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# Nash Region of the Linear Deterministic Interference Channel with Noisy Output Feedback

Victor Quintero, Samir M. Perlaza, Jean-Marie Gorce, and H. Vincent Poor

**Abstract**—In this paper, the  $\eta$ -Nash equilibrium ( $\eta$ -NE) region of the two-user linear deterministic interference channel (IC) with noisy channel-output feedback is characterized for all  $\eta > 0$ . The  $\eta$ -NE region, a subset of the capacity region, contains the set of all achievable information rate pairs that are stable in the sense of an  $\eta$ -NE. More specifically, given an  $\eta$ -NE coding scheme, there does not exist an alternative coding scheme for either transmitter-receiver pair that increases the individual rate by more than  $\eta$  bits per channel use. Existing results such as the  $\eta$ -NE region of the linear deterministic IC without feedback and with perfect output feedback are obtained as particular cases of the result presented in this paper.

**Index Terms**—Nash equilibrium, Linear Deterministic Interference Channel.

## I. SYSTEM MODEL

Consider the two-user decentralized linear deterministic interference channel with noisy channel-output feedback (D-LD-IC-NOF) depicted in Figure 1. For all  $i \in \{1, 2\}$ , with  $j \in \{1, 2\} \setminus \{i\}$ , the number of bit-pipes between transmitter  $i$  and its intended receiver is denoted by  $\vec{n}_{ii}$ ; the number of bit-pipes between transmitter  $i$  and its non-intended receiver is denoted by  $n_{ji}$ ; and the number of bit-pipes between receiver  $i$  and its corresponding transmitter is denoted by  $\overleftarrow{n}_{ii}$ . These six non-negative integer parameters describe the D-LD-IC-NOF in Figure 1.

At transmitter  $i$ , the channel-input  $\mathbf{X}_{i,n}$  at channel use  $n$ , with  $n \in \{1, 2, \dots, N_i\}$ , is a  $q$ -dimensional binary vector  $\mathbf{X}_{i,n} = (X_{i,n}^{(1)}, X_{i,n}^{(2)}, \dots, X_{i,n}^{(q)})^T \in \mathcal{X}_i$ , with  $\mathcal{X}_i = \{0, 1\}^q$ ,

$$q = \max(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}), \quad (1)$$

and  $N_i \in \mathbb{N}$  is the block-length of transmitter-receiver pair  $i$ . At receiver  $i$ , the channel-output  $\vec{\mathbf{Y}}_{i,n}$  at channel use  $n$ , with  $n \in \{1, 2, \dots, \max(N_1, N_2)\}$ , is also a  $q$ -dimensional binary vector  $\vec{\mathbf{Y}}_{i,n} = (\vec{Y}_{i,n}^{(1)}, \vec{Y}_{i,n}^{(2)}, \dots, \vec{Y}_{i,n}^{(q)})^T$ . Let  $\mathbf{S}$  be a  $q \times q$  binary lower shift matrix. The input-output relation during channel use  $n$  is given by

$$\vec{\mathbf{Y}}_{i,n} = \mathbf{S}^{q-\vec{n}_{ii}} \mathbf{X}_{i,n} + \mathbf{S}^{q-n_{ij}} \mathbf{X}_{j,n}, \quad (2)$$

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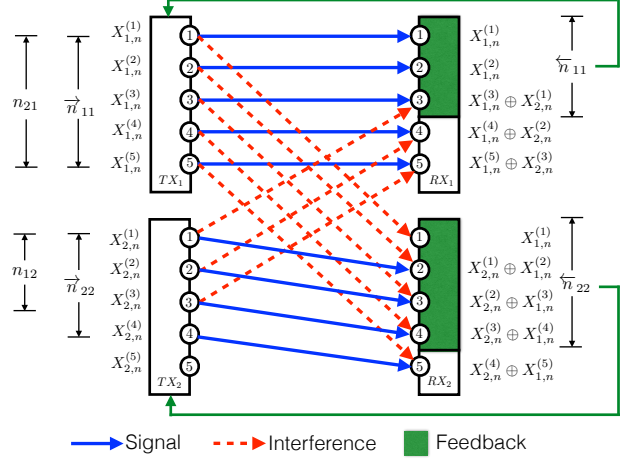


Fig. 1. Two-user linear deterministic interference channel with noisy channel-output feedback at channel use  $n$ .

where  $\mathbf{X}_{i,n} = (0, 0, \dots, 0)^T$  for all  $n > N_i$ . The feedback signal  $\vec{\mathbf{Y}}_{i,n}$  available at transmitter  $i$  at the end of channel use  $n$  is

$$\vec{\mathbf{Y}}_{i,n} = \mathbf{S}^{(\max(\vec{n}_{ii}, n_{ij}) - \overleftarrow{n}_{ii})^+} \vec{\mathbf{Y}}_{i,n-d}, \quad (3)$$

where  $d$  is a finite delay, additions and multiplications are defined over the binary field, and  $(\cdot)^+$  is the positive part operator.

