Controllability of bilinear quantum systems in explicit times via explicit control fields

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Abstract

We consider the bilinear Schrödinger equation on a bounded one-dimensional domain and we provide explicit times such that the global exact controllability is verified. In addition, we show how to construct controls for the global approximate controllability.

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1 Introduction

In non relativistic quantum mechanics any pure state of a system is mathematically represented by a wave function ψ in the unit sphere of a Hilbert space $\mathscr{H}$.
We consider the evolution of a particle confined in a one dimensional bounded region and subjected to an external electromagnetic field that plays the role of a control. A standard choice for such a setting is $\mathscr{H} = L^2((0,1),\mathbb{C})$, while the field is represented by an operator B and by a real function u which accounts its intensity. In this framework, the evolution of ψ is modeled by the bilinear Schrödinger equation contained in the following Cauchy problem

$$\text{(BSE)} \qquad \begin{cases} i\partial_t\psi(t) = A\psi(t) + u(t)B\psi(t), & t \in (0,T),\ T>0. \\ \psi(0,x) = \psi^0(x). \end{cases}$$

The operator $A = -\Delta$ is the Laplacian with Dirichlet homogeneous boundary conditions ($D(A) = H^2 \cap H_0^1$), B is a bounded symmetric operator,

$u \in L^2((0,T),\mathbb{R})$ is a control function and ψ^0 the initial state of the system. We call Γ_t^u the unitary propagator of the (BSE) when it is defined.

A natural question of practical implications is whether there exists u steering the quantum system from any initial state to any target one and how to build explicitly this control function.

The (BSE) is said to be globally (locally) exactly controllable in $\mathcal{M} \subset \mathscr{H}$ if, for any $\psi^1, \psi^2 \in \mathcal{M}$ (in a neighborhood of $\mathcal{M}$) unitarily equivalent, there exists $u \in L^2((0,T),\mathbb{R})$ so that the bilinear Schrödinger equation drives ψ^1 to ψ^2 in a time $T > 0$.
The (BSE) is said to be globally approximate controllable in $\mathcal{M} \subset \mathscr{H}$ if, for any $\psi^1, \psi^2 \in \mathcal{M}$ unitarily equivalent, the bilinear Schrödinger equation steers ψ^1 as close as desired to ψ^2 with suitable $u \in L^2((0,T),\mathbb{R})$ and $T > 0$.

The controllability of the (BSE) has already been studied in the literature and we start by mentioning [BMS82] by Ball, Mardsen and Slemrod. The work shows the well-posedness of the (BSE) in $\mathscr{H}$ for controls belonging to $L^1_{loc}((0,T),\mathbb{R})$ and an important non-controllability result. In particular, let S be the unit sphere in $\mathscr{H}$ and

$$Z(\psi_0) := \{\psi \in \mathscr{H} |\ \exists T > 0,\ \exists r > 1, \exists u \in L^r_{loc}((0,T),\mathbb{R}) :\ \psi = \Gamma_u^T \psi_0\}.$$

For every $\psi_0 \in S$, the attainable set $Z(\psi_0)$ is contained in a countable union of compact sets and it has dense complement in S.
However, despite this non-controllability feature, many authors have addressed the problem for weaker notions of controllability.
For instance in [BL10], Beauchard and Laurent prove the well-posedness of the bilinear Schrödinger equation in $H^3_{(0)} := D(A^{\frac{3}{2}})$, when B is a multiplication operator for $\mu \in H^3((0,1),\mathbb{R})$.
Another outcome ensured by Beauchard and Laurent is the local exact controllability. They show that for suitable $B = \mu \in H^3((0,1),\mathbb{R})$, the bilinear Schrödinger equation is locally exactly controllable in a neighborhood of the first eigenfunction of A in $H^3_{(0)}$.
Global approximate controllability results for the (BSE) are provided with different techniques. Adiabatic arguments are considered by Boscain, Chittaro, Gauthier, Mason, Rossi and Sigalotti in [BCMS12] and [BGRS15]
The result is achieved with Lyapunov techniques by Mirrahimi in [Mir09], while by Nersesyan in [Ner10].
Lie-Galerking arguments are used by Boscain, Boussaïd, Caponigro, Chambrion, Mason and Sigalotti in [CMSB09], [BCCS12], [BdCC13] and [BCS14].

The most of the existing results focus on proving the controllablity of the (BSE) without precising the relative controls and times.
To explicit those elements for the global exact controllability, the main obstacle is that the common techniques adopted to prove the local exact controllability do not provide explicit neighborhoods where the result is valid.

Hence, when the outcome is extended to the global exact controllability, any track of the dynamics time and of the relative control is lost.
For this reason, we prove the local exact controllability of the (BSE) for specific neighborhoods and times, which leads to the global exact controllability in explicit times and partially explicit control functions.

In more technical terms, the main novelties of the work are the following.
First, for any suitable couple of eigenfunctions ϕ_j and ϕ_k of A, we construct controls and times such that the relative dynamics of the (BSE) drives ϕ_j close to ϕ_k as much desired with respect to the $H^3_{(0)}-$norm.
Second, we estimate a neighborhood of ϕ_k in $H^3_{(0)}$ where the local exact controllability is satisfied in a given time.
Third, by gathering the two previous results, we define a dynamics steering any eigenstate of A to any other in an explicit time.
In conclusion, we apply the proved results to an example.

The work represents a preliminary step for the application of the control theory to the physical systems modeled by the bilinear Schrödinger equation. Nevertheless, many improvements are still required and the provided estimates are far from being optimal. For example in Section 5, we consider an electron trapped in a one-dimensional guide of length $\sim 10^{-3}$ meters and subjected to an external electromagnetic field. We show a suitable control field driving the state of the electron from the first excited state to the ground state in a time $T \sim 10^{184}$ seconds. The achieved time is way too large for any practical implementation, however future optimization may lead to more reasonable estimates as we explain afterwards.
Moreover, the work allows to study the (BSE) from a numerical point of view. Indeed, as the control function u is explicit for the global approximate controllability, it is possible to implement algorithm confuting the provided estimates, as the one on the time T.

1.1 Framework and main results

Let us consider the (BSE) in the Hilbert space $\mathscr{H} = L^2((0,1),\mathbb{C})$. We denote $\langle\cdot,\cdot\rangle$ the scalar product in $\mathscr{H}$ and $\|\cdot\|$ the corresponding norm. Let $\{\phi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis composed by eigenfunctions of A associated to the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ ($\lambda_k = \pi^2k^2$) and

$$\phi_j(t) = e^{-iAt}\phi_j = e^{-i\lambda_j t}\phi_j. \tag{1}$$

We define the following spaces for $s > 0$

$$h^s(\mathbb{C}) = \Big\{\{x_j\}_{j\in\mathbb{N}} \subset \mathbb{C} \Big| \sum_{j=1}^{\infty} |j^s x_j|^2 < \infty\Big\}, \qquad \|\cdot\|_{h^s} = \Big(\sum_{k=1}^{\infty} |k^s\,\cdot|^2\Big)^{\frac{1}{2}},$$

$$H^s_{(0)} = H^s_{(0)}((0,1),\mathbb{C}) := D(A^{\frac{s}{2}}), \qquad \|\cdot\|_{(s)} = \Big(\sum_{k=1}^{\infty} |k^s\langle\phi_k,\cdot\rangle|^2\Big)^{\frac{1}{2}}.$$

Let $H^s := H^s((0,1),\mathbb{C})$ and $H^s_0 := H^s_0((0,1),\mathbb{C})$ with $s>0$ and

$$|||\cdot||| := |||\cdot|||_{L(\mathscr{H},\mathscr{H})}, \quad |||\cdot|||_{(s)} := |||\cdot|||_{L(H^s_{(0)},H^s_{(0)})}, \quad |||\cdot|||_3 := |||\cdot|||_{L(H^3_{(0)},H^3\cap H^1_0)}.$$

In the current chapter, we consider the space $H^3\cap H^1_0$ equipped with the norm $\|\cdot\|_{H^3\cap H^1_0} = \sqrt{\sum_{j=1}^3 \|\partial_x^j\cdot\|^2}$.

Assumptions (I)**.** The bounded symmetric operator B satisfies the following conditions.

1. For every $k\in\mathbb{N}$, there exists $C_k>0$ such that $|\langle\phi_j,B\phi_k\rangle| \geq \frac{C_k}{j^3}$ for every $j\in\mathbb{N}$.
2. $Ran(B|_{D(A)}) \subseteq D(A)$ and $Ran(B|_{H^3_{(0)}}) \subseteq H^3\cap H^1_0$.

Before proceeding with the main results of the work, we need to introduce few further notations.

Let $B_{j,k} := \langle\phi_j, B\phi_k\rangle$ with $k,j\in\mathbb{N}$. We have $\{B_{j,k}\}_{j\in\mathbb{N}}, \{B_{j,k}\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$ and we denote, for $n\in\mathbb{N}$,

$$T^* := \frac{\pi}{|B_{k,j}|}, \qquad b := |||B|||_{(2)}^{6}\,|||B|||\,|||B|||_3^{16}\max\big\{|||B|||, |||B|||_3\big\},$$

$$E(j,k) := e^{\frac{6|||B|||_{(2)}}{|B_{j,k}|}}|k^2-j^2|^5 C_k^{-16}k^{24}|B_{j,k}|^{-7}\max\{j,k\}^{24},$$

$$u_n(t) := \frac{\cos\big((k^2-j^2)\pi^2 t\big)}{n}, \qquad C' := \sup_{(l,m)\in\Lambda'}\left\{\left|\sin\left(\pi\frac{|l^2-m^2|}{|k^2-j^2|}\right)\right|^{-1}\right\},$$

$$\Lambda' := \big\{(l,m)\in\mathbb{N}^2 :\ \{l,m\}\cap\{j,k\}\neq\emptyset,\ |l^2-m^2| \leq \frac{3}{2}|k^2-j^2|,$$
$$|l^2-m^2|\neq|k^2-j^2|,\ \langle\phi_l,B\phi_m\rangle\neq 0\big\}.$$

The following theorem represents the main result of the work, which ensure the global exact controllability between eigenfunctions. We underline that the control time is explicit and u_n defines a dynamics steering the initial data to the target one up the well-known distance when n is sufficiently large.

Theorem 1.1. *Let $j,n\in\mathbb{N}$ and $k\in\mathbb{N}$ be such that $k\neq j$ and*

$$m^2-k^2 \neq k^2-l^2, \qquad \forall m,l\in\mathbb{N},\ m,l\neq k. \tag{2}$$

Let B satisfy Assumptions I. If $n\geq 6^{42}\pi^{12}b\,(1+C')E(j,k)$, then there exists $\theta\in\mathbb{R}$ such that

$$\big\|\Gamma^{u_n}_{nT^*}\phi_j - e^{i\theta}\phi_k\big\|_{(3)} \leq C_k^2(6^2k^3\,|||B|||_3^2)^{-1}$$

with C_k defined in Assumptions I. Moreover, there exists $u \in L^2((0,\frac{4}{\pi}),\mathbb{R})$ such that

$$\|u\|_{L^2((0,\frac{4}{\pi}),\mathbb{R})} \leq \frac{C_k}{3\,|||\,B\,|||_3^2 k^3}, \qquad \Gamma^u_{\frac{4}{\pi}}\Gamma^{u_n}_{nT^*}\phi_j = e^{i\theta}\phi_k.$$

Proof. See Section 3. □

Examples of $k \in \mathbb{N}$ satisfying the relation (15) are $k \leq 3$. However, Theorem 1.1 can be generalized for every $k \in \mathbb{N}$ by defining, for every ϕ_j and ϕ_k, a dynamics steering ϕ_j in ϕ_k and passing through ϕ_1.
Moreover, the choice of $\{u_n\}_{n\in\mathbb{N}}$ can be replaced by other $\frac{2\pi}{|\lambda_k-\lambda_j|}-$periodic controls by refering to [Cha12], which is used in the proof of the theorem.

Theorem 1.1 is not optimal and its purpose is to exhibit readable results for generic B, j and k. For any specific choice of B, j and k, it is possible to retrace the proof in order to obtain sharper bounds by using stronger intermediate estimates. We briefly treat the example of $B : \psi \to x^2\psi$, $j = 2$ and $k = 1$ in Section 5. In addition, we present in Section 4 how to compute and remove the phase appearing in the target state, even though this is not particularly relevant from a physical point of view.

