathcal{O}} p & \text{in } \mathbb{R}^{\mathfrak{M}} \times (0, T), \\ \partial_t p + \mathcal{A}^{\mathfrak{M}}p + bp = 0 & \text{in } \mathbb{R}^{\mathfrak{M}} \times (0, T), \\ z = p = 0 & \text{on } \partial\mathfrak{M} \times (0, T), \\ z(T) = 0, \quad p(0) = p_0, \end{cases}$$

with a constant $C > 0$ that only depends on T, ω, \mathcal{O} and on the $L^\infty(0, T; \mathbb{R}^{\mathfrak{M}})$ norms of a and b .

Note that there is an additional term in the right-hand side of the inequality (1.15) as compared with the similar estimate in the continuous setting (1.7) (see also [25, Eq. (8)]). In fact, because of

the presence of this term we refer to it as a *relaxed* observability inequality. Indeed, as discussed in [6], [9], in some cases this term cannot be avoided. This is for instance connected to an obstruction of the null controllability of the semi discrete heat equation, as pointed out by a counter-example in dimension 2 due to O. Kavian, see for instance [27]. The study of relaxed observability estimates for discretized parabolic equations was initiated in [22]. We refer to [5] for a review.

Actually, with the inequality (1.15) we are able to prove that there exists $v \in L^2(0, T; \mathbb{R}^m)$ with $\|v\|_{L^2(\omega \times (0, T))} \leq C$, for some positive constant C not depending on \mathfrak{M} , such that

$$|q(0)|_{L^2(\Omega)} \leq C \sqrt{\phi(h)} \|\xi\|_{L^2(e_{\mathcal{M}})},$$

where $L^2(e_{\mathcal{M}})$ is the weighted space (1.14) and $h \mapsto \phi(h)$ is a function of the discretization parameter such that

$$\liminf_{h \rightarrow 0} \frac{\phi(h)}{e^{-C/h}} > 0. \quad (1.16)$$

This means that we do not exactly achieve null controllability at the discrete level. Nevertheless, we are able to reach small targets $q(0)$ whose size goes to zero as the mesh size $h \rightarrow 0$, at a prescribed rate $\sqrt{\phi(h)}$, with controls that remain uniformly bounded with respect to h . We refer to Section 5 where the choice of $h \mapsto \phi(h)$ is discussed and illustrated in practice.

Thus we speak of $\phi(h)$ -insensitizing controls, which should not be confused with the notion of ε -insensitivity (as discussed in [2], [21]): here, the size of the neighborhood reached by the solution at time T is not fixed, but is a function of the discretization step, which is freely chosen as soon as (1.16) holds.

We now state our main insensitivity result whose proof is given in Section 3.

Theorem 1.4 *Let $f \in C^1(\mathbb{R})$ be globally Lipschitz with $f(0) = 0$. Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exists a positive constant \mathcal{M} depending on Ω , ω , \mathcal{O} and T such that for any mesh \mathfrak{M} with h sufficiently small, for $y_0^m = 0$ and for any $\xi \in L^2(e_{\mathcal{M}})$ and any function ϕ verifying (1.16), one can find a semi discrete control function $v \in L^2(0, T; \mathbb{R}^m)$ uniformly bounded as*

$$\|v\|_{L^2(Q)} \leq C_{obs} \|\xi\|_{L^2(e_{\mathcal{M}})},$$

with C_{obs} given in (2.3), and such that the functional given by (1.9) is $\phi(h)$ -insensitized.

Remark 1.5 *Some remarks are in order:*

- *Roughly speaking, the condition $y_0^m = 0$ is due to the fact that the first equation in (1.10) is forward in time and the second one is backward in time. Most of the results regarding insensitizing controls assume this condition. We refer the reader to [26] for a compendium on the possible initial conditions that can be insensitized. As suggested on that work, the answer is not obvious.*
- *Note that the case $f(0) \neq 0$ is not allowed since it would be equivalent to adding a constant to the source term ξ , but this is not compatible with the condition $\xi \in L^2(e_{\mathcal{M}})$.*
- *The assumption $\omega \cap \mathcal{O} \neq \emptyset$ is essential to prove an observability inequality (see Eq. (2.2) below), which is the main ingredient in the proof of Theorem 1.4. In the continuous and linear case, there are some results on the controllability of non-scalar parabolic systems when $\omega \cap \mathcal{O} = \emptyset$. In [1], the authors proved several null controllability results for a 1-D coupled parabolic system in which both equations are forward in time. In that work, some new interesting phenomena appear, such as the minimal time for controllability or the geometrical dependence of the sets ω and \mathcal{O} .*
- *Also, in [21] the authors prove that in the continuous insensitizing problem for the pure heat equation, the assumption on $\omega \cap \mathcal{O}$ may be omitted as soon as we restrict ourselves to an ε -insensitizing result. The exact insensitivity problem in the general linear/semilinear case remains today as an open problem, both in the continuous and semi discrete case.*
- *Additionally, we may ask to find a control v to ensure simultaneous $\phi(h)$ -null and $\phi(h)$ -insensitizing controls, that is, to impose that the solution (y, q) to (1.10) satisfies*

$$|y(T)|_{L^2(\Omega)} + |q(0)|_{L^2(\Omega)} \leq C \sqrt{\phi(h)} \left(\iint_Q e^{\frac{\mathcal{M}'}{T-t}} |\xi|^2 \right)^{1/2}.$$

for a constant \mathcal{M}' possibly different from \mathcal{M} . As in the continuous problem, this is possible by using the same kind of discrete Carleman estimates that we will use below. Observe however that we need to impose an extra condition on ξ at time $t = T$. See Section 5.2 for some numerical results in this direction.

2 The semi discrete relaxed observability inequality

In this section we prove an observability inequality that is the semi discrete counterpart of the presented in [25] or [4]. This result will be the main tool in the proof of Theorem 1.4. As mentioned above, the $\phi(h)$ -insensitivity problem is equivalent to find a uniformly bounded control v such that

$$|q(0)|_{L^2(\Omega)} \leq C\sqrt{\phi(h)}\|\xi\|_{L^2(\epsilon_{\mathcal{M}})},$$

where (y, q) is the solution to (1.10). It is well known that controllability properties for system (1.10) are related to the observability of the linear adjoint system, in this case, given by

$$\begin{cases} -\partial_t z + \mathcal{A}^{\text{op}} z + az = \mathbf{1}_{\mathcal{O}} p & \text{in } \mathbb{R}^{\text{op}} \times (0, T), \\ \partial_t p + \mathcal{A}^{\text{op}} p + bp = 0 & \text{in } \mathbb{R}^{\text{op}} \times (0, T), \\ z = p = 0 & \text{on } \partial\mathbb{M} \times (0, T), \\ z(T) = 0, \quad p(0) = p_0. \end{cases} \quad (2.1)$$

Thus, the main result in this section is the following:

Theorem 2.1 *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exist positive constants h_0, C_0, C_1 and C_2 such that for all $T > 0$ and all potential functions a and b , under the condition $h \leq \min(h_0, h_1)$ with*

$$h_1 = C_0 \left(1 + \frac{1}{T} + (\|a\|_{\infty}^{2/3} + \|b\|_{\infty}^{2/3}) \right)^{-1},$$

for every $p_0 \in \mathbb{R}^{\text{op}}$, the corresponding solution (z, p) to (2.1) satisfies

$$\iint_Q \exp(-\frac{\mathcal{M}}{t}) |z|^2 dx dt \leq C_{obs}^2 \left(\|z\|_{L^2(\omega \times (0, T))}^2 + e^{-\frac{C_1}{h}} |p_0|_{L^2(\Omega)}^2 \right), \quad (2.2)$$

where

$$C_{obs} = \exp \left[C_2 \left(1 + \frac{1}{T} + \|a\|_{\infty}^{2/3} + \|b\|_{\infty}^{2/3} + T(1 + \|a\|_{\infty} + \|b\|_{\infty}) \right) \right], \quad (2.3)$$

and

$$\mathcal{M} = C_2 \left[1 + T + T(\|a\|_{\infty}^{2/3} + \|b\|_{\infty}^{2/3}) \right]. \quad (2.4)$$

The main tool to prove this theorem is a uniform Carleman estimate for semi discrete parabolic operators. This strategy was originally developed in [9]. The goal is to mimic at the discrete level various techniques from the analysis of PDE control problems.

To this end, it is necessary to introduce an auxiliary function ψ fulfilling the following assumption.

Assumption 2.2 *Let \mathcal{B}_0 be a nonempty open set of $\bar{\Omega}$. Let $\tilde{\Omega}$ be a smooth open and connected neighborhood of $\bar{\Omega}$ in \mathbb{R}^n . The function $x \mapsto \psi(x)$ is in $C^p(\tilde{\Omega}, \mathbb{R})$, p sufficiently large, and satisfies for some $c > 0$*

$$\begin{aligned} \psi &> 0 \quad \text{in } \tilde{\Omega}, & |\nabla \psi| &\geq c \quad \text{in } \tilde{\Omega} \setminus \mathcal{B}_0, \\ \text{and } \partial_{n_x} \psi(x) &\leq -c < 0, & \text{for } x &\in V_{\partial\Omega}, \end{aligned}$$

where $V_{\partial\Omega}$ is a sufficiently small neighborhood of $\partial\Omega$ in $\tilde{\Omega}$, in which the outward unit normal n_x is extended from $\partial\Omega$.

The construction of such function in general smooth domains is classical. Interested readers can see [14, 9] for additional remarks on this function. In our present 1D case, one can simply take a point $x_0 \in \mathcal{B}_0$ and consider $\psi(x) = C - (x - x_0)^2$ for $C > 0$ large enough.

Now, let $K > \|\psi\|_{\infty}$ and set

$$\begin{aligned} \varphi(x) &= e^{\lambda\psi(x)} - e^{\lambda K} < 0, \\ r(t, x) &= e^{s(t)\varphi(x)}, \quad \rho(t, x) = (r(t, x))^{-1} \end{aligned} \quad (2.5)$$

with

$$\begin{aligned} s(t) &= \tau\theta(t), \quad \tau > 0, \\ \theta(t) &= \frac{1}{(t + \delta T)(T + \delta T - t)} \end{aligned}$$

for $0 < \delta < 1/2$. The parameter δ is introduced to avoid singularities at time $t = 0$ and $t = T$ and will be chosen in the course of the proof of the Carleman estimate to be somehow proportional to the mesh size h . Further comments are provided in [9].

We recall below the Carleman estimate for semi discrete parabolic operators of the form $\mathcal{P}_{\pm}^{\mathfrak{M}} = \partial_t \pm \mathcal{A}^{\mathfrak{M}}$. We use the following notation, for any $u \in \mathcal{C}^1([0, T]; \mathbb{R}^{\mathfrak{M} \cup \partial \mathfrak{M}})$, to abridge the estimates:

$$\begin{aligned} J_{\tau}(u) := & \tau^{-1} \left(\|\theta^{-1/2} e^{\tau \theta \varphi} \overline{D}(Du)\|_{L^2(Q)}^2 + \|\theta^{-1/2} e^{\tau \theta \varphi} \partial_t u\|_{L^2(Q)}^2 \right) \\ & + \tau \left(\|\theta^{1/2} e^{\tau \theta \varphi} Du\|_{L^2(Q)}^2 + \|\theta^{1/2} e^{\tau \theta \varphi} \overline{D}u\|_{L^2(Q)}^2 \right) + \tau^3 \|\theta^{3/2} e^{\tau \theta \varphi} u\|_{L^2(Q)}^2 \end{aligned} \quad (2.6)$$

Theorem 2.3 *Let \mathcal{B}_0 be a nonempty open set of Ω and a function ψ satisfying Assumption 2.2. We define φ according to (2.5).*

Let \mathcal{B} be another open subset of Ω such that $\mathcal{B}_0 \subset \subset \mathcal{B}$. For the parameter $\lambda \geq 1$ sufficiently large, there exist $C, \tau_0 \geq 1, h_0 > 0, \varepsilon_0 > 0$, depending on $\mathcal{B}, \mathcal{B}_0$ and λ such that

$$\begin{aligned} J_{\tau}(u) \leq & C \left(\|e^{\tau \theta \varphi} \mathcal{P}_{\pm}^{\mathfrak{M}} u\|_{L^2(Q)}^2 + \tau^3 \|\theta^{3/2} e^{\tau \theta \varphi} u\|_{L^2(\mathcal{B} \times (0, T))}^2 \right) \\ & + Ch^{-2} \left(\left| e^{\tau \theta \varphi} u \right|_{t=0} \right|_{L^2(\Omega)}^2 + \left| e^{\tau \theta \varphi} u \right|_{t=T} \right|_{L^2(\Omega)}^2 \Big), \end{aligned}$$

for all $\tau \geq \tau_0(T + T^2)$, $0 < h \leq h_0$, $0 < \delta \leq 1/2$, $\tau h(\delta T^2)^{-1} \leq \varepsilon_0$, and $u \in \mathcal{C}^1([0, T]; \mathbb{R}^{\mathfrak{M} \cup \partial \mathfrak{M}})$ satisfying $u^{\partial \mathfrak{M}}(t) = 0$ for any $t \in [0, T]$.

Remark 2.4 *Unlike [9], note that we have added $\tau^{-1} \|\theta^{-1/2} e^{\tau \theta \varphi} \overline{D}(Du)\|_{L^2(Q)}^2$ in the term $J_{\tau}(u)$ of the left-hand side of the Carleman inequality. This simply follows from the fact that $\overline{D}(Du) = \mathcal{P}_{\pm}^{\mathfrak{M}} u \pm \partial_t u$ and*

$$\tau^{-1} \|\theta^{-1/2} e^{\tau \theta \varphi} \overline{D}(Du)\|_{L^2(Q)}^2 \leq 2\tau^{-1} \|\theta^{-1/2} e^{\tau \theta \varphi} \mathcal{P}_{\pm}^{\mathfrak{M}} u\|_{L^2(Q)}^2 + 2\tau^{-1} \|\theta^{-1/2} e^{\tau \theta \varphi} \partial_t u\|_{L^2(Q)}^2.$$

Now we are in position to prove the observability inequality. To manipulate the operators such as D, \overline{D} and also to provide estimates for the successive application of such operators on the weight functions, we have summarized the main discrete calculus rules in Appendix A. We state only the most useful results to accomplish the proof of Theorem 2.1. For a rigorous discussion on these features we refer the reader to [6], [9].

Proof of Theorem 2.1. The structure of the proof is similar to [25] and [4]. We have divided the proof in four steps. We keep track of the dependences of the constants. We start by considering a non empty $\mathcal{B}_0 \subset \subset \omega \cap \mathcal{O}$ and the associated weight functions as in the previous theorem.

Step 1. We set $B_2 = \omega \cap \mathcal{O}$. Let us consider now an open set B_1 such that $\mathcal{B}_0 \subset \subset B_1 \subset \subset B_2$. We begin by applying Theorem 2.3 to the solution p of (2.1) with $\mathcal{P}_{+}^{\mathfrak{M}} p = -bp$ and $\mathcal{B} = B_1$, to get

$$J_{\tau}(p) \leq C \left(\|e^{\tau \theta \varphi} bp\|_{L^2(Q)}^2 + \tau^3 \|\theta^{3/2} e^{\tau \theta \varphi} p\|_{L^2(B_1 \times (0, T))}^2 \right) + Ch^{-2} \left(\left| e^{\tau \theta \varphi} p \right|_{t=0} \right|_{L^2(\Omega)}^2 + \left| e^{\tau \theta \varphi} p \right|_{t=T} \right|_{L^2(\Omega)}^2 \Big),$$

for all $\tau \geq \tau_0(T + T^2)$, $0 < h \leq h_0$ and $\tau h(\delta T^2)^{-1} \leq \varepsilon_0$. As $1 \leq C\theta T^2$, the term with the coefficient b can be eliminated

$$J_{\tau}(p) \leq C\tau^3 \|\theta^{3/2} e^{\tau \theta \varphi} p\|_{L^2(B_1 \times (0, T))}^2 + Ch^{-2} \left(\left| e^{\tau \theta \varphi} p \right|_{t=0} \right|_{L^2(\Omega)}^2 + \left| e^{\tau \theta \varphi} p \right|_{t=T} \right|_{L^2(\Omega)}^2 \Big) \quad (2.7)$$

for $\tau_1 \geq \tau_0$ sufficiently large and $\tau \geq \tau_1(T + T^2 + T^2 \|b\|_{\infty}^{2/3})$.