Without any loss of generality, the feedback delay is assumed to be equal to one channel use. Let  $\mathcal{W}_i$  be the set of message indices of transmitter  $i$ . Transmitter  $i$  sends the message index  $W_i \in \mathcal{W}_i$  by transmitting the codeword  $\mathbf{X}_i = (\mathbf{X}_{i,1}, \mathbf{X}_{i,2}, \dots, \mathbf{X}_{i,N_i}) \in \mathcal{X}_i^{N_i}$ , which is a binary  $q \times N_i$  matrix. The encoder of transmitter  $i$  can be modeled as a set of deterministic mappings  $f_{i,1}^{(N)}, f_{i,2}^{(N)}, \dots, f_{i,N_i}^{(N)}$ , with  $f_{i,1}^{(N)} : \mathcal{W}_i \times \mathbb{N} \rightarrow \{0, 1\}^q$  and for all  $n \in \{2, 3, \dots, N_i\}$ ,  $f_{i,n}^{(N)} : \mathcal{W}_i \times \mathbb{N} \times \{0, 1\}^{q \times (n-1)} \rightarrow \{0, 1\}^q$ , such that

$$\mathbf{X}_{i,1} = f_{i,1}^{(N)}(W_i, \Omega_i) \text{ and} \quad (4a)$$

$$\mathbf{X}_{i,n} = f_{i,n}^{(N)}(W_i, \Omega_i, \vec{\mathbf{Y}}_{i,1}, \vec{\mathbf{Y}}_{i,2}, \dots, \vec{\mathbf{Y}}_{i,n-1}), \quad (4b)$$

where  $\Omega_i$  is a randomly generated index known by both transmitter  $i$  and receiver  $i$ , while unknown by transmitter  $j$  and receiver  $j$ . The decoder of receiver  $i$  is defined by a deterministic function  $\psi_i^{(N)} : \{0, 1\}^{q \times N} \times \mathbb{N} \rightarrow \mathcal{W}_i$ . At the end of the communication, receiver  $i$  uses the  $q \times N$  binary matrix  $(\vec{\mathbf{Y}}_{i,1}, \vec{\mathbf{Y}}_{i,2}, \dots, \vec{\mathbf{Y}}_{i,N})$  and  $\Omega_i$  to obtain an estimate  $\widehat{W}_i \in \mathcal{W}_i$  of the message index  $W_i$ , i.e.,

$\widehat{W}_i = \psi_i^{(N)}(\vec{Y}_{i,1}, \vec{Y}_{i,2}, \dots, \vec{Y}_{i,N}, \Omega_i)$ . Let  $W_i$  be written as  $c_{i,1} c_{i,2} \dots c_{i,M_i}$  in binary form, with  $M_i = \lceil \log_2 |\mathcal{W}_i| \rceil$ . Let also  $\widehat{W}_i$  be written as  $\widehat{c}_{i,1} \widehat{c}_{i,2} \dots \widehat{c}_{i,M_i}$  in binary form.

A transmit-receive configuration for transmitter-receiver pair  $i$ , denoted by  $s_i$ , can be described in terms of the block-length  $N_i$ , the number of bits per block  $M_i$ , the channel-input alphabet  $\mathcal{X}_i$ , the codebook, the encoding functions  $f_{i,1}^{(N)}, f_{i,2}^{(N)}, \dots, f_{i,N_i}^{(N)}$ , the decoding function  $\psi_i^{(N)}$ , etc.

The average bit error probability at decoder  $i$  given the configurations  $s_1$  and  $s_2$ , denoted by  $p_i(s_1, s_2)$ , is given by

$$p_i(s_1, s_2) = \frac{1}{M_i} \sum_{\ell=1}^{M_i} \mathbb{1}_{\{\widehat{c}_{i,\ell} \neq c_{i,\ell}\}}. \quad (5)$$

Within this context, a rate pair  $(R_1, R_2) \in \mathbb{R}_+^2$  is said to be achievable if it complies with the following definition.

**Definition 1 (Achievable Rate Pairs):** A rate pair  $(R_1, R_2) \in \mathbb{R}_+^2$  is achievable if there exists at least one pair of configurations  $(s_1, s_2)$  such that the decoding bit error probabilities  $p_1(s_1, s_2)$  and  $p_2(s_1, s_2)$  can be made arbitrarily small by letting the block-lengths  $N_1$  and  $N_2$  grow to infinity.

The aim of transmitter  $i$  is to autonomously choose its transmit-receive configuration  $s_i$ , in order to maximize its achievable rate  $R_i$ . Note that the rate achieved by transmitter-receiver  $i$  depends on both configurations  $s_1$  and  $s_2$  due to mutual interference. This reveals the competitive interaction between both links in the decentralized interference channel. The following section models this interaction using tools from game theory.

## II. THE TWO-USER INTERFERENCE CHANNEL AS A GAME

The competitive interaction between the two transmitter-receiver pairs in the decentralized interference channel can be modeled by the following game in normal-form:

$$\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k \in \mathcal{K}}, \{u_k\}_{k \in \mathcal{K}}). \quad (6)$$

The set  $\mathcal{K} = \{1, 2\}$  is the set of players, that is, the set of transmitter-receiver pairs. The sets  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are the sets of actions of players 1 and 2, respectively. An action of a player  $i \in \mathcal{K}$ , which is denoted by  $s_i \in \mathcal{A}_i$ , is basically its transmit-receive configuration as described in Section I. The utility function of player  $i$  is  $u_i : \mathcal{A}_1 \times \mathcal{A}_2 \rightarrow \mathbb{R}_+$  and it is defined as the information rate of transmitter  $i$ ,

$$u_i(s_1, s_2) = \begin{cases} R_i = \frac{M_i}{N_i}, & \text{if } p_i(s_1, s_2) < \epsilon \\ 0, & \text{otherwise,} \end{cases} \quad (7)$$

where  $\epsilon > 0$  is an arbitrarily small number.