1.2 Well-posedness

As mentioned in the introduction, Beauchard and Laurent prove in [BL10] the well-posedness of the (BSE) in $H^3_{(0)}$ when B is a multiplication operator for a suitable function $\mu \in H^3((0,1),\mathbb{R})$. We rephrase the result in the following proposition.

Proposition 1.2. *[BL10, Lemma 1; Proposition 2]*
1) *Let $T > 0$ and $\widetilde{f} \in L^2((0,T), H^1_0 \cap H^3)$. The function $G : t \mapsto \int_0^t e^{iAs}\widetilde{f}(s)ds$ belongs to $C^0([0,T], H^3_{(0)})$. Moreover,*

$$\|G\|_{L^\infty((0,T),H^3_{(0)})} \leq c_1(T)\|\widetilde{f}\|_{L^2((0,T),H^3\cap H^1_{(0)})},$$

where the constant $c_1(T)$ is uniformly bounded with T in bounded intervals.
2) *Let $\mu \in H^3((0,1),\mathbb{R})$, $T > 0$, $\psi^0 \in H^3_{(0)}$ and $u \in L^2((0,T),\mathbb{R})$. There exists a unique mild solution of the (BSE) in $H^3_{(0)}$ when $B = \mu$, i.e. $\psi \in C^0([0,T], H^3_{(0)})$ such that*

$$(3)\ \ \psi(t,x) = e^{-iAt}\psi^0(x) - i\int_0^t e^{-iA(t-s)}(u(s)\mu(x)\psi(s,x))ds, \qquad \forall t \in [0,T].$$

Moreover, for every $R > 0$, there exists $C = C(T,\mu,R) > 0$ such that, for every $\psi^0 \in H^3_{(0)}$, if $\|u\|_{L^2((0,T),\mathbb{R})} < R$, then the solution satisfies

$$\|\psi\|_{C^0([0,T],H^3_{(0)})} \leq C\|\psi^0\|_{(3)}, \qquad \|\psi(t)\|_{\mathscr{H}} = \|\psi^0\|_{\mathscr{H}} \quad \forall t \in [0,T].$$

Remark 1.3. *The outcome of Proposition* 1.2 *is not only valid for multiplication operators, but also for suitable* B*. Indeed, the same proofs of* [*BL10*, *Lemma* 1] *and* [*BL10*, *Proposition* 2] *lead to the well-posedness of the* (BSE) *when* B *is a bounded symmetric operator such that*

$$B \in L(H^3_{(0)}, H^3 \cap H^1_0), \qquad B \in L(H^2_{(0)}),$$

which are verified if B *satisfies Assumptions I, thanks to* [*Duc*, *Remark* 1.1].

1.3 Scheme of the work

In Section 2, Proposition 2.1 ensures the local exact controllability in $H^3_{(0)}$ and we exhibit a neighborhood where it is verified in Proposition 2.2.
We prove Theorem 1.1 in Section 3, while in Section 4, we estimate the phase appearing in the statement of Theorem 1.1.
In Section 5, we apply the main results to a physical system and, in Section 6, we conclude with few comments on the outcomes of Theorem 1.1.
We provide some intermediate results in Appendix A, while in Appendix B, we expose some tools required in the work.

2 Local exact controllability in $H^3_{(0)}$

Let us provide a brief proof of the local exact controllability in $H^3_{(0)}$ by rephrasing the existing results of local controllability as [BL10], [Mor14] and [MN15]. Our purpose is to introduce the tools that we use in the proof of Theorem 1.1. For $\psi \in H^3_{(0)}$ and $\epsilon > 0$, we define

$$\widetilde{B}_{H^3_{(0)}}(\psi, \epsilon) := \big\{ \widetilde{\psi} \in H^3_{(0)} \big|\ \|\widetilde{\psi}\| = \|\psi\|,\ \|\widetilde{\psi} - \psi\|_{(3)} < \epsilon \big\}. \tag{4}$$

Proposition 2.1. *Let* B *satisfy Assumptions I. For every* $l \in \mathbb{N}$ *such that*

$$m^2 - l^2 \neq l^2 - n^2, \qquad \forall m, n \in \mathbb{N},\ m, n \neq l, \tag{5}$$

there exist $T > 0$ *and* $\epsilon > 0$ *such that, for every* $\psi \in \widetilde{B}_{H^3_{(0)}}(\phi_l(T), \epsilon)$*, there exists a control function* $u \in L^2((0,T), \mathbb{R})$ *such that* $\psi = \Gamma^u_T \phi_l$.

Proof. The statement of Proposition 2.1 is equivalent to the local surjectivity of the map $\Gamma^{(\cdot)}_T \phi_l : u \in L^2((0,T), \mathbb{R}) \longmapsto \Gamma^u_T \phi_l \in H^3_{(0)}$ for $T > 0$. We consider

$$\Gamma^u_t \phi_l = \sum_{k=1}^{\infty} \phi_k(t) \langle \phi_k(t), \Gamma^u_t \phi_l \rangle$$

and the function $\alpha_l(u) = \{\alpha_{k,l}(u)\}_{k \in \mathbb{N}}$ such that $\alpha_{k,l}(u) = \langle \phi_k(T), \Gamma^u_T \phi_l \rangle$ with $k \in \mathbb{N}$. We know that $\Gamma^u_T \phi_l \in H^3_{(0)}$ for every $u \in L^2((0,T), \mathbb{R})$ and

then $\alpha_l(u) \in h^3(\mathbb{C})$ for every $u \in L^2((0,T),\mathbb{R})$. Proving Proposition 2.1 is equivalent to prove the local surjectivity for $T > 0$ large enough of the map

$$\alpha_l : L^2((0,T),\mathbb{R}) \longrightarrow Q := \{\mathbf{x} := \{x_k\}_{k\in\mathbb{N}} \in h^3(\mathbb{C}) \mid \|\mathbf{x}\|_{\ell^2} = 1\}.$$

To this end, we use the Generalized Inverse Function Theorem ([Lue69, Theorem 1; p. 240]) and we study the surjectivity of the Fréchet derivative of α_l, $\gamma_l(v) := (d_u\alpha_l(0)) \cdot \ v$, the sequence with elements $\gamma_{k,l}(v) := -i\int_0^T v(s)e^{i(\lambda_k-\lambda_l)s}dsB_{k,l}$ with $k \in \mathbb{N}$. We notice that

$$\gamma_l : L^2((0,T),\mathbb{R}) \longrightarrow T_{\delta_l}Q = \{\{x_k\}_{k\in\mathbb{N}} \in h^3(\mathbb{C}) \mid ix_l \in \mathbb{R}\}$$

and the surjectivity of γ_l consists in the solvability of the moment problem

$$\frac{x_k}{B_{k,l}} = -i\int_0^T u(s)e^{i(\lambda_k-\lambda_l)s}ds, \qquad \forall\{x_k\}_{k\in\mathbb{N}} \in T_{\delta_l}Q. \tag{6}$$

As B is symmetric, we have $B_{l,l} \in \mathbb{R}$ and $i\big(x_l/B_{l,l}\big) \in \mathbb{R}$. Moreover, $\{x_kB_{k,l}^{-1}\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$ since $\{x_k\}_{k\in\mathbb{N}} \in h^3(\mathbb{C})$ and thanks to Assumptions I. Thanks to the relation (5), for every $k,j \in \mathbb{N}$ with $k,j \neq l$, we know that $\lambda_k - \lambda_l = \pi^2(k^2-l^2) \neq \pi^2(l^2-j^2) = \lambda_l - \lambda_j$. The solvability of (6) for $u \in L^2((0,T),\mathbb{R})$ is guaranteed by Remark $B.1$, which follows from Ingham's Theorem ([KL05, $Theorem$ 4.3]) for $T > \frac{2\pi}{\mathscr{G}}$ and $\mathscr{G} := \pi^2$. In particular, for X defined in Remark $B.1$, the map $\gamma_l : X \longrightarrow T_{\delta_l}Q$ is an homeomorphism, thus $\gamma_l : L^2((0,T),\mathbb{R}) \to T_{\delta_l}Q$ is surjective in $T_{\delta_l}Q$ for T large enough. □

2.1 Local exact controllability neighborhood estimate

Let C_l and $\widetilde{B}_{H^3_{(0)}}(\cdot,\cdot)$ be respectively defined in Assumptions I and (4). The following proposition ensures the local exact controllablity in an explicit neighborood of $H^3_{(0)}$ and for a specific time. The result is crucial for the proof of Theorem 1.1, which is proved by extending it.

Proposition 2.2. *Let B satisfy Assumptions I and $l \in \mathbb{N}$ be such that*

$$m^2 - l^2 \neq l^2 - n^2, \qquad \forall m,n \in \mathbb{N},\ m,n \neq l. \tag{7}$$

For every

$$\psi \in \widetilde{B}_{H^3_{(0)}}\Big(\phi_l\big(4/\pi\big), C_l^2/(6^2l^3 \,|\!|\!|\, B \,|\!|\!|\, \tfrac{2}{3})\Big),$$

there exists $u \in L^2\big((0,\frac{4}{\pi}),\mathbb{R}\big)$ such that $\psi = \Gamma^u_{\frac{4}{\pi}}\phi_l$.

Proof. Let us define the following notations

$$|\!|\!|\cdot|\!|\!|_{L(L^2((0,T),\mathbb{R}),H^3_{(0)})} = |\!|\!|\cdot|\!|\!|_{(L^2_t,H^3_x)}, \quad |\!|\!|\cdot|\!|\!|_{L(H^3_{(0)},L^2((0,T),\mathbb{R}))} = |\!|\!|\cdot|\!|\!|_{(H^3_x,L^2_t)},$$

$$\|\cdot\|_{L^\infty((0,T),H^3_{(0)})} = \|\cdot\|_{L^\infty_tH^3_x}, \quad \|\cdot\|_{L^2((0,T),\mathbb{R})} = \|\cdot\|_2.$$

Let $T > \frac{2\pi}{\mathscr{G}}$ for $\mathscr{G} = \pi^2$ (as in the proof of Proposition 2.1). We consider the space X defined in Remark $B.1$ (Appendix B) and equipped with the L^2-norm. The local exact controllability provided in the proof of Proposition 2.1 is equivalent to the local surjectivity of the map

$$A_l(\cdot) := \Gamma_T^{(\cdot)}\phi_l : L^2((0,T),\mathbb{R}) \to \{\psi \in H^3_{(0)} \ : \ \|\psi\|_{\mathscr{H}} = 1\}$$

such that $A_l(u) = e^{-i\lambda_l T}\phi_l - i\int_0^T e^{-iA(T-s)}u(s)B\Gamma_s^u\phi_l ds$. Indeed, from the proof of Proposition 2.1, the map

$$F_l(\cdot) : X \to \{\psi \in H^3_{(0)} : \ \{\langle\phi_j(T),\psi\rangle\}_{j\in\mathbb{N}} \in T_{\delta_l}Q\} = \{\psi \in H^3_{(0)} : \ i\langle\phi_l(T),\psi\rangle \in \mathbb{R}\}$$

such that $F_l(u) := \big((d_vA_l(v=0))\cdot u\big)$ is an homeomorphism, which implies the local surjectivity of A_l thanks to the Generalized Inverse Function Theorem ([Lue69, *Theorem* 1; *p.* 240]).