Next, we apply Theorem 2.3 to the solution z to (2.1) with $\mathcal{B} = B_1$ and $\mathcal{P}_{-}^{\mathfrak{M}} = az - \mathbf{1}_{\mathcal{O}} p$, hence

$$J_{\tau}(z) \leq C \left(\|e^{\tau \theta \varphi} az\|_{L^2(Q)}^2 + \|e^{\tau \theta \varphi} p\|_{L^2(\mathcal{O} \times (0, T))}^2 + \tau^3 \|\theta^{3/2} e^{\tau \theta \varphi} z\|_{L^2(B_1 \times (0, T))}^2 \right) + Ch^{-2} \left| e^{\tau \theta \varphi} z \right|_{t=0} \right|_{L^2(\Omega)}^2,$$

where we have used the fact that $z(T) = 0$. Reasoning as before, it is not difficult to see that the term containing the coefficient a can also be absorbed as follows

$$J_{\tau}(z) \leq C \left(\|e^{\tau \theta \varphi} p\|_{L^2(\mathcal{O} \times (0, T))}^2 + \tau^3 \|\theta^{3/2} e^{\tau \theta \varphi} z\|_{L^2(B_1 \times (0, T))}^2 \right) + Ch^{-2} \left| e^{\tau \theta \varphi} z \right|_{t=0} \right|_{L^2(\Omega)}^2, \quad (2.8)$$

for $\tau \geq \tau_2(T + T^2 + T^2\|a\|_\infty^{2/3})$. Then, combining (2.7) and (2.8), we readily obtain

$$J_\tau(z) + J_\tau(p) \leq C \left(\tau^3 \|\theta^{3/2} e^{\tau\theta\varphi} z\|_{L^2(B_1 \times (0, T))}^2 + \tau^3 \|\theta^{3/2} e^{\tau\theta\varphi} p\|_{L^2(B_1 \times (0, T))}^2 \right) \\ + Ch^{-2} \left(\left| e^{\tau\theta\varphi} p|_{t=0} \right|_{L^2(\Omega)}^2 + \left| e^{\tau\theta\varphi} p|_{t=T} \right|_{L^2(\Omega)}^2 + \left| e^{\tau\theta\varphi} z|_{t=0} \right|_{L^2(\Omega)}^2 \right), \quad (2.9)$$

for all τ_3 sufficiently large and

$$\tau \geq \tau_3(T + T^2 + T^2(\|a\|_\infty^{2/3} + \|b\|_\infty^{2/3})). \quad (2.10)$$

Step 2. We proceed to obtain an inequality which bounds the observation term in B_1 containing p , by an observation term with respect to z in the larger domain B_2 . For this, we consider a function $\eta \in C^\infty(\Omega)$ such that

$$0 \leq \eta \leq 1 \text{ in } \Omega, \quad \eta = 1 \text{ in } B_1, \quad \text{supp } \eta \subset B_2 \subset \omega \cap \mathcal{O}. \quad (2.11)$$

By the properties of the discretization, we observe that we can ensure that the following bounds holds uniformly with respect to h

$$\frac{\overline{D}(\eta)}{\eta^{1/2}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\overline{D}\eta}{\eta^{1/2}} \in L^\infty(\Omega). \quad (2.12)$$

Let τ be as in (2.10). We multiply the equation satisfied by z in (2.1) by $\eta s^3 r^2 p$. Then, we have

$$\iint_{B_1 \times (0, T)} s^3 r^2 |p|^2 \leq \iint_{\mathcal{O} \times (0, T)} \eta s^3 r^2 |p|^2 \\ = \iint_Q (a - b) z \eta s^3 r^2 p + \iint_Q (-\partial_t z + \mathcal{A}^{\text{wt}} z + bz) \eta s^3 r^2 p \\ = \sum_{i=1}^4 I_n, \quad (2.13)$$

where we recall that $s = \tau\theta$ and $r = e^{s\varphi}$.

Let us estimate each I_n , $1 \leq n \leq 3$. We keep the term I_4 as it will be useful later. Hereinafter, C will denote a generic positive constant which may change from line to line. First, using Hölder and Young inequalities we have

$$I_1 = \iint_Q (a - b) z \eta s^3 r^2 p \\ \leq \gamma_0 \iint_Q \eta s^3 r^2 |p|^2 + \frac{1}{4\gamma_0} (\|a\|_\infty^2 + \|b\|_\infty^2) \iint_Q \eta s^3 r^2 |z|^2, \quad (2.14)$$

for any $\gamma_0 > 0$. On the other hand, integrating with respect to t we obtain that

$$I_2 = - \iint_Q \partial_t z \eta s^3 r^2 p \\ = - \int_\Omega z \eta s^3 r^2 p \Big|_0^T + \iint_Q z \eta \partial_t (s^3 r^2 p) \\ = \int_\Omega z(0) \eta s^3(0) r^2(0) p(0) + \iint_Q z \eta \partial_t (s^3 r^2) p + \iint_Q z \eta s^3 r^2 \partial_t p \\ := I_{21} + I_{22} + I_{23}, \quad (2.15)$$

where we have used the fact that $z(T) = 0$.

Remark 2.5 *Unlike the continuous case, note that $r(0) \neq 0$, so we have the additional term I_{21} .*

First, we estimate I_{21} as follows

$$\begin{aligned} I_{21} &= \int_{\Omega} z(0) \tau^3 \left(\frac{1}{(\delta T)(T + \delta T)} \right)^3 e^{\frac{2\tau\varphi(x)}{(\delta T)(T + \delta T)}} p(0) \\ &\leq \int_{\Omega} |z(0)| \frac{\tau^3}{\delta^3 T^6} e^{-\frac{C\tau}{\delta T^2}} |p(0)|. \end{aligned}$$

Therefore

$$|I_{21}| \leq \frac{1}{2} \frac{\tau^2}{\delta^2 T^4} \int_{\Omega} |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} + \frac{1}{2} \frac{\tau^4}{\delta^4 T^8} \int_{\Omega} |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}},$$

where we have applied Young and Hölder inequalities. From the conditions of Theorem 2.3, we have $\frac{\tau h}{\delta T^2} \leq \varepsilon_0$, then

$$|I_{21}| \leq Ch^{-2} \int_{\Omega} |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} + Ch^{-4} \int_{\Omega} |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \quad (2.16)$$

Now, we estimate I_{22} . We compute $\partial_t(\theta^3 r^2) = (\partial_t \theta^3) r^2 + \theta^3 (\partial_t r^2)$ and since $\tau \geq CT$, we have

$$\begin{aligned} |\partial_t(\theta^3 r^2)| &\leq 3\theta^4 T r^2 + 2\theta^5 r^2 \tau T |\varphi(x)| \\ &\leq C\theta^4 \tau r^2 + C\theta^5 \tau^2 r^2. \end{aligned}$$

With the estimate above, we deduce

$$\begin{aligned} |I_{22}| &\leq \tau^3 \iint_Q \eta |z| |\partial_t(\theta^3 r^2)| |p| \\ &\leq C\tau^3 \iint_Q \eta |z| (\theta^4 \tau r^2 + \theta^5 \tau^2 r^2) |p| \\ &= C \iint_Q \eta |z| (s^4 r^2 + s^5 r^2) |p|. \end{aligned}$$

Applying Hölder and Young inequalities, we get

$$|I_{22}| \leq \gamma_0 \iint_Q s^3 r^2 \eta |p|^2 + \frac{C}{\gamma_0} \iint_Q s^7 r^2 \eta |z|^2. \quad (2.17)$$

We keep the term I_{23} as it will be useful later.

In order to estimate I_3 , we integrate by parts using the discrete integration formula (Proposition A.4)

$$\begin{aligned} I_3 &= \iint_Q (\mathcal{A}^{\text{m}} z) \eta s^3 r^2 p = - \iint_Q \overline{D}(Dz) \eta s^3 r^2 p \\ &= - \iint_Q s^3 z \overline{D}(D(\eta r^2 p)). \end{aligned}$$

We compute with (A.4)

$$\overline{D}(D(\eta r^2 p)) = \eta r^2 \overline{D}(Dp) + \overline{D}(D(\eta r^2)) p + 2\overline{D}(\eta r^2) \overline{D}p + \frac{h^2}{2} (\overline{D}(Dp)) (\overline{D}(D(\eta r^2)))$$

Thus,

$$\begin{aligned} I_3 &= - \iint_Q s^3 z \left(\eta r^2 \overline{D}(Dp) + \overline{D}(D(\eta r^2)) p + \frac{h^2}{2} (\overline{D}(Dp)) (\overline{D}(D(\eta r^2))) \right) - 2 \iint_Q s^3 z \overline{D}(\eta r^2) \overline{D}p \\ &=: I_{31} + I_{32}. \end{aligned}$$

We proceed to estimate I_{31} . By using (A.4), (A.1) and (A.3) we obtain

$$\begin{aligned} \overline{D}(D(\eta r^2)) &= \eta \overline{D}(Dr^2) + r^2 \overline{D}(D\eta) + 2\overline{D}\eta \overline{D}r^2 + \frac{h^2}{2} (\overline{D}(Dr^2)) (\overline{D}(D\eta)), \\ \overline{D}(Dr^2) &= 2r \overline{D}(Dr) + 2(\overline{D}r)^2 + \frac{h^2}{2} (\overline{D}(Dr))^2, \\ \overline{D}r^2 &= 2r \overline{D}r + h^2 \overline{D}r (\overline{D}Dr), \end{aligned}$$

so that after a straightforward computation

$$\begin{aligned}
\overline{D}(D(\eta r^2)) &= r^2 \overline{D}(D\eta) + 2\eta r \overline{D}(Dr) + 2\eta \overline{D}r^2 + \frac{h^2}{2} \eta \overline{D}(Dr)^2 \\
&\quad + h^2 r \overline{D}(D\eta) \overline{D}(Dr) + h^2 \overline{D}(D\eta) \overline{D}r^2 + \frac{h^4}{4} \overline{D}(D\eta) \overline{D}(Dr)^2 \\
&\quad + 4r \overline{D}\eta \overline{D}r + 2h^2 \overline{D}\eta \overline{D}(Dr) \overline{D}r \\
&=: \tilde{I}_1(r).
\end{aligned}$$

Thus, we can group together all the terms of I_{31} as follows

$$\begin{aligned}
I_{31} &= - \iint_Q s^3 z \eta r^2 \overline{D}(Dp) - \iint_Q s^3 z p \tilde{I}_1(r) - \frac{h^2}{2} \iint_Q s^3 z \overline{D}(Dp) \tilde{I}_1(r) \\
&=: I_{31}^{(1)} + I_{31}^{(2)} + I_{31}^{(3)}.
\end{aligned} \tag{2.18}$$

We will keep the first term of the above expression. In order to estimate the second one, we take into account the result of Proposition A.6, the properties (2.12) and the relation between τ , h and δ that gives, for any $t \in (0, T)$,

$$s(t)h \leq \tau\theta(t)h \leq 2\frac{\tau h}{\delta T^2} \leq 2\varepsilon_0.$$

Therefore, we obtain that

$$|\tilde{I}_1(r)| \leq Cr^2 \sqrt{\eta} + C\eta r^2 s^2 + Cr^2 \sqrt{\eta} s, \tag{2.19}$$

where C only depends on λ (which is fixed) and ε_0 . Since η is supported in B_2 , we can use the Cauchy-Schwarz and Young inequalities together with (2.19) so that, for any $\gamma_0 > 0$ and $\gamma_1 > 0$, we get

$$|I_{31}^{(2)}| \leq \gamma_0 \iint_Q s^3 r^2 \eta |p|^2 + \frac{C}{\gamma_0} \iint_{B_2 \times (0, T)} s^7 r^2 |z|^2, \tag{2.20}$$

$$|I_{31}^{(3)}| \leq \gamma_1 \iint_Q s^{-1} r^2 \eta |\overline{D}(Dp)|^2 + \frac{C}{\gamma_1} \iint_{B_2 \times (0, T)} s^{11} r^2 |z|^2. \tag{2.21}$$

Arguing as in the previous steps, we compute

$$\begin{aligned}
2\overline{D}(\eta r^2) &= 2\left(\eta \overline{D}r^2 + r^2 \overline{D}\eta + \frac{h^2}{2} \overline{D}(D\eta) \overline{D}r^2 + \frac{h^2}{2} \overline{D}\eta \overline{D}(Dr^2)\right) \\
&= 2\left(2\eta r \overline{D}r + h^2 \eta \overline{D}(Dr) \overline{D}r + r^2 \overline{D}\eta + h^2 \overline{D}(D\eta) r \overline{D}r\right. \\
&\quad + \frac{h^4}{2} \overline{D}(D\eta) \overline{D}(Dr) \overline{D}r + h^2 \overline{D}\eta r \overline{D}(Dr) + \frac{h^4}{4} \overline{D}\eta \overline{D}(Dr)^2 \\
&\quad \left. + h^2 \overline{D}\eta \overline{D}r^2\right) \\
&=: \tilde{I}_2(r),
\end{aligned}$$

and we obtain, for some $C > 0$ depending only on ε_0 and λ that

$$|\tilde{I}_2(r)| \leq C\eta r^2 s + C\sqrt{\eta} r^2.$$

Replacing the above expression in I_{32} we obtain

$$I_{32} = - \iint_Q s^3 z \overline{D}p \tilde{I}_2(r),$$

and we finally get that

$$|I_{32}| \leq \gamma_2 \iint_Q s r^2 \eta |\overline{D}p|^2 + \frac{C}{\gamma_2} \iint_{B_2 \times (0, T)} s^7 r^2 |z|^2, \tag{2.22}$$

for any $\gamma_2 > 0$. Notice that the sum of the terms I_4 , I_{23} and $I_{31}^{(1)}$ (see eq. (2.13), (2.15) and (2.18)) exactly cancels thanks to the equation satisfied by p , that is

$$I_4 + I_{23} + I_{31}^{(1)} = \iint_Q z\eta s^3 r^2 (\partial_t p - \overline{D}(Dp) + bp) = 0.$$

By means of equations (2.16) and (2.17) we get

$$\begin{aligned} |I_2| &\leq \gamma_0 \iint_Q s^3 r^2 \eta |p|^2 + Ch^{-2} \int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \\ &\quad + \frac{C}{\gamma_0} \iint_Q s^7 r^2 \eta |z|^2 + Ch^{-4} \int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \end{aligned} \quad (2.23)$$

We put together (2.20), (2.21) and (2.22), obtaining

$$\begin{aligned} |I_{31}^{(2)} + I_{31}^{(3)} + I_{32}| &\leq \gamma_0 \iint_Q s^3 r^2 \eta |p|^2 + \gamma_1 \iint_Q s^{-1} r^2 \eta |\overline{D}(Dp)|^2 + \gamma_2 \iint_Q sr^2 \eta |\overline{D}p|^2 \\ &\quad + C \left(\frac{1}{\gamma_0} + \frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) \iint_{B_2 \times (0, T)} s^{11} r^2 |z|^2. \end{aligned} \quad (2.24)$$

Taking estimates (2.14), (2.23) and (2.24) in equation (2.13) and using (2.11), we obtain

$$\begin{aligned} \iint_{B_1 \times (0, T)} s^3 r^2 |p|^2 &\leq 3\gamma_0 \iint_Q s^3 r^2 |p|^2 + \gamma_1 \iint_Q s^{-1} r^2 |\overline{D}(Dp)|^2 + \gamma_2 \iint_Q sr^2 |\overline{D}p|^2 \\ &\quad + C \left(\frac{1}{\gamma_0} + \frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) \iint_{B_2 \times (0, T)} r^2 [s^3 (\|a\|_\infty^2 + \|b\|_\infty^2) |z|^2 + s^{11} |z|^2] \\ &\quad + Ch^{-2} \int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} + Ch^{-4} \int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \end{aligned}$$

Thus, replacing the above expression in (2.9) and taking γ_i small enough, we select τ as in (2.10) to obtain

$$\begin{aligned} J_\tau(z) + J_\tau(p) &\leq C \iint_{B_2 \times (0, T)} s^{11} r^2 |z|^2 + Ch^{-2} \int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} + Ch^{-4} \int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \\ &\quad + Ch^{-2} \left(\left| e^{\tau\theta\varphi} p|_{t=0} \right|_{L^2(\Omega)}^2 + \left| e^{\tau\theta\varphi} p|_{t=T} \right|_{L^2(\Omega)}^2 + \left| e^{\tau\theta\varphi} z|_{t=0} \right|_{L^2(\Omega)}^2 \right). \end{aligned}$$

Returning to the original notation, we rewrite the above inequality as

$$\begin{aligned} J_\tau(z) + J_\tau(p) &\leq C \iint_{B_2 \times (0, T)} e^{2\theta\tau\varphi} \tau^{11} \theta^{11} |z|^2 + Ch^{-4} \int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}} + Ch^{-2} \int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \\ &\quad + Ch^{-2} \left(\left| e^{\tau\theta\varphi} p|_{t=0} \right|_{L^2(\Omega)}^2 + \left| e^{\tau\theta\varphi} p|_{t=T} \right|_{L^2(\Omega)}^2 + \left| e^{\tau\theta\varphi} z|_{t=0} \right|_{L^2(\Omega)}^2 \right), \end{aligned} \quad (2.25)$$

valid for every

$$\tau \geq \tau_3 \left(T + T^2 + T^2 (\|a\|_\infty^{2/3} + \|b\|_\infty^{2/3}) \right)$$

with τ_3 large enough.