This game formulation was first proposed in [1] and [2]. A class of transmit-receive configurations  $\mathbf{s}^* = (s_1^*, s_2^*) \in \mathcal{A}_1 \times \mathcal{A}_2$  that are particularly important in the analysis of this game is referred to as the set of  $\eta$ -Nash equilibria ( $\eta$ -NE), with  $\eta > 0$ . This type of configuration satisfies the following definition.

**Definition 2 ( $\eta$ -Nash equilibrium):** In the game  $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k \in \mathcal{K}}, \{u_k\}_{k \in \mathcal{K}})$ , an action profile  $(s_1^*, s_2^*)$  is an  $\eta$ -Nash equilibrium if for all  $i \in \mathcal{K}$  and for all  $s_i \in \mathcal{A}_i$ , there exists an  $\eta > 0$  such that

$$u_i(s_i, s_j^*) \leq u_i(s_i^*, s_j^*) + \eta. \quad (8)$$

Let  $(s_1^*, s_2^*)$  be an  $\eta$ -Nash equilibrium action profile of the game in (6). Then, none of the transmitters can increase its own information transmission rate more than  $\eta$  bits per channel use by changing its own transmit-receive configuration and keeping the average bit error probability arbitrarily close to zero. Note that for  $\eta$  sufficiently large, from Definition 2, any pair of configurations can be an  $\eta$ -NE. Alternatively, for  $\eta = 0$ , the classical definition of Nash equilibrium is obtained [3]. In this case, if a pair of configurations is a Nash equilibrium ( $\eta = 0$ ), then each individual configuration is optimal with respect to each other. Hence, the interest is to describe the set of all possible  $\eta$ -NE rate pairs  $(R_1, R_2)$  of the game in (6) with the smallest  $\eta$  for which there exists at least one equilibrium configuration pair. The set of rate pairs that can be achieved at an  $\eta$ -NE is known as the  $\eta$ -Nash equilibrium region.

**Definition 3 ( $\eta$ -NE Region):** Let  $\eta > 0$  be fixed. An achievable rate pair  $(R_1, R_2)$  is said to be in the  $\eta$ -NE region of the game  $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k \in \mathcal{K}}, \{u_k\}_{k \in \mathcal{K}})$  if there exists a pair  $(s_1^*, s_2^*) \in \mathcal{A}_1 \times \mathcal{A}_2$  that is an  $\eta$ -NE and the following holds:

$$u_1(s_1^*, s_2^*) = R_1 \quad \text{and} \quad u_2(s_1^*, s_2^*) = R_2. \quad (9)$$

The following section characterizes the  $\eta$ -NE region (Def. 3) of the two-user D-LD-IC-NOF in (6), denoted by  $\mathcal{N}_\eta(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ , for fixed parameters  $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) \in \mathbb{N}^6$  and for all  $\eta > 0$ .

## III. MAIN RESULTS

The  $\eta$ -NE region is characterized in terms of two regions: the capacity region, denoted by  $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$  and a convex region, denoted by  $\mathcal{B}_\eta(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ . In the following, the tuple  $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$  is used only when needed.

The capacity region  $\mathcal{C}$  of the two-user LD-IC-NOF is described in Theorem 1 in [4], which is a generalization of previous works in [5] and [6]. For all  $\eta > 0$ , the convex region  $\mathcal{B}_\eta$  is defined as follows:

$$\mathcal{B}_\eta = \left\{ (R_1, R_2) : L_i \leq R_i \leq U_i, \text{ for all } i \in \{1, 2\} \right\}, \quad (10)$$

where,

$$L_i = \left( (\vec{n}_{ii} - n_{ij})^+ - \eta \right)^+ \quad \text{and} \quad (11a)$$

$$U_i = \max(\vec{n}_{ii}, n_{ij}) - \left( \min\left( (\vec{n}_{jj} - n_{ji})^+, n_{ij} \right) \right. \quad (11b)$$

$$\left. - \left( \min\left( (\vec{n}_{jj} - n_{ji})^+, n_{ji} \right) - \left( \max(\vec{n}_{jj}, n_{ji}) - \overleftarrow{n}_{jj} \right)^+ \right) \right)^+ + \eta,$$

with  $i \in \{1, 2\}$  and  $j \in \{1, 2\} \setminus \{i\}$ . Theorem 1 uses the region  $\mathcal{B}_\eta$  in (10) and the capacity region  $\mathcal{C}$  to describe the  $\eta$ -NE region  $\mathcal{N}_\eta$ .

**Theorem 1:** Let  $\eta > 0$  be fixed. The  $\eta$ -NE region  $\mathcal{N}_\eta$  of the two-user D-LD-IC-NOF with parameters  $\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}$  and  $\overleftarrow{n}_{22}$ , is  $\mathcal{N}_\eta = \mathcal{C} \cap \mathcal{B}_\eta$ .

