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Energy statistics in open harmonic networks

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Abstract. We relate the large time asymptotics of the energy statistics in open harmonic networks to the variance-gamma distribution and prove a full Large Deviation Principle. We consider both Hamiltonian and stochastic dynamics, the later case including electronic RC networks. We compare our theoretical predictions with the experimental data obtained by Ciliberto et al. [CINT].

1 Introduction

This note grew out of a set of remarks concerning energy fluctuations in thermal equilibrium for open harmonic Hamiltonian or stochastic networks made in [JPS1, JPS2] (see Remark 5 in Section 3.2 of [JPS1] and Remark 3.25 in Section 3.6 of [JPS2]) and our attempts to interpret the experimental results of [CINT] in this context. From a theoretical point of view, it turned out that for this very specific question one can go far beyond the large deviation results of [JPS1, JPS2] and the above mentioned remarks: one can compute the statistics of energy fluctuations in steady state network in a closed form in terms of the variance-gamma distribution. Equally surprisingly, for some specific electronic RC-circuit this statistics and the resulting variance-gamma distribution were precisely measured in experiments of Ciliberto, Imparato, Naert, and Tanase [CINT], although the result was not explicitly reported (the focus of the experiment was the measurement of the non-equilibrium entropy production statistics). The purpose of this note is to comment on the theoretical derivation of the heat and work statistics in harmonic networks and RC-circuits, and in the latter case compare the theoretical prediction with experimental results of [CINT].

To describe more precisely the line of thought that has led to this note, consider a stationary stochastic process Z_t, $t \geq 0$, and set

$$Q_t = Z_t - Z_0.$$

Obviously, $\langle Q_t \rangle = 0$, where $\langle \cdot \rangle$ denotes expectation, and one may ask about order one fluctuations of $\frac{1}{t} Q_t$. A standard way to approach the respective large deviations problem is to study the limit

$$q(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log \left\langle \mathrm{e}^{\alpha Q_t} \right\rangle$$

for α in $\mathbb{R}$. Of course, if Z_0 is almost surely bounded, then $\frac{1}{t}Q_t$ has no order one fluctuations and $q(\alpha) = 0$ for all α. However, in some physically important cases with unbounded Z_0, one has $q(\alpha) = 0$ for $|\alpha| \leq \alpha_c$ and $q(\alpha) = \infty$ for $|\alpha| > \alpha_c$, where the critical α_c is finite. This was precisely the case in the above mentioned remarks in [JPS1, JPS2], where $Z_t - Z_0$ was related to the energy flux across a finite part of harmonic Hamiltonian network in thermal equilibrium. If the Large Deviations Principle for $\frac{1}{t}Q_t$ is controlled by the Legendre transform $I(\theta) = \alpha_c|\theta|$ of $q(\alpha)$, then for any open set $O \subset \mathbb{R}$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{Q}_t(tO) = -\inf_{\theta\in O} I(\theta), \tag{1.1}$$

where $\mathbb{Q}_t$ denotes the probability distribution of Q_t. The rigorous justification of this step in [JPS1, JPS2] required an application of a somewhat subtle extension of the Gärtner-Ellis theorem. The conclusion of the remarks is that a physically important quantity like energy variation per unit time in thermal equilibrium, which has expectation zero, may exhibit order one fluctuations due to the fact that the phase space space is unbounded.

The experimental results of [CINT] indicated that a similar phenomenon happens and can be precisely measured in a suitable electronic RC-circuit. In examining the connection we realized that the large deviation result (1.1) can be considerably refined and that one can express the large time statistics of the energy fluctuations in a finite part of an open harmonic network in closed form in terms of the variance-gamma distribution. That in turn reflected on the experimental findings of [CINT]. In their setting, the large time statistics of energy change turns into the large time statistics of the total power dissipated in the resistors and is also described by the variance-gamma distribution. This allowed us to compare the theoretical prediction of the variance-gamma distribution with the experimental results of [CINT]. The comparison is presented in Subsection 4.3. The theoretical prediction is nearly perfectly matched with experimental results.

Although the mathematical analysis presented in this note is elementary, the conclusions shed a new light on some important aspects of statistical mechanics of open system and can be viewed as a fine statistical characterization of the quantities entering the first law of thermodynamic for finite subsystems of extended harmonic systems. For example, we show that the total kinetic energy of a finite subsystem verifies a universal LDP (1.1) with $\alpha_c = (k_B T)^{-1}$. These points requires further study and we plan to return to them in the future.

The note is organized as follows. In Section 2 we review the derivation of the variance-gamma distribution and describe its basic properties. Hamiltonian harmonic networks are discussed in Section 3. Stochastic harmonic networks and electronic RC-circuits are discussed in Section 4.

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2 Probabilistic setting

2.1 Variance-gamma distribution of Gaussian vectors

Let X and Y be independent $\mathbb{R}^n$-valued Gaussian vectors with zero mean and covariance $M > 0$, and let $L > 0$ be a matrix on $\mathbb{R}^n$. The probability distribution of the random variable

$$Q = X \cdot LX - Y \cdot LY$$

is called *variance-gamma* distribution of the pair (L, M). We shall denote it by $\mathbb{Q}_{\rm vg}$. Obviously,

$$\mathbb{Q}_{\rm vg}(]-\infty, s]) = \int_{x \cdot Ny \le s} \mathrm{d}\mathcal{N}(x)\, \mathrm{d}\mathcal{N}(y)$$

where $\mathcal{N}$ denotes the standard (zero mean, unit covariance) Gaussian measure on $\mathbb{R}^n$ and the matrix N is given by

$$N = 2L^{1/2} M L^{1/2}. \tag{2.2}$$

In this subsection we shall compute the density of $\mathbb{Q}_{\rm vg}$ in terms of the modified Bessel functions of the second kind

$$K_\nu(z) = \int_0^\infty \mathrm{e}^{-z \cosh t} \cosh(\nu t) \mathrm{d}t, \qquad \left(\nu \in \mathbb{C}, |\arg z| < \frac{\pi}{2}\right), \tag{2.3}$$

and the function

$$\mathbb{R}^n \ni x \mapsto \kappa(x) = \left(\sum_{j=1}^n \lambda_j^2 x_j^2\right)^{1/2},$$

where $0 < \lambda_1 \le \lambda_2 \cdots \le \lambda_n$ denote the repeated eigenvalues of the matrix (2.2).