Step 3. Here, we use standard energy estimates for the heat equation to bound the last four terms in inequality (2.25).

As $\theta(T) = \theta(0) = (T^2(1 + \delta)\delta)^{-1}$, we have $e^{\tau\theta\varphi}|_{t=0} = e^{\tau\theta\varphi}|_{t=T} \leq e^{C\frac{\tau}{\delta T^2} \sup_{x \in \Omega} \varphi}$ and we compute

$$\begin{aligned} J_\tau(z) + J_\tau(p) &\leq C \iint_{B_2 \times (0, T)} e^{2\theta\tau\varphi} \tau^{11} \theta^{11} |z|^2 + Ch^{-2} \int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \\ &\quad + Ch^{-4} \int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}} + Ch^{-2} \int_\Omega |p(T)|^2 e^{-\frac{C\tau}{\delta T^2}}, \end{aligned} \quad (2.26)$$

as $\sup_{x \in \bar{\Omega}} \varphi < 0$. From energy estimates for p solution to the second equation in system (2.1), for $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$|p(t_2)|_{L^2(\Omega)}^2 \leq e^{2\|b\|_\infty(t_2-t_1)} |p(t_1)|_{L^2(\Omega)}^2. \quad (2.27)$$

In particular, we obtain

$$\int_{\Omega} |p(T)|^2 e^{-\frac{C\tau}{\delta T^2}} \leq C \int_{\Omega} |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \quad (2.28)$$

On the other hand, from energy estimates for z solution to the first equation in (2.1), we get for $t \in [0, T]$

$$|z(t)|_{L^2(\Omega)}^2 \leq \int_t^T e^{2(1+\|a\|_\infty)(s-t)} |p(s)|_{L^2(\mathcal{O})}^2 ds,$$

whence

$$\int_{\Omega} |z(0)|^2 \leq C \iint_{\mathcal{O} \times (0, T)} |p|^2.$$

Using (2.27) it is not difficult to see that

$$\int_{\Omega} |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \leq C \int_{\Omega} |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \quad (2.29)$$

Replacing accordingly (2.28) and (2.29) in inequality (2.26) we obtain

$$J_\tau(z) + J_\tau(p) \leq C \iint_{B_2 \times (0, T)} e^{2\theta\tau\varphi} \tau^{11} \theta^{11} |z|^2 + Ch^{-4} \int_{\Omega} |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \quad (2.30)$$

Step 4. In the last part of the proof, we use energy estimates and inequality (2.30) to obtain a modified Carleman inequality with weight functions not decaying at $t = T$. This is possible since we have the condition $z(T) = 0$.

Let us first fix

$$\tau = \tau_3 \left(T + T^2 + T^2 (\|a\|_\infty^{2/3} + \|b\|_\infty^{2/3}) \right), \quad (2.31)$$

and let us consider

$$l(t) = \begin{cases} (t + \delta T)(T + \delta T - t) & \text{for } 0 \leq t \leq T/2, \\ (T/2 + \delta T)^2 & \text{for } T/2 \leq t \leq T, \end{cases}$$

and the following associated function

$$\sigma(t) = \frac{1}{l(t)}.$$

By construction, $\theta(t) = \sigma(t)$ for $t \in [0, T/2]$, so that by using (2.6) and (2.30), we have

$$\begin{aligned} \int_0^{T/2} \int_{\Omega} e^{2\tau\sigma\varphi} \sigma^3 |z|^2 + \int_0^{T/2} \int_{\Omega} e^{2\tau\sigma\varphi} \sigma^3 |p|^2 &= \int_0^{T/2} \int_{\Omega} e^{2\tau\theta\varphi} \theta^3 |z|^2 + \int_0^{T/2} \int_{\Omega} e^{2\tau\theta\varphi} \theta^3 |p|^2 \\ &\leq J_\tau(z) + J_\tau(p) \\ &\leq C \iint_{B_2 \times (0, T)} e^{2\tau\theta\varphi} \tau^{11} \theta^{11} |z|^2 \\ &\quad + Ch^{-4} \int_{\Omega} |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \end{aligned} \quad (2.32)$$

Now, consider a function $\nu \in C^1([0, T])$ such that

$$\nu = 0 \text{ in } [0, T/4], \quad \nu = 1 \text{ in } [T/2, T], \quad |\nu'| \leq C/T.$$

We set $\tilde{z}(t) = e^{-\|a\|_\infty(T-t)} \nu(t) z$ and $\tilde{p} = e^{-\|b\|_\infty t} \nu(t) p$ and we observe that they solve the following equations

$$\begin{cases} -\partial_t \tilde{z} + \mathcal{A}^{\text{int}} \tilde{z} + (a + \|a\|_\infty) \tilde{z} = \mathbf{1}_{\mathcal{O}} e^{\|b\|_\infty t - \|a\|_\infty(T-t)} \tilde{p} + \nu'(t) e^{-\|a\|_\infty(T-t)} z & \text{in } \mathbb{R}^{\text{int}} \times (0, T), \\ \partial_t \tilde{p} + \mathcal{A}^{\text{ext}} \tilde{p} + (b + \|b\|_\infty) \tilde{p} = \nu'(t) e^{-\|b\|_\infty t} p & \text{in } \mathbb{R}^{\text{ext}} \times (0, T), \\ \tilde{z} = \tilde{p} = 0 & \text{on } \partial \mathfrak{M} \times (0, T), \\ \tilde{z}(T) = 0, \quad \tilde{p}(0) = 0. \end{cases}$$

Since $b + \|b\|_\infty \geq 0$, the energy estimate for \tilde{p} leads to

$$\frac{1}{2}|p(T)|_{L^2(\Omega)}^2 + \int_0^T (\mathcal{A}^{\text{or}} \tilde{p}, \tilde{p})_{L^2(\Omega)} dt \leq \frac{C}{T} \int_{\frac{T}{4}}^{\frac{T}{2}} |(p, \tilde{p})_{L^2(\Omega)}| dt.$$

and by the discrete Poincaré inequality (1.13) we deduce that

$$\|\tilde{p}\|_{L^2(\Omega \times (0, T))} \leq \frac{C}{T} \|p\|_{L^2(\Omega \times (T/4, T/2))}. \quad (2.33)$$

The energy estimate for \tilde{z} reads

$$\frac{1}{2}|\tilde{z}(0)|_{L^2(\Omega)}^2 + \int_0^T (\mathcal{A}^{\text{or}} \tilde{z}, \tilde{z})_{L^2(\Omega)} dt \leq e^{\|b\|_\infty T} \int_0^T |(\tilde{p}, \tilde{z})_{L^2(\Omega)}| dt + \frac{C}{T} \int_{\frac{T}{4}}^{\frac{T}{2}} |(z, \tilde{z})_{L^2(\Omega)}| dt,$$

which leads to

$$\|\tilde{z}\|_{L^2(\Omega \times (0, T))} \leq e^{\|b\|_\infty T} \|\tilde{p}\|_{L^2(\Omega \times (0, T))} + \frac{C}{T} \|z\|_{L^2(\Omega \times (T/4, T/2))}. \quad (2.34)$$

Combining (2.33) and (2.34) and bearing in mind the definitions of \tilde{z} , \tilde{p} and the properties of ν we get

$$\|z\|_{L^2(\Omega \times (T/2, T))} \leq \frac{C}{T} e^{(\|a\|_\infty + \|b\|_\infty)T} (\|z\|_{L^2(\Omega \times (T/4, T/2))} + \|p\|_{L^2(\Omega \times (T/4, T/2))}).$$

Since σ is constant and smaller than $4/T^2$ on $(T/2, T)$ and $\varphi < 0$ we can introduce the weight function on the left-hand side of the above inequality to obtain for some $C_T > 0$ depending only on T ,

$$\int_{T/2}^T \int_\Omega e^{2\tau\sigma\varphi} \sigma^3 |z|^2 \leq C_T e^{2(\|a\|_\infty + \|b\|_\infty)T} (\|z\|_{L^2(\Omega \times (T/4, T/2))}^2 + \|p\|_{L^2(\Omega \times (T/4, T/2))}^2).$$

Observe that the function σ satisfies $c_1/T^2 \leq \sigma \leq c_2/T^2$ in $[T/4, T/2]$, for some universal constants $c_1, c_2 > 0$. Setting $c_0 := -\inf_\Omega \varphi > 0$, we can introduce the weight functions in the right-hand side terms as follows

$$\int_{T/2}^T \int_\Omega e^{2\tau\sigma\varphi} \sigma^3 |z|^2 \leq C_T e^{2(\|a\|_\infty + \|b\|_\infty)T} e^{2c_2 c_0 \frac{\tau}{T^2}} \int_{T/4}^{T/2} \int_\Omega e^{2\tau\sigma\varphi} \sigma^3 (|z|^2 + |p|^2),$$

and then use (2.32) to obtain that

$$\int_{T/2}^T \int_\Omega e^{2\tau\sigma\varphi} \sigma^3 |z|^2 \leq C_T e^{2(\|a\|_\infty + \|b\|_\infty)T + \frac{1}{T}} e^{2c_2 c_0 \frac{\tau}{T^2}} \left(\iint_{B_2 \times (0, T)} e^{2\tau\theta\varphi} \tau^{11} \sigma^{11} |z|^2 + Ch^{-4} \int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \right).$$

It can be readily verified by means of the definition of σ that

$$C e^{-\frac{8c_0\tau}{Tt}} \leq C e^{-2c_0\tau\gamma} \leq e^{2\tau\sigma\varphi} \sigma^3,$$

where

$$\gamma = \begin{cases} 1/(t(T-t)) & 0 \leq t \leq T/2, \\ 4/T^2 & T/2 \leq t \leq T. \end{cases}$$

This, together with the fact that $\sigma \geq (T + \delta T^2)^{-1}$ yields

$$\int_0^T \int_\Omega e^{-\frac{8c_0\tau}{Tt}} |z|^2 \leq C T^6 e^{2(\|a\|_\infty + \|b\|_\infty)T + \frac{1}{T}} e^{2c_2 c_0 \frac{\tau}{T^2}} \left(\iint_{B_2 \times (0, T)} e^{2\tau\theta\varphi} \tau^{11} \sigma^{11} |z|^2 + Ch^{-4} \int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \right).$$

Setting now $\bar{c}_0 := -\sup_\Omega \varphi > 0$, we have

$$e^{2\tau\theta\varphi} \tau^{11} \theta^{11} \leq e^{-2\bar{c}_0\tau\theta} \tau^{11} \theta^{11} \leq \frac{C}{\bar{c}_0^{11}},$$

for some universal $C > 0$. It follows that

$$\iint_Q e^{-\frac{8c_0\tau}{Tt}} |z|^2 \leq C e^{C((1+\|a\|_\infty + \|b\|_\infty)T + \frac{1}{T} + \frac{\tau}{T^2})} \left(\iint_{B_2 \times (0, T)} |z|^2 + Ch^{-4} \int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \right). \quad (2.35)$$

To conclude the proof, we recall the conditions from Theorem 2.3:

$$\frac{\tau h}{\delta T^2} \leq \varepsilon_0 \quad \text{and} \quad h \leq h_0.$$

They need to be fulfilled along with $\delta \leq \delta_1$, and we recall that τ was defined in (2.31). We thus define h_1 as

$$h_1 := \frac{\varepsilon_0}{\tau_4} \delta_1 \left(1 + \frac{1}{T} + \|a\|_\infty^{2/3} + \|b\|_\infty^{2/3} \right)^{-1}.$$

Then we choose $h \leq \min\{h_0, h_1\}$ and $\delta = h\delta_1/h_1 \leq \delta_1$. With these choices, we can ensure the equality $\frac{\tau h}{\delta T^2} = \varepsilon_0$ and moreover, from (2.35) we have

$$\iint_Q e^{-\frac{8\varepsilon_0\tau}{Tt}} |z|^2 \leq e^{C'((1+\|a\|_\infty+\|b\|_\infty)T+\frac{\tau}{T^2})} \left(\iint_{B_2 \times (0,T)} |z|^2 + \int_\Omega |p(0)|^2 e^{-\frac{C'\varepsilon_0}{h}} \right).$$

Finally, using the formula (2.31) for τ and recalling that $B_2 \subset \omega$, our claim is proved. ■

3 Proof of Theorem 1.4

We devote this section to prove the existence of controls insensitizing the L^2 -norm of the observation of the solution of (1.10). The proof follows the same spirit as other well-known results for controllability of nonlinear systems (see [12], [13], [25], ...). We start with the existence of $\phi(h)$ -insensitizing controls for a linearized version of (1.10), that is, for given $a \in L^\infty(0, T; \mathbb{R}^m)$, $b \in L^\infty(0, T; \mathbb{R}^m)$ and $\xi \in L^2(0, T; \mathbb{R}^m)$, we consider the linear system

$$\begin{cases} \partial_t y + \mathcal{A}^m y + ay = \mathbf{1}_\omega v + \xi & \text{in } \mathbb{R}^m \times (0, T), \\ -\partial_t q + \mathcal{A}^m q + bq = \mathbf{1}_Q y & \text{in } \mathbb{R}^m \times (0, T), \\ y = q = 0 & \text{on } \partial\mathfrak{M} \times (0, T), \\ y(0) = 0, \quad q(T) = 0, \end{cases} \quad (3.1)$$

and the corresponding adjoint system (2.1).

The following result holds:

Proposition 3.1 *Let us consider $T > 0$ and \mathfrak{M} a mesh satisfying $h \leq \min\{h_0, h_1\}$ with h_0, h_1 as given in Theorem 2.1. Let \mathcal{M} be defined as in (2.4).*

There exists a continuous linear map $L_{(T;a,b)} : L^2(e_{\mathcal{M}}) \rightarrow L^2(0, T; \mathbb{R}^m)$ such that for all source term $\xi \in L^2(0, T; \mathbb{R}^m)$ satisfying

$$\|\xi\|_{L^2(e_{\mathcal{M}})} < \infty, \quad (3.2)$$

the semi discrete control v given by $v = L_{(T;a,b)}(\xi)$ is such that the solution to (3.1) satisfies

$$|q(0)|_{L^2(\Omega)} \leq C_{obs} e^{-\frac{C}{h}} \|\xi\|_{L^2(e_{\mathcal{M}})},$$

and

$$\|v\|_{L^2(Q)} \leq C_{obs} \|\xi\|_{L^2(e_{\mathcal{M}})},$$

with C_{obs} as given in Theorem 2.1.

Proof. Consider the adjoint system (2.1). The relaxed observability inequality of Theorem 2.1 gives

$$\iint_Q \exp(-\frac{M}{t}) |z|^2 dx dt \leq C_{obs}^2 \left(\|z\|_{L^2(\omega \times (0,T))}^2 + \phi(h) |p_0|_{L^2(\Omega)}^2 \right), \quad (3.3)$$

with $\phi(h) = e^{-C_1/h}$. We introduce the functional

$$J(p_0) = \frac{1}{2} \iint_{\omega \times (0,T)} |z|^2 + \iint_Q z\xi + \frac{\phi(h)}{2} |p_0|_{L^2(\Omega)}^2. \quad (3.4)$$

The functional J is continuous, strictly convex and coercive on a finite dimensional space, thus it admits a unique minimizer that we denote as p_0^{opt} . We denote by (z^{opt}, p^{opt}) the associated solution of the adjoint problem (2.1) with this initial data.