Thanks [GC97, *Lemma* 2.3; *p.* 42], we estimate the radius of the neighborhood where A_l is surjective. We consider that X and $\{\psi \in H^3_{(0)} : i\langle\phi_l(T),\psi\rangle \in \mathbb{R}\}$ are Banach spaces and $F_l : X \to \{\psi \in H^3_{(0)} : \ i\langle\phi_l(T),\psi\rangle \in \mathbb{R}\}$ is an homeomorphism for T large enough. We compute a constant $M > 0$ such that

$$\|F_l(u) - F_l(v)\|_{H^3_{(0)}} \geq M\|u-v\|_{L^2((0,T),\mathbb{R})}, \qquad \forall u,v \in X. \tag{8}$$

Fixed $T > 0$ large enough, we provide a neighborhood $U \subset X$ and a constant $M_1 < M$ such that

$$\|(A_l - F_l)(u) - (A_l - F_l)(v)\|_{H^3_{(0)}} \leq M_1\|u-v\|_{L^2((0,T),\mathbb{R})}, \qquad \forall u,v \in U.$$

Thanks to [GC97, *Lemma* 2.3; *p.* 42], the map $A_l : U \longrightarrow A_l(U)$ is an homeomorphism and, from the proof of [GC97, *Lemma* 2.3; *p.* 42], we deduce that if $U \supset \{u \in X : \|u\|_{L^2((0,T),\mathbb{R})} \leq r\}$ with $r > 0$, then

$$A_l(U) \supset \{\psi \in H^3_{(0)} : \|\psi - \phi_l(T)\|_{H^3_{(0)}} \leq r(M - M_1)\}.$$

1) We compute $M > 0$ such that (8) is verified. Let us suppose $\|\!|\!| B |\!|\!|_3 = 1$. The surjectivity of F_l in $H^3_{(0)}$ is equivalent to the surjectivity of γ_l in h^3 and, for every $\psi \in H^3_{(0)}$, there exist $T > 0$ and $u \in X$ such that, for every $j \in \mathbb{N}$

$$\langle\phi_j(T),\psi\rangle = \gamma_{j,l}(u) = \langle\phi_j(T),F_l(u)\rangle. \qquad F_l^{-1}(\psi) = u. \tag{9}$$

From the relation (32)(Remark $B.1$), there exists $\widetilde{C}(T) > 0$ such that

$$\|F_l^{-1}(\psi)\|_2^2 = \|u\|_2^2 \leq \widetilde{C}(T)^2\sum_{j=1}^{\infty}\Big|\frac{\gamma_{j,l}(u)}{B_{j,l}}\Big|^2 \leq \frac{\widetilde{C}(T)^2}{C_l^2}\sum_{j=1}^{\infty}|j^3\gamma_{j,l}(u)|^2 \leq \frac{\widetilde{C}(T)^2}{C_l^2}\|\psi\|_{(3)}^2.$$

In conclusion, we set $M = C_l/\widetilde{C}(T)$ since, for every $\psi, \varphi \in H^3_{(0)}$, there exist $v, w \in X$ such that $\psi = F_l(v)$, $\varphi = F_l(w)$ and

$$\|v-w\|_2 \le \|F_l^{-1}(\psi-\varphi)\|_2 \le \|\!|\!| F_l^{-1} |\!|\!|_{(H^3_x,L^2_t)} \|\psi-\varphi\|_{(3)} \le \frac{\widetilde{C}(T)}{C_l}\|\psi-\varphi\|_{(3)}.$$

2) Let $H_l(u) := -\int_0^T e^{-iA(T-s)}u(s)B\int_0^s e^{-iA(s-\tau)}u(\tau)B\Gamma^u_\tau\phi_l d\tau ds$ with $u \in X$. Thanks to the Duhamel's formula,

$$A_l(u) = \Gamma^u_T\phi_l = e^{-i\lambda_l T}\phi_l - i\int_0^T e^{-iA(T-s)}u(s)Be^{-i\lambda_l s}\phi_l ds$$
$$-\int_0^T e^{-iA(T-s)}u(s)B\Big(\int_0^s e^{-iA(s-\tau)}u(\tau)B\Gamma^u_\tau\phi_l d\tau\Big)ds = e^{-i\lambda_l T}\phi_l + F_l(u) + H_l(u).$$

We exhibit a ball $U \subset X$ with center $u = 0$ such that, for every $u \in U$, the map $A_l : u \in U \mapsto \Gamma^u_T\phi_l \in A_l(U)$ is an homeomorphism thanks to [GC97, *Lemma* 2.3; *p.* 42] and

$$\|(A_l - F_l)(u) - (A_l - F_l)(v)\|_{H^3_{(0)}} = \|H_l(u) - H_l(v)\|_{H^3_{(0)}}, \qquad \forall u, v \in X.$$

We define U as the neighborhood such that there exists $M_1 \le M/2$ so that $\|H_l(u) - H_l(v)\|_{(3)} \le M_1\|u-v\|_{L^2}$ for every $u, v \in U$. First, we notice that

$$\begin{aligned} H_l(u) - H_l(v) = &-\int_0^T e^{-iA(T-s)}(u(s)-v(s))B\Big(\int_0^s e^{-iA(s-\tau)}u(\tau)B\Gamma^u_\tau\phi_l d\tau\Big)ds \\ &-\int_0^T e^{-iA(T-s)}v(s)B\Big(\int_0^s e^{-iA(s-\tau)}(u(\tau)-v(\tau))B\Gamma^u_\tau\phi_l d\tau\Big)ds \\ &-\int_0^T e^{-iA(T-s)}v(s)B\Big(\int_0^s e^{-iA(s-\tau)}v(\tau)B(\Gamma^u_\tau\phi_l - \Gamma^v_\tau\phi_l)d\tau\Big)ds. \end{aligned}$$

Thanks to Proposition 1.2 and Remark 1.3, there exists a constant $C(T) > 0$ such that, for every $\psi \in H^3 \cap H^1_0$ and $u \in L^2((0,T),\mathbb{R})$,

$$\Big\|\int_0^T e^{-iA(T-s)}u(s)B\psi ds\Big\|_{(3)} \le C(T)\|u\|_2 \|\!|\!| B |\!|\!|_3 \|\psi\|_{L^\infty_t H^3_x}. \tag{10}$$

As we assumed $\|\!|\!| B |\!|\!|_3 = 1$, we have

$$\begin{aligned} \|\Gamma^v_t\phi_l - \Gamma^u_t\phi_l\|_{L^\infty_t H^3_x} &\le \Big\|\int_0^t e^{-iA(t-s)}B(v\Gamma^v_t\phi_l - u\Gamma^u_t\phi_l)\Big\|_{L^\infty_t H^3_x} \\ &\le C(T)\|\!|\!| B |\!|\!|_3 \|v\Gamma^v_t\phi_l - u\Gamma^u_t\phi_l\|_{L^\infty_t H^3_x} \\ &\le C(T)\|v-u\|_2\|\Gamma^u_t\phi_l\|_{L^\infty_t H^3_x} + C(T)\|v\|_2\|\Gamma^v_t - \Gamma^u_t\|_{L^\infty_t H^3_x}. \end{aligned}$$

Let $\mu > 1$. If $U = \{u \in X : \ \|u\|_2 \le (\mu C(T))^{-1}\}$, then

$$\|\Gamma^v_t\phi_l - \Gamma^u_t\phi_l\|_{L^\infty_t H^3_x} \le \frac{\mu C(T)}{\mu - 1}\|v-u\|_2\|\Gamma^u_t\phi_l\|_{L^\infty_t H^3_x}$$

for every $u, v \in U$. Now,

$$\|H_l(u) - H_l(v)\|_{(3)} \leq C(T)^2 \|v-u\|_2 (\|u\|_2 + \|v\|_2) \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3}$$
$$+ C(T)^2 \|v\|_2^2 \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \leq \frac{(2\mu-1)}{(\mu-1)\mu} C(T) \|v-u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3}.$$

Thanks to the relation (10) and to the Duhamel's formula,

$$\|\Gamma_T^u \phi_l\|_{L_t^\infty H_x^3} \leq \|\phi_l\|_{(3)} + C(T)\|u\|_2 \,|||\, B \,|||\,_3 \|\Gamma_T^u \phi_l\|_{L_t^\infty H_x^3}.$$

We obtain $\|\Gamma_T^u \phi_l\|_{L_t^\infty H_x^3} \leq \frac{\|\phi_l\|_{(3)}}{1 - C(T)\|u\|_2 |||\, B \,|||\,_3} \leq \frac{\mu l^3}{\mu - 1}$ and

$$\Longrightarrow \qquad \|H_l(u) - H_l(v)\|_{(3)} \leq \frac{2\mu - 1}{(\mu-1)^2} l^3 C(T) \|v-u\|_2.$$

To apply [GC97, *Lemma* 2.3; *p.* 42], we set $M_1 = \frac{2\mu-1}{(\mu-1)^2} l^3 C(T)$ and we estimate μ such that $M_1 \leq M/2$. The last inequality is true when

$$\mu \geq a_l + \sqrt{a_l(a_l+1)} + 1, \qquad a_l := 2C(T)\widetilde{C}(T) l^3 C_l^{-1}. \tag{11}$$

Let us establish an upper bound for $C(T)\widetilde{C}(T)$ by studying the constants C_1, C_2 appearing in Ingham's Theorem ([KL05, *Theorem* 4.3]). First, we refer to Remark B.1 (Appendix B) and we set $T = \frac{4\pi}{\mathscr{G}} = \frac{4}{\pi}$ for $\mathscr{G} = \pi^2$. Let $|I'| := \frac{\mathscr{G}}{\pi} T = 4$, $\beta = \frac{\pi^2}{4}$, $G(0) = \frac{\pi}{2}$, $I_0 = [-1, +1]$, $m = \big(|I'||I_0|^{-1}\big) = 2$, $\alpha = 4R^2$, $\widehat{G}(0) = \frac{(R^2-1)\pi}{2}$ and $R = \frac{|I'|}{2} = 2$. By substituting the constants in the proof of Ingham's Theorem [KL05, *pp.* $62-65$]), we obtain $C_2 = \frac{2m\pi G(0)\pi}{\beta \mathscr{G}} = \frac{8}{\pi}$ and $C_1 = \frac{2\pi \widehat{G}(0)\pi}{\alpha \mathscr{G}} = \frac{3\pi}{16}$. The proof of Proposition 1.2 (presented in [BL10]) and the relation (32) (Remark B.1; Appendix B) imply

$$C\big(4/\pi\big) = 3\pi^{-3} \max\big\{\sqrt{2}C_2, \sqrt{4/\pi}\big\} = 24\sqrt{2}\pi^{-4}. \tag{12}$$

In addition, we have $\widetilde{C}\Big(\frac{4}{\pi}\Big) = 2C_1^{-1}$ and $C\Big(\frac{4}{\pi}\Big)\widetilde{C}\Big(\frac{4}{\pi}\Big) \leq \frac{6}{5}$. Now, we know that $a_l = \frac{2C(T)\widetilde{C}(T) l^3}{C_l}$ and $a_l \leq \frac{12}{5}\widetilde{a}_l$ for $\widetilde{a}_l := l^3/C_l$ (C_l is defined in Assumptions I). Moreover, $C_l \leq |\langle \phi_1, B\phi_l\rangle| \leq |||\, B \,||| = 1$, which ensures that $\widetilde{a}_l > 1$. We need to define μ such that (11) is verified and

$$a_l + \sqrt{a_l(a_l+1)} + 1 \leq \Big(\frac{12}{5}\widetilde{a}_l + \sqrt{\frac{12}{5}\widetilde{a}_l\Big(\frac{12}{5}\widetilde{a}_l + 1\Big)} + 1\Big) \leq \frac{34}{5}\widetilde{a}_l = \frac{34}{5}\frac{l^3}{C_l}.$$

If we choose $\mu = \frac{34}{5}\frac{l^3}{C_l}$, then $\mu \geq a_l + \sqrt{a_l(a_l+1)} + 1$ as required in (11). We recall $U = \big\{u \in X : \ \|u\|_2 \leq (\mu C(4/\pi))^{-1}\big\}$ and, as $M_1 \leq M/2$, [GC97, *Lemma* 2.3; *p.* 42] is satisfied.
In conclusion, $A_l : U \subseteq L^2((0, 4/\pi), \mathbb{R}) \to A(U) \subseteq H^3_{(0)}$ is an homeomorphism.