Proposition 2.1 *The density of $\mathbb{Q}_{\rm vg}$ is given by*

$$f_{\rm vg}(s) = |s|^{(n-1)/2} \int_{S^{n-1}} K_{(n-1)/2}\left(\frac{|s|}{\kappa(\hat{k})}\right) \frac{\mathrm{d}\sigma(\hat{k})}{(2\pi\kappa(\hat{k}))^{(n+1)/2}}, \tag{2.4}$$

where σ denotes the Lebesgue measure on the unit sphere S^{n-1} in $\mathbb{R}^n$.

Remark 1. Although this result is well-known, due to the lack of a convenient reference we give a proof below.

Remark 2. It follows immediately from the definition (2.3) that $K_\nu(x) > 0$ for $x > 0$ and $\nu \in \mathbb{R}$. Hence (2.4) implies that $f_{\rm vg}(s) > 0$ for $s \neq 0$. For in depth discussion of Bessel functions we refer the reader to Watson's treatise [W]. In this note we shall make use of the following facts:

(1) The only singularity of K_ν is a logarithmic branching point at $z = 0$, with the leading asymptotics

$$K_\nu(z) \sim \frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu}, \qquad (\mathrm{Re}\,\nu > 0, z \to 0).$$

Moreover, the function $[0, \infty[\ni x \mapsto x^\nu K_\nu(x)$ is monotone decreasing. It follows that $f_{\rm vg}$ is a monotone decreasing function of $|s|$ satisfying

$$0 < f_{\rm vg}(s) \le f_{\rm vg}(0) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{4(2\pi)^{(n+1)/2}} \int_{S^{n-1}} \frac{\mathrm{d}\sigma(\hat{k})}{\kappa(\hat{k})}.$$

(2) For fixed ν the asymptotic expansion

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} \mathrm{e}^{-z} \sum_{k=0}^{\infty} \frac{a_k(\nu)}{z^k}, \qquad (|z| \to \infty, |\arg z| < 3\pi/2) \tag{2.5}$$

holds with $a_0(\nu) = 1$ and

$$a_k(\nu) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\cdots(4\nu^2 - (2k-1)^2)}{k!\, 8^k}.$$

(3) For half-integer $\nu = m + \frac{1}{2}$, $m \in \mathbb{N}$, one has

$$K_\nu(z) = \left(\frac{2z}{\pi}\right)^{1/2} k_m(z). \tag{2.6}$$

where k_m denotes the modified spherical Bessel function of the second kind

$$k_m(z) = \frac{\pi}{2}(-z)^m \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \frac{\mathrm{e}^{-z}}{z}.$$

Remark 3. If $N = \lambda I_n$, where I_n denotes the identity matrix on $\mathbb{R}^n$ and $\lambda > 0$, then the integrand in (2.4) is a constant and thus

$$f_{\mathrm{vg}}(s) = |S^{n-1}|\, K_{(n-1)/2}\left(\frac{|s|}{\lambda}\right) \frac{|s|^{(n-1)/2}}{(2\pi\lambda)^{(n+1)/2}}, \tag{2.7}$$

where $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ is the area of S^{n-1}.

Remark 4. In the special case $n = 2$ and $N = \lambda I_n$ it follows from (2.7) and (2.6) that

$$f_{\mathrm{vg}}(s) = \frac{\mathrm{e}^{-|s|/\lambda}}{2\lambda}. \tag{2.8}$$

More generally, for $n = 2$, (2.4) reduces to

$$f_{\mathrm{vg}}(s) = \int_0^{2\pi} \frac{\mathrm{e}^{-|s|/\kappa(\varphi)}}{4\pi\kappa(\varphi)} \mathrm{d}\varphi,$$

with $\kappa(\varphi) = \sqrt{\lambda_1^2 \sin^2(\varphi) + \lambda_2^2 \cos^2(\varphi)}$. Elementary manipulations lead to

$$f_{\mathrm{vg}}(s) = \frac{\theta}{2\pi} \int_0^{\pi} \sqrt{\frac{1-\varepsilon^2}{1+\varepsilon\cos\varphi}}\, \mathrm{e}^{-\theta|s|\sqrt{1+\varepsilon\cos\varphi}} \mathrm{d}\varphi, \tag{2.9}$$

where

$$\varepsilon = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2 + \lambda_1^2}, \qquad \theta = \left(\frac{1}{2}\left(\lambda_1^{-2} + \lambda_2^{-2}\right)\right)^{1/2}.$$

Proof of Proposition 2.1. Setting $U = N^{-1/2}L^{1/2}(X - Y)$ and $V = N^{-1/2}L^{1/2}(X + Y)$ one has $Q = U \cdot NV$. Note that U and V are independent identically distributed Gaussian random vectors

with zero mean and unit covariance. It follows that the characteristic function of $\mathbb{Q}_{\rm vg}$ can be written as

$$\chi_{\rm vg}(\alpha) = \int_{\mathbb{R}^{2n}} {\rm e}^{{\rm i}\alpha x\cdot Ny} {\rm d}\mathcal{N}(x)\,{\rm d}\mathcal{N}(y). \tag{2.10}$$

Performing the Gaussian integral over y and invoking the invariance of $\mathcal{N}$ under orthogonal transformations further yields

$$\chi_{\rm vg}(\alpha) = \int_{\mathbb{R}^n} {\rm e}^{-\frac{\alpha^2}{2}x\cdot N^2 x}{\rm d}\mathcal{N}(x) = \int_{\mathbb{R}^n} {\rm e}^{-\alpha^2\kappa(x)^2/2}{\rm d}\mathcal{N}(x).$$

It follows that

$$f_{\rm vg}(s) = \int_{\mathbb{R}} {\rm e}^{-{\rm i}\alpha s}\left(\int_{\mathbb{R}^n} {\rm e}^{-\alpha^2\kappa(x)^2/2}{\rm d}\mathcal{N}(x)\right)\frac{{\rm d}\alpha}{2\pi}.$$