We compute the Euler-Lagrange equation for this minimization problem, namely

$$\iint_{\omega \times (0, T)} z^{opt} z + \iint_Q \xi z + \phi(h) \langle p_0^{opt}, p_0 \rangle_{L^2(\Omega)} = 0, \quad \forall p_0 \in \mathbb{R}^{\mathfrak{M}}, \quad (3.5)$$

where (z, p) is the solution associated with the data p_0 . We set the control $v = L_{(T; a, b)}(\xi) = \mathbf{1}_\omega z^{opt}$ and consider the solution (y, q) to the controlled problem

$$\begin{cases} \partial_t y + \mathcal{A}^{\mathfrak{M}} y + ay = \mathbf{1}_\omega z^{opt} + \xi, & \text{in } \mathbb{R}^{\mathfrak{M}} \times (0, T), \\ -\partial_t q + \mathcal{A}^{\mathfrak{M}} y + bq = \mathbf{1}_\mathcal{O} y, & \text{in } \mathbb{R}^{\mathfrak{M}} \times (0, T), \\ y = q = 0, & \text{on } \partial\mathfrak{M} \times (0, T), \\ y(0) = 0, \quad q(T) = 0. \end{cases}$$

Multiplying the above equation by (z, p) and integrating by parts we obtain

$$(q(0), p_0)_{L^2(\Omega)} = \iint_{\omega \times (0, T)} z^{opt} z + \iint_Q \xi z,$$

for any $p_0 \in \mathbb{R}^{\mathfrak{M}}$. Substituting this expression in (3.5) we deduce that

$$q(0) = -\phi(h) p_0^{opt}. \quad (3.6)$$

On the other hand, we take $p_0 = p_0^{opt}$ in (3.5), to get

$$\|z^{opt}\|_{L^2(\omega \times (0, T))}^2 + \phi(h) |p_0^{opt}|_{L^2(\Omega)}^2 = - \iint_Q \xi z^{opt}.$$

Since ξ satisfies (3.2), we introduce the weight function in the right-hand side of the above inequality, thus

$$\|z^{opt}\|_{L^2(\omega \times (0, T))}^2 + \phi(h) |p_0^{opt}|_{L^2(\Omega)}^2 \leq \left(\iint_Q e^{\frac{\mathcal{M}}{t}} |\xi|^2 \right)^{1/2} \left(\iint_Q e^{-\frac{\mathcal{M}}{t}} |z^{opt}|^2 \right)^{1/2}.$$

With the observability inequality (3.3) we have

$$\|z^{opt}\|_{L^2(\omega \times (0, T))}^2 + \phi(h) |p_0^{opt}|_{L^2(\Omega)}^2 \leq C_{obs}^2 \iint_Q e^{\frac{\mathcal{M}}{t}} |\xi|^2.$$

This yields

$$\|v\|_{L^2(\omega \times (0, T))} = \|z^{opt}\|_{L^2(\omega \times (0, T))} \leq C_{obs} \left(\iint_Q e^{\frac{\mathcal{M}}{t}} |\xi|^2 \right)^{1/2}$$

and

$$\sqrt{\phi(h)} |p_0^{opt}|_{L^2(\Omega)} \leq C_{obs} \left(\iint_Q e^{\frac{\mathcal{M}}{t}} |\xi|^2 \right)^{1/2}. \quad (3.7)$$

Hence, the linear map

$$\begin{aligned} L_{(T; a, b)} : L^2(e_{\mathcal{M}}) &\rightarrow L^2(\omega \times (0, T)), \\ \xi &\mapsto v, \end{aligned}$$

is well defined and continuous. Finally, with (3.6) and (3.7) we get

$$|q(0)|_{L^2(\Omega)} \leq C_{obs} e^{-C/h} \left(\iint_Q e^{\frac{\mathcal{M}}{t}} |\xi|^2 \right)^{1/2},$$

which concludes the proof. \blacksquare

Proof of Theorem 1.4. Let us define

$$g(s) := \begin{cases} \frac{f(s)}{s} & \text{if } s \neq 0, \\ f'(0) & \text{if } s = 0. \end{cases} \quad (3.8)$$

The assumption on f guarantees that g and f' are both continuous and bounded functions. We set $\mathcal{Z} = L^2(0, T; \mathbb{R}^{\mathfrak{m}})$. For $\zeta \in \mathcal{Z}$ we consider the semi discrete linear controlled system

$$\begin{cases} \partial_t y + \mathcal{A}^{\mathfrak{m}} y + g(\zeta)y = \mathbf{1}_\omega v + \xi & \text{in } \mathbb{R}^{\mathfrak{m}} \times (0, T), \\ -\partial_t q + \mathcal{A}^{\mathfrak{m}} q + f'(\zeta)q = \mathbf{1}_O y & \text{in } \mathbb{R}^{\mathfrak{m}} \times (0, T), \\ y = q = 0 & \text{on } \partial\mathfrak{M} \times (0, T), \\ y(0) = 0, \quad q(T) = 0. \end{cases} \quad (3.9)$$

We set $a_\zeta = g(\zeta)$ and $b_\zeta = f'(\zeta)$, so that we have

$$\|a_\zeta\|_\infty + \|b_\zeta\|_\infty \leq K := 2\|f'\|_\infty, \quad \forall \zeta \in \mathcal{Z}. \quad (3.10)$$

Then, we apply Proposition 3.1, with h chosen sufficiently small, i.e. $h \leq \min(h_0, h_1)$ with

$$h_1 = C \left(1 + \frac{1}{T} + K^{2/3}\right)^{-1},$$

and denote by $v_\zeta = L_{(T; a_\zeta, b_\zeta)}(\xi)$ and (y_ζ, q_ζ) the associated control function and controlled solution. We have

$$\|q_\zeta(0)\|_{L^2(\Omega)} \leq C e^{-C_1/h} \|\xi\|_{L^2(e_{\mathcal{M}})}, \quad \|v_\zeta\|_{L^2(Q)} \leq C \|\xi\|_{L^2(e_{\mathcal{M}})}. \quad (3.11)$$

where $C_1 > 0$ and $C = \exp\left[C\left(1 + \frac{1}{T} + K^{2/3} + T(1 + K)\right)\right]$ are uniform with respect to ζ and to the discretization parameter h . We have thus built a map

$$\begin{aligned} \Lambda : \mathcal{Z} &\rightarrow \mathcal{Z}, \\ \zeta &\mapsto y_\zeta, \end{aligned}$$

where y_ζ is the solution to (3.9) associated to $a_\zeta = g(\zeta)$ and $b_\zeta = f'(\zeta)$, with v_ζ as in (3.11).

By classical energy estimates for the semi discrete parabolic equations we obtain

$$\begin{aligned} \|y_\zeta\|_{L^2(Q)} &\leq e^{C(1+T+T(\|a_\zeta\|_\infty + \|b_\zeta\|_\infty))} (\|\xi\|_{L^2(Q)} + \|\mathbf{1}_\omega v_\zeta\|_{L^2(Q)}) \\ &\leq C e^{C(1+T+T(\|a_\zeta\|_\infty + \|b_\zeta\|_\infty))} (\|\xi\|_{L^2(e_{\mathcal{M}})} + \|\mathbf{1}_\omega v_\zeta\|_{L^2(Q)}), \end{aligned}$$

and taking into account (3.10) and (3.11), we deduce that the image of Λ is bounded, which implies in particular that there exists a closed convex and bounded set in \mathcal{Z} which is fixed by Λ . Following the methods of [2] and [12], it can be verified that Λ is continuous and compact from \mathcal{Z} into itself, by the Ascoli theorem. Therefore, applying Schauder's fixed point theorem, we obtain that there exists $y \in \mathcal{Z}$ such that $\Lambda(y) = y$. Setting $v = L_{(T; a_y, b_y)}(\xi)$ we obtain

$$\begin{cases} \partial_t y + \mathcal{A}^{\mathfrak{m}} y + f(y) = \mathbf{1}_\omega v + \xi & \text{in } \mathbb{R}^{\mathfrak{m}} \times (0, T), \\ -\partial_t q + \mathcal{A}^{\mathfrak{m}} q + f'(y)q = \mathbf{1}_O y & \text{in } \mathbb{R}^{\mathfrak{m}} \times (0, T), \\ y = q = 0 & \text{on } \partial\mathfrak{M} \times (0, T), \\ y(0) = 0, \quad q(T) = 0, \end{cases}$$

which concludes the proof as we have found a control v that drives the solution of the semilinear semi discrete parabolic system to a final state $q(0)$ satisfying the estimates (3.11). ■

4 The fully discrete insensitizing control problem

As noted in Proposition 1.1, the insensitizing problem is equivalent to a null control problem for a cascade system of equations. In the present section we consider a fully discrete (time and space) version of our problem. We shall compute the suitable fully discrete version of the cascade system which is equivalent to the insensitizing property as well as its associated adjoint system. By using the penalized HUM approach, we can characterize and build the optimal control satisfying a convenient minimization problem that will furnish computable controls satisfying the expected properties.

4.1 Fully discrete null-controllability formulation

We consider a standard fully discrete scheme for our semilinear parabolic equation with unknown data. More precisely, for any mesh \mathfrak{M} and any integer $M > 0$ given, we set $\delta t = T/M$ and we introduce the following semi-implicit Euler scheme with respect to the time variable

$$\begin{cases} \frac{y^{n+1} - y^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} y^{n+1} + f(y^n) = \mathbf{1}_\omega v^{n+1} + \xi^{n+1}, & \forall n \in \llbracket 0, M-1 \rrbracket, \\ y^0 = y_0 + \tau w_0, \end{cases} \quad (4.1)$$

where $v_{\delta t}$ is an element of the fully discrete function space defined by

$$L_{\delta t}^2(0, T; \mathbb{R}^{\mathfrak{M}}) := \{v_{\delta t} = (v^n)_{1 \leq n \leq M}, \quad v^n \in \mathbb{R}^{\mathfrak{M}}, \quad \forall n \in \llbracket 1, M \rrbracket\},$$

and endowed with the norm

$$\|v_{\delta t}\|_{L_{\delta t}^2(0, T; \mathbb{R}^{\mathfrak{M}})} := \left(\sum_{n=1}^M \delta t |v^n|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Note that we consider an explicit discretization for the nonlinear term in (4.1) to ensure that we can compute the solution of the system by simply solving a set of linear equations at each time iteration with the same underlying matrix $I + \delta t \mathcal{A}^{\mathfrak{M}}$. Therefore, a direct solver can be efficiently used. Since the nonlinear function f is assumed to be globally Lipschitz continuous, this discretization is stable as soon as we assume the following condition

$$\delta t \text{Lip}(f) < 1. \quad (4.2)$$

Consider now the fully discrete version of the functional (1.9) defined by

$$\Psi(y_{\delta t}) := \frac{1}{2} \sum_{n=1}^M \delta t \int_{\mathcal{O}} |y^n|^2, \quad \forall y_{\delta t} \in L_{\delta t}^2(0, T; \mathbb{R}^{\mathfrak{M}}). \quad (4.3)$$

Then, our desire is to insensitize the functional (4.3) computed on the solutions of (4.1) with respect to perturbations of the initial data. This means that we find $v_{\delta t}$ such that

$$\left. \frac{\partial \Psi(y_{\delta t}[y_0, \xi_{\delta t}, v_{\delta t}, w_0, \tau])}{\partial \tau} \right|_{\tau=0} = 0, \quad \forall w_0 \in \mathbb{R}^{\mathfrak{M}}, \quad (4.4)$$

where $y_{\delta t}[y_0, \xi_{\delta t}, v_{\delta t}, w_0, \tau]$ is the solution of (4.1).

Similarly to the semi discrete and continuous cases, we have the following characterization for an insensitizing control.

Proposition 4.1 *We assume that the time step satisfies the stability condition (4.2). We consider the following cascade system of fully discrete semilinear parabolic equations*

$$\begin{cases} \frac{y^{n+1} - y^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} y^{n+1} + f(y^n) = \mathbf{1}_\omega v^{n+1} + \xi^{n+1}, & \forall n \in \llbracket 0, M-1 \rrbracket, \\ y^0 = y_0, \end{cases} \quad (4.5)$$

$$\begin{cases} \frac{q^n - q^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}} q^n + f'(y^n) q^{n+1} = \mathbf{1}_{\mathcal{O}} y^n, & \forall n \in \llbracket 1, M \rrbracket, \\ q^{M+1} = 0. \end{cases} \quad (4.6)$$

Then, the insensitizing condition (4.4) is equivalent to

$$q^1 = 0.$$

Proof. With the definition (4.3), the derivative of $\Psi(y_{\delta t}[y_0, \xi_{\delta t}, v_{\delta t}, w_0, \tau])$ with respect to τ evaluated at $\tau = 0$ shows that (4.4) is precisely equivalent to the equality

$$\sum_{n=1}^M \delta t \int_{\mathcal{O}} y^n w^n = 0, \quad (4.7)$$

for every $w_0 \in \mathbb{R}^{\mathfrak{m}}$, where $(y^n)_n \in L^2_{\delta t}(0, T; \mathbb{R}^{\mathfrak{m}})$ is the solution of (4.1) corresponding to $\tau = 0$ and $(w^n)_n \in L^2_{\delta t}(0, T; \mathbb{R}^{\mathfrak{m}})$ is the derivative of the solution of (4.1) at $\tau = 0$. More precisely, $(y^n)_{1 \leq n \leq M}$ solves

$$\begin{cases} \frac{y^{n+1} - y^n}{\delta t} + \mathcal{A}^{\mathfrak{m}} y^{n+1} + f(y^n) = \mathbf{1}_\omega v^{n+1} + \xi^{n+1}, & n \in \llbracket 0, M-1 \rrbracket, \\ y^0 = y_0, \end{cases}$$

and $(w^n)_{1 \leq n \leq M}$ solves

$$\begin{cases} \frac{w^{n+1} - w^n}{\delta t} + \mathcal{A}^{\mathfrak{m}} w^{n+1} + f'(y^n) w^n = 0, & n \in \llbracket 0, M-1 \rrbracket, \\ w^0 = w_0. \end{cases} \quad (4.8)$$

We multiply (4.8) by a sequence $(q^{n+1})_{0 \leq n \leq M-1}$ in $L^2_{\delta t}(0, T; \mathbb{R}^{\mathfrak{m}})$, that is,

$$\sum_{n=0}^{M-1} \delta t \left(\frac{w^{n+1} - w^n}{\delta t} + \mathcal{A}^{\mathfrak{m}} w^{n+1} + f'(y^n) w^n, q^{n+1} \right)_{L^2(\Omega)} = 0.$$

After rearranging some terms and from the fact that $\mathcal{A}^{\mathfrak{m}}$ is a symmetric operator we obtain

$$\begin{aligned} - (w^0, q^1 - \delta t f'(y^0) q^1)_{L^2(\Omega)} + \sum_{n=1}^{M-1} \delta t \left(w^n, \frac{q^n - q^{n+1}}{\delta t} + f'(y^n) q^{n+1} \right)_{L^2(\Omega)} \\ + (w^M, q^M)_{L^2(\Omega)} + \sum_{n=1}^M \delta t (w^n, \mathcal{A}^{\mathfrak{m}} q^n)_{L^2(\Omega)} = 0. \end{aligned}$$

Adding and subtracting the term $(w^M, q^M - q^{M+1} + \delta t f'(y^M) q^{M+1})_{L^2(\Omega)}$ in the above expression we get

$$\begin{aligned} - (w^0, q^1 - \delta t f'(y^0) q^1)_{L^2(\Omega)} + \sum_{n=1}^M \delta t \left(w^n, \frac{q^n - q^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{m}} q^n + f'(y^n) q^{n+1} \right)_{L^2(\Omega)} \\ + (w^M, q^{M+1} - \delta t f'(y^M) q^{M+1})_{L^2(\Omega)} = 0. \end{aligned}$$

It follows that, if the sequence $(q^n)_n$ solves the adjoint problem (4.6), we obtain that

$$\sum_{n=1}^M \delta t (w^n, y^n)_{L^2(\mathcal{O})} = (w^0, q^1 - \delta t f'(y^0) q^1)_{L^2(\Omega)}.$$

Hence (4.7) is equivalent to ask

$$(w_0, q^1 - \delta t f'(y^0) q^1)_{L^2(\Omega)} = 0 \quad \forall w_0 \in \mathbb{R}^{\mathfrak{m}},$$

and by using the condition (4.2), this is equivalent to

$$q^1 = 0.$$

The proof is complete. ■

Remark 4.2 We would like to emphasize the fact that the fully discrete equation satisfied by $(q^n)_n$ as well as the insensitizing condition $q^1 = 0$ cannot be chosen at the user convenience and replaced by any consistent time discrete version of (1.5). Indeed, the precise form of those equations depend on the discretization chosen for the state equation and for the fully discrete functional Ψ .

As an example, following similar computations as the ones in the previous proposition one can prove that if we replace (4.3) by

$$\Psi(y_{\delta t}) := \frac{1}{2} \sum_{n=1}^M \delta t \int_{\mathcal{O}} \left| \frac{y^n + y^{n-1}}{2} \right|^2, \quad \forall y_{\delta t} \in L^2_{\delta t}(0, T; \mathbb{R}^{\mathfrak{m}}),$$

and keeping the state equation (4.1), then the insensitizing condition becomes

$$q^0 = 0,$$

where $(q^n)_{0 \leq n \leq M}$ satisfies

$$\begin{cases} \frac{q^n - q^{n+1}}{\delta t} + \mathcal{A}^{\text{m}} q^n + f'(y^n) q^{n+1} = \mathbf{1}_O \frac{y^{n-1} + 2y^n + y^{n+1}}{4}, & \forall n \in \llbracket 0, M \rrbracket, \\ q^{M+1} = 0, \end{cases}$$

with the convention $y^{-1} = -y^0$ and $y^{M+1} = -y^M$.