3) We show a neighborhood of ϕ_l in $H^3_{(0)}$ contained in $A_l(U)$. Let

$$B_X(x,r) := \{\widetilde{x} \in X \mid \|\widetilde{x} - x\|_{L^2((0,\frac{4}{\pi}),\mathbb{R})} \leq r\}.$$

We notice that $\mu C\left(\frac{4}{\pi}\right) < 3\frac{l^3}{C_l}$ and we set $\widetilde{U} = B_X\left(0, \frac{C_l}{3l^3}\right) \subset U$. From the proof of [GC97, *Lemma* 2.3; *p.* 42], we know that $A_l(U)$ contains a ball of center $A_l(0) = \phi_l\left(\frac{4}{\pi}\right)$ and radius $(M - M_1)\frac{C_l}{3l^3}$. Thanks to the relation $M_1 \leq M/2$, we know that

$$M - M_1 \geq \frac{1}{2}M \geq \frac{C_l}{2\widetilde{C}\left(\frac{4}{\pi}\right)} \geq \frac{3\pi C_l}{2^5}, \qquad \frac{1}{3}(M - M_1) > 6^{-2}C_l,$$

$$A_l\left(B_X\left(0, \frac{C_l}{3l^3}\right)\right) \supseteq \widetilde{B}_{H^3_{(0)}}\left(A_l(0), (M - M_1)\frac{C_l}{3l^3}\right) \supseteq \widetilde{B}_{H^3_{(0)}}\left(\phi_l\left(\frac{4}{\pi}\right), \frac{C_l^2}{6^2 l^3}\right).$$

In the first part of the proof, we suppose $\|\!|\!| B |\!|\!|_3 = 1$, but we can generalize the result for $\|\!|\!| B |\!|\!|_3 \neq 1$ thanks to the identity $A + uB = A + u\|\!|\!| B |\!|\!|_3 \frac{B}{\|\!|\!| B |\!|\!|_3}$. To this purpose, we consider the operator $\frac{B}{\|\!|\!| B |\!|\!|_3}$ and the control $u\|\!|\!| B |\!|\!|_3$ and we substitute to C_l with $C_l \|\!|\!| B |\!|\!|_3^{-1}$ (defined in Assumptions I). We also notice that if $\|\!|\!| B |\!|\!|_3 u \in B_X\left(0, \frac{C_l}{3l^3\|\!|\!| B |\!|\!|_3}\right)$, then $u \in B_X\left(0, \frac{C_l}{3l^3\|\!|\!| B |\!|\!|_3^2}\right)$. In conclusion, we obtain

$$\forall \psi \in \widetilde{B}_{H^3_{(0)}}\left(\phi_l\left(\frac{4}{\pi}\right), \frac{C_l^2}{6^2 l^3 \|\!|\!| B |\!|\!|_3^2}\right),\ \exists\, u \in B_X\left(0, \frac{C_l}{3l^3 \|\!|\!| B |\!|\!|_3^2}\right) :\ A_l(u) = \psi. \quad \square$$

3 Proof of Theorem 1.1

Proposition 3.1. *Let B satisfy Assumptions I. For every $j, k \in \mathbb{N}$, $j \neq k$, and $n \in \mathbb{N}$ such that $n \geq 6^{42}\pi^{12} b\ (1 + C')E(j,k)$, there exists $\theta \in \mathbb{R}$ so that*

$$\|\Gamma^{u_n}_{nT^*}\phi_j - e^{i\theta}\phi_k\|^8_{(3)} \leq$$

$$\frac{6^{26}\pi^{12} e^{\frac{6\|\!|\!| B |\!|\!|_{(2)}}{B_{j,k}}}(1 + C')\|\!|\!| B |\!|\!|^6_{(2)}\|\!|\!| B |\!|\!|\max\{\|\!|\!| B |\!|\!|, \|\!|\!| B |\!|\!|_3\}|k^2 - j^2|^5 \max\{j,k\}^{24}}{|B_{j,k}|^7 n}.$$

Proof. See Appendix *A*. $\square$

Proof of Theorem 1.1. The proof follows from Proposition 2.2 and Proposition 3.1. Let $R''_n := \|\Gamma^{u_n}_{nT^*}\phi_j - e^{i\theta}\phi_k\|^8_{(3)}$ be estimated in Proposition 3.1. We know $\lim_{n\to\infty} R''_n = 0$ and there exist n^* and θ (depending on n^*) such that

$$(13)\quad \Gamma^{u_{n^*}}_{n^*T^*}\phi_j \in \widetilde{B}_{H^3_{(0)}}\left(e^{i\theta}\phi_k, C_k^2(6^2k^3\|\!|\!| B |\!|\!|_3^2)^{-1}\right), \quad R''_{n^*} \leq \frac{C_k^{16}}{6^2 k^{24}\|\!|\!| B |\!|\!|_3^{16}}.$$

For $0 \leq s < 3$ and $j, k \in \mathbb{N}$, we know that $\| \! | B | \! \|_{(s)} \geq C_k$ and $\| \! | B | \! \|_{(s)} \geq |B_{j,k}|$. For $b := \| \! | B | \! \|_{(2)}^{6} \| \! | B | \! \| \, \| \! | B | \! \|_{3}^{16} \max\left\{ \| \! | B | \! \|, \| \! | B | \! \|_{3} \right\}$, the relation (13) is valid when

$$n^* \geq \frac{6^{42}\pi^{12} e^{\frac{6 \| \! | B | \! \|_{(2)}}{B_{j,k}}} b(1+C')|k^2-j^2|^5 k^{24} \max\{j,k\}^{24}}{C_k^{16}|B_{j,k}|^7}.$$

For $u_{n^*}(t) = \frac{\cos\left((k^2-j^2)\pi^2 t\right)}{n^*}$ and $n^*T^* = n^* \frac{\pi}{|B_{j,k}|}$, thanks to Proposition 2.2 and to the time reversibility of the (BSE) (see [Duc, *Section* 1.3]), there exists $u \in L^2((0, \frac{4}{\pi}), \mathbb{R})$ such that

$$\Gamma^{u}_{\frac{4}{\pi}} \Gamma^{u_{n^*}}_{n^*T^*} \phi_j = e^{i\theta}\phi_k. \tag{14}$$ □

4 Computing the phase

Let $k, j, N \in \mathbb{N}$. We define the $N \times N$ matrix M^N such that, for $l, m \in \mathbb{N}$,

$$M^N_{l,m} = \langle \phi_l, M^N \phi_m \rangle = \frac{B_{l,m}}{I} \int_0^I e^{i\pi^2(l^2-m^2)v(x)} dx, \qquad \text{if} \quad \frac{|l^2-m^2|}{|k^2-j^2|} \in \mathbb{N},$$

for $v(t)$ the reciprocal function of $t \mapsto \int_0^t |\cos(\pi^2(k^2-j^2)s)| ds$, otherwise $M^N_{l,m} = 0$. Let $\theta^N \in \mathbb{R}^+$ be the smallest value such that $e^{i\theta^N} = \langle \phi_k, e^{2|B_{k,j}|^{-1} M^N} \phi_j \rangle$ and

$$\widetilde{T}^N = \frac{\theta^N}{(j\pi)^2}.$$

In the following proposition, we provide a similar result of Proposition 3.1 without the presence of the phase ambiguity in the target state. The proposition leads to the subsequent Corollary which ensures a similar result to Theorem 1.1.

Proposition 4.1. *Let B satisfy Assumptions I. Let $j, k \in \mathbb{N}$, $j \neq k$, and $n \in \mathbb{N}$ such that $n \geq 6^{42}\ 10\ \pi^{12} b\ (1+C')\, \| \! | B | \! \|\, E(j,k) |B_{j,k}|^{-1}$. For $N \in \mathbb{N}$ such that*

$$\frac{2}{|B_{j,k}|}\Big(\Big(\sum_{l=N+1}^{\infty} |B_{l,k}|^2 \Big)^{\frac{1}{2}} + \Big(\sum_{l=N+1}^{\infty} |B_{l,j}|^2 \Big)^{\frac{1}{2}} \Big) \leq \frac{4 \| \! | B | \! \|}{n\pi^2 |k^2-j^2|},$$

we have that

$$\|\Gamma^{u_n}_{nT^*} \Gamma^0_{\widetilde{T}^N} \phi_j - \phi_k\|^8_{(3)} \leq$$

$$\frac{10\ 6^{26}\pi^{12}(1+C')\, \| \! | B | \! \|_{(2)}^{6}\, \| \! | B | \! \|^{2} \max\{ \| \! | B | \! \|, \| \! | B | \! \|_{3}\} |k^2-j^2|^5 \max\{j,k\}^{24}}{|B_{j,k}|^8 n}.$$

By following the proof Theorem 1.1, we have that Proposition 2.2 and Proposition 4.1 imply the next corollary.

Corollary 4.2. *Let $j, n, N \in \mathbb{N}$ and $k \in \mathbb{N}$ be such that $k \neq j$ and*

$$m^2 - k^2 \neq k^2 - l^2, \qquad \forall m, l \in \mathbb{N},\ m, l \neq k. \tag{15}$$

Let $n \geq 6^{42}\ 10\ \pi^{12} b\ (1 + C')\ |\!|\!| B |\!|\!|\ E(j,k) |B_{j,k}|^{-1}$. If $N \geq \max\{j, k\}$ and

$$\frac{2}{|B_{j,k}|}\Big(\Big(\sum_{l=N+1}^{\infty} |B_{l,k}|^2\Big)^{\frac{1}{2}} + \Big(\sum_{l=N+1}^{\infty} |B_{l,j}|^2\Big)^{\frac{1}{2}}\Big) \leq \frac{4\,|\!|\!| B |\!|\!|}{n\pi^2 |k^2 - j^2|}, \tag{16}$$

then

$$\big\|\Gamma^{u_n}_{nT^*}\Gamma^0_{\widetilde{T}^N}\phi_j - \phi_k\big\|_{(3)} \leq C_k^2 (6^2 k^3\ |\!|\!| B |\!|\!|\ {}^2_3)^{-1}.$$

Moreover, there exists $u \in L^2((0, \frac{4}{\pi}), \mathbb{R})$ such that $\|u\|_{L^2((0,\frac{4}{\pi}),\mathbb{R})} \leq \frac{C_k}{3\ |\!|\!| B |\!|\!|\ {}^2_3 k^3}$ and

$$\Gamma^u_{\frac{4}{\pi}}\Gamma^{u_n}_{nT^*}\Gamma^0_{\widetilde{T}^N}\phi_j = \phi_k.$$

5 Application of the main result

In the current section, we briefly propose a possible application of Theorem 1.1. Let us consider an electron trapped in a one-dimensional guide of length $\sim 10^{-3}\ m$ and represented by the quantum state ψ. We suppose that the electron is subjected to an external time-depending electromagnetic field $\mathcal{V}(\tau)$ with $\tau \in (0, \mathrm{T})$ and T a positive time. Let $m_e \sim 10^{-30}\ Kg$ be the mass of the electron and $\hbar \sim 10^{-34}\ \frac{m^2 \cdot Kg}{s}$ with $\hbar$ the reduced Planck constant. The dynamics of ψ is modeled by the Schrödinger equation

$$i\hbar \cdot \frac{d}{d\tau}\psi(\tau) = -\frac{\hbar^2}{m_e} \cdot \frac{d^2}{d\mathrm{x}}\psi(\tau) + \mathcal{V}(\tau)\psi(\tau), \qquad \tau \in (0, \mathrm{T}). \tag{17}$$

We substitute

$$x := \mathrm{x} \cdot 10^3\ m^{-1}, \qquad t := \tau \cdot 10^2\ s^{-1}, \qquad \psi(t, x) := \psi(\tau, \mathrm{x}).$$

The quantity $V(t) := \mathcal{V}(\tau)\ 10^{32}\ \frac{s^2}{m^2 \cdot Kg}$ and the coordinates $(t, x) \in (0, T) \times (0, 1)$ with $T := \mathrm{T} \cdot 10^2\ s^{-1}$ are dimensionless. By computation, we obtain

$$i\ \frac{d}{dt}\psi(t) = -\frac{d^2}{dx^2}\psi(t) + V\psi(t), \qquad t \in (0, T).$$