Performing the Gaussian integral over α, we derive

$$f_{\rm vg}(s) = (2\pi)^{-1/2}\int_{\mathbb{R}^n} {\rm e}^{-s^2/2\kappa(x)^2}\frac{{\rm d}\mathcal{N}(x)}{\kappa(x)},$$

and passing to the spherical coordinates $r = \|x\|$ and $\hat{k} = x/\|x\|$ we obtain

$$f_{\rm vg}(s) = (2\pi)^{-(n+1)/2}\int_{S^{n-1}}\left(\int_0^\infty {\rm e}^{-\frac{1}{2}\left(r^2+r^{-2}s^2\kappa(\hat{k})^{-2}\right)}r^{n-2}{\rm d}r\right)\frac{{\rm d}\sigma(\hat{k})}{\kappa(\hat{k})}.$$

The change of variable $r^2 = |s|{\rm e}^t/\kappa(\hat{k})$ gives

$$\int_0^\infty {\rm e}^{-\frac{1}{2}\left(r^2+r^{-2}s^2\kappa(\hat{k})^{-2}\right)}r^{n-2}{\rm d}r = \left(\frac{|s|}{\kappa(\hat{k})}\right)^{(n-1)/2}\int_0^\infty {\rm e}^{-\frac{|s|}{\kappa(\hat{k})}\cosh(t)}\cosh((n-1)t/2){\rm d}t,$$

and comparison with (2.3) yields the result. □

2.2 Variance-gamma distribution of a Gaussian process

Consider a stationary $\mathbb{R}^n$-valued Gaussian stochastic process $X_t = (X_t^{(1)},\ldots,X_t^{(n)})$, $t \geq 0$, with mean zero and covariance $M > 0$. Given a matrix $L > 0$ on $\mathbb{R}^n$ we set

$$Q_t = X_t\cdot LX_t - X_0\cdot LX_0,$$

and denote by $\mathbb{Q}_t$ the law of Q_t.

Setting again $N = 2L^{1/2}ML^{1/2}$ and $U_t = N^{-1/2}L^{1/2}(X_t - X_0)$, $V_t = N^{-1/2}L^{1/2}(X_t + X_0)$, one has $Q_t = U_t\cdot NV_t$. The Gaussian random vectors U_t and V_t are identically distributed, with zero mean, but they do not need to be independent. To deal with this point we introduce the $n\times n$-matrix $\Delta(t)$ with entries

$$\Delta_{ij}(t) = \langle X_t^{(i)}X_0^{(j)}\rangle$$

and formulate

Assumption (A)

$$\lim_{t\to\infty}\Delta(t) = 0.$$

Proposition 2.2 *Suppose that* **(A)** *holds. Then* $\mathbb{Q}_t \to \mathbb{Q}_{\rm vg}$ *weakly as* $t \to \infty$*. Moreover, this convergence is accompanied with the following Large Deviations Principle: for any open set* $O \subset \mathbb{R}$*,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{Q}_t(tO) = -\inf_{\theta\in O} I(\theta), \tag{2.11}$$

where the rate function is given by

$$I(\theta) = \frac{|\theta|}{\lambda_n}.$$

Proof. In terms of the $\mathbb{R}^{2n}$-valued random variable $Z_t = (U_t, V_t)$ one has

$$Q_t = Z_t \cdot \widetilde{N} Z_t, \qquad \widetilde{N} = \frac{1}{2}\begin{bmatrix} 0 & N \\ N & 0 \end{bmatrix}.$$

Z_t is Gaussian with mean zero and covariance

$$\widetilde{M}_t = \begin{bmatrix} I_n - A(t) & B(t) \\ B^T(t) & I_n + A(t) \end{bmatrix},$$

where $A(t) = A^T(t)$ and $B(t) = -B^T(t)$ are the $n \times n$ matrices given by

$$A(t) = N^{-1/2} L^{1/2} (\Delta(t) + \Delta^T(t)) L^{1/2} N^{-1/2}, \qquad B(t) = N^{-1/2} L^{1/2} (\Delta(t) - \Delta^T(t)) L^{1/2} N^{-1/2}.$$

The characteristic function of $\mathbb{Q}_t$ is

$$\chi_t(\alpha) = \int_{\mathbb{R}^{2n}} \mathrm{e}^{\mathrm{i}\alpha z\cdot\widetilde{N} z} \mathrm{e}^{-\frac{1}{2} z\cdot\widetilde{M}_t z} \frac{\mathrm{d}z}{\det(2\pi\widetilde{M}_t)^{1/2}}.$$

Assumption **(A)** gives that

$$\lim_{t\to\infty} \widetilde{M}_t = I_{2n}, \tag{2.12}$$

and comparing with (2.10) we deduce that for any $\alpha \in \mathbb{R}$,

$$\lim_{t\to\infty} \chi_t(\alpha) = \chi_{\rm vg}(\alpha).$$

The weak convergence $\mathbb{Q}_t \to \mathbb{Q}_{\rm vg}$ then follows from Lévy's continuity theorem.

To prove the LDP, it suffices to consider the case $O =]a, b[$. If $0 \in O$, then tO is an increasing function of t and since $\mathbb{Q}_{\rm vg}$ has a strictly positive density, the weak convergence $\mathbb{Q}_t \to \mathbb{Q}_{\rm vg}$ implies that

$$\mathbb{Q}_t(tO) \geq \mathbb{Q}_t(O) \to \mathbb{Q}_{\rm vg}(O) = c > 0$$

for $1 \leq t \to \infty$. Hence,

$$0 \geq \frac{1}{t} \log \mathbb{Q}_t(tO) \geq \frac{1}{t} \log c \to 0$$

and the result follows. So we may assume that $0 \notin O$. We consider the case $0 < a < b$, the other case is similar.