As for the semi discrete model, we do not expect to be able to solve the previous problem but rather a relaxed version of it, that is to find a fully discrete control such that the solution of (4.5)-(4.6) satisfies

$$|q^1|_{L^2(\Omega)} \leq C \sqrt{\phi(h)}, \quad \text{and} \quad \|v_{\delta t}\|_{L^2(0, T; \mathbb{R}^{\text{m}})} \leq C, \quad (4.9)$$

for a suitably chosen function $h \mapsto \phi(h)$ and some C depending only on uniform bounds for the source term (and/or the initial data) in some appropriate norms, but not on the discretization parameters h and δt .

4.2 Linearized problem

In order to solve the nonlinear null-control problem (or its relaxed version, that is satisfying (4.9)) we will use a fixed point procedure. To this end we first need to consider the linearized problem defined as follows. We suppose given a set of discrete functions $(a^n)_{0 \leq n \leq M-1} \in (\mathbb{R}^{\text{m}})^M$ and $(b^n)_{1 \leq n \leq M} \in (\mathbb{R}^{\text{m}})^M$ and we conventionally set $b^0 = 0$. We will deal with the following linearized controlled cascade system

$$\begin{cases} \frac{y^{n+1} - y^n}{\delta t} + \mathcal{A}^{\text{m}} y^{n+1} + a^n y^n = \mathbf{1}_\omega v^{n+1} + \xi^{n+1}, & \forall n \in \llbracket 0, M-1 \rrbracket, \\ y^0 = y_0, \end{cases} \quad (4.10)$$

$$\begin{cases} \frac{q^n - q^{n+1}}{\delta t} + \mathcal{A}^{\text{m}} q^n + b^n q^{n+1} = \mathbf{1}_O y^n, & \forall n \in \llbracket 1, M \rrbracket, \\ q^{M+1} = 0. \end{cases} \quad (4.11)$$

We would like to build a control $v_{\delta t}$ to ensure that $q^1 = 0$, or at least that q^1 solves some estimate similar to (4.9).

With the above notation and following the methodology of the penalized HUM (see for instance [15, 16, 5]), we introduce for some penalization parameter $\varepsilon > 0$ (to be determined later) the following primal fully discrete functional

$$F_{\varepsilon, h, \delta t}(v_{\delta t}) := \frac{1}{2} \sum_{n=1}^M \delta t |v^n|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon} |q^1|_{L^2(\Omega)}^2, \quad \forall v_{\delta t} \in L^2_{\delta t}(0, T; \mathbb{R}^{\text{m}}),$$

that we wish to minimize onto the whole fully discrete control space $L^2_{\delta t}(0, T; \mathbb{R}^{\text{m}})$ and where q^1 is taken from the solution of (4.10)-(4.11) associated with the control $v_{\delta t}$. Note that, since q^1 is an affine function of $v_{\delta t}$, it is straightforward to prove that this functional has a unique minimizer without any assumption on the various parameters of the problem. This is one of the main interest of the penalized HUM approach: the optimal penalized control always exist and is unique and studying the controllability properties of the system simply amounts to analyzing the behavior of this control with respect to the penalization parameter ε , in connection with the discretization parameters.

Let us first identify the dual functional for the above optimization problem.

Proposition 4.3 *For any $\varepsilon > 0$, and any $p_0 \in \mathbb{R}^{\text{m}}$, we define the functional*

$$J_{\varepsilon, h, \delta t}(p_0) := \frac{1}{2} \sum_{n=1}^M \delta t |\mathbf{1}_\omega z^n|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} |p_0|_{L^2(\Omega)}^2 + \sum_{n=1}^M \delta t (\xi^n, z^n)_{L^2(\Omega)} + (y_0 - \delta t a^0 y_0, z^1)_{L^2(\Omega)}, \quad (4.12)$$

where the sequence $(z^n, p^n)_n$ is the solution to the following adjoint problem

$$\begin{cases} \frac{z^n - z^{n+1}}{\delta t} + \mathcal{A}^{\text{or}} z^n + a^n z^{n+1} = \mathbf{1}_{\mathcal{O}} p^n, & n \in \llbracket 1, M \rrbracket, \\ z^{M+1} = 0, \end{cases} \quad (4.13)$$

$$\begin{cases} \frac{p^{n+1} - p^n}{\delta t} + \mathcal{A}^{\text{or}} p^{n+1} + b^n p^n = 0, & n \in \llbracket 0, M-1 \rrbracket, \\ p^0 = p_0. \end{cases} \quad (4.14)$$

The functionals $F_{\varepsilon, h, \delta t}$ and $J_{\varepsilon, h, \delta t}$ are dual one from each other in the sense that their respective minimizers $v_{\varepsilon, h, \delta t} \in L^2_{\delta t}(0, T; \mathbb{R}^{\text{or}})$ and $p_{0, \varepsilon, h, \delta t} \in \mathbb{R}^{\text{or}}$ satisfy

$$\inf_{L^2_{\delta t}(0, T; \mathbb{R}^{\text{or}})} F_{\varepsilon, h, \delta t} = F_{\varepsilon, h, \delta t}(v_{\varepsilon, h, \delta t}) = -J_{\varepsilon, h, \delta t}(p_{0, \varepsilon, h, \delta t}) = -\inf_{\mathbb{R}^{\text{or}}} J_{\varepsilon, h, \delta t},$$

and

$$v_{\varepsilon, h, \delta t} = (\mathbf{1}_{\omega} z^n_{\varepsilon, h, \delta t})_{1 \leq n \leq M}, \quad (4.15)$$

where $(z^n_{\varepsilon, h, \delta t})_n$ is the solution to (4.13)-(4.14) with initial data $p_0 = p_{0, \varepsilon, h, \delta t}$.

Moreover, the value of the target q^1 associated with the penalized HUM control $v_{\varepsilon, h, \delta t}$ is given by

$$q^1 = -\varepsilon p_{0, \varepsilon, h, \delta t}.$$

Remark 4.4 Even though our theoretical results proved in the previous sections only concern the case where $y_0 = 0$ (see Remark 1.5), we have considered here the general case where $y_0 \neq 0$ in (4.5) in order to perform numerical computations also in that case. That is the reason why y_0 appears in the definition of the dual functional. As a consequence, we will be able to illustrate numerically different results already known about the class of initial data that can be insensitized. Note that when $y_0 = 0$, the functional (4.12) is in fact the fully discrete version of (3.4)

Proof. For any $v_{\delta t}, \xi_{\delta t} \in L^2_{\delta t}(0, T; \mathbb{R}^{\text{or}})$, and any $y_0 \in \mathbb{R}^{\text{or}}$, we denote by $L(v_{\delta t}, \xi_{\delta t}, y_0)$ the value q^1 of the corresponding solution of (4.10)-(4.11). We can write

$$\begin{aligned} F_{\varepsilon, h, \delta t}(v_{\delta t}) &= \frac{1}{2} \|v_{\delta t}\|_{L^2_{\delta t}(0, T; \mathbb{R}^{\text{or}})}^2 + \frac{1}{2\varepsilon} |L(v_{\delta t}, \xi_{\delta t}, y_0)|_{L^2(\Omega)}^2 \\ &= \frac{1}{2} \|v_{\delta t}\|_{L^2_{\delta t}(0, T; \mathbb{R}^{\text{or}})}^2 + \frac{1}{2\varepsilon} |L(v_{\delta t}, 0, 0) + L(0, \xi_{\delta t}, y_0)|_{L^2(\Omega)}^2, \end{aligned}$$

where we used the linearity of the operator L in the last equality.

The Fenchel-Rockafellar duality theorem (see [11]) gives that

$$\inf_{v_{\delta t} \in L^2_{\delta t}(0, T; \mathbb{R}^{\text{or}})} F_{\varepsilon, h, \delta t}(v_{\delta t}) = -\inf_{p_0 \in \mathbb{R}^{\text{or}}} J_{\varepsilon, h, \delta t}(p_0),$$

where

$$J_{\varepsilon, h, \delta t}(p_0) := \frac{1}{2} \|L(\cdot, 0, 0)^* p_0\|_{L^2_{\delta t}(0, T; \mathbb{R}^{\text{or}})}^2 + \frac{\varepsilon}{2} |p_0|_{L^2(\Omega)}^2 + (p_0, L(0, \xi_{\delta t}, y_0))_{L^2(\Omega)}. \quad (4.16)$$

It remains to check that this formula is equivalent to (4.12) which amounts in particular to compute the adjoint of $L(\cdot, 0, 0)$. To this end, for any p_0 we denote by $(p^n)_n$ and $(z^n)_n$ the solutions of (4.13)-(4.14) and by $(y^n)_n$ and $(q^n)_n$ the solutions of (4.10)-(4.11) associated with the control $v_{\delta t}$ the source term $\xi_{\delta t}$ and the initial data y_0 .

- Step 1 : We multiply by p^n the equation satisfied by q^n , we use that $q^{M+1} = 0$, and then the equation satisfied by $(p^n)_n$ to obtain

$$\begin{aligned} \sum_{n=1}^M \delta t (y^n, p^n)_{L^2(\mathcal{O})} &= \sum_{n=1}^M \delta t \left(p^n, \frac{q^n - q^{n+1}}{\delta t} + A q^n + b^n q^{n+1} \right)_{L^2(\Omega)} \\ &= \sum_{n=1}^M \delta t \left(q^n, \frac{p^n - p^{n-1}}{\delta t} + A p^n + b^{n-1} p^{n-1} \right)_{L^2(\Omega)} + (p^0, q^1)_{L^2(\Omega)} \\ &= (p_0, q^1)_{L^2(\Omega)}. \end{aligned}$$

- Step 2 : We multiply by y^n the equation satisfied by z^n . Then we use the fact that $z^{M+1} = 0$, and finally the equation satisfied by $(y^n)_n$ to obtain

$$\begin{aligned}
\sum_{n=1}^M \delta t (y^n, p^n)_{L^2(\mathcal{O})} &= \sum_{n=1}^M \delta t \left(y^n, \frac{z^n - z^{n+1}}{\delta t} + Az^n + a^n z^{n+1} \right)_{L^2(\Omega)} \\
&= \sum_{n=1}^M \delta t \left(z^n, \frac{y^n - y^{n-1}}{\delta t} + Ay^n + a^{n-1} y^{n-1} \right)_{L^2(\Omega)} + (z^1, y_0 - \delta t a^0 y_0)_{L^2(\Omega)} \\
&= \sum_{n=1}^M \delta t (z^n, v^n)_{L^2(\omega)} + \sum_{n=1}^M \delta t (z^n, \xi^n)_{L^2(\Omega)} + (z^1, y_0 - \delta t a^0 y_0)_{L^2(\Omega)}.
\end{aligned}$$

Comparing the two formulas above, we conclude that

$$(L(v_{\delta t}, \xi_{\delta t}, y_0), p_0)_{L^2(\Omega)} = \sum_{n=1}^M \delta t (z^n, v^n)_{L^2(\omega)} + \sum_{n=1}^M \delta t (z^n, \xi^n)_{L^2(\Omega)} + (z^1, y_0 - \delta t a^0 y_0)_{L^2(\Omega)}.$$

In particular, this proves that

$$L(\cdot, 0, 0)^* = (1_\omega z^n)_{1 \leq n \leq M},$$

and that

$$(L(0, \xi_{\delta t}, y_0), p_0)_{L^2(\Omega)} = \sum_{n=1}^M \delta t (z^n, \xi^n)_{L^2(\Omega)} + (z^1, y_0 - \delta t a^0 y_0)_{L^2(\Omega)}.$$

Substituting those expressions in (4.16) concludes the proof of the claim.

Moreover, the Euler-Lagrange equation satisfied by the minimizer $p_{0,\varepsilon,h,\delta t}$ of the quadratic functional $J_{\varepsilon,h,\delta t}$ can be written as follows

$$L(L(\cdot, 0, 0)^* p_{0,\varepsilon,h,\delta t}, 0, 0) + \varepsilon p_{0,\varepsilon,h,\delta t} + L(0, \xi_{\delta t}, y_0) = 0, \quad (4.17)$$

which gives with (4.15)

$$L(v_{\varepsilon,h,\delta t}, \xi_{\delta t}, y_0) = -\varepsilon p_{0,\varepsilon,h,\delta t}.$$

The proof is complete. ■

In general, it is well known that we cannot expect, for a given bounded family of initial data and source terms, that the fully discrete penalized controls are uniformly bounded when the discretization parameters h , δt and the penalization parameter tend to zero independently, see for instance [5].

Due to the additional term in the relaxed observability estimate, we can however expect to obtain uniform bounds if one consider a penalization parameter $\varepsilon = \phi(h)$ that tends to 0 in connection with the mesh size not too fast compared to some exponential and if the time step δt satisfy some very weak condition $\delta t \leq \zeta(h)$ where ζ typically tends to zero logarithmically when $h \rightarrow 0$ (see [8]).

We do not provide a detailed analysis of the fully discrete case in this paper (we postpone this study to future work) but we can already make the following remarks.

- Assume that, for some bounded family (depending on \mathfrak{M} and δt) of initial data y_0 and/or source term ξ , we have the following fully discrete observability estimate

$$\sum_{n=1}^M \delta t (\xi^n, z^n)_{L^2(\Omega)} + (y_0 - \delta t a_0 y_0, z^1)_{L^2(\Omega)} \leq C \left(\sum_{n=1}^M \delta t |\mathbf{1}_\omega z^n|_{L^2(\Omega)}^2 + \phi(h) |p_0|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (4.18)$$

for any solution (z, p) of (4.13)-(4.14), and any $\delta t \leq \psi(h)$.

Then, Proposition 4.3 shows that the fully discrete control $v_{\phi(h),h,\zeta(h)}$ given in (4.15) with $\varepsilon = \phi(h)$ and $\delta t = \zeta(h)$ remains bounded as $h \rightarrow 0$ and that the associated value of the target q^1 satisfies

$$|q^1|_{L^2(\Omega)} \leq C\sqrt{\varepsilon} = C\sqrt{\phi(h)}.$$

This proves the $\phi(h)$ -insensitizing property.

- The estimate (4.18) depends on the particular source term and initial data which is considered. One way to obtain more generic results is to prove observability inequality accounting only on the adjoint states (z, p) and not on a particular choice of the data. For instance, in view of our results on the semi discrete case of Section 2, we may hope to be able to prove that

$$\sum_{n=1}^M \delta t \exp(-\mathcal{M}/(n\delta t)) |z^n|_{L^2(\Omega)}^2 \leq C_{obs}^2 \left(\sum_{n=1}^M \delta t |\mathbf{1}_\omega z^n|_{L^2(\Omega)}^2 + e^{-\frac{C_1}{h}} |p_0|_{L^2(\Omega)}^2 \right), \quad (4.19)$$

for any solution (z, p) of (4.13)-(4.14), and any $\delta t \leq \zeta(h)$.

Observe that, using the Cauchy-Schwarz inequality, (4.19) implies (4.18) as soon as we consider $y_0 = 0$ (this is somehow natural in this problem as we have already explained) and a family of discrete source terms $\xi \in L^2_{\delta t}(0, T; \mathbb{R}^m)$ that are bounded in $L^2(e_{\mathcal{M}})$. We refer to [8] for the proof of inequalities similar to (4.19) in the framework of the null-control problem. However the proofs given in this reference rely on the discrete Lebeau-Robbiano strategy which is not useful for dealing with insensitizing control problems. Proving (4.19) is thus still an open problem.

4.3 Computational method

We devote this section to address the actual computation of the fully discrete insensitizing controls for the linearized problem. As noted in Proposition 4.3, such controls are the minimizers of $F_{\varepsilon, h, \delta t}$ but may be also be computed by minimizing the dual functionals $J_{\varepsilon, h, \delta t}$. Since the dual functionals are defined on the finite dimensional space \mathbb{R}^m , instead of the larger space $L^2_{\delta t}(0, T; \mathbb{R}^m)$, it is somehow more convenient to apply optimization algorithms to the dual functional. For a given set of parameters $\varepsilon, h, \delta t$ and data $(a^n)_n, (b^n)_n, \xi_{\delta t}, y_0$, our problem is to solve the linear equation (4.17). Since it is a symmetric positive definite problem, we usually solve it by a conjugate gradient algorithm in \mathbb{R}^m that needs, at each iteration, the computation of the linear operator $LL^* + \varepsilon \text{Id}$ (to simplify, we have denoted by L the operator $L(\cdot, 0, 0)$).

The actual computation of the term LL^* applied to some $p_0 \in \mathbb{R}^m$ must be regarded as follows.