If the control potential $V(t, x)$ is equal to $u(t)x^2$ where x^2 represents the action of the field and u its time-dependent intensity, then we obtain the bilinear Schrödinger equation

$$i\partial_t\psi(t, x) = A\psi(t, x) + u(t)x^2\psi(t, x).$$

We exhibit u driving the state of the particle from the first excited state to the ground state. For this reason, we retrace the proof of the first point of Theorem 1.1 with $B : \psi \mapsto x^2\psi$. Let ϕ_1 and ϕ_2 be eigenstates of A. We define a control function driving ϕ_2 in ϕ_1. First, for $\langle \phi_j, x^2\phi_k\rangle = 2\int_0^1 x^2 \sin(\pi j x)\sin(\pi k x)dx$, we notice that Assumptions I are satisfied since

$$|\langle \phi_j, x^2\phi_k\rangle| = \Big|\frac{(-1)^{j-k}}{(j-k)^2\pi^2} - \frac{(-1)^{j+k}}{(j+k)^2\pi^2}\Big| = \frac{4jk}{(j^2-k^2)^2\pi^2}, \qquad j \neq k,$$
$$|\langle \phi_k, x^2\phi_k\rangle| = \Big|\frac{1}{3} - \frac{1}{2k^2\pi^2}\Big|, \qquad\qquad k \in \mathbb{N}.$$

For every $\psi \in H^3_{(0)}$, we have $x^2\psi \in H^3 \cap H^1_0$, $\|\partial_x\psi\| \leq \|\partial_x^2\psi\|$ and, from the Poincaré inequality, $\|\psi\| \leq \pi^{-1}\|\partial_x\psi\|$ and $\|\partial_x^2\psi\| \leq \pi^{-1}\|\partial_x^3\psi\|$. Thus,

$$|\!|\!| B |\!|\!|_3^2 = \sup_{\substack{\psi\in H^3_{(0)}\\ \|\psi\|_{(3)}\leq 1}} (\|\partial_x x^2\psi\|^2 + \|\partial_x^2 x^2\psi\|^2 + \|\partial_x^3 x^2\psi\|^2) \leq \Big(\frac{2\sqrt{5}+\sqrt{3}\pi}{\sqrt{15}\pi^2}\Big)^2$$
$$+ \Big(\frac{2\sqrt{15}+4\sqrt{5}\pi+\sqrt{3}\pi}{\sqrt{15}\pi^2}\Big)^2 + \Big(\frac{6\sqrt{15}+6\sqrt{5}\pi+\sqrt{3}\pi}{\sqrt{15}\pi}\Big)^2 \leq 5,93.$$

Equivalently $|\!|\!| B |\!|\!|_{(2)} \leq 3,4$ and $|\!|\!| B |\!|\!| = 1/\sqrt{5}$. Moreover, $C' = 0$ and

$$|B_{1,1}| = C_1 = \frac{2\pi-3}{6\pi^2}, \qquad |B_{1,2}| = C_2 = \frac{8}{9\pi^2}, \qquad I = \frac{4}{3\pi^2}.$$

If we retrace the proof of Proposition 2.2 by substituting the previous constants, then we obtain that in $\widetilde{B}_{H^3_{(0)}}\left(\phi_1,\ 7,76\cdot 10^{-6}\right)$ the local exact controllability is verified. Let $T = \frac{2}{3\pi}$, $u(t) = \cos(3\pi^2 t)$, $T^* = \frac{9\pi^3}{8}$, $K = \frac{9\pi^2}{4}$. By repeating the proof of Theorem 1.1 and Proposition 3.1, we see that, for $u_n := \frac{u}{n}$, there exists $\theta \in \mathbb{R}$ such that

$$\|e^{i\theta}\phi_1 - \Gamma^{u_n}_{nT^*}\phi_2\|^8_{(3)} \leq 2,61\cdot 10^{140} n^{-1}.$$

In the neighborhood $\widetilde{B}_{H^3_{(0)}}\left(\phi_1,\ 2,4\cdot 10^{-6}\right)$, the local exact controllability is verified and the first point of Theorem 1.1 is satisfied for $n = 2,38\cdot 10^{185}$. In conclusion, there exists $\theta \in \mathbb{R}$ such that for

$$u(t) = (2,38\cdot 10^{185})^{-1}\cos(3\pi^2 t), \qquad T = (2,38\cdot 10^{185})\frac{9\pi^3}{8},$$

there holds $\left\|e^{i\theta}\phi_1 - \Gamma^u_T\phi_2\right\|_{(3)} \leq 2,4\cdot 10^{-6}$. In addition, there exists $\tilde{u} \in L^2((0,\frac{4}{\pi}),\mathbb{R})$ such that

$$\Gamma^u_T\Gamma^{\tilde{u}}_{\frac{4}{\pi}}\phi_2 = e^{i\theta}\phi_1.$$

In conclusion, the dynamics of (17) drives the state of the electron from the first excited state to the ground state in a time T $\sim 10^{184}$ s.

6 Conclusion

The estimates introduced in the work are far from being optimal and one might be interested in optimizing them in order to study meaningful physical systems.
The first try is to repeat the steps of the proof of Theorem 1.1 for specific B, j and k, but other possible ideas can be the following. For instance, Theorem 1.1 can be stated for other $\frac{2\pi}{|\lambda_k-\lambda_j|}-$periodic controls by using the theory exposed in [Cha12].
Moreover, in Proposition 2.2, one can look for a larger neighborhood of validity of the local exact controllability. A try is to use "Haraux's Theorem" ([KL05, *Theorem* 4.6]) instead of Ingham's Theorem ([KL05, *Theorem* 4.3]) and change the time $T = \frac{4}{\pi}$.

Even though an optimization work is required for any physycal application, numerical results can be provided by following the developed techniques. Indeed, as the dynamics time and the relative control function are explicit for the global approximate controllability, suitable algorithms can be implement in order to confute the provided estimates.

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A Explicit controls and times for the global approximate controllability

For $j, k \in \mathbb{N}$, we recall the definition of $B_{j,k} = \langle \phi_j, B\phi_k \rangle$ and we denote

$$T^* = \frac{\pi}{|B_{j,k}|}, \qquad T = \frac{2\pi}{|\lambda_k - \lambda_j|}, \qquad u_n(t) = \frac{\cos\big((k^2-j^2)\pi^2 t\big)}{n},$$

$$I = \frac{4}{|\lambda_k - \lambda_j|}, \qquad K = \frac{2}{|B_{j,k}|}, \qquad C' = \sup_{(l,m)\in\Lambda'} \Big\{ \Big| \sin\Big(\pi \frac{|\lambda_l - \lambda_m|}{|\lambda_k - \lambda_j|}\Big)\Big|^{-1} \Big\},$$

$$\Lambda' = \big\{(l,m) \in \mathbb{N}^2 : \ \{l,m\} \cap \{j,k\} \neq \emptyset, \ |\lambda_l - \lambda_m| \leq \frac{3}{2}|\lambda_k - \lambda_j|, \\ |\lambda_l - \lambda_m| \neq |\lambda_k - \lambda_j|, \ B_{l,m} \neq 0\big\}.$$

In the following proposition, we provide a global approximate controllability result with explicit controls and times with respect to the $\mathscr{H}$-norm.

Proposition A.1. *Let B satisfy Assumptions I. For every $j, k \in \mathbb{N}$, $j \neq k$, and $n \in \mathbb{N}$ such that*

$$n \geq \frac{3(1+C')|B_{j,k}|^{-1} \,|\!|\!|\, B \,|\!|\!|^{\,2}}{|k^2 - j^2|}, \tag{18}$$

there exist $T_n \in (nT^* - T, nT^* + T)$ *and* $\theta \in \mathbb{R}$ *such that*

$$\|\Gamma_{T_n}^{u_n}\phi_j - e^{i\theta}\phi_k\|_{\mathscr{H}}^2 \leq \frac{3^2|B_{j,k}|^{-1}(1+C')\,|||\,B\,|||^{\,2}}{n|k^2-j^2|}.$$

Proof. Thanks to [Cha12, *Proposition* 6], for any $n \in \mathbb{N}$, there exists $T_n \in (nT^* - T, nT^* + T)$ such that

$$\frac{1-|\langle\phi_k, \Gamma_{T_n}^{u_n}\phi_j\rangle|}{1+2K\,|||\,B\,|||} \leq \frac{(1+C')\,|||\,(\phi_j\langle\phi_j,\cdot\rangle + \phi_k\langle\phi_k,\cdot\rangle)B\,|||\,I}{n}.$$

We point out that the definition of T^* provided in [Cha12, *Proposition* 6] is incorrect and the formulation that we provide can be deduced from [Cha12, *Proposition* 2]. In addition, we have $1-|\langle\phi_k, \Gamma_{T_n}^{u_n}\phi_j\rangle| \leq \frac{(1+2K\,|||\,B\,|||\,)(1+C')\,|||\,B\,|||\,I}{n} =: R_n$ and

$$\sum_{l\neq k}|\langle\phi_l, \Gamma_{T_n}^{u_n}\phi_j\rangle - \langle\phi_l,\phi_k\rangle|^2 \leq \big(1-|\langle\phi_k, \Gamma_{T_n}^{u_n}\phi_j\rangle|\big)\big(1+|\langle\phi_k, \Gamma_{T_n}^{u_n}\phi_j\rangle|\big) \leq 2R_n.$$

Afterwards, fixed $n \in \mathbb{N}$, there exists $\theta \in \mathbb{R}$ (depending on n) such that $|\langle\phi_k, e^{i\theta}\phi_k\rangle - \langle\phi_k, \Gamma_{T_n}^{u_n}\phi_j\rangle|^2 \leq R_n^2$ and $R_n' := \|e^{i\theta}\phi_k - \Gamma_{T_n}^{u_n}\phi_j\|^2 \leq 2R_n + R_n^2$. As $|B_{j,k}|^{-1}\,|||\,B\,|||\, = \frac{|||\,B\,|||}{|\langle\phi_j, B\phi_k\rangle|} \geq 1$, we have

$$R_n = \frac{(1+2K\,|||\,B\,|||\,)(1+C')\,|||\,B\,|||\,I}{n} \leq \frac{3(1+C')|B_{j,k}|^{-1}\,|||\,B\,|||^{\,2}}{n|k^2-j^2|}.$$

If $n \geq \frac{3(1+C')|B_{j,k}|^{-1}\,|||\,B\,|||^{\,2}}{|k^2-j^2|}$ for $j \neq k$, then $R_n \leq 1$, $R_n^2 \leq R_n$ and

$$\|e^{i\theta}\phi_k - \Gamma_{T_n}^{u_n}\phi_j\|^2 \leq 2R_n + R_n^2 \leq 3R_n \leq \frac{3^2|B_{j,k}|^{-1}(1+C')\,|||\,B\,|||^{\,2}}{n|k^2-j^2|}. \quad \square$$

Proposition A.2. *Let* B *satisfy Assumptions I. For every* $j,k \in \mathbb{N}$, $j \neq k$, *and* $n \in \mathbb{N}$ *satisfying* (18) *such that*

$$n \geq 2\,|||\,B\,|||_{(2)}, \tag{19}$$

there exists $T_n \in (nT^* - T, nT^* + T)$ *and* $\theta \in \mathbb{R}$ *such that*

$$\|\Gamma_{T_n}^{u_n}\phi_j - e^{i\theta}\phi_k\|_{(3)}^8 \leq \frac{2^{18}3^{26}\pi^{12}(1+C')\,|||\,B\,|||_{(2)}^{\,6}\,|||\,B\,|||^{\,2}|k^2-j^2|^5\max\{j,k\}^{24}}{|B_{j,k}|^7 n}.$$