From the convergence (2.12) we infer that for large enough $t > 0$ there exists $0 < \delta_t < 1/2$ such that

$$\lim_{t\to\infty} \delta_t = 0, \qquad (1-\delta_t) I_{2n} \leq \widetilde{M}_t \leq (1+\delta_t) I_{2n}. \tag{2.13}$$

Denoting by $\mathcal{N}_{\pm\delta}$ the Gaussian measure on $\mathbb{R}^{2n}$ with zero mean and covariance $(1\pm\delta)I_{2n}$ and setting

$$\mathbb{Q}_{\pm\delta}(A) = \int_{z\cdot\widetilde{N}z\in A} \mathrm{d}\mathcal{N}_{\pm\delta}(z),$$

it follows from (2.13) that

$$3^{-n}\mathbb{Q}_{-\delta_t} \le \mathbb{Q}_t \le 3^n\mathbb{Q}_{+\delta_t}$$

for large enough $t>0$. Further noticing that

$$\mathbb{Q}_{\pm\delta_t}(tO) = \mathbb{Q}_{\mathrm{vg}}(\tau_t O), \qquad \tau_t = \frac{t}{1\pm\delta_t},$$

and $\tau_t/t \to 1$ as $t\to\infty$ we conclude that it suffices to show that

$$\lim_{\tau\to\infty}\frac{1}{\tau}\log\mathbb{Q}_{\mathrm{vg}}(\tau O) = -\frac{a}{\lambda_n}. \tag{2.14}$$

We proceed to prove (2.14).

Obviously,

$$\mathbb{Q}_{\mathrm{vg}}(\tau O) = \tau\int_a^b f_{\mathrm{vg}}(\tau s)\mathrm{d}s,$$

and from the asymptotic expansion (2.5) we infer that there is a constant $C>0$ such that for τ large enough, all $s\in[a,b]$, and all $\hat{k}\in S^{n-1}$,

$$C^{-1}\frac{1}{\sqrt{\tau}}\mathrm{e}^{-\frac{\tau|s|}{\kappa(\hat{k})}} \le K_{(n-1)/2}\left(\frac{\tau|s|}{\kappa(\hat{k})}\right) \le C\frac{1}{\sqrt{\tau}}\mathrm{e}^{-\frac{\tau|s|}{\kappa(\hat{k})}}. \tag{2.15}$$

Since $\max_{\hat{k}\in S^{n-1}}\kappa(\hat{k}) = \lambda_n$, the formula (2.4) and the upper bound in (2.15) give, for large τ,

$$\begin{aligned}\mathbb{Q}_{\mathrm{vg}}(\tau O) &\le C\sqrt{\tau}\left(\int_{S^{n-1}}\frac{\mathrm{d}\sigma(\hat{k})}{(2\pi\kappa(\hat{k}))^{(n+1)/2}}\right)\int_a^b s^{(n-1)/2}\mathrm{e}^{-\tau s/\lambda_n}\mathrm{d}s\\ &\le \left[C\left(\int_{S^{n-1}}\frac{\mathrm{d}\sigma(\hat{k})}{(2\pi\kappa(\hat{k}))^{(n+1)/2}}\right)(b-a)a^{(n-1)/2}\right]\sqrt{\tau}\mathrm{e}^{-\tau a/\lambda_n}\end{aligned}$$

from which the upper bound

$$\limsup_{\tau\to\infty}\frac{1}{\tau}\log\mathbb{Q}_{\mathrm{vg}}(\tau O) \le -\frac{a}{\lambda_n},$$

follows. To prove the corresponding lower bound, fix $0<\epsilon<b-a$ and pick a neighborhood $\mathcal{O}_\epsilon$ of $(0,0,\dots,1)$ in S^{n-1} such that

$$\inf_{\hat{k}\in\mathcal{O}_\epsilon}\kappa(\hat{k}) \ge \lambda_n-\epsilon.$$

Replacing integration over S^{n-1} with integration over $\mathcal{O}_\epsilon$ in (2.4), the lower bound in (2.15) gives

$$\begin{aligned}\mathbb{Q}_{\mathrm{vg}}(\tau O) &\ge C^{-1}\sqrt{\tau}\left(\int_{\mathcal{O}_\epsilon}\frac{\mathrm{d}\sigma(\hat{k})}{(2\pi\kappa(\hat{k}))^{(n+1)/2}}\right)\int_a^b s^{(n-1)/2}\mathrm{e}^{-\tau s/(\lambda_n-\epsilon)}\mathrm{d}s\\ &\ge C^{-1}\sqrt{\tau}\left(\int_{\mathcal{O}_\epsilon}\frac{\mathrm{d}\sigma(\hat{k})}{(2\pi\kappa(\hat{k}))^{(n+1)/2}}\right)\int_a^{a+\epsilon} s^{(n-1)/2}\mathrm{e}^{-\tau s/(\lambda_n-\epsilon)}\mathrm{d}s\\ &\ge \left[\epsilon C^{-1}\left(\int_{\mathcal{O}_\epsilon}\frac{\mathrm{d}\sigma(\hat{k})}{(2\pi\kappa(\hat{k}))^{(n+1)/2}}\right)(a+\epsilon)^{(n-1)/2}\right]\sqrt{\tau}\mathrm{e}^{-\tau(a+\epsilon)/(\lambda_n-\epsilon)},\end{aligned}$$

from which we infer

$$\liminf_{\tau\to\infty} \frac{1}{\tau} \log \mathbb{Q}_{\mathrm{vg}}(\tau O) \geq -\frac{a+\epsilon}{\lambda_n - \epsilon}.$$

Taking $\epsilon \downarrow 0$ we derive the result. □

3 Hamiltonian harmonic networks

3.1 Setup

A Hamiltonian harmonic network is specified by a quadruple $(G, m, \omega, \mathcal{C})$, where G is a countable set describing vertices of the network, $m = (m_x)_{x\in G}$ and $\omega = (\omega_x)_{x\in G}$ are elements of $\ell^\infty_{\mathbb{R}}(G)$ describing the masses and frequencies of the oscillators, and $\mathcal{C}$ is a non-negative bounded operator on $\ell^2_{\mathbb{R}}(G)$ describing the coupling between the oscillators. We denote by $\mathcal{C}_{xy} = \langle \delta_x, \mathcal{C}\delta_y\rangle$ the matrix elements of $\mathcal{C}$ w.r.t. the standard basis $\{\delta_x\}_{x\in G}$ of $\ell^2_{\mathbb{R}}(G)$.