1. In a first step, we solve the adjoint problem with the initial datum p_0 . This is achieved in two steps. We begin by solving the homogeneous forward system

$$\begin{cases} \frac{p^{n+1} - p^n}{\delta t} + \mathcal{A}^m p^{n+1} + b^n p^n = 0, & n \in \llbracket 0, M-1 \rrbracket, \\ p^0 = p_0. \end{cases}$$

Then, we solve the backward system for z with second member $\mathbf{1}_\mathcal{O} p^n$

$$\begin{cases} \frac{z^n - z^{n+1}}{\delta t} + \mathcal{A}^m z^n + a^n z^{n+1} = \mathbf{1}_\mathcal{O} p^n, & n \in \llbracket 1, M \rrbracket, \\ z^{M+1} = 0. \end{cases}$$

2. We compute the restriction of the solution $(z^n)_n$ to the control domain ω by setting $v^n = \mathbf{1}_\omega z^n$. This gives a control in $L^2_{\delta t}(0, T; \mathbb{R}^m)$.
3. Afterwards, we proceed to compute the solution $(y^n)_n$ with this particular control and without initial data and source term. More precisely, we solve

$$\begin{cases} \frac{y^{n+1} - y^n}{\delta t} + \mathcal{A}^m y^{n+1} + a^n y^n = \mathbf{1}_\omega z^{n+1}, & \forall n \in \llbracket 0, M-1 \rrbracket, \\ y^0 = 0. \end{cases}$$

Finally, we solve for the backward problem for q with second member $\mathbf{1}_\mathcal{O} y^n$

$$\begin{cases} \frac{q^n - q^{n+1}}{\delta t} + \mathcal{A}^m q^n + b^n q^{n+1} = \mathbf{1}_\mathcal{O} y^n, & \forall n \in \llbracket 1, M \rrbracket, \\ q^{M+1} = 0. \end{cases}$$

The value of $L(L^* p_0)$ is then given by q^1 .

Remark 4.5 *Note that the procedure to compute the control for a given problem basically requires to solve four parabolic systems at each iteration of the minimization algorithm: a forward parabolic equation with the zero-order term a^n (resp. b^n) for y (resp. for p), one backward parabolic equation with the zero-order term a^n (resp. b^n) for z (resp. for q).*

4.4 Semilinear problem

We may now come back to solving the nonlinear problem. For a given mesh, a given time step and a given penalization parameter ε (that may depend on h as discussed previously), we propose to simply use a fixed point procedure, with a relaxation parameter $\theta \in (0, 1]$ based on an iteration $\bar{y}_{\delta t} = (\bar{y}^n)_n \mapsto y_{\delta t} = (y^n)_n$ described as follows

- Given a state $\bar{y}_{\delta t} = (\bar{y}^n)_n$ satisfying $\bar{y}^0 = y_0$, we set

$$a^n = g(\bar{y}^n), \forall n \in \llbracket 0, M-1 \rrbracket, \quad b^n = f'(\bar{y}^n), \forall n \in \llbracket 1, M \rrbracket, \quad b^0 = 0,$$

where the function g is defined in (3.8).

- We solve the penalized insensitizing control problem for the linear system (4.10)-(4.11) associated with those coefficients. This is done by the conjugate gradient method described in the previous section. The controlled solution is denoted by $y_{\delta t}$.
- If $\bar{y}_{\delta t} - y_{\delta t}$ is small enough then we stop the nonlinear solver and take the (linear) HUM control $v_{\delta t} = (v^n)_n$ computed during the previous step as an insensitizing control for the nonlinear equation.
- Otherwise, we step over a new iteration by using as a new guess the state

$$\bar{y}_{\delta t} \leftarrow \theta y_{\delta t} + (1 - \theta) \bar{y}_{\delta t}.$$

In the nonlinear test cases presented below we have used $\theta = 0.8$ and less than 10 nonlinear iterations were necessary to achieve the convergence criterion

$$\|y_{\delta t} - \bar{y}_{\delta t}\|_{\infty} / \|\bar{y}_{\delta t}\|_{\infty} < 10^{-5}.$$

5 Numerical experiments

We present here some results obtained with the methodology described and analyzed above to the problem of insensitizing control in various situations. In accordance with the discussion in Section 4, we use the standard finite-difference scheme on a uniform mesh of the domain $\Omega = (0, 1)$ for the space discretization and the semi-implicit scheme in the time variable. We denote by N the number of points in the mesh and by M the number of time intervals. It has been observed in [5] that the results in those kind of problems does not depend too much on the time step, as soon as it is chosen to ensure at least the same accuracy as the space discretization. The same observation can be done here so that we will always take $M = 2000$ in order to concentrate the discussion on the dependency of the results with respect to the mesh size h . Observe that, with such a choice, the stability condition (4.2) is actually satisfied in all the presented cases.

In all the results presented below, the chosen horizon time is $T = 1$ and the underlying elliptic operator \mathcal{A}^{m} is a discretization of the operator $-0.1\partial_x^2$. Note that the presence of a diffusion coefficient which is not equal to 1 does not change anything to our analysis.

The methodology is the one described in Section 4.4 for the nonlinear case and in Section 4.3 for the linear case.

5.1 The insensitizing problem

As previously noticed, the first positive result on the existence of insensitizing controls for (1.1) was developed in [25] in the case where $y_0 = 0$, $\xi \in L^2(e_{\mathcal{M}})$ and $\omega \cap \Omega \neq \emptyset$. Our main result in this paper (Theorem 1.4) was precisely a semi discrete version of this result. We also discussed the extensions to the fully discrete case in Section 4.

Our goal here is to provide illustrations of those results obtained by our simulation tool in the case where all those assumptions are fulfilled but also to investigate numerically some situations that are not covered by our analysis.

5.1.1 Illustration of positive insensitizing results

We begin by testing the case of a localized control domain $\omega = (0, 0.5)$, an observatory domain $\mathcal{O} = (0.3, 0.8)$ and $y_0 \equiv 0$. The source term ξ is selected as the space independent function $\xi(x, t) = \mathbf{1}_{[0.4, 1]}(t)$. This ensures in particular that $\xi \in L^2(e_{\mathcal{M}})$ for any value of \mathcal{M} . Moreover we choose the nonlinear term to be $f(y) = -0.1 \sin(y)$. All the assumptions of our main result are thus satisfied.

As discussed above, we choose the penalization term ε as a function of h . More precisely, we choose $\varepsilon = \phi(h) = h^4$ in all the simulations presented in this paper. This choice is consistent with the order of approximation of the finite difference scheme since we expect to obtain $|q^1|_{L^2(\Omega)} \leq C\sqrt{\phi(h)} \sim Ch^2$. We refer the reader to [5] for a more detailed discussion on the selection of the function $\phi(h)$ in the context of the null-controllability problem.

We first plot on Figure 1 the solutions (y, q) without any control. We observe that $y = 0$ on $(0, 0.4)$ since the initial data is zero and the source term vanishes on that time interval. Moreover, the adjoint state $q(0)$ is clearly not zero which proves that, without any control, the functional Ψ is indeed very sensitive to perturbations of the initial data (see also Figure 4).

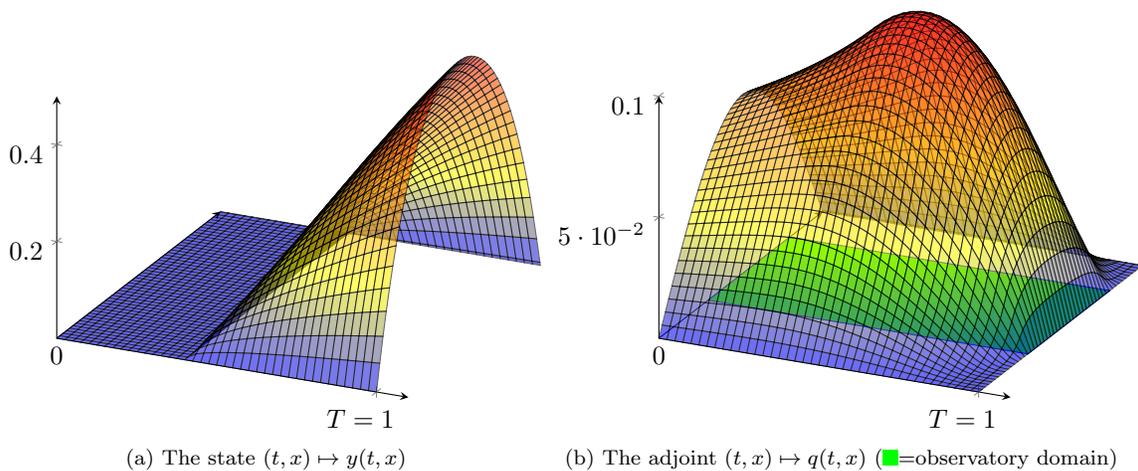


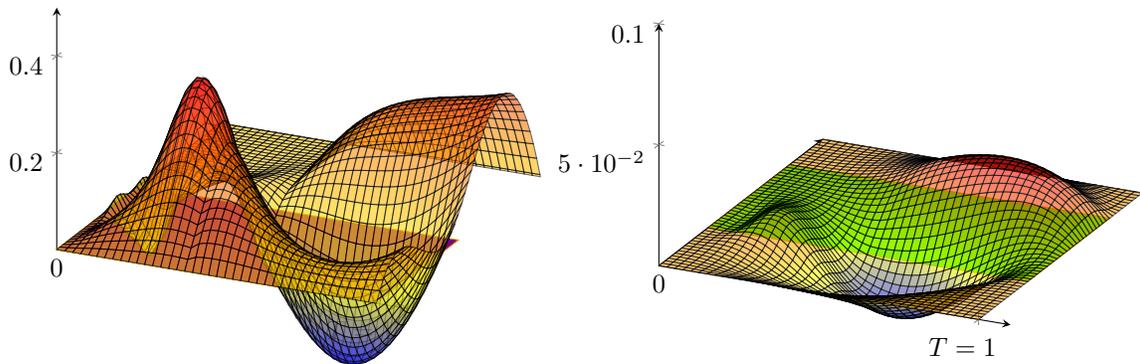
Figure 1: $f(y) = -0.1 \sin(y)$, $y_0 = 0$, $\xi(x, t) = \mathbf{1}_{[0.4, 1]}(t)$. Uncontrolled solution.

In Figure 2, we plot the solution (y, q) obtained with the HUM control v computed by the algorithm described in previous sections. We observe that, due to the action of the control, y is no more equal to 0 on the time interval $(0, 0.4)$ and that the adjoint state at the initial time $q(0)$ is very small (see the discussion below), which illustrates that the functional Ψ is now insensitized to the initial data perturbations (see also Figure 4).

As far as the asymptotic behavior of the method is concerned, we present in Figure 3 the behavior of various quantities of interest when the mesh size goes to 0. More precisely, we observe that the control cost $\|v_{h, \delta t}\|_{L^2_{\delta t}(0, T; \mathbb{R}^{2M})}$ as well as the optimal energy $\inf F_{\phi(h), h, \delta t}$ both remain bounded as $h \rightarrow 0$. In the meantime, we see that $|q^1|_{L^2(\Omega)}$ behaves like $\sim C\sqrt{\phi(h)} = Ch^2$ as predicted by the theory.

Finally, to illustrate the insensitizing property for the functional Ψ defined in (1.3), we plot the values of $\Psi(y)$ for solutions of our problem associated with various perturbed initial data $y(0) = y_0 + \tau w_0$ in the case without control and in the case with the computed control acting on ω . In Figure 4, we can observe the expected behaviour:

- In the controlled case, the value of Ψ is minimal for $\tau = 0$ and for any choice of w_0 .
- In the uncontrolled case, the values of Ψ around $\tau = 0$ strongly depend on τ and w_0 , which proves that Ψ is sensitive to the perturbations of the initial data.



(a) The state $(t, x) \mapsto y(t, x)$ (■=control domain) (b) The adjoint $(t, x) \mapsto q(t, x)$ (■=observatory domain)

Figure 2: $f(y) = -0.1 \sin(y)$, $y_0 = 0$, $\xi(x, t) = \mathbf{1}_{[0.4, 1]}(t)$. Controlled solution.

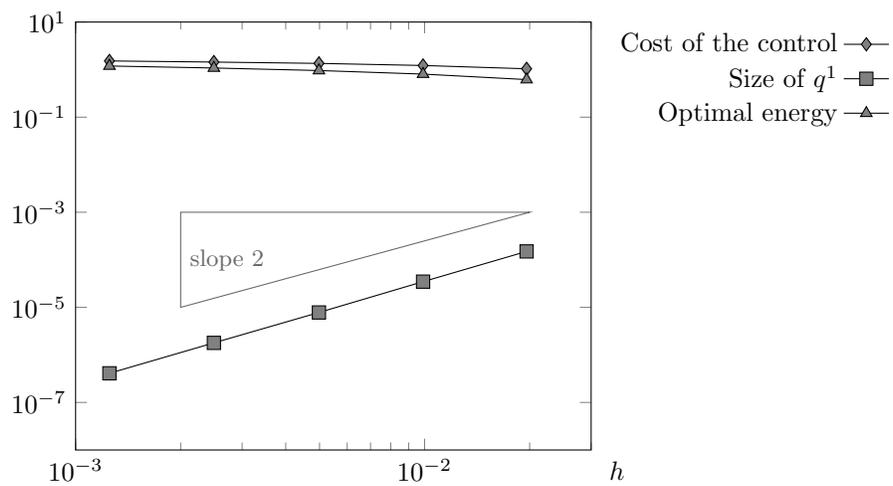


Figure 3: Convergence properties of the method for insensitizing problem.

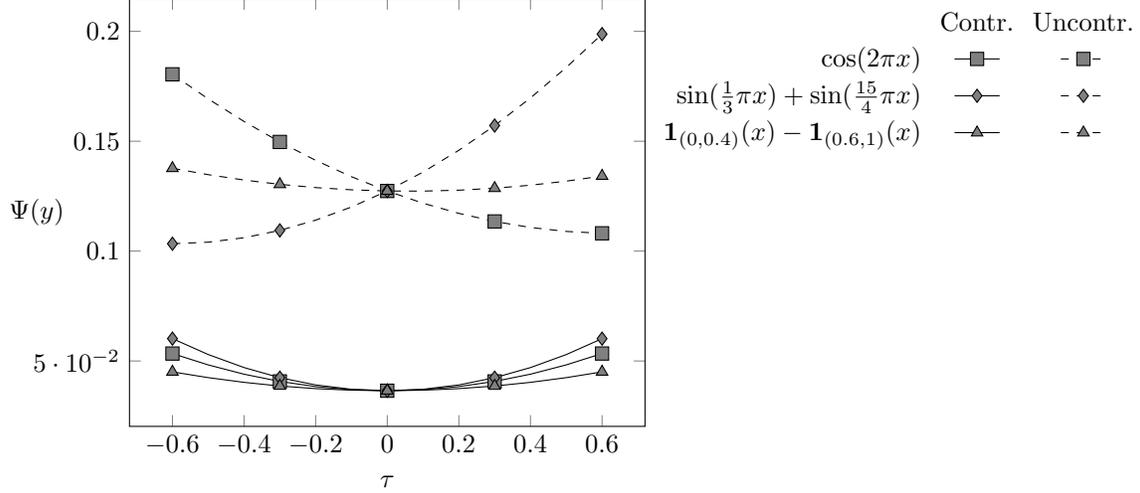


Figure 4: Value of $\Psi(y)$ for different parameters τ and initial perturbations w_0 .

5.1.2 The class of initial data that can be insensitized

The insensitizing results in Theorem 1.4 and in [25, Theorem 1] use the fact that $y_0 \equiv 0$. Actually, there are very few results identifying the class of initial data that can be insensitized. In [26], the authors studied this question under particular geometric configurations of the subdomain \mathcal{O} to be insensitized and of the control set ω . To simplify a little, we only consider now the linear case, that is when $f(y) = 0$.

The case $\mathcal{O} \subset \omega$: In that situation, one may obtain through classical energy estimates, the following inequality

$$\int_{\Omega} |\partial_x z(x, 0)|^2 \leq C \iint_{\omega \times (0, T)} (|\partial_t z|^2 + |\partial_x^2 z|^2),$$

for solutions to the adjoint system

$$\begin{cases} -\partial_t z - 0.1 \partial_x^2 z = \mathbf{1}_{\mathcal{O}} p, & \text{in } \Omega \times (0, T), \\ \partial_t p - 0.1 \partial_x^2 p = 0, & \text{in } \Omega \times (0, T), \\ z = p = 0, & \text{on } \partial\Omega \times (0, T), \\ z(T) = 0, \quad p(0) = p_0. \end{cases}$$

This estimate of a Sobolev norm on $z(0)$ in terms of the observation for derivatives of z in ω implies that the insensitization property can be achieved for any initial data in $L^2(\Omega)$ as soon as we allow the controls to belong to some negative Sobolev space. It is not known if the result still holds for L^2 controls.

We would like to illustrate this issue in Figure 5. For this experiment we have used that $\omega = (0.3, 0.8)$, $\xi = 0$ and $y_0(x) = \mathbf{1}_{(0.2, 0.7)}(x)$.