Proof. **1) Propagation of regularity from $H^2_{(0)}$ to $H^4_{(0)}$:** In the first part of the proof, we show that the propagator Γ^u_T preserves $H^4_{(0)}$ and $B \in L(H^2_{(0)})$. Let us denote $\|f\|_{BV(T)} := \|f\|_{BV((0,T),\mathbb{R})} = \sup_{\{t_j\}_{j\leq n}\in P}\sum_{j=1}^n |f(t_j) - f(t_{j-1})|$ for $f \in BV((0,T),\mathbb{R})$, where P is the set of the partitions of $(0,T)$ such that $t_0 = 0 < t_1 < ... < t_n = T$.
Let $\epsilon > 0$, $\lambda_\epsilon = |\!|\!| B |\!|\!|_{(2)}\epsilon^{-1}$, $\tilde{\lambda}_\epsilon = \lambda_\epsilon + \|u_n\|_{L^\infty((0,T),\mathbb{R})}|\!|\!| B |\!|\!|_{(2)}$ and $\widehat{H}^4_{(0)} := D(A(i\tilde{\lambda}_\epsilon - A))$. We refer to [Kat53] and we prove that the propagator $U^{u_n}_t$ generated by

$$A + u_n(t)B - i\|u_n\|_{L^\infty((0,T),\mathbb{R})}|\!|\!| B |\!|\!|_{(2)}$$

satisfies the condition $\|U^{u_n}_t\psi\|_{(4)} \leq C\|\psi\|_{(4)}$ for every $\psi \in H^4_{(0)}$ and suitable $C > 0$. Indeed, $-i(A + u_n(t)B - i\|u_n\|_{L^\infty((0,T),\mathbb{R})}|\!|\!| B |\!|\!|_{(2)})$ is not just dissipative in $H^2_{(0)}$, but also maximal dissipative thanks to Kato-Rellich's Theorem [Dav95, *Theorem* 1.4.2]. Now, Hille-Yosida Theorem implies that the semi-group generated by $-i(A + u_n(t)B - i\|u_n\|_{L^\infty((0,T),\mathbb{R})}|\!|\!| B |\!|\!|_{(2)})$ is a semi-group of contraction and the techniques adopted in the proofs of [Kat53, *Theorem* 2; *Theorem* 3] are valid. For $n \geq 3\epsilon$, $|\!|\!| u_n(t)B(i\tilde{\lambda}_\epsilon - A)^{-1} |\!|\!|_{(2)} \leq \frac{|\!|\!| B |\!|\!|_{(2)}|\!|\!| (i\tilde{\lambda}_\epsilon - A)^{-1} |\!|\!|_{(2)}}{n} \leq \frac{|\!|\!| B |\!|\!|_{(2)}}{n\lambda_\epsilon} < 1$ and we introduce

$$\begin{aligned} M &:= \sup_{t\in[0,T_n]} |\!|\!| (i\tilde{\lambda}_\epsilon - A - u_n(t)B)^{-1} |\!|\!|_{L(H^2_{(0)},\widehat{H}^4_{(0)})} \\ &= \sup_{t\in[0,T_n]} |\!|\!| (I - u_n(t)B(i\tilde{\lambda}_\epsilon - A)^{-1})^{-1} |\!|\!|_{(2)} \\ &= \sup_{t\in[0,T_n]} |\!|\!| \sum_{l=1}^{+\infty}(u_n(t)B(i\tilde{\lambda}_\epsilon - A)^{-1})^l |\!|\!|_{(2)} = \frac{n}{n-\epsilon} < 2. \end{aligned} \tag{20}$$

We know that $\|k + f(\cdot)\|_{BV((0,T),\mathbb{R})} = \|f\|_{BV((0,T),\mathbb{R})}$ for every $f \in BV((0,T),\mathbb{R})$ and $k \in \mathbb{R}$. The same idea leads to

$$N := |\!|\!| i\tilde{\lambda}_\epsilon - A - u_n(\cdot)B |\!|\!|_{BV\left([0,T_n],L(\widehat{H}^4_{(0)},H^2_{(0)})\right)} = \|u_n\|_{BV(T_n)}|\!|\!| B |\!|\!|_{L(\widehat{H}^4_{(0)},H^2_{(0)}))}.$$

Thanks to [Kat53, *Section* 3.10], there holds

$$\begin{aligned} &\|(A + u_n(T_n)B - i\tilde{\lambda}_\epsilon)U^{u_n}_{T_n}\phi_j\|_{(2)} \leq Me^{MN}\|(A - i\tilde{\lambda}_\epsilon)\phi_j\|_{(2)} \\ &\leq Me^{MN}(\lambda_j + \tilde{\lambda}_\epsilon)j^2 \leq Me^{MN}(\pi^2 + \lambda_\epsilon + \|u\|_{L^\infty((0,T),\mathbb{R})}|\!|\!| B |\!|\!|_{(2)})j^4. \end{aligned}$$

Now, as $\|(A - i\tilde{\lambda}_\epsilon)\psi\|^2_{(2)} = \|A\psi\|^2_{(2)} + \tilde{\lambda}^2_\epsilon\|\psi\|^2_{(2)}$, for every $\psi \in \widehat{H}^4_{(0)}$,

$$\|B\psi\|^2_{(2)} \leq \left(\epsilon\|A\psi\|_{(2)} + |\!|\!| B |\!|\!|_{(2)}\|\psi\|_{(2)}\right)^2 \leq 2\epsilon^2\left(\|A\psi\|^2_{(2)} + \tilde{\lambda}^2_\epsilon\|\psi\|^2_{(2)}\right) \leq 2\epsilon^2\|\psi\|^2_{\widehat{H}^4_{(0)}}$$

and $N \leq \epsilon\sqrt{2}\|u_n\|_{BV(T_n)}$. In addition, thanks to the relation (20),

$$|\!|\!| A(A + u_n(T_n)B - i\tilde{\lambda}_\epsilon)^{-1} |\!|\!|_{(2)} \leq M + |\!|\!| \tilde{\lambda}_\epsilon(A - i\tilde{\lambda}_\epsilon)^{-1} |\!|\!|_{(2)}M \leq 4.$$

For every $j \in \mathbb{N}$, we know that

$$\|U_t^{u_n}\phi_j\|_{(4)} = e^{-t\|u_n\|_{L^\infty((0,T),\mathbb{R})}|\!|\!|B|\!|\!|_{(2)}}\|\Gamma_t^{u_n}\phi_j\|_{(4)} \leq e^{-\frac{t}{n}|\!|\!|B|\!|\!|_{(2)}}\|\Gamma_t^{u_n}\phi_j\|_{(4)}$$

and, for n satisfying (19) and $\epsilon = (2\sqrt{2}\|u_n\|_{BV(T_n)})^{-1}$, we have

(21)

$$\begin{aligned}
&\|\Gamma_{T_n}^{u_n}\phi_j\|_{(4)} = \|A\Gamma_{T_n}^{u_n}\phi_j\|_{(2)} \leq 4e^{\frac{T_n}{n}|\!|\!|B|\!|\!|_{(2)}}\|(A + u_n(T_n)B - i\tilde{\lambda}_\epsilon)U_{T_n}^{u_n}\phi_j\|_{(2)} \\
&\leq 8e^{\frac{T_n}{n}|\!|\!|B|\!|\!|_{(2)}+2\sqrt{2}\epsilon\|u_n\|_{BV(T_n)}}(\pi^2 + \tilde{\lambda}_\epsilon)j^4 \\
&\leq 8e^{\frac{|\!|\!|B|\!|\!|_{(2)}}{|B_{j,k}|}+\frac{2|\!|\!|B|\!|\!|_{(2)}}{n\pi|k^2-j^2|}+2\sqrt{2}\epsilon\|u_n\|_{BV(T_n)}}(\pi^2 + |\!|\!|B|\!|\!|_{(2)}(\epsilon^{-1} + \|u_n\|_{L^\infty((0,T),\mathbb{R})}))j^4 \\
&\leq 8e^{\frac{|\!|\!|B|\!|\!|_{(2)}}{|B_{j,k}|}+\frac{1}{2}+2\sqrt{2}\epsilon\|u_n\|_{BV(T_n)}}(\pi^2 + |\!|\!|B|\!|\!|_{(2)}(\epsilon^{-1} + n^{-1}))j^4. \\
&\leq 8e^{\frac{|\!|\!|B|\!|\!|_{(2)}}{|B_{j,k}|}+3/2}\big(1 + 2\sqrt{2}\|u_n\|_{BV(T_n)}|\!|\!|B|\!|\!|_{(2)} + |\!|\!|B|\!|\!|_{(2)}n^{-1}\big)j^4.
\end{aligned}$$

The interval $[0, nT^* + T]$ contains less than d quarters of period of the function u_n for $d := 2\pi^2 n|k^2 - j^2||B_{j,k}|^{-1} + 4$ since

$$u_n(nT^*+T) = \frac{1}{n}\sin\big(\pi^2(k^2-j^2)(nT^*+T)\big) \quad \Rightarrow \quad d = \big(\pi^2(k^2-j^2)(nT^*+T)\big)\frac{2}{\pi}.$$

From (19), $n \geq |\!|\!|B|\!|\!|(5\pi^{-2}|j^2 - k^2|^{-1})$, $\pi^2 n|k^2 - j^2||B_{j,k}|^{-1} \geq 5$ and

$$\|u_n\|_{BV(T_n)} \leq \|u_n\|_{BV(nT^*+T)} \leq (d+1)/n \leq 3\pi^2|k^2 - j^2||B_{j,k}|^{-1} \tag{22}$$

(also the assumption $n \geq 3\epsilon$ is verified). Thanks to $|\!|\!|B|\!|\!|_{(2)} \geq |B_{j,k}|$ and $|\!|\!|B|\!|\!| \geq |B_{j,k}|$, the relation (21) becomes

(23)

$$\begin{aligned}
&\|\Gamma_{T_n}^{u_n}\phi_j\|_{(4)} \leq 8e^{\frac{|\!|\!|B|\!|\!|_{(2)}}{|B_{j,k}|}+3/2}(\pi^2 + 3\cdot 2\sqrt{2}\pi^2|\!|\!|B|\!|\!|_{(2)}|k^2 - j^2||B_{j,k}|^{-1} \\
&+ 5^{-1}\pi^2|\!|\!|B|\!|\!|_{(2)}|\!|\!|B|\!|\!|^{-1}|j^2 - k^2|)j^4 \leq 2^2 3^4\pi^2 e^{\frac{|\!|\!|B|\!|\!|_{(2)}}{|B_{j,k}|}}|\!|\!|B|\!|\!|_{(2)}|k^2 - j^2||B_{j,k}|^{-1}j^4.
\end{aligned}$$

When $u \in BV(T)$, the propagator Γ_T^u preserves $H^4_{(0)}$ if $B \in L(H^2_{(0)})$.

2) Conclusion: Let $f_n := e^{i\theta}\phi_k - \Gamma_{T_n}^{u_n}\phi_j$. First, we point out that, for every $s > 0$, we have $\|f_n\|^2_{(s)} \leq (k^s + \|\Gamma_{T_n}^{u_n}\phi_j\|_{(s)})^2$. As $\phi_j, \phi_k \in H^s_{(0)}$, for every $s > 0$, the point **1)** ensures that $\Gamma_T^u\phi_j$ and $\Gamma_T^u\phi_j$ belong to $H^4_{(0)}$ for $u \in BV(0,T)$. Thanks to the Cauchy-Schwarz inequality,

$$\|A^{\frac{3}{2}}f_n\|^4 = \big(\langle A^{\frac{3}{2}}f_n, A^{\frac{3}{2}}f_n\rangle\big)^2 \leq \big(\langle A^2 f_n, Af_n\rangle\big)^2 \leq \|A^2 f_n\|^2\|Af_n\|^2,$$

$$\|Af_n\|^2 = \langle Af_n, Af_n\rangle \leq \|A^2 f_n\|\|f_n\|, \qquad \Longrightarrow \qquad \|f_n\|^8_{(3)} \leq \|f_n\|^2\|f_n\|^6_{(4)}.$$

For R_n defined in the proof of Proposition $A.1$, the relation (23) implies

$$\|f_n\|^8_{(3)} \le 3R_n(2^3 3^4\pi^2 e^{\frac{|\!|\!| B |\!|\!|_{(2)}}{|B_{j,k}|}} |\!|\!| B |\!|\!|_{(2)}|k^2-j^2||B_{j,k}|^{-1}\max\{j,k\}^4)^6$$
$$\le \frac{2^{18}3^{26}\pi^{12}(1+C')e^{\frac{6\,|\!|\!| B |\!|\!|_{(2)}}{|B_{j,k}|}} |\!|\!| B |\!|\!|^6_{(2)} |\!|\!| B |\!|\!|^{\,2}|k^2-j^2|^5\max\{j,k\}^{24}}{n|B_{j,k}|^7}.$$