We identify m and ω with the induced multiplication operators on $\ell^2_{\mathbb{R}}(G)$ and assume that, as such, m and ω are strictly positive[1]. The finite energy configurations of the network are described by vectors in the Hilbert space

$$\mathcal{K} = \ell^2_{\mathbb{R}}(G) \oplus \ell^2_{\mathbb{R}}(G) \simeq \ell^2_{\mathbb{R}}(G) \otimes \mathbb{R}^2,$$

where the pair $(p, q) \in \mathcal{K}$ describes the momenta and positions of the oscillators. The Hamiltonian of the network is

$$H(p,q) = \sum_{x\in G} \left(\frac{p_x^2}{2m_x} + \frac{m_x\omega_x^2 q_x^2}{2} \right) + \frac{1}{2} \sum_{x,y\in G} \mathcal{C}_{xy} q_x q_y,$$

which is the quadratic form associated to the linear map $H : \mathcal{K} \to \mathcal{K}$ defined by

$$H = \frac{1}{2} \begin{bmatrix} m^{-1} & 0 \\ 0 & m\omega^2 + \mathcal{C} \end{bmatrix}.$$

H is a bounded strictly positive operator on $\mathcal{K}$ so that

$$\langle \phi, \psi\rangle_H = \langle \phi, H\psi\rangle.$$

defines an inner product on $\mathcal{K}$. We denote by $\mathcal{K}_H$ the Euclidean space $\mathcal{K}$ equipped with this inner product.

The Hamilton equation of motion writes $\dot{\phi} = \mathcal{L}\phi$ with

$$\mathcal{L} = \begin{bmatrix} 0 & -(m\omega^2 + \mathcal{C}) \\ m^{-1} & 0 \end{bmatrix}.$$

Note that $\mathcal{L}$ is a bounded skew-symmetric operator on $\mathcal{K}_H$. It follows that the finite energy trajectories of the network are described by the action of the one parameter group of orthogonal transformations $\mathbb{R} \ni t \mapsto \mathrm{e}^{t\mathcal{L}}$ of $\mathcal{K}_H$.

In the sequel we shall make use of the following dynamical assumption:

> **Assumption (B)** The spectrum of the operator $\mathcal{L}$, acting on $\mathcal{K}_H$, is purely absolutely continuous

[1] The operator A acting on a Hilbert space is strictly positive whenever $\langle \phi, A\phi\rangle \geq \lambda\|\phi\|^2$ for all ϕ and some $\lambda > 0$.

It follows from Assumption **(B)** that for all $\phi, \psi \in \mathcal{K}_H$,

$$\lim_{t\to\infty} \langle \phi, \mathrm{e}^{t\mathcal{L}}\psi\rangle_H = 0. \tag{3.16}$$

3.2 Internal and external work, heat

We fix a finite subset $G_0 \subset G$ and define the total/kinetic/potential energy of the corresponding subnetwork by

$$\begin{aligned} H_{G_0}(p,q) &= \sum_{x\in G_0}\left(\frac{p_x^2}{2m_x} + \frac{m_x\omega_x^2 q_x^2}{2}\right) + \frac{1}{2}\sum_{x,y\in G_0}\mathcal{C}_{xy}q_xq_y, \\ K_{G_0}(p) &= \sum_{x\in G_0}\frac{p_x^2}{2m_x}, \\ V_{G_0}(q) &= \sum_{x\in G_0}\frac{m_x\omega_x^2 q_x^2}{2} + \frac{1}{2}\sum_{x,y\in G_0}\mathcal{C}_{xy}q_xq_y. \end{aligned}$$

To avoid trivialities, we shall assume that there exists $x \in G_0$ and $y \in G_0^{\mathrm{c}}$ such that $\mathcal{C}_{xy} \neq 0$.

The force acting on a vertex $x \in G_0$ decomposes into an internal component

$$F_x^{\mathrm{int}}(q) = -m_x\omega_x^2 q_x - \sum_{y\in G_0}\mathcal{C}_{xy}q_y$$

and an external one

$$F_x^{\mathrm{ext}}(q) = -\sum_{y\in G_0^{\mathrm{c}}}\mathcal{C}_{xy}q_y.$$

The work done by the external forces over the time interval $[0,t]$ is equal to the change in internal energy of the subnetwork

$$W^{\mathrm{ext}}(t) = \sum_{x\in G_0}\int_0^t F_x^{\mathrm{ext}}(q(s))\dot{q}_x(s)\mathrm{d}s = H_{G_0}(p(t),q(t)) - H_{G_0}(p(0),q(0)). \tag{3.17}$$

Adding to it the work done by the internal forces we get the total work done on the subnetwork which equals the change in its kinetic energy:

$$\begin{aligned} W^{\mathrm{ext}}(t) + W^{\mathrm{int}}(t) &= \sum_{x\in G_0}\int_0^t \left(F_x^{\mathrm{int}}(q(s)) + F_x^{\mathrm{ext}}(q(s))\right)\dot{q}_x(s)\mathrm{d}s \\ &= K_{G_0}(p(t)) - K_{G_0}(p(0)). \end{aligned}$$

The change in potential energy of the subnetwork is $V_{G_0}(q(t)) - V_{G_0}(q(0))$ and it is the opposite of the work done by the internal forces,

$$-W^{\mathrm{int}}(t) = -\sum_{x\in G_0}\int_0^t F_x^{\mathrm{int}}(q(s))\dot{q}_x(s)\mathrm{d}s = V_{G_0}(q(t)) - V_{G_0}(q(0)).$$

Thinking of the subnetwork as an open system connected to a reservoir, we can interpret Eq. (3.17) as a statement of the first law of thermodynamics: the increase of the internal energy of the subnetwork equals the amount of heat ΔQ_t extracted from the reservoir.

3.3 The equilibrium case

In this subsection we assume that the network is in thermal equilibrium at temperature $T > 0$. This means that $X_t = (p(t), q(t))$ is a stationary Gaussian process over the Hilbert space $\mathcal{K}$ with mean zero and covariance

$$M = k_{\mathrm{B}}T \begin{bmatrix} m & 0 \\ 0 & (m\omega^2 + \mathcal{C})^{-1} \end{bmatrix}, \tag{3.18}$$

where k_{B} is the Boltzmann constant.