We compare the behavior as $h \rightarrow 0$ of the computed solutions in the case where $\mathcal{O} = (0.4, 0.6)$ (Subfigure 5a) and in the case $\mathcal{O} = (0, 0.6)$ (Subfigure 5b).

In the first case, we observe a similar situation as in Figure 3 namely the boundedness of the control cost and of the optimal energy as well as the convergence of the target q^1 to zero like $h \mapsto \sqrt{\phi(h)}$. This seems to confirm that such a system is insensitizable with L^2 controls.

The results are very different in the second situation, where the cost of the control increases like $h \mapsto h^{-1}$, the optimal energy like $h \mapsto h^{-2}$ whereas the target q^1 tends to zero like h . This seems to confirm that uniform relaxed observability estimates do not hold for this system and that we can only achieve approximate insensitizing controllability in general if $\mathcal{O} \not\subset \omega$.

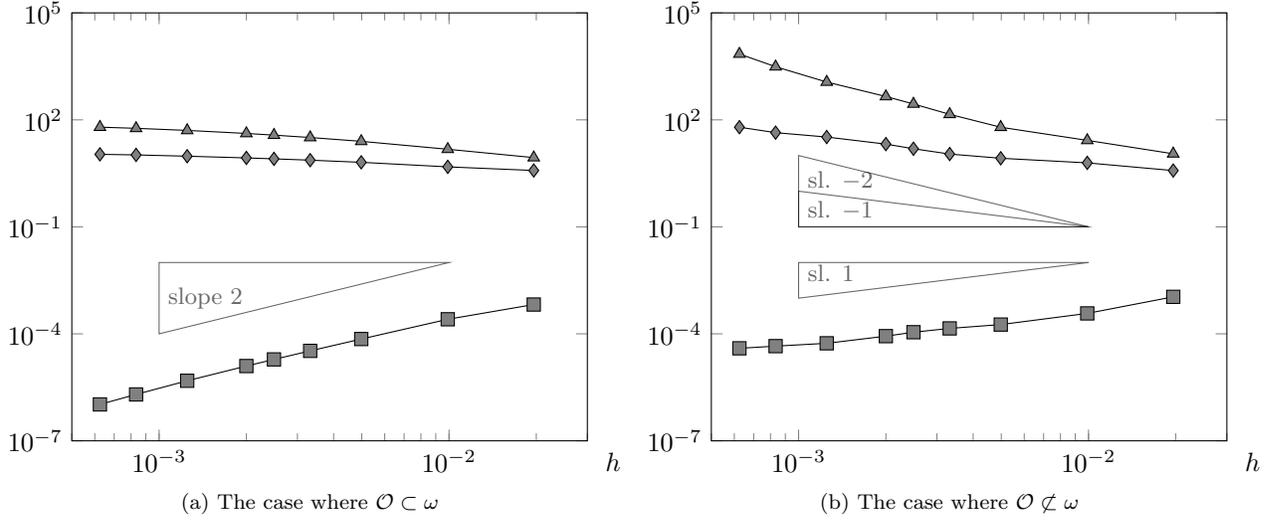


Figure 5: $y_0(x) = \mathbf{1}_{(0.2,0.7)}(x)$, $\xi = 0$, $f = 0$, $\omega = (0.3, 0.8)$. Same legend as in Figure 3.

The case $\mathcal{O} = \Omega$: In this situation, it is known from [26, Theorem 2.2] that it is possible to insensitize any initial data of the form $y_0 = \sum_{j=1}^{\infty} b_j \varphi_j$ as soon as

$$\sum_{j=1}^{\infty} e^{B\sqrt{\lambda_j}} b_j^2 < \infty, \quad B > 0, \quad (5.1)$$

where λ_j and φ_j are the eigenvalues and eigenfunctions of the Dirichlet Laplacian, respectively. This property can be understood as regularity/compatibility conditions for an initial data to be insensitized.

In Figure 6, we present some experiments with different initial data. In Subfigures 6a and 6b, we select initial data satisfying condition (5.1) and, as expected, we observe that the convergence ratio of q^1 is $\sqrt{\phi(h)} = h^2$ as well as the boundedness of the control cost. In SubFigure 6c we select an initial data that does not fulfill (5.1): we observe that the size of the target actually goes to 0 but at a lower rate $h \mapsto h$, while the optimal energy is blowing up as $h \mapsto h^{-2}$. This is again a numerical evidence that, for such data that does not fulfill (5.1), the system seems to be approximately, but not exactly insensitizable.

5.1.3 The influence of the source term ξ

It has been widely discussed if the hypothesis on the source term, namely $\xi \in L^2(e_{\mathcal{M}})$ for some $\mathcal{M} > 0$, is indeed necessary for the insensitizing property to hold. We propose in Figure 7 different simulations for space independent source terms $\xi(x, t) = \exp(-\frac{\mathcal{M}}{t})$ with different values of \mathcal{M} .

We observe that for the larger values of \mathcal{M} we maintain the insensitability result, but as these values decrease to 0 the convergence rate of the target q^1 becomes close to h and not h^2 as expected if the uniform relaxed observability would hold. Similarly, the optimal energy and the control cost seem to blow up like h^{-1} and h^{-2} respectively. As discussed in [5], such behaviors might correspond to the case where the continuous problem is approximately but not null controllable (and it may even depend on the time T). In the insensitizing framework, this means that we are in the context of ε -insensitizing (see for instance [2]). In these cases, further investigation is desirable.

5.1.4 The case $\omega \cap \mathcal{O} = \emptyset$

As in other insensitizing results (see e.g. [25], [3], [20], ...), we strongly used in our proofs the fact that $\omega \cap \mathcal{O} \neq \emptyset$ in order to locally estimate p in terms of z and thus to keep only one observation term in z in ω . Without this hypothesis, we would not be able to obtain the observability inequality (2.2).

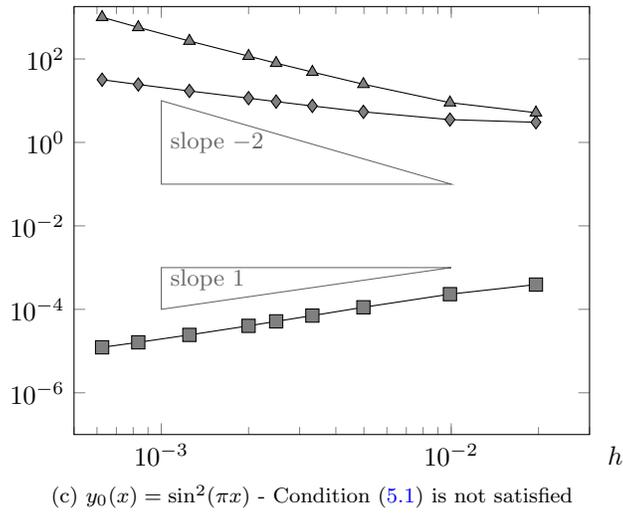
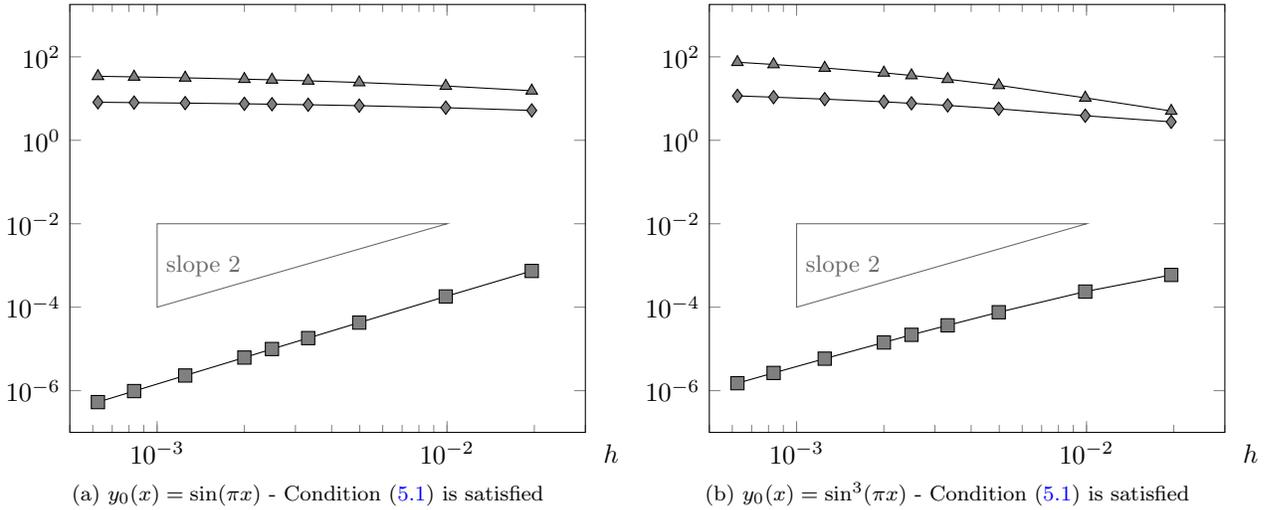
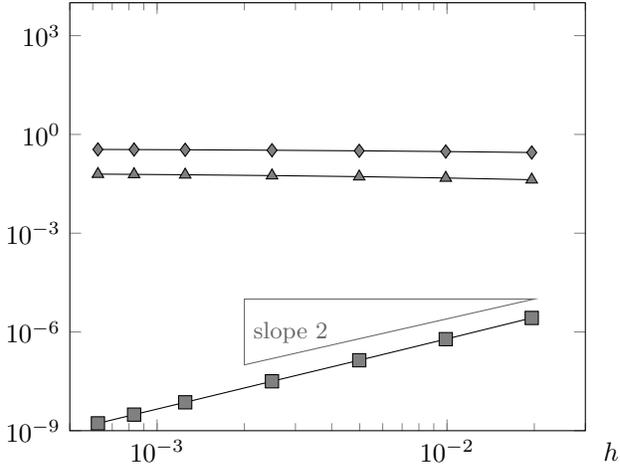
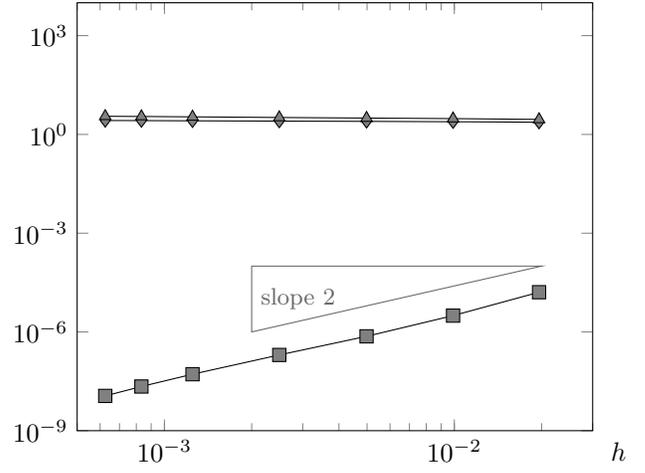


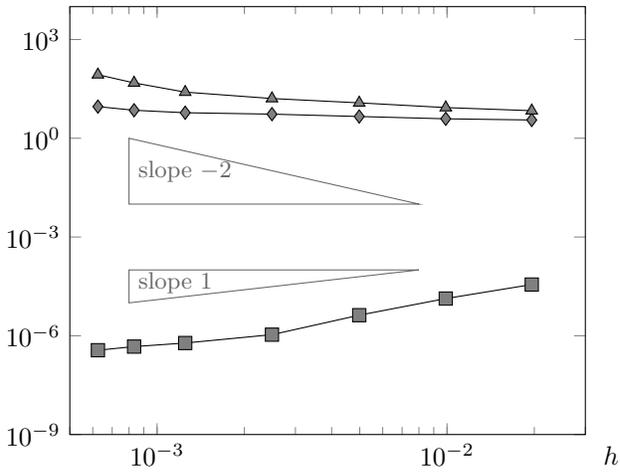
Figure 6: The case where $\mathcal{O} = \Omega$ with $\xi = 0$, $\omega = (0, 0.5)$. Same legend as in Figure 3.



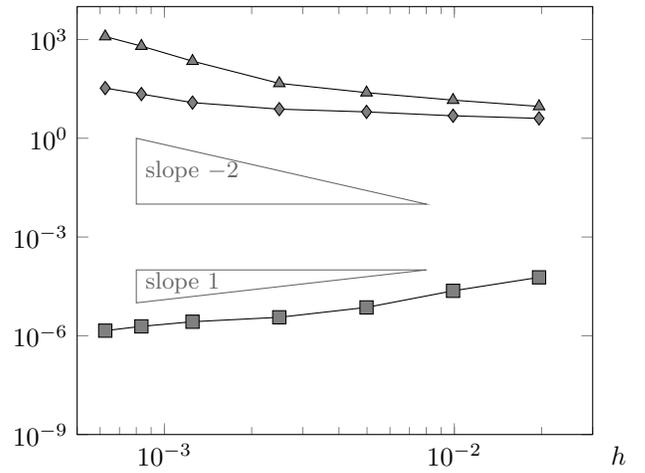
(a) $\mathcal{M} = 1$



(b) $\mathcal{M} = 0.1$



(c) $\mathcal{M} = 0.01$



(d) $\mathcal{M} = 0.0$

Figure 7: Different values of \mathcal{M} in the source term. Same legend as in Figure 3.

In [21], the authors proved that, for system (1.1), the functional Ψ defined in (1.3) can be actually ε -insensitized when $\omega \cap \mathcal{O} = \emptyset$, for any $y_0 \in L^2(\Omega)$ and any $\xi \in L^2(Q)$. Using our computational code, we are able to test different geometric configurations of ω and \mathcal{O} and then to begin investigating the open problems in that field.

For instance, we choose $\omega = (0, 0.5)$, $\mathcal{O} = (0.8, 1)$, $y_0(x) = \sin^2(\pi x)$, $f(y) = 0$ and $\xi(x, t) = 0$. In Figure 8, we observe that the size of the computed target $|q^1|_{L^2(\Omega)}$ decreases to 0 like $h^{0.6}$ instead of the optimal rate $h \mapsto h^2 = \sqrt{\phi(h)}$. Since only a result of ε -insensitizing is known for the continuous case, this result may express the fact that the problem may not be exactly insensitizable or that the numerical approximation may require a stronger condition on the penalization function ϕ (see [5]). Moreover, new phenomena (such as a minimal controllability time) associated to the fact that $\omega \cap \mathcal{O} = \emptyset$ may arise. This is for instance the case for the null-controllability of coupled parabolic systems, see [1]. In any case, further investigation is desirable and the numerical simulations may help to make progresses in that direction.

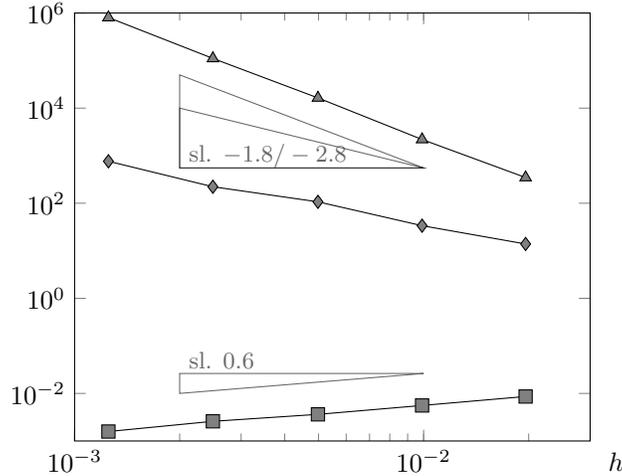


Figure 8: The case where $\mathcal{O} \cap \omega = \emptyset$. Same legend as in Figure 3.

5.1.5 A quadratic case

To conclude this section, we propose to test our computational code for a quadratic nonlinearity $f(y) = -y^2$. None of our theoretical result apply to this case and it is in fact known from [4] that, even for slightly sublinear functions f , such equations may not be insensitizable. Actually, the situation is even worse since those authors show that, for well-chosen nonlinearities f , whatever the control v is, the solution of the state equation blows up before the time T , which implies in particular that the insensitizing problem is not even meaningful in that case.

Our goal here is to show that, even if theoretical tools are lacking for studying the general nonlinear case, we may use numerical simulations to investigate the behavior of the system.

We propose here to deal with the initial data $y_0 = 0$, the source term $\xi(x, t) = 8 \times \mathbf{1}_{[0.2, 1]}$, with a control domain $\omega = (0, 0.7)$ and an observatory domain $\mathcal{O} = (0.3, 0.9)$.

With this choice of parameters, it can be shown that the uncontrolled state equation is blowing up before the final time $T = 1$ (we estimate the blow-up time to be around 0.8). However, with our algorithm we were able to produce discrete controls such that the controlled state equation is well-posed on $(0, T)$ and which is insensitized around the initial data 0. In other words the control v here has two functions: it stabilizes the nonlinear state equation on the chosen time interval and simultaneously, it ensures the insensitizing property for our functional Ψ .