Figure 2 shows the capacity region  $\mathcal{C}$  and the  $\eta$ -NE region  $\mathcal{N}_\eta$  of a channel with parameters  $\vec{n}_{11} = 7, \vec{n}_{22} = 6, n_{12} = 4, n_{21} = 4$  and different values for  $\overleftarrow{n}_{11}$  and  $\overleftarrow{n}_{22}$ , with  $\eta$  chosen arbitrarily small. Note that when  $\overleftarrow{n}_{11} \in$

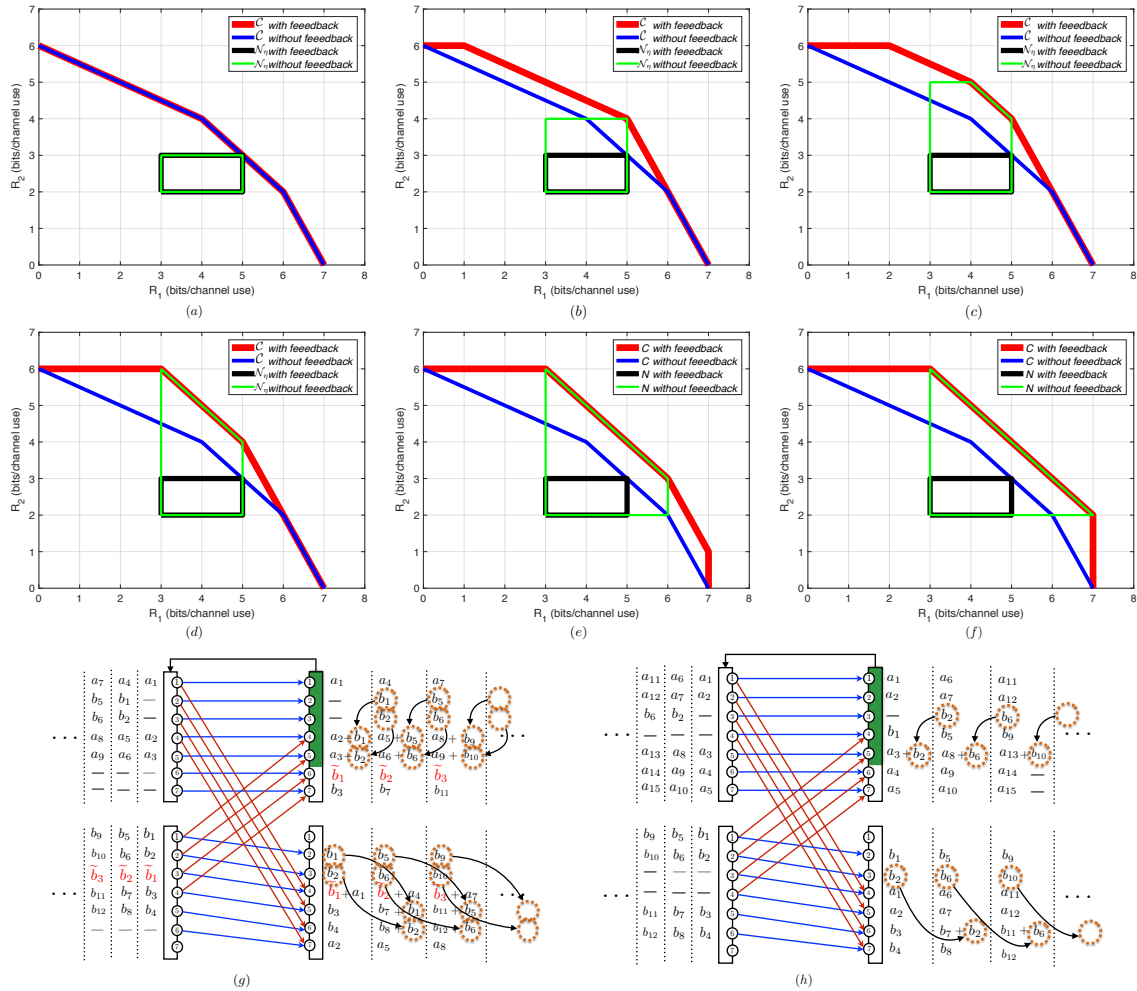


Fig. 2. Capacity region  $\mathcal{C}(7, 6, 4, 4, 0, 0)$  (thin blue line) and  $\eta$ -NE region  $\mathcal{N}_\eta(7, 6, 4, 4, 0, 0)$  (thick black line) with  $\eta$  chosen arbitrarily small. Fig. 2a shows the capacity region  $\mathcal{C}(7, 6, 4, 4, \bar{n}_{11}, \bar{n}_{22})$  (thick red line) and the  $\eta$ -NE region  $\mathcal{N}_\eta(7, 6, 4, 4, \bar{n}_{11}, \bar{n}_{22})$  (thin green line), with  $\bar{n}_{11} \in \{0, 1, 2, 3, 4\}$  and  $\bar{n}_{22} \in \{0, 1, 2, 3, 4\}$ . Fig. 2b shows the capacity region  $\mathcal{C}(7, 6, 4, 4, 5, \bar{n}_{22})$  (thick red line) and the  $\eta$ -NE region  $\mathcal{N}_\eta(7, 6, 4, 4, 5, \bar{n}_{22})$  (thin green line), with  $\bar{n}_{22} \in \{0, 1, 2, 3, 4\}$ . Fig. 2c shows the capacity region  $\mathcal{C}(7, 6, 4, 4, 6, \bar{n}_{22})$  (thick red line) and the  $\eta$ -NE region  $\mathcal{N}_\eta(7, 6, 4, 4, 6, \bar{n}_{22})$  (thin green line), with  $\bar{n}_{22} \in \{0, 1, 2, 3, 4\}$ . Fig. 2d shows the capacity region  $\mathcal{C}(7, 6, 4, 4, 7, \bar{n}_{22})$  (thick red line) and the  $\eta$ -NE region  $\mathcal{N}_\eta(7, 6, 4, 4, 7, \bar{n}_{22})$  (thin green line), with  $\bar{n}_{22} \in \{0, 1, 2, 3, 4\}$ . Fig. 2e shows the capacity region  $\mathcal{C}(7, 6, 4, 4, 7, 5)$  (thick red line) and the  $\eta$ -NE region  $\mathcal{N}_\eta(7, 6, 4, 4, 7, 5)$  (thin green line). Fig. 2f shows the capacity region  $\mathcal{C}(7, 6, 4, 4, 7, 6)$  (thick red line) and the  $\eta$ -NE region  $\mathcal{N}_\eta(7, 6, 4, 4, 7, 6)$  (thin green line). Fig. 2g and Fig. 2h illustrate the achievability scheme for the equilibrium rate pair (3, 4) and (5, 4) in  $\mathcal{N}_\eta(7, 6, 4, 4, 5, 0)$ .

$\{0, 1, 2, 3, 4\}$  and  $\bar{n}_{22} \in \{0, 1, 2, 3, 4\}$  (Figure 2a), it follows that  $\mathcal{N}_\eta(7, 6, 4, 4, \bar{n}_{11}, \bar{n}_{22}) = \mathcal{N}_\eta(7, 6, 4, 4, 0, 0)$ . Thus, in this case the use of feedback in any of the transmitter-receiver pairs does not enlarge the  $\eta$ -Nash region. Alternatively, when  $\bar{n}_{11} > 4$  and  $\bar{n}_{22} \in \{0, 1, 2, 3, 4\}$  (Figures 2b, 2c and 2d), the resulting  $\eta$ -Nash region is strictly larger than in the previous case. A similar effect