Since

$$\langle p_x(t) p_y \rangle = \langle \mathrm{e}^{t\mathcal{L}}(\delta_x \oplus 0), M(\delta_y \oplus 0) \rangle,$$

Assumption **(B)** implies (recall (3.16)) that

$$\lim_{t\to\infty} \langle p_x(t) p_y \rangle = 0.$$

In the same way one shows that

$$\lim_{t\to\infty} \langle p_x(t) q_y \rangle = \lim_{t\to\infty} \langle q_x(t) p_y \rangle = \lim_{t\to\infty} \langle q_x(t) q_y \rangle = 0,$$

and so the process $(X_t)_{t\geq 0}$ satisfies Assumption **(A)**. Applying Proposition 2.2 to the quadratic form K_{G_0} we infer that the increase in kinetic energy of the subnetwork converges in law, as $t \to \infty$, to the variance-gamma distribution of the pair (M, K_{G_0}). One easily shows that

$$N = 2K_{G_0}^{1/2} M K_{G_0}^{1/2} = k_{\mathrm{B}}T\, I_{|G_0|},$$

and so, by Remark 3, the limiting law has density (2.7) with $\lambda = k_{\mathrm{B}}T$ and $n = |G_0|$. The LDP that accompanies this convergence holds with the rate function

$$I(\theta) = \frac{|\theta|}{k_{\mathrm{B}}T}.$$

Applying Proposition 2.2 to H_{G_0} we deduce in the same way that the heat ΔQ_t absorbed by the subnetwork over the time interval $[0, t]$ converges in law, as $t \to \infty$, to the variance-gamma distribution of the pair (M, H_{G_0}). In this case

$$N = 2H_{G_0}^{1/2} M H_{G_0}^{1/2} = k_{\mathrm{B}}T \begin{bmatrix} I_{|G_0|} & 0 \\ 0 & V_{G_0}^{1/2} V^{-1} V_{G_0}^{1/2} \end{bmatrix},$$

where $V = m\omega^2 + \mathcal{C}$ and V_{G_0} denote its restriction to G_0 with Dirichlet boundary conditions on the complement G_0^{c}. The density of the limiting law is given by (2.4) and, in this case, depends on the details of the interaction. It is interesting to notice that, by Schur's complement formula,

$$V_{G_0}^{1/2} V^{-1} V_{G_0}^{1/2} = (I_{|G_0|} - V_{G_0}^{-1/2} \mathcal{C} V_{G_0^{\mathrm{c}}}^{-1} \mathcal{C} V_{G_0}^{-1/2})^{-1} > I_{|G_0|}$$

so that the LDP for heat holds with rate function

$$I(\theta) = \frac{|\theta|}{(1-\vartheta) k_{\mathrm{B}}T},$$

where $\vartheta = \| V_{G_0}^{-1/2} \mathcal{C} V_{G_0^{\mathrm{c}}}^{-1} \mathcal{C} V_{G_0}^{-1/2} \| > 0$.

3.4 The non-equilibrium case

Let

$$G = \bigcup_a G_a$$

be a finite partition of G and

$$\mathcal{K} = \bigoplus_a \mathcal{K}_a$$

the induced decomposition of Hilbert space $\mathcal{K}$. We restrict $\mathcal{C}$ to $\ell^2_{\mathbb{R}}(G_a)$ by setting $\langle \delta_x, \mathcal{C}_a \delta_y \rangle = \mathcal{C}_{xy}$ if $x, y \in G_a$ and zero otherwise. This leads to (non-interacting) subnetworks $(G_a, m_a, \omega_a, \mathcal{C}_a)$. Suppose that the a-th subnetwork is in thermal equilibrium at temperature T_a, let M_a be the corresponding covariance and set

$$M = \bigoplus_a M_a.$$

Suppose also that the full network was initially described by a Gaussian random field over $\mathcal{K}$ with mean zero and covariance M. The initial probability measure $\mathbb{P}$ evolves under the interacting dynamics determined by $\mathcal{L}$ and at time t, $\mathbb{P}_t$ is a Gaussian probability measure with mean zero and covariance

$$M_t = \mathrm{e}^{t\mathcal{L}} M \mathrm{e}^{t\mathcal{L}^*}.$$

Set

$$\widehat{\mathcal{L}} = \bigoplus_a \mathcal{L}_a,$$

where $\mathcal{L}_a$ denotes the generator of the dynamics of the a-th subnetwork. Assuming that $\mathcal{C} - \bigoplus_a \mathcal{C}_a$ is a trace class operator and that Condition **(B)** holds, a standard trace-class scattering argument [LS] gives that the strong limits

$$W_+ = \mathrm{s} - \lim_{t \to \infty} \mathrm{e}^{-t\widehat{\mathcal{L}}} \mathrm{e}^{t\mathcal{L}}$$

exists. Hence,

$$M_+ = \mathrm{s} - \lim_{t \to \infty} M_t$$

also exists and $M_+ = W_+ M W_+^*$. The Gaussian measure $\mathbb{P}_+$ over $\mathcal{K}$ with mean zero and covariance M_+ is the non-equilibrium steady state of the network associated to the initial state determined by M. Obviously, $\mathbb{P}_t \to \mathbb{P}_+$ weakly as $t \to \infty$.

One can now repeat the analysis of the previous section for the stationary Gaussian process $X_t = (p(t), q(t))$ with mean zero and covariance M_+. Both the change in kinetic and total energy of a finite subnetwork converge in law to a variance-gamma distribution as $t \to \infty$. However, in this case there is no simple expression for the λ_j's. These eigenvalues depend on detailed dynamical properties of the model encoded in the scattering matrix W_+. Even in the simplest example of a harmonic crystal analyzed in [JPS2, Section 3.2], not much is known about the λ_j's outside perturbative regimes.