In Figure 9, we observe the same expected behavior as in the linear or globally Lipschitz case. The evolution in time of the L^2 -norm of the state y , the adjoint state q and of the control v is given in Figure 10 whereas the complete shape of the solution is shown in Figure 11.

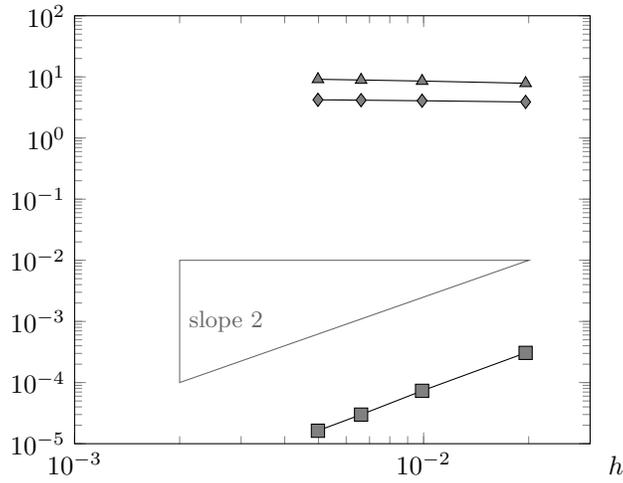


Figure 9: Convergence properties for the quadratic case. Same legend as in Figure 3.

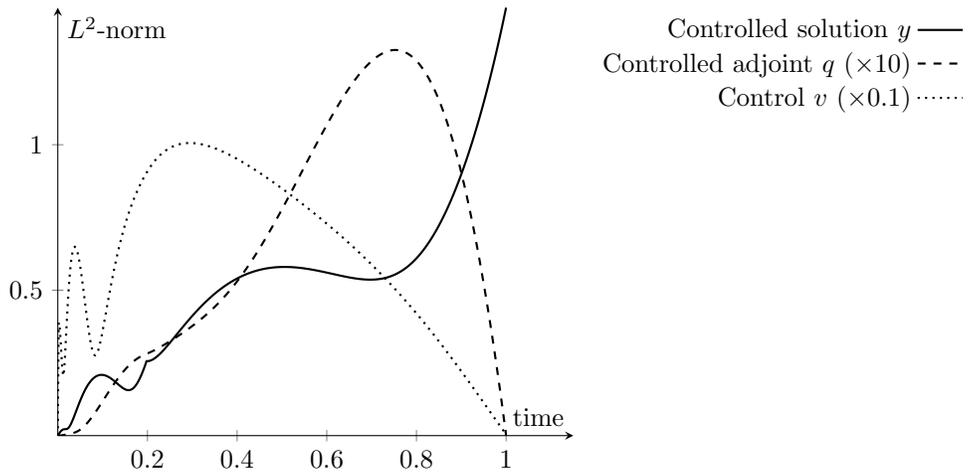


Figure 10: $f(y) = -y^2$, $y_0 = 0$, $\xi(x, t) = 8 \times \mathbf{1}_{[0.2, 1]}(t)$. Time evolution

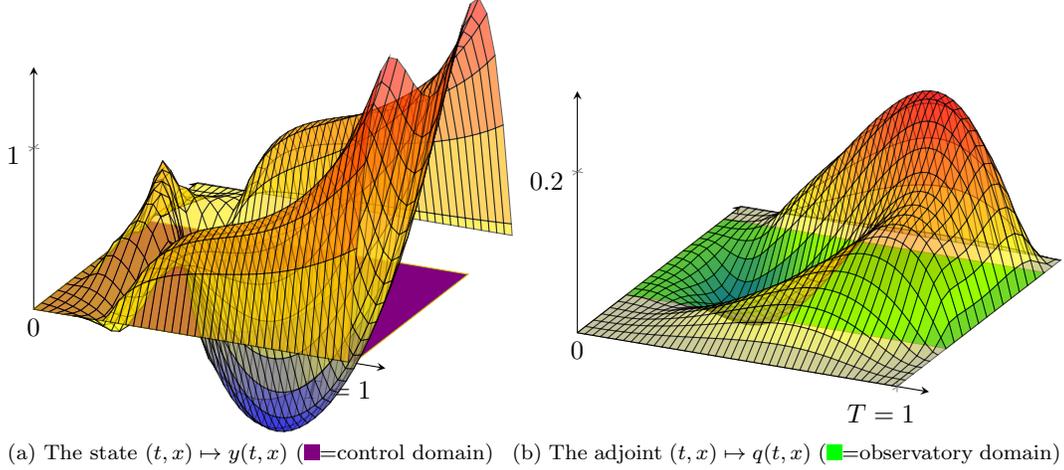


Figure 11: $f(y) = -y^2$, $y_0 = 0$, $\xi(x, t) = 8 \times \mathbf{1}_{[0.2, 1]}(t)$. Controlled solution

5.2 Simultaneous insensitizing and null control

In this section we give a short insight of a slightly more general issue than the one of insensitizing controls. Indeed, in the continuous case, we can ask for simultaneous null and insensitizing controls, that is, we look for a control $v \in L^2(\omega \times (0, T))$ such that we have simultaneously the null-controllability condition at time T ,

$$y(T) = 0,$$

and the insensitizing condition

$$q(0) = 0.$$

In the semi discrete case, we have an analogous concept. We describe it only in the linear case, for simplicity. Consider the linear semi discrete system

$$\begin{cases} \partial_t y + \mathcal{A}^{\text{sd}} y = \mathbf{1}_\omega v + \xi & \text{in } \mathbb{R}^{\text{sd}} \times (0, T), \\ -\partial_t q + \mathcal{A}^{\text{sd}} q = \mathbf{1}_O y & \text{in } \mathbb{R}^{\text{sd}} \times (0, T), \\ y = q = 0 & \text{on } \partial\mathfrak{M} \times (0, T), \\ y(0) = 0, \quad q(T) = 0. \end{cases} \quad (5.2)$$

Following the proof of Theorem 2.1, we can obtain the observability inequality

$$\iint_Q e^{-\frac{M}{t(T-t)}} |z|^2 \leq C \left(\iint_{\omega \times (0, T)} |z|^2 + e^{-C/h} (|z_F|_{L^2(\Omega)}^2 + |p_0|_{L^2(\Omega)}^2) \right), \quad (5.3)$$

for any solution (z, p) to the following adjoint system

$$\begin{cases} -\partial_t z + \mathcal{A}^{\text{sd}} z = \mathbf{1}_O p & \text{in } \mathbb{R}^{\text{sd}} \times (0, T), \\ \partial_t p + \mathcal{A}^{\text{sd}} p = 0 & \text{in } \mathbb{R}^{\text{sd}} \times (0, T), \\ z = p = 0 & \text{on } \partial\mathfrak{M} \times (0, T), \\ z(T) = z_F, \quad p(0) = p_0. \end{cases}$$

In this system, we notice that $z(T)$ is not supposed to vanish, which is the main difference with the previous case.

Remark 5.1 Note that the weight function in the left-hand side of (5.3) vanishes at $t = 0$ and $t = T$.

Adapting the results of Section 3, we can prove the simultaneous insensitizing and null control by minimizing with respect to $(z_F, p_0) \in (\mathbb{R}^{\text{sd}})^2$ the dual functional

$$J(z_F, p_0) = \frac{1}{2} \iint_{\omega \times (0, T)} |z|^2 + \iint_Q z \xi + \frac{\phi(h)}{2} (|z_F|_{L^2(\Omega)}^2 + |p_0|_{L^2(\Omega)}^2), \quad (5.4)$$

instead of the one defined in (3.4).

Assuming that $\xi \in L^2(Q)$ is such that

$$\iint_Q e^{\frac{M}{T-t}} |\xi|^2 < +\infty, \quad (5.5)$$

and by using the inequality (5.3), we can then prove that the control v built by the minimization of (5.4) yields a solution (y, q) of system (5.2) satisfying

$$\begin{aligned} |y(T)|_{L^2(\Omega)} + |q(0)|_{L^2(\Omega)} &\leq C e^{-C/h} \left(\iint_Q e^{\frac{M}{T-t}} |\xi|^2 \right)^{\frac{1}{2}}, \\ \|v\|_{L^2(Q)} &\leq C \left(\iint_Q e^{\frac{M}{T-t}} |\xi|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

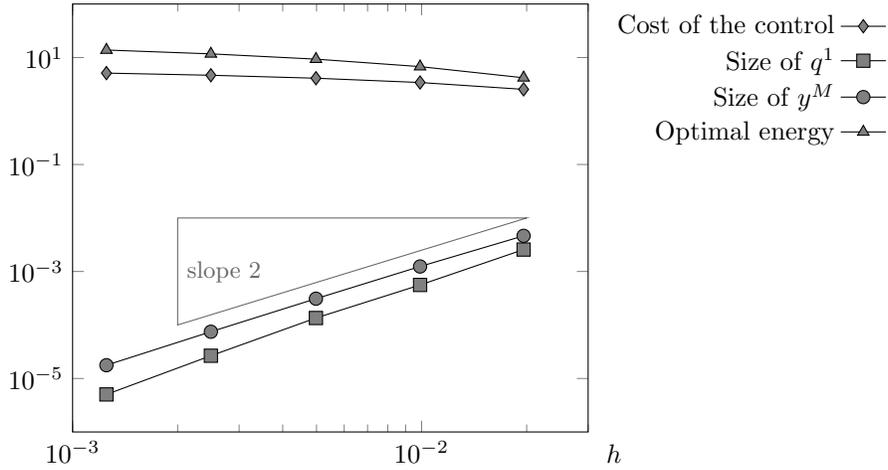


Figure 12: Simultaneous insensitizing and null-control.

In this case, we can make numerical simulations to illustrate the simultaneous null and insensitizing controls. As before, we take $\omega = (0, 0.5)$, $\mathcal{O} = (0.3, 0.8)$ and $y_0(x) = 0$. For this test, we choose the source term as

$$\xi(x, t) = \mathbf{1}_{[0.2, 0.8]}(t),$$

which verifies the integrability condition (5.5). In Figure 12, we observe that the size of the computed targets $y(T)$ and $q(0)$ behaves as expected, i.e., $\sqrt{\phi(h)} = h^2$. Moreover, the norm of the computed control remains bounded as $h \rightarrow 0$.

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A Some discrete calculus results

The objective of this appendix is to provide a summary of calculus rules for discrete operators such as D , \bar{D} and also to provide estimates for successive applications of such operators on the weight functions. We state here the results without proof. For a detailed reading we refer to [6].

To avoid cumbersome notation we introduce the following continuous difference and averaging operators. For a function f defined on \mathbb{R} we set:

$$\begin{aligned}\tau^+ f(x) &:= f(x + \frac{h}{2}), & \tau^- f(x) &:= f(x - \frac{h}{2}), \\ \mathbf{D}f &:= \frac{1}{h}(\tau^+ - \tau^-)f, & \mathbf{A}f = \hat{f} &:= \frac{1}{2}(\tau^+ + \tau^-)f.\end{aligned}$$

Discrete versions of the results we give below will be natural, indeed, with the notation given in the introduction, for a function f continuously defined on \mathbb{R} , the discrete function Df is in fact $\mathbf{D}f$ sampled on the dual mesh $\overline{\mathfrak{M}}$, and $\overline{D}f$ is $\mathbf{D}f$ sampled on the primal mesh \mathfrak{M} . We use similar meanings for averaging symbols \tilde{f} , \bar{f} (see (1.12), (1.11)), and for more general combinations: for instance $\widetilde{\overline{D}D}f$ will be the function $\widetilde{\overline{D}\mathbf{D}}f$ sampled on $\overline{\mathfrak{M}}$.

A.1 Discrete calculus formulae

Lemma A.1 *Let the functions f_1 and f_2 be continuously defined over \mathbb{R} . We have*

$$\mathbf{D}(f_1 f_2) = \mathbf{D}(f_1)\hat{f}_2 + \hat{f}_1 \mathbf{D}f_2.$$

The translation of the result to discrete functions $f_1, f_2 \in \mathbb{R}^{\mathfrak{M}}$ and $g_1, g_2 \in \mathbb{R}^{\overline{\mathfrak{M}}}$ is

$$D(f_1 f_2) = D(f_1)\tilde{f}_2 + \tilde{f}_1 D(f_2), \quad \overline{D}(g_1 g_2) = \overline{D}(g_1)\bar{g}_2 + \bar{g}_1 \overline{D}(g_2), \quad (\text{A.1})$$

$$\widetilde{\overline{D}D}(f_1 f_2) = (\widetilde{\overline{D}D}f_1)\bar{f}_2 + \bar{f}_1 (\widetilde{\overline{D}D}f_2) + 2\overline{D}f_1 \overline{D}f_2. \quad (\text{A.2})$$

Lemma A.2 *Let the functions f_1 and f_2 be continuously defined over \mathbb{R} . We have*

$$\widehat{f_1 f_2} = \hat{f}_1 \hat{f}_2 + \frac{h^2}{4} \mathbf{D}(f_1) \mathbf{D}(f_2)$$

The translation of the result to discrete functions $f_1, f_2 \in \mathbb{R}^{\mathfrak{M}}$ and $g_1, g_2 \in \mathbb{R}^{\overline{\mathfrak{M}}}$ is

$$\widetilde{\widehat{f_1 f_2}} = \tilde{f}_1 \tilde{f}_2 + \frac{h^2}{4} D(f_1) D(f_2), \quad \overline{\widehat{g_1 g_2}} = \bar{g}_1 \bar{g}_2 + \frac{h^2}{4} \overline{D}(g_1) \overline{D}(g_2). \quad (\text{A.3})$$

Lemma A.3 *Let the function f be continuously defined over \mathbb{R} . We have*

$$A^2 f := \hat{\hat{f}} = f + \frac{h^2}{4} \mathbf{D}\mathbf{D}f$$

In particular, (A.2) can also be written as follows

$$\widetilde{\overline{D}(D(f_1 f_2))} = (\widetilde{\overline{D}(Df_1)})\bar{f}_2 + \bar{f}_1 (\widetilde{\overline{D}(Df_2)}) + 2\overline{D}f_1 \overline{D}f_2 + \frac{h^2}{2} (\widetilde{\overline{D}(Df_1)}) (\widetilde{\overline{D}(Df_2)}). \quad (\text{A.4})$$

The following proposition covers discrete integration by parts:

Proposition A.4 *Let $f \in \mathbb{R}^{\mathfrak{M} \cup \partial \mathfrak{M}}$ and $g \in \mathbb{R}^{\overline{\mathfrak{M}}}$. Then,*

$$\begin{aligned}\int_{\Omega} f(\overline{D}g) &= - \int_{\Omega} (Df)g + f_{N+1}g_{N+\frac{1}{2}} - f_0 g_{\frac{1}{2}}, \\ \int_{\Omega} f\bar{g} &= \int_{\Omega} \tilde{f}g - \frac{h}{2} f_{N+1}g_{N+\frac{1}{2}} - \frac{h}{2} f_0 g_{\frac{1}{2}}.\end{aligned}$$

A.2 Some results related to the weight functions

We present here two technical results related to discrete operations performed on the Carleman weight functions. These are of particular interest in the demonstration of Theorem 2.1. We refer the reader to [6], [9] for a complete review of the results and their proofs.

Lemma A.5 *Let f be a smooth function defined on \mathbb{R} . We have*

$$\begin{aligned} D^j f &= \partial_x^j f + C_j' h^2 \int_{-1}^1 (1 - |\sigma|)^{j+1} \partial_x^{j+2} f(\cdot + l_j \sigma h) d\sigma, \\ A^j f &= f + C_j h^2 \int_{-1}^1 (1 - |\sigma|) \partial_x^2 f(\cdot + l_j \sigma h) d\sigma, \quad j = 1, 2, \quad l_1 = \frac{1}{2}, \quad l_2 = 1. \end{aligned}$$

We set $r = e^{s\varphi}$ and $\rho = r^{-1}$. The positive parameters s and h will be large and small respectively. We highlight the dependence on s , h and λ in the following estimate. We assume $s \geq 1$ and $\lambda \geq 1$.

Proposition A.6 *Provided $sh \leq \mathfrak{R}$, we have*

$$\begin{aligned} rA^j D\rho &= r\partial_x^j \rho + s\mathcal{O}_{\lambda, \mathfrak{R}}(sh^2) = s\mathcal{O}_{\lambda, \mathfrak{R}}(1), \quad j = 0, 1, \\ rD^2 \rho &= r\partial_x^2 \rho + s^2\mathcal{O}_{\lambda, \mathfrak{R}}(sh^2) = s^2\mathcal{O}_{\lambda, \mathfrak{R}}(1). \end{aligned}$$

The same estimates hold with ρ and r interchanged.

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