is observed in Figures 2e and 2f. This observation implies the existence of a threshold on each feedback parameter  $\bar{n}_{11}$  and  $\bar{n}_{22}$  beyond which the  $\eta$ -Nash region is enlarged. The exact values of  $\bar{n}_{11}$  and  $\bar{n}_{22}$ , given a fixed tuple  $(\bar{n}_{11}, \bar{n}_{22}, n_{12}, n_{21})$ , beyond which the  $\eta$ -Nash region can be enlarged is presented in [7]. Figure 2g and Figure 2h show the coding schemes to achieve the rate pairs (3, 4) and (5, 4), respectively, when  $\bar{n}_{11} = 5$  and  $\bar{n}_{22} = 0$ . In Figure 2g, note that common randomness is used by transmitter-receiver pair 2 to prevent transmitter-receiver pair 1 from increasing its individual rate. More specifically, the bits  $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \dots$  are known by both transmitter 2 and receiver 2. The use of common randomness is also observed in [8], [9] and [10]. Common randomness reflects a competitive

behavior between both transmitter-receiver pairs. In Figure 2g, common randomness is not used by transmitter-receiver pair 2 and thus, transmitter-receiver pair 1 achieves a higher rate at an  $\eta$ -NE with respect to the previous example. This suggests a more altruistic behavior.

The  $\eta$ -NE region  $\mathcal{N}_\eta$  without feedback, i.e., when  $\bar{n}_{11} = 0$  and  $\bar{n}_{22} = 0$  (Theorem 1 in [8]), is  $\mathcal{N}_\eta(\bar{n}_{11}, \bar{n}_{22}, n_{12}, n_{21}, 0, 0)$ . The  $\eta$ -NE region with perfect feedback i.e.,  $\bar{n}_{11} = \max(\bar{n}_{11}, n_{12})$  and  $\bar{n}_{22} = \max(\bar{n}_{22}, n_{21})$  (Theorem 1 in [9]), is  $\mathcal{N}_\eta(\bar{n}_{11}, \bar{n}_{22}, n_{12}, n_{21}, \max(\bar{n}_{11}, n_{12}), \max(\bar{n}_{22}, n_{21}))$ . From the comments above, it is interesting to highlight the following inclusions:

$$\begin{aligned} \mathcal{N}_\eta(\bar{n}_{11}, \bar{n}_{22}, n_{12}, n_{21}, 0, 0) &\subseteq \\ \mathcal{N}_\eta(\bar{n}_{11}, \bar{n}_{22}, n_{12}, n_{21}, \bar{n}_{11}, \bar{n}_{22}) &\subseteq \\ \mathcal{N}_\eta(\bar{n}_{11}, \bar{n}_{22}, n_{12}, n_{21}, \max(\bar{n}_{11}, n_{12}), \max(\bar{n}_{22}, n_{21})) & \end{aligned} \quad (12)$$

for all  $\eta > 0$ . The inclusions above might appear trivial, however, enlarging the set of actions often leads to paradoxes

(Braess Paradox [11]) in which the new game possesses equilibria at which players obtain smaller individual benefits and/or smaller total benefit. Nonetheless, letting both transmitter-receiver pairs to use feedback does not induce this type of paradoxes with respect to the case without feedback.