4 Stochastic networks

4.1 Preliminaries

In this section we shall consider $\mathbb{R}^n$-valued Gaussian stochastic processes $(Z_t)_{t\geq 0}$ that arise as solutions of SDE's of the form

$$\mathrm{d}Z_t = AZ_t + B\mathrm{d}w_t, \tag{4.19}$$

where A and B are $n \times n$ and $n \times m$ matrices and $w(t)$ a standard $\mathbb{R}^m$-valued Wiener process. The solution of the Cauchy problem (4.19) with initial condition Z_0 is

$$Z_t = \mathrm{e}^{tA} Z_0 + \int_0^t \mathrm{e}^{(t-s)A} B\mathrm{d}w_s.$$

We shall assume that A is stable, that is, that all its eigenvalues have strictly negative real part, and set

$$M = \int_0^\infty \mathrm{e}^{sA} BB^* \mathrm{e}^{sA^*} \mathrm{d}s. \tag{4.20}$$

Note that M is non-negative and satisfies the Lyapunov equation

$$AM + MA^* + BB^* = 0.$$

The pair (A, B) is called controllable if the smallest A-invariant subspace that contains $\operatorname{Ran} B$ is equal to $\mathbb{R}^n$. If (A, B) is controllable, then $M > 0$ and the centered Gaussian measure $\mathbb{P}$ on $\mathbb{R}^n$ with covariance M is the unique stationary measure for the process $(Z_t)_{t\geq 0}$. Moreover

$$\Delta_{ij}(t) = \langle Z_t^{(i)} Z_0^{(j)} \rangle = (\mathrm{e}^{tA} M)_{ij},$$

satisfies Assumption **(A)**.

We now turn to two examples of the above framework: stochastic harmonic networks and stochastic RC-circuit.

4.2 Stochastic harmonic networks

We start with a Hamiltonian harmonic network $(G, m, \omega, \mathcal{C})$ where the vertex set G is assumed to be *finite*. To each $x \in G$ we associate two parameters describing a thermal reservoir coupled to the oscillator at x: the temperature $T_x > 0$ and the relaxation rate $\gamma_x \geq 0$ ($\gamma_x = 0$ meaning: no reservoir attached to x). The external force acting on the oscillator at x has the usual Langevin form

$$F_x(p, q) = (2\gamma_x k_\mathrm{B} T_x)^{1/2} \dot{w}_x - \gamma_x p_x,$$

where the $\dot{w}_x$ are independent standard white noises. The equation of motion of stochastic network is

$$m_x \dot{q}_x = p_x, \qquad \dot{p}_x = -m_x \omega_x^2 q_x - \sum_{y\in G} \mathcal{C}_{xy} q_y + F_x(p, q).$$

With $Z_t = (p(t), q(t))$ and obvious notation it can be cast to the form (4.19) with

$$A = \begin{bmatrix} -\gamma & -(m\omega^2 + \mathcal{C}) \\ m^{-1} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} (2\gamma k_\mathrm{B} T)^{1/2} & 0 \\ 0 & 0 \end{bmatrix}.$$

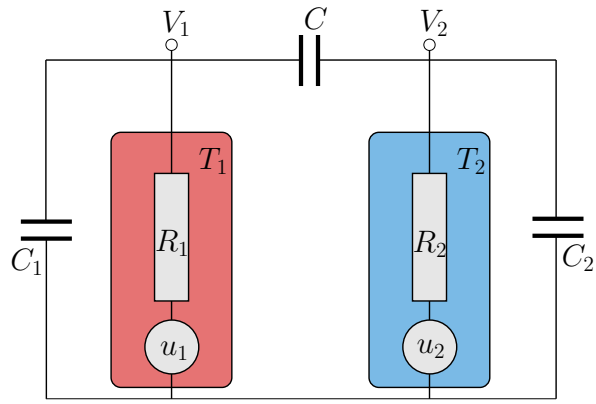

Figure 1: The RC-circuit of [CINT]: The resistances R_1 and R_2 are kept at temperatures T_1 and T_2 and are coupled via capacitance C. C_1 and C_2 are the inner circuit capacitances. u_1 and u_2 denote the fluctuating Johnson-Nyquist voltage. The voltages V_1 and V_2 are amplified by two low noise amplifiers.

The matrix A is stable and we shall assume that the pair (A, B) is controllable. Distributing Z_0 according to the centered Gaussian measure with covariance M given by (4.20) yields a stationary Gaussian process satisfying Assumption **(A)**. Note that in the equilibrium case, $T_x = T$ for all $x \in G$, M is given by (3.18) which is now a finite matrix.

The variations of the total/kinetic energy of the network can be analyzed in the same way as in Subsections 3.3 and 3.4. In particular, in both cases one has the convergence in law to an appropriate variance-gamma distribution. The advantage of the stochastic setting over the Hamiltonian setting described in Section 3 is that the covariances are finite-dimensional matrices and thus accessible to numerical computations. In some important examples they can be explicitly computed. In the next subsection we analyze one such example.

4.3 Stochastic RC-circuit

The RC-circuit subject of the experimental investigations of [CINT] is described in Figure 1. We set

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} C + C_2 & -C \\ -C & C + C_1 \end{bmatrix}, \qquad \mathcal{R} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}, \qquad \mathcal{T} = 2k_{\mathrm{B}} \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}.$$

The circuit analysis gives that the voltages satisfy the SDE

$$\mathrm{d}V_t = AV_t\mathrm{d}t + B\mathrm{d}w_t,$$

where $A = -\mathcal{C}^{-1}\mathcal{R}^{-1}$ and $B = \mathcal{C}^{-1}(\mathcal{T}\mathcal{R}^{-1})^{1/2}$. One easily verifies that the matrix A is stable and that the pair (A, B) is controllable. The stochastic term is due to Nyquist's thermal noise in the resistors. We denote by $Q_1(t)$ and $Q_2(t)$ the powers dissipated in the resistances R_1 and R_2 at time t. They satisfy, in Stratonovich sense,

$$\mathrm{d}Q_1(t) = V_1(t)((C_1 + C)\mathrm{d}V_1(t) - C\mathrm{d}V_2(t)),$$

$$\mathrm{d}Q_2(t) = V_2(t)(-\mathrm{d}V_1(t) + (C_2 + C)\mathrm{d}V_2(t)).$$

We are interested in the total dissipated power defined by

$$Q_t = Q_1(t) + Q_2(t),$$

where we set $Q_0 = 0$. By direct integration,

$$Q_t = \frac{1}{2}\left(V_t \cdot \mathcal{C}V_t - V_0 \cdot \mathcal{C}V_0\right).$$

The eigenvalues of

$$N = \int_0^\infty \mathcal{C}^{1/2} \mathrm{e}^{sA} BB^* \mathrm{e}^{sA^*} \mathcal{C}^{1/2} \mathrm{d}s$$

can be computed by diagonalizing A and evaluating the integral. The eigenvalues are given by

$$\lambda_\pm = \frac{k_\mathrm{B}}{2}\left(T_1 + T_2 \pm |T_1 - T_2|\sqrt{1-\Lambda^2}\right),$$

where

$$\Lambda = \frac{\sqrt{R_1 R_2}\, C}{\frac{1}{2}(R_1+R_2)C + \frac{1}{2}(R_1C_1 + R_2C_2)} \in]0,1[.$$

Recalling Remark 4 after Proposition 2.1, we can now complete the theoretical analysis of the total dissipated power.