□

Proof of Proposition 3.1. We notice that the hypotheses of Proposition $A.2$ are verified. We estimate $\sup_{t\in[nT^*-T,nT^*+T]}\|\Gamma^{u_n}_t\phi_j - \Gamma^{u_n}_{T_n}\phi_j\|_{(3)}$ and we consider the arguments leading to (21). The uniformly bounded constant $C(\cdot)$ is increasing and (12) implies $\sup_{t\in[nT^*-T,nT^*+T]} C(|t-T_n|) \le C(2T) \le C(4/\pi) = \frac{24\sqrt{2}}{\pi^4}$. Thanks to Proposition 1.2 and Remark 1.3,

$$\sup\Big\{\sup_{t\in[nT^*-T,T_n]}\|\Gamma^{u_n}_t\phi_j - \Gamma^{u_n}_{T_n-t}\Gamma^{u_n}_t\phi_j\|_{(3)}, \sup_{t\in[T_n,nT^*+T]}\|\Gamma^{u_n}_{t-T_n}\Gamma^{u_n}_{T_n}\phi_j - \Gamma^{u_n}_{T_n}\phi_j\|_{(3)}\Big\}$$
$$\le C\Big(\frac{4}{\pi}\Big)\,|\!|\!| B |\!|\!|_3\int_{nT^*-T}^{nT^*+T}|u_n(s)|ds\ \sup\Big\{\|\Gamma^{u_n}_{T_n}\phi_j\|_{(4)}, \sup_{t\in[nT^*-T,T_n]}\|\Gamma^{u_n}_t\phi_j\|_{(4)}\Big\}.$$

The techniques adopted in (21) lead to

$$\sup_{t\in[nT^*-T,T_n]}\|\Gamma^{u_n}_t\phi_j\|_{(4)} \le 8e^{\frac{|\!|\!| B |\!|\!|_{(2)}}{|B_{j,k}|}+3/2}\big(1+2\sqrt{2}\|u_n\|_{BV(T_n)}\,|\!|\!| B |\!|\!|_{(2)}\big)j^4,$$
$$\Longrightarrow \sup_{t\in(nT^*-T,nT^*+T)}\|\Gamma^{u_n}_t\phi_j - \Gamma^{u_n}_{T_n}\phi_j\|_{(3)}$$
$$\le C\Big(\frac{4}{\pi}\Big)e^{\frac{|\!|\!| B |\!|\!|_{(2)}}{|B_{j,k}|}+3/2}\,|\!|\!| B |\!|\!|_3\frac{2T}{n}\ 2^2 3^4\pi^2\,|\!|\!| B |\!|\!|_{(2)}|k^2-j^2||B_{j,k}|^{-1}j^4$$
$$\le e^{\frac{|\!|\!| B |\!|\!|_{(2)}}{|B_{j,k}|}+3/2}\,\frac{|\!|\!| B |\!|\!|_3}{n}\ 6^3\pi^2\,|\!|\!| B |\!|\!|_{(2)}|B_{j,k}|^{-1}j^4.$$

Now, for $R''_n := \|\Gamma^{u_n}_{nT^*}\phi_j - e^{i\theta}\phi_k\|^8_{(3)}$,

$$\begin{aligned} R''_n &\le 2^7\sup_{t\in(nT^*-T,nT^*+T)}\|\Gamma^{u_n}_t\phi_j - \Gamma^{u_n}_{T_n}\phi_j\|^8_{(3)} + 2^7\|\Gamma^{u_n}_{T_n}\phi_j - e^{i\theta}\phi_k\|^8_{(3)} \\ &\le 2^7\Big(e^{\frac{|\!|\!| B |\!|\!|_{(2)}}{|B_{j,k}|}+3/2}\,|\!|\!| B |\!|\!|_3\ 6^3\pi^2\,|\!|\!| B |\!|\!|_{(2)}|B_{j,k}|^{-1}n^{-1}j^4\Big)^8 + 2^7\|f_n\|^8_{(3)}. \end{aligned} \tag{24}$$

We underile that $|\!|\!| B |\!|\!|\,, |\!|\!| B |\!|\!|_{(2)} \ge |B_{j,k}|$ for every $j,k\in\mathbb{N}$. If

$$n \ge 6^{42}\pi^{12}b\ (1+C')E(j,k) \ge e^{\frac{|\!|\!| B |\!|\!|_{(2)}}{|B_{j,k}|}+3/2}\,|\!|\!| B |\!|\!|_3\pi^2\ 6^3\,|\!|\!| B |\!|\!|_{(2)}|B_{j,k}|^{-1}j^4,$$

$$\Longrightarrow \qquad R_n'' \leq 2^7 \left(e^{\frac{|\!|\!| B |\!|\!|_{(2)}}{|B_{j,k}|}+3/2} \, |\!|\!| B |\!|\!|_3 \, 6^3\pi^2 \, |\!|\!| B |\!|\!|_{(2)} |B_{j,k}|^{-1} n^{-1} j^4 + \|f_n\|_{(3)}^8 \right)$$

$$\leq \frac{6^{26}\pi^{12}(1+C')e^{\frac{6\,|\!|\!| B |\!|\!|_{(2)}}{|B_{j,k}|}} \, |\!|\!| B |\!|\!|_{(2)}^6 \, |\!|\!| B |\!|\!| \, \max\{ |\!|\!| B |\!|\!|, |\!|\!| B |\!|\!|_3 \} |k^2-j^2|^5 \max\{j,k\}^{24}}{|B_{j,k}|^7 n}.$$

□

Proof of Proposition 4.1. The proof follows from [Cha12] that defines the phase θ introduced in the propositions 3.1, $A.1$ and $A.2$. By referring to [Cha12, *Section* 3.1], we estimate $N \geq \max\{j,k\}$ such that

$$K\|(1-\pi_N)B(\phi_j\langle\phi_j,\cdot\rangle + \phi_k\langle\phi_k,\cdot\rangle)\| \leq R_n, \tag{25}$$

for $\pi_N(\cdot) := \sum_{k=1}^N \phi_k\langle\phi_k,\cdot\rangle$. We have

$$K\|(1-\pi_N)B(\phi_j\langle\phi_j,\cdot\rangle + \phi_k\langle\phi_k,\cdot\rangle)\| \leq K\|(1-\pi_N)B(\phi_j\langle\phi_j,\cdot\rangle)\|$$

$$+ K\|(1-\pi_N)B(\phi_k\langle\phi_k,\cdot\rangle)\| \leq \frac{2}{|B_{j,k}|}\Big(\Big(\sum_{l=N+1}^{\infty}|B_{l,k}|^2\Big)^{\frac{1}{2}} + \Big(\sum_{l=N+1}^{\infty}|B_{l,j}|^2\Big)^{\frac{1}{2}}\Big) \leq R_n.$$

As $\frac{4\,|\!|\!| B |\!|\!|}{n\pi^2|k^2-j^2|} \leq R_n$, we impose that $N \geq \max\{j,k\}$ is such that

$$\frac{2}{|B_{j,k}|}\Big(\Big(\sum_{l=N+1}^{\infty}|B_{l,k}|^2\Big)^{\frac{1}{2}} + \Big(\sum_{l=N+1}^{\infty}|B_{l,j}|^2\Big)^{\frac{1}{2}}\Big) \leq \frac{4\,|\!|\!| B |\!|\!|}{n\pi^2|k^2-j^2|}.$$

Let $X_{(N)}^u(t,s)$ be the finite-dimensional propagator defined in the first part of [Cha12, *Section* 2.1]. Thanks to the proof of [Cha12, *Proposition* 2], there exists $T_n \in (nT^*-T, nT^*+T)$ such that

$$|\langle\phi_k, e^{KM^N}\phi_j\rangle - \langle\phi_k, X_{(N)}^{u_n}(T_n,0)\phi_j\rangle| \leq R_n, \qquad \forall n \in \mathbb{N}. \tag{26}$$

We point out that $|\langle\phi_k, e^{KM^N}\phi_j\rangle| = 1$ since $M^N = i\widetilde{M}$ for $\widetilde{M}$ a $N\times N$ matrix with real entries (see also [Cha12, *p.* 5]). Now, $\theta^N \in \mathbb{R}^+$ is the smallest value such that $e^{i\theta^N} = \langle\phi_k, e^{KM^N}\phi_j\rangle$, which follows from [Cha12, *relation* 11]. Indeed, $e^{tM^N}z_n(0)$ appearing in the mentioned equation corresponds to the free finite-dimensional propagator after a time reparameterization and the averaging procedure performed in the first part of [Cha12, *Section* 2]. Moreover, from [Cha12, *relation* (14)] and the following one, the time reparameterization maps K in T_n. Now, we use [Cha12, *relations* (18), (19)] as in [Cha12, *relation* (20)] and we obtain

$$|\langle\phi_k, X_{(N)}^{u_n}(T_n,0)\phi_j\rangle - \langle\phi_k, \Gamma_{T_n}^{u_n}\phi_j\rangle| \leq K\|(1-\pi_N)B(\phi_j\langle\phi_j,\cdot\rangle + \phi_k\langle\phi_k,\cdot\rangle)\| \tag{27}$$

$$+ 4KR_n\|(1-\pi_N)B\pi_N\| \leq R_n + 8|B_{j,k}|^{-1}\,|\!|\!| B |\!|\!|\, R_n \leq 9|B_{j,k}|^{-1}\,|\!|\!| B |\!|\!|\, R_n.$$

Hence, from (26) and (27), it follows

$$\begin{aligned}
&1-|\langle\phi_k,\Gamma_{T_n}^{u_n}\phi_j\rangle|\le|e^{i\theta^N}-\langle\phi_k,\Gamma_{T_n}^{u_n}\phi_j\rangle|\\
&\le|e^{i\theta^N}-\langle\phi_k,X_{(N)}^{u_n}(T_n,0)\phi_j\rangle|+|\langle\phi_k,X_{(N)}^{u_n}(T_n,0)\phi_j\rangle-\langle\phi_k,\Gamma_{T_n}^{u_n}\phi_j\rangle|\\
&\le R_n+9|B_{j,k}|^{-1}\,|\!|\!|\,B\,|\!|\!|\,R_n\le 10|B_{j,k}|^{-1}\,|\!|\!|\,B\,|\!|\!|\,R_n=:\widetilde{R}_n.
\end{aligned}$$

In conclusion, $e^{-i\theta^N}\phi_j=\Gamma_{\widetilde{T}^N}^0\phi_j$ for $\widetilde{T}^N=\lambda_j^{-1}\theta^N$ and we substitute R_n with $\widetilde{R}_n$ in the proofs of the propositions 3.1, $A.1$ and $A.2$ which leads to change the relation (24) as follows

$$\begin{aligned}
&\|\Gamma_{nT^*}^{u_n}\Gamma_{\widetilde{T}^N}^0\phi_j-\phi_k\|_{(3)}^8=\|\Gamma_{nT^*}^{u_n}\phi_j-e^{i\theta^N}\phi_k\|_{(3)}^8\\
&\le\frac{10\;6^{26}\pi^{12}(1+C')\,|\!|\!|\,B\,|\!|\!|_{(2)}^{\,6}\,|\!|\!|\,B\,|\!|\!|^{\,2}\max\{\,|\!|\!|\,B\,|\!|\!|\,,\,|\!|\!|\,B\,|\!|\!|_{\,3}\}|k^2-j^2|^5\max\{j,k\}^{24}}{|B_{j,k}|^8 n}.
\end{aligned}$$