#### IV. PROOFS

To prove Theorem 1, the first step is to show that a rate pair  $(R_1, R_2)$ , with  $R_i < L_i$  or  $R_i > U_i$  for at least one  $i \in \{1, 2\}$ , is not achievable at an  $\eta$ -equilibrium for all  $\eta > 0$ . That is,

$$\mathcal{N}_\eta \subseteq \mathcal{C} \cap \mathcal{B}_\eta. \quad (13)$$

The second step is to show that, for all  $\eta > 0$ , any point in  $\mathcal{C} \cap \mathcal{B}_\eta$  can be achievable at an  $\eta$ -equilibrium. That is,

$$\mathcal{N}_\eta \supseteq \mathcal{C} \cap \mathcal{B}_\eta, \quad (14)$$

which proves the equality  $\mathcal{N}_\eta = \mathcal{C} \cap \mathcal{B}_\eta$ .

*a) Proof of (13):* The proof of (13) is completed by the following lemmas.

*Lemma 1:* A rate pair  $(R_1, R_2) \in \mathcal{C}$ , with either  $R_1 < L_1$  or  $R_2 < L_2$  is not achievable at an  $\eta$ -equilibrium for all  $\eta > 0$ .

*Proof:* The proof of Lemma 1 is presented in [7]. ■ The intuition behind this proof is that the rate  $R_i = (\vec{n}_{ii} - n_{ij})^+$  is always achievable independently of the coding scheme of transmitter-receiver pair  $j$ . To achieve  $R_i = (\vec{n}_{ii} - n_{ij})^+$  transmitter  $i$  uses the most significant bit-pipes, which are interference free, to transmit new bits at each channel use  $n$ .

*Lemma 2:* A rate pair  $(R_1, R_2) \in \mathcal{C}$ , with either  $R_1 > U_1$  or  $R_2 > U_2$  is not achievable at an  $\eta$ -equilibrium for all  $\eta > 0$ .

*Proof:* The proof of Lemma 2 is presented in [7]. ■ This proof is based on the fact that at an  $\eta$ -NE, transmitter  $j$  might re-transmit some of the bits previously transmitted by transmitter  $i$ . The interference produced by those re-transmitted bits at receiver  $i$  can be eliminated if they were received interference free during previous channel uses. This allows transmitter  $i$  to use the bit-pipes interfered with by those re-transmitted bits to send new information bits at each channel use. The key point of this proof is to show that the maximum number of bits that can be re-transmitted at an  $\eta$ -NE is upper bounded.

*b) Proof of (14):* Consider a modification of the coding scheme with noisy feedback presented in [4], which combines rate splitting [12], block Markov superposition coding [13] and backward decoding [14]. The novelty with respect to [4] consists of allowing users to introduce common randomness as suggested in [8] and [9].

Consider without any loss of generality that  $N = N_1 = N_2$ . Let  $W_i^{(t)} \in \{1, 2, \dots, 2^{NR_i}\}$  and  $\Omega_i^{(t)} \in \{1, 2, \dots, 2^{NR_{i,R}}\}$  denote the message index and the random message index sent by transmitter  $i$  during the  $t$ -th block, with  $t \in \{1, 2, \dots, T\}$ , respectively. Following a rate-splitting argument, assume that  $(W_i^{(t)}, \Omega_i^{(t)})$  is represented by the indices  $(W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}, W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)}, W_{i,P}^{(t)}) \in \{1, 2, \dots, 2^{NR_{i,C1}}\} \times \{1, 2, \dots, 2^{NR_{i,R1}}\} \times \{1, 2, \dots, 2^{NR_{i,C2}}\} \times \{1, 2, \dots, 2^{NR_{i,P}}\}$ , where

$R_i = R_{i,C1} + R_{i,C2} + R_{i,P}$  and  $R_{i,R} = R_{i,R1} + R_{i,R2}$ . The rate  $R_{i,R}$  is the number of transmitted bits that are known by both transmitter  $i$  and receiver  $i$  per channel use, and thus it does not have an impact on the information rate  $R_i$ .