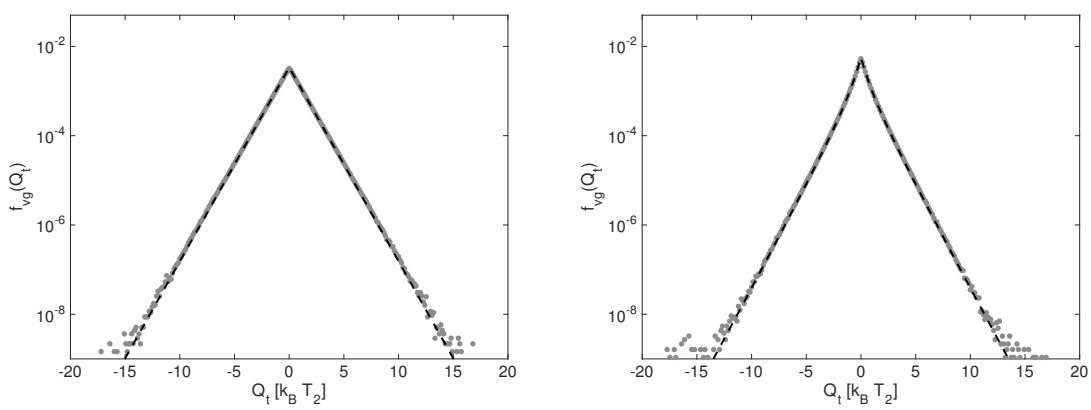

Figure 2: Empirical and theoretical distribution of Q_t with the experimental parameters of [CINT]: $R_1 = R_2 = 10^8\,\Omega$, $C = 10^{-10}\,F$, $C_1 = 6.8\times10^{-10}\,F$ and $C_2 = 4.2\times10^{-10}\,F$. The heat units are in multiples of $k_\mathrm{B}T_2$. For the empirical distribution $t = 0.2$ s, the theoretical distribution is the limiting $t \to \infty$ variance-gamma distribution[3]. Both at equilibrium (left pane, $T_1 = T_2 = 296K$) and out of equilibrium (right pane, $T_1 = 88K$, $T_2 = 296K$), the theoretical prediction (black dashed lines) matches almost perfectly the empirical distribution computed from the experimental data (grey dots). Note that we have taken the theoretical prediction as a reference for the normalization of the empirical distribution. The discrepancy between the theoretical and empirical distribution for $|Q_t| \geq 10k_\mathrm{B}T_2$ is due to the lack of experimental data in this energy range.

[3]Since the largest eigenvalue of A is smaller than -12 and $\exp(-.2\times12) \leq .091$, the experimental data is comparable to the limit $t \to \infty$.

Thermal equilibrium case $T_1 = T_2 = T$. Then $\lambda_- = \lambda_+ = k_\mathrm{B}T$ and it follows from Proposition 2.2 and Eq. (2.8) that Q_t converges in law as $t \to \infty$ to the probability distribution with density

$$f_\mathrm{eq}(s) = \frac{\mathrm{e}^{-|s|/k_\mathrm{B}T}}{2k_\mathrm{B}T}.$$

Non-equilibrum case $T_1 < T_2$. Then $k_\mathrm{B}T_1 < \lambda_- < \frac{k_\mathrm{B}}{2}(T_1 + T_2) < \lambda_+ < k_\mathrm{B}T_2$ and Proposition 2.2 and Eq. (2.9) yield that Q_t converges in law as $t \to \infty$ to the probability distribution with density

$$f_\mathrm{neq}(s) = \frac{\theta}{2\pi}\int_0^\pi \sqrt{\frac{1-\varepsilon^2}{1+\varepsilon\cos\varphi}}\mathrm{e}^{-\theta|s|\sqrt{1+\varepsilon\cos\varphi}}\mathrm{d}\varphi,$$

where

$$\varepsilon = \frac{\lambda_+^2 - \lambda_-^2}{\lambda_+^2 + \lambda_-^2} = \frac{\sqrt{1-\Lambda^2}\,|T_1^2 - T_2^2|}{(T_1^2 + T_2^2) - \frac{1}{2}\Lambda^2(T_1 - T_2)^2},$$

$$\theta = \left(\frac{1}{2}\left(\lambda_-^{-2} + \lambda_+^{-2}\right)\right)^{1/2} = \frac{1}{k_\mathrm{B}}\frac{\sqrt{\frac{1}{2}(T_1^2 + T_2^2) - \frac{\Lambda^2}{4}(T_1 - T_2)^2}}{T_1T_2 + \frac{\Lambda^2}{4}(T_1 - T_2)^2}.$$

Figure 2 summarizes the comparison of these theoretical predictions with the experimental data from [CINT].

References

[CINT] Ciliberto, S., Imparato, A., Naert, A., Tanase, M.: Heat flux and entropy produced by thermal fluctuations. Phys. Rev. Lett. **110**, 180601 (2013).

[JPS1] Jakšić, V., Pillet, C-A., Shirikyan, A.: Entropic fluctuations in thermally driven harmonic networks. J. Stat. Phys. **166**, 926–1015 (2017).

[JPS2] Jakšić, V., Pillet, C-A., Shirikyan, A.: Entropic Fluctuations in Gaussian Dynamical Systems. Rep. Math. Phys. **77**, 335–376 (2016).

[LS] Lebowitz, J.L., Spohn, H.: Stationary non-equilibrium states of infinite harmonic systems. Commun. Math. Phys. **54**, 97–120 (1977).

[W] Watson, G.N.: *A Treatise on the Theory of Bessel Functions.* (2nd. ed.). Cambridge University Press, Cambridge, 1966.