□

B Moment problem

In this appendix, we briefly adapt some results concerning the solvability of the moment problem (as the relation (6)). Let [BL10, *Proposition* 19; 2)] be satisfied and $\{f_k\}_{k\in\mathbb{Z}}$ be a Riesz Basis (see [BL10, *Definition* 2]) in

$$X=\overline{span\{f_k:\;k\in\mathbb{Z}\}}^{\,\mathscr{H}}\subseteq\mathscr{H},$$

with $\mathscr{H}$ and Hilbert space. For $\{v_k\}_{k\in\mathbb{Z}}$ the unique biorthogonal family to $\{f_k\}_{k\in\mathbb{Z}}$ ([BL10, *Remark* 7]), $\{v_k\}_{k\in\mathbb{Z}}$ is also a Riesz Basis of X ([BL10, *Remark* 9]). If $\{f_k\}_{k\in\mathbb{Z}}$ is the image of an orthonormal family $\{e_k\}_{k\in\mathbb{Z}}\subset\mathscr{H}$ by an isomorphism $V:\mathscr{H}\to\mathscr{H}$, then $\{v_k\}_{k\in\mathbb{Z}}$ is the image of $\{e_k\}_{k\in\mathbb{Z}}\subset\mathscr{H}$ by the isomorphism $(V^*)^{-1}:\mathscr{H}\to\mathscr{H}$. Indeed, for every $k,n\in\mathbb{Z}$, we have

$$\delta_{k,j}=\langle v_k,f_j\rangle_{\mathscr{H}}=\langle v_k,V(e_j)\rangle_{\mathscr{H}}=\langle V^*(v_k),e_j\rangle_{\mathscr{H}}$$

that implies $(V^*)^{-1}(e_k)=v_k$ for every $k\in\mathbb{Z}$. We point out that in [BL10, *relation* (71)] there is a misprint as there exist $C_1,C_2>0$ such that

$$C_1\sum_{k\in\mathbb{Z}}|x_k|^2\le\|u\|_{\mathscr{H}}^2\le C_2\sum_{k\in\mathbb{Z}}|x_k|^2, \tag{28}$$

for every $u(t)=\sum_{k\in\mathbb{Z}}x_kf_k(t)$ with $\{x_k\}_{k\in\mathbb{N}}\ell^2(\mathbb{C})$. The arguments of the proof of [BL10, *Proposition* 19; 2)] and the relations

$$(V^*)^{-1}=(V^{-1})^*,\quad|\!|\!|\,V^*\,|\!|\!|_{\,L(\mathscr{H})}=|\!|\!|\,V\,|\!|\!|_{\,L(\mathscr{H})},\quad|\!|\!|\,(V^{-1})^*\,|\!|\!|_{\,L(\mathscr{H})}=|\!|\!|\,V^{-1}\,|\!|\!|_{\,L(\mathscr{H})}$$

lead to a similar relation as $C_2^{-2}\sum_{k\in\mathbb{Z}}|x_k|^2 \leq \|u\|^2_{\mathscr{H}} \leq C_1^{-2}\sum_{k\in\mathbb{Z}}|x_k|^2$ for every $u(t)=\sum_{k\in\mathbb{Z}}x_k v_k(t)$ with $\{x_k\}_{k\in\mathbb{N}}\ell^2(\mathbb{C})$. The constants $C_1, C_2 > 0$ are the same of (28). Moreover, for every $u\in X$, we know that $u=\sum_{k\in\mathbb{Z}}v_k\langle f_k,u\rangle_{\mathscr{H}}$ since $\{f_k\}_{k\in\mathbb{Z}}$ and $\{v_k\}_{k\in\mathbb{Z}}$ are reciprocally biorthonoromal (see [BL10, *Remark* 9]) and

$$C_2^{-1}\Big(\sum_{k\in\mathbb{Z}}|\langle f_k,u\rangle_{\mathscr{H}}|^2\Big)^{\frac{1}{2}} \leq \|u\|_{\mathscr{H}} \leq C_1^{-1}\Big(\sum_{k\in\mathbb{Z}}|\langle f_k,u\rangle_{\mathscr{H}}|^2\Big)^{\frac{1}{2}}. \tag{29}$$

When Ingham's Thoerem [KL05, *Theorem* 4.3] is verified, for $T>0$ large enough, the family of functions $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a Riesz Basis in $X=\overline{span\{e^{i\lambda_k(\cdot)}:\ k\in\mathbb{Z}\}}^{L^2}\subseteq L^2((0,T),\mathbb{C})$. For $\{v_k\}_{k\in\mathbb{Z}}$ the unique biorthogonal family to $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ and $\mathscr{H}=L^2((0,T),\mathbb{C})$, we have

$$C_2^{-1}\Big(\sum_{k\in\mathbb{Z}}|\langle e^{i\lambda_k(\cdot)},u\rangle_{\mathscr{H}}|^2\Big)^{\frac{1}{2}} \leq \|u\|_{\mathscr{H}} \leq C_1^{-1}\Big(\sum_{k\in\mathbb{Z}}|\langle e^{i\lambda_k(\cdot)},u\rangle_{\mathscr{H}}|^2\Big)^{\frac{1}{2}}. \tag{30}$$

Then, the map $F: u\in X \longmapsto \big\{\langle e^{i\lambda_k(\cdot)},u\rangle_{\mathscr{H}}\big\}_{k\in\mathbb{Z}}\in\ell^2(\mathbb{C})$ is invertible. For every $\{x_k\}_{k\in\mathbb{Z}}\in\ell^2(\mathbb{C})$, there exists a unique $u\in X$ such that

$$x_k=\int_0^T u(s)e^{-i\lambda_k s}ds, \qquad \forall k\in\mathbb{Z}.$$

Remark B.1. *We refer to the proof of Proposition* 2.1 *and we consider* $\{\lambda_k\}_{k\in\mathbb{N}}=\{\pi^2(k^2-l^2)\}_{k\in\mathbb{N}}$ *for* $l\in\mathbb{N}$ *such that*

$$\lambda_k-\lambda_l=\pi^2(k^2-l^2)\neq\pi^2(l^2-j^2)=\lambda_l-\lambda_j, \qquad \forall k,j\in\mathbb{N}. \tag{31}$$

For $k>0$, *we call* $\omega_k=-\lambda_k$, *while we impose* $\omega_k=\lambda_{-k}$ *for* $k<0$ *and* $k\neq -l$. *We call* $\mathbb{Z}^*=\mathbb{Z}\setminus\{0\}$. *The sequence* $\{\omega_k\}_{k\in\mathbb{Z}^*\setminus\{-l\}}$ *satisfies the hypotheses of* [*KL05*, *Theorem* 4.3] *thanks to the relation* (31), *which implies* $G:=\inf_{k\neq j}|\omega_k-\omega_j|\geq\pi^2$. *Given* $\{x_k\}_{k\in\mathbb{N}}\in\ell^2(\mathbb{C})$, *we introduce* $\{\widetilde{x}_k\}_{k\in\mathbb{Z}^*\setminus\{-l\}}\in\ell^2(\mathbb{C})$ *such that* $\widetilde{x}_k=x_k$ *for* $k>0$, *while* $\widetilde{x}_k=\overline{x}_{-k}$ *for* $k<0$ *and* $k\neq -l$. *For* $T>2\pi/G$, *there exists* $u\in L^2((0,T),\mathbb{C})$ *such that* $\widetilde{x}_k=\int_0^T u(s)e^{-i\omega_k s}ds$ *for every* $k\in\mathbb{Z}^*\setminus\{-l\}$. *Then*

$$\begin{cases} x_k=\int_0^T u(s)e^{i\lambda_k s}ds=\int_0^T\overline{u}(s)e^{i\lambda_k s}ds, & k\in\mathbb{N}\setminus\{l\},\\ x_k=\int_0^T u(s)ds, & k=l,\end{cases}$$

which implies that, if $x_l\in\mathbb{R}$, *then* u *is real. For* $\{v_k\}_{k\in\mathbb{N}}$ *the biorthogonal family to* $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{N}}$, *we have* $v_l\in\mathbb{R}$ *and* $\{\overline{v}_k\}_{k\in\mathbb{N}}$ *is the biorthogonal family to* $\{e^{-i\lambda_k(\cdot)}\}_{k\in\mathbb{N}}$. *Thus,* $u(t)=\sum_{k\in\mathbb{N}}\widetilde{x}_k v_k(t)+\sum_{k\in\mathbb{N}\setminus\{l\}}\widetilde{x}_{-k}\overline{v_k}(t)=x_l v_l(t)+2\sum_{k\in\mathbb{N}\setminus l}\Re(x_k v_k(t))$ *and* (30) *leads to*

$$C_2^{-1}\Big(\sum_{k\in\mathbb{N}}|x_k|^2\Big)^{\frac{1}{2}}\leq\|u\|_{L^2((0,T),\mathbb{R})}\leq 2C_1^{-1}\Big(\sum_{k\in\mathbb{N}}|x_k|^2\Big)^{\frac{1}{2}}. \tag{32}$$

For $\mathbf{x} := \{x_k\}_{k\in\mathbb{Z}^*\setminus\{-l\}}$ *belonging to* $\ell^2_l(\mathbb{C}) := \{\{x_k\}_{k\in\mathbb{Z}^*\setminus\{-l\}} : \{x_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C});\ x_{-k} = \overline{x_k},\ -k \in \mathbb{N}\setminus\{l\};\ x_l \in \mathbb{R}\}$*, we define*

$$u_{\mathbf{x}}(t) = x_l v_l + 2\sum_{k\in\mathbb{N}\setminus\{l\}} \Re(x_k v_k), \qquad X := \{u_{\mathbf{x}} :\ \mathbf{x} \in \ell^2_l(\mathbb{C})\}.$$

From (32)*,* $J : u \in X \longmapsto \{\langle u, e^{i\omega_k(\cdot)}\rangle\}_{k\in\mathbb{Z}^*\setminus\{-l\}} \in \ell^2_l(\mathbb{C})$ *is an homeomorphism (for* $\{\omega_k\}_{k\in\mathbb{N}}$ *defined above), which implies that* $\widetilde{J} : u \in X \longmapsto \{\langle u, e^{i\lambda_k(\cdot)}\rangle\}_{k\in\mathbb{N}} \in \{\{x_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C}) :\ x_l \in \mathbb{R}\}$ *is also an homeomorphism.*

References

[BCCS12] U. Boscain, M. Caponigro, T. Chambrion, and M. Sigalotti. A weak spectral condition for the controllability of the bilinear Schrödinger equation with application to the control of a rotating planar molecule. *Comm. Math. Phys.*, 311(2):423–455, 2012.

[BCMS12] U. V. Boscain, F. Chittaro, P. Mason, and M. Sigalotti. Adiabatic control of the Schrödinger equation via conical intersections of the eigenvalues. *IEEE Trans. Automat. Control*, 57(8):1970–1983, 2012.

[BCS14] U. Boscain, M. Caponigro, and M. Sigalotti. Multi-input Schrödinger equation: controllability, tracking, and application to the quantum angular momentum. *J. Differential Equations*, 256(11):3524–3551, 2014.

[BdCC13] N. Boussaï d, M. Caponigro, and T. Chambrion. Weakly coupled systems in quantum control. *IEEE Trans. Automat. Control*, 58(9):2205–2216, 2013.

[BGRS15] U. Boscain, J. P. Gauthier, F. Rossi, and M. Sigalotti. Approximate controllability, exact controllability, and conical eigenvalue intersections for quantum mechanical systems. *Comm. Math. Phys.*, 333(3):1225–1239, 2015.

[BL10] K. Beauchard and C. Laurent. Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control. *J. Math. Pures Appl. (9)*, 94(5):520–554, 2010.

[BMS82] J. M. Ball, J. E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. *SIAM J. Control Optim.*, 20(4):575–597, 1982.

[Cha12] T. Chambrion. Periodic excitations of bilinear quantum systems. *Automatica J. IFAC*, 48(9):2040–2046, 2012.

[CMSB09] T. Chambrion, P. Mason, M. Sigalotti, and U. Boscain. Controllability of the discrete-spectrum Schrödinger equation driven by an external field. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(1):329–349, 2009.

[Dav95] E. B. Davies. *Spectral theory and differential operators*, volume 42 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[Duc] A. Duca. Simultaneous global exact controllability in projection. *submitted: https://arxiv.org/abs/1703.00966*.

[GC97] C.-M. Marle G. Christol, A. Cot. *Calcul différentiel*. Ellipses, Paris, 1997.

[Kat53] T. Kato. Integration of the equation of evolution in a Banach space. *J. Math. Soc. Japan*, 5:208–234, 1953.

[KL05] V. Komornik and P. Loreti. *Fourier series in control theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2005.

[Lue69] D. G. Luenberger. *Optimization by vector space methods*. John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[Mir09] M. Mirrahimi. Lyapunov control of a quantum particle in a decaying potential. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1743–1765, 2009.

[MN15] M. Morancey and V. Nersesyan. Simultaneous global exact controllability of an arbitrary number of 1D bilinear Schrödinger equations. *J. Math. Pures Appl. (9)*, 103(1):228–254, 2015.

[Mor14] M. Morancey. Simultaneous local exact controllability of 1D bilinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(3):501–529, 2014.

[Ner10] V. Nersesyan. Global approximate controllability for Schrödinger equation in higher Sobolev norms and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(3):901–915, 2010.