The codeword generation follows a four-level superposition coding scheme. The indices  $W_{i,C1}^{(t-1)}$  and  $\Omega_{i,R1}^{(t-1)}$  are assumed to be decoded at transmitter  $j$  via the feedback link of transmitter-receiver pair  $j$  at the end of the transmission of block  $t-1$ . Therefore, at the beginning of block  $t$ , each transmitter possesses the knowledge of the indices  $W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}$  and  $\Omega_{2,R1}^{(t-1)}$ . In the case of the first block  $t=1$ , the indices  $W_{1,C1}^{(0)}, \Omega_{1,R1}^{(0)}, W_{2,C1}^{(0)}$  and  $\Omega_{1,R2}^{(0)}$  are assumed to be known by all transmitters and receivers. Using these indices both transmitters are able to identify the same codeword in the first code-layer. This first code-layer, which is common for both transmitter-receiver pairs, is a sub-codebook of  $2^{N(R_{1,C1}+R_{2,C1}+R_{1,R1}+R_{2,R1})}$  codewords. Denote by  $\mathbf{u}(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)})$  the corresponding codeword in the first code-layer. The second codeword is chosen by transmitter  $i$  using  $(W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)})$  from the second code-layer, which is a sub-codebook of  $2^{N(R_{i,C1}+R_{i,R1})}$  codewords corresponding to the codeword  $\mathbf{u}(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)})$ . Denote by  $\mathbf{u}_i(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)})$  the corresponding codeword in the second code-layer. The third codeword is chosen by transmitter  $i$  using  $(W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)})$  from the third code-layer, which is a sub-codebook of  $2^{N(R_{i,C2}+R_{i,R2})}$  codewords corresponding to the codeword  $\mathbf{u}_i(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)})$ . Denote by  $\mathbf{v}_i(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}, W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)})$  the corresponding codeword in the third code-layer. The fourth codeword is chosen by transmitter  $i$  using  $W_{i,P}^{(t)}$  from the fourth code-layer, which is a sub-codebook of  $2^{NR_{i,P}}$  codewords corresponding to the codeword  $\mathbf{v}_i(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}, W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)})$ . Denote by  $\mathbf{x}_{i,P}(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}, W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)}, W_{i,P}^{(t)})$  the corresponding codeword in the fourth code-layer. Finally, the codeword  $\mathbf{x}_i(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}, W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)}, W_{i,P}^{(t)})$  to be sent during block  $t \in \{1, 2, \dots, T\}$  is a simple concatenation of the previous codewords, i.e.,  $\mathbf{x}_i = (\mathbf{u}_i^\top, \mathbf{v}_i^\top, \mathbf{x}_{i,P}^\top)^\top \in \{0, 1\}^{q \times N}$ , where the message indices have been dropped for ease of notation.

The decoder follows a backward decoding scheme. In the following, this coding scheme is referred to as a randomized Han-Kobayashi coding scheme with noisy feedback (R-HK-NOF) and it is described in [7]. The rest of the proof consists of showing that the R-HK-NOF coding scheme is capable of achieving an  $\eta$ -NE with  $(R_1, R_2) \in \mathcal{C} \cap \mathcal{B}_\eta$  for all  $\eta > 0$ , subject to a proper choice of the rates  $R_{i,R1}$  and  $R_{i,R2}$ , for all  $i \in \{1, 2\}$ .

*Lemma 3:* The achievable region of the randomized Han-Kobayashi coding scheme for the D-LD-IC-NOF is the set

of non-negative rates  $(R_{1,C1}, R_{1,R1}, R_{1,C2}, R_{1,R2}, R_{1,P}, R_{2,C1}, R_{2,R1}, R_{2,C2}, R_{2,R2}, R_{2,P})$  that satisfy the following conditions for all  $i \in \{1, 2\}$  and  $j \in \{1, 2\} \setminus \{i\}$ :

$$R_{j,C1} + R_{j,R1} \leq \theta_{1,i}, \quad (15a)$$

$$R_i + R_{j,C} + R_{j,R} \leq \theta_{2,i}, \quad (15b)$$

$$R_{j,C2} + R_{j,R2} \leq \theta_{3,i}, \quad (15c)$$

$$R_{i,P} \leq \theta_{4,i}, \quad (15d)$$

$$R_{i,P} + R_{j,C2} + R_{j,R2} \leq \theta_{5,i}, \quad (15e)$$

$$R_{i,C2} + R_{i,P} \leq \theta_{6,i}, \text{ and} \quad (15f)$$

$$R_{i,C2} + R_{i,P} + R_{j,C2} + R_{j,R2} \leq \theta_{7,i}, \quad (15g)$$

where,

$$\theta_{1,i} = (n_{ij} - (\max(\vec{n}_{ii}, n_{ij}) - \overleftarrow{n}_{ii})^+)^+, \quad (16a)$$

$$\theta_{2,i} = \max(\vec{n}_{ii}, n_{ij}), \quad (16b)$$

$$\theta_{3,i} = \min(n_{ij}, (\max(\vec{n}_{ii}, n_{ij}) - \overleftarrow{n}_{ii})^+), \quad (16c)$$

$$\theta_{4,i} = (\vec{n}_{ii} - n_{ji})^+, \quad (16d)$$

$$\theta_{5,i} = \max\left((\vec{n}_{ii} - n_{ji})^+, \min(n_{ij}, (\max(\vec{n}_{ii}, n_{ij}) - \overleftarrow{n}_{ii})^+)\right), \quad (16e)$$

$$\theta_{6,i} = \min(n_{ji}, (\max(\vec{n}_{jj}, n_{ji}) - \overleftarrow{n}_{jj})^+) - \min((n_{ji} - \vec{n}_{ii})^+, (\max(\vec{n}_{jj}, n_{ji}) - \overleftarrow{n}_{jj})^+) + (\vec{n}_{ii} - n_{ji})^+, \text{ and} \quad (16f)$$

$$\theta_{7,i} = \max\left(\min(n_{ij}, (\max(\vec{n}_{ii}, n_{ij}) - \overleftarrow{n}_{ii})^+), \min(n_{ji}, (\max(\vec{n}_{jj}, n_{ji}) - \overleftarrow{n}_{jj})^+) - \min((n_{ji} - \vec{n}_{ii})^+, (\max(\vec{n}_{jj}, n_{ji}) - \overleftarrow{n}_{jj})^+) + (\vec{n}_{ii} - n_{ji})^+\right). \quad (16g)$$

*Proof:* The proof of Lemma 3 is presented in [7]. ■

The set of inequalities in (15) can be written in terms of the transmission rates  $R_1 = R_{1,C1} + R_{1,C2} + R_{1,P}$  and  $R_2 = R_{2,C1} + R_{2,C2} + R_{2,P}$  to observe that the R-HK-NOF achieves all the rates  $(R_1, R_2) \in \mathcal{C}$ , when  $R_{1,R} = R_{2,R} = 0$ .

The following lemma shows that when both transmitter-receiver links use the R-HK-NOF scheme and one of them unilaterally changes its coding scheme, it obtains a rate improvement that can be upper bounded.

*Lemma 4:* Let  $\eta > 0$  be fixed and let the rate tuple  $\mathbf{R} = (R_{1,C} - \frac{\eta}{6}, R_{1,R} - \frac{\eta}{6}, R_{1,P} - \frac{\eta}{6}, R_{2,C} - \frac{\eta}{6}, R_{2,R} - \frac{\eta}{6}, R_{2,P} - \frac{\eta}{6})$  be achievable with the R-HK-NOF such that  $R_1 = R_{1,P} + R_{1,C} - \frac{1}{3}\eta$  and  $R_2 = R_{2,P} + R_{2,C} - \frac{1}{3}\eta$ . Then, any unilateral deviation of transmitter-receiver pair  $i$  by using any other coding scheme leads to a transmission rate  $R'_i$  that satisfies  $R'_i \leq \max(\vec{n}_{ii}, n_{ij}) - (R_{j,C} + R_{j,R}) + \frac{2}{3}\eta$ .

*Proof:* The proof of Lemma 4 is presented in [7]. ■

Lemma 4 reveals the relevance of the random symbols  $\Omega_1$  and  $\Omega_2$  used by the R-HK-NOF. Even though the random symbols used by transmitter  $j$  do not increase the effective transmission rate of transmitter-receiver pair  $j$ , they strongly limit the rate improvement transmitter-receiver pair  $i$  can obtain by deviating from the R-HK-NOF coding scheme. This observation can be used to show that the R-HK-NOF can be an  $\eta$ -NE, when both  $R_{1,R}$  and  $R_{2,R}$  are properly chosen. The following lemma formalizes this intuition.

*Lemma 5:* Let  $\eta > 0$  be fixed and let the rate tuple  $\mathbf{R} = (R_{1,C} - \frac{\eta}{6}, R_{1,R} - \frac{\eta}{6}, R_{1,P} - \frac{\eta}{6}, R_{2,C} - \frac{\eta}{6}, R_{2,R} - \frac{\eta}{6}, R_{2,P} - \frac{\eta}{6})$  be achieved by using the R-HK-NOF, with

$$R_{i,C} + R_{i,P} + R_{j,C} + R_{j,R} = \max(\vec{n}_{ii}, n_{ij}) + \frac{2}{3}\eta, \quad (17)$$

for all  $i \in \{1, 2\}$ . Then, the rate pair  $(R_1, R_2)$ , with  $R_i = R_{i,C} + R_{i,P} - \frac{1}{3}\eta$  is achievable at an  $\eta$ -Nash equilibrium.

*Proof:* The proof of Lemma 5 is presented in [7]. ■

The following lemma shows that all the rate pairs  $(R_1, R_2) \in \mathcal{C} \cap \mathcal{B}_\eta$  are achievable by the R-HK-NOF coding scheme at an  $\eta$ -NE, for all  $\eta > 0$ .

*Lemma 6:* Let  $\eta > 0$  be fixed. Then, for all rate pairs  $(R_1, R_2) \in \mathcal{C} \cap \mathcal{B}_\eta$ , there always exists at least one  $\eta$ -NE transmit-recv configuration pair  $(s_1^*, s_2^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ , such that  $u_1(s_1^*, s_2^*) = R_1$  and  $u_2(s_1^*, s_2^*) = R_2$ .

*Proof:* The proof of Lemma 6 is presented in [7]. ■

This proof consists of showing that the set of inequalities in (15) and (17) leads to a set of rate pairs identical to  $\mathcal{C} \cap \mathcal{B}_\eta$ . This concludes the proof of Theorem 1.

## V. CONCLUSIONS

In this paper, the  $\eta$ -NE region of the D-LD-IC-NOF has been characterized for all  $\eta > 0$ . This region contains the  $\eta$ -NE region without feedback studied in [8] and is contained within the  $\eta$ -NE region with perfect channel-output feedback studied in [9].

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