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# Construction of the control function for the global exact controllability and further estimates

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## Abstract

We consider the one dimensional bilinear Schrödinger equation in a bounded domain. We exhibit how to construct controls and times for the global exact controllability.

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## 1 Introduction

In non relativistic quantum mechanics any pure state of a system is mathematically represented by a wave function  $\psi$  in the unit sphere of a Hilbert space  $\mathcal{H}$ . If we consider of a particle constrained in a one dimensional bounded region in presence of an external field (e.g a laser) then we choose  $\mathcal{H} = L^2((0, 1), \mathbb{C})$  while the field is represented by an operator  $B$  and by a real function  $u$  which accounts its intensity. In this framework the evolution

of  $\psi$  is modeled by the Cauchy problem

$$(1) \quad \begin{cases} i\partial_t\psi(t, x) = A\psi(t, x) + u(t)B\psi(t, x), & x \in (0, 1), t \in (0, T). \\ \psi(0, x) = \psi^0(x). \end{cases}$$

where  $A = -\Delta$  is the Laplacian with Dirichlet homogeneous boundary conditions ( $D(A) = H^2 \cap H_0^1$ ),  $B$  is a bounded symmetric operator,  $u$  is a control function and  $\psi^0(x)$  the initial state of the system.

A natural question of practical implications is whether there exists  $u$  steering the quantum system from any given initial wave function to any given target one and how to build explicitly this control function.

We say that Problem (1) is globally (locally) exactly controllable in a normed space  $\mathcal{M} \subset \mathcal{H}$  if for any  $\psi_1, \psi_2 \in \mathcal{M}$  (in a neighborhood of  $\mathcal{M}$ ) there exists a control so that the dynamics of (1) steers  $\psi_1$  into  $\psi_2$ .

We also say that Problem (1) is globally approximately controllable in  $\mathcal{M}$  if for any  $\psi_1, \psi_2 \in \mathcal{M}$  there exists a control such that the related dynamics drives  $\psi_1$  infinitesimally close to  $\psi_2$ .

The global (local) exact controllability is said to be simultaneous if, given two arbitrary sequences in  $\mathcal{M}$  (in a neighborhood of  $\mathcal{M}$ ), there exists a control mapping each element of the first into the corresponding of the second.

The controllability of Problem (1) has been widely studied in the literature starting by the seminal work on bilinear systems of Ball, Mardsen and Slemrod [2]. We refer the reader to [11, *Proposition 1*] for two important consequences of the results proved in [2]: well-posedness conditions and a non-controllability result, both for Problem (1) in  $\mathcal{H}$ . For further characterization on the well-posedness and the time reversibility of the problem see also [11, *Section 1.2*] and [11, *Section 1.3*]. However despite this non-controllability feature many authors have addressed the problem for weaker notions of controllability.

For instance in [3] and [4], local exact controllability results are ensured in  $H_{(0)}^s := D(|A|^{\frac{s}{2}})$  for some  $s \geq 3$ .

Global approximate controllability in a Hilbert space has been studied in [7], [8], [9] and in  $H_{(0)}^s$  for  $s > 0$  in [5], [6], [10], [17] and [18].

In [16] simultaneous local exact controllability up to phase shifts in  $H_{(0)}^3$  is provided, while [15] proves the simultaneous global exact controllability for n-tuples in  $H_{(V)}^4 := D(|A + V|^{\frac{4}{2}})$  is ensured.

In [11] the author proves the simultaneous global exact controllability in projection for sequences of elements in  $H_{(0)}^3$ .

The main novelties of the work are the following. First, we estimate the controllability time for a dynamics steering a given eigenfunction of  $A$  into another. Second, we explicit the control function in such dynamics driving the initial eigenvalue into the second one up to well-known distance.

In details we prove the following results.

First, we show how to construct a neighborhood in  $H_{(0)}^3$  of any eigenfunction of  $A$  in which the local exact controllability is satisfied.

Second, for any couple of eigenfunctions  $\phi_j$  and  $\phi_k$ , we study how to construct controls such that the relative dynamics of (1) drives  $\phi_j$  close to  $\phi_k$  as much desired with respect to the  $H_{(0)}^3$ -norm.

Third by gathering the two previous results we define a dynamics steering any eigenstate of  $A$  into any other and we provide the exact time required to get to the target state.

In more technical terms, for any  $\phi_j$  and  $\phi_k$  we show how to construct a sequence of control functions  $u_n$  and a sequence of times  $T_n > 0$  such that

$$\exists \theta \in \mathbb{R} \quad : \quad \lim_{n \rightarrow \infty} \|\Gamma_{T_n}^{u_n} \phi_j - e^{i\theta} \phi_k\|_{H_{(0)}^3} = 0$$

for  $\Gamma_t^u$  the unitary propagator of Problem (1). We also establish a neighborhood of  $\phi_k$  with radius  $r$  where the local exact controllability is satisfied and such that there exist  $n^* \in \mathbb{N}$  and  $u \in L^2((0, \frac{4}{3\pi}), \mathbb{R})$  so that

$$\|\Gamma_{T_{n^*}}^{u_{n^*}} \phi_j - e^{i\theta} \phi_k\|_{H_{(0)}^3} < r, \quad \Gamma_{\frac{4}{3\pi}}^u \Gamma_{T_{n^*}}^{u_{n^*}} \phi_j = e^{i\theta} \phi_k.$$

In conclusion we show how to remove the phase ambiguity and we provide a time  $T_1 > 0$  such that

$$\Gamma_{\frac{4}{3\pi}}^u \Gamma_{T_{n^*}}^{u'_{n^*}} \Gamma_{T_1}^0 \phi_j = \phi_k.$$

This work represents a step for using the control theory into the experimentation of the quantum systems modeled by the bilinear Schrödinger equation. Indeed almost the entirety of the previous works focus the attention in proving the existence of controls and times so that the controllability is satisfied without making the two explicit. The main reason is that it was not still clear how to establish the radius of the neighborhood where the local exact controllability is verified.

## 1.1 Scheme of the work

In Section 2 we expose the main results of the work in Theorem 1 which is divided in two parts.

In Section 3, Theorem 2 ensures the local exact controllability in  $H_{(0)}^3$  and we provide the proof of Theorem 1 in Section 4.

In Section 5 we present an example in which we apply the techniques developed and in Section 6 we discuss some open problems.

## 2 Framework and main results

We consider an orthonormal basis  $\{\phi_j\}_{j \in \mathbb{N}}$  composed by eigenfunctions of  $A$  associated with the eigenvalues  $\{\lambda_j\}_{j \in \mathbb{N}}$  and

$$(2) \quad \phi_j(t, x) = e^{-iAt} \phi_j(x) = e^{-i\lambda_j t} \phi_j(x).$$

Let the spaces for  $s \geq 0$

$$H_{(0)}^s = H_{(0)}^s((0, 1), \mathbb{C}) := D(A^{\frac{s}{2}}), \quad \|\cdot\|_{(s)} := \|\cdot\|_{H_{(0)}^s} = \left( \sum_{k=1}^{\infty} |k^s \langle \cdot, \phi_k \rangle|^2 \right)^{\frac{1}{2}},$$

$$\ell^\infty(\mathcal{H}) = \left\{ \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H} \mid \sup_{j \in \mathbb{N}} \|\psi_j\| < \infty \right\},$$

$$\ell^2(\mathcal{H}) = \left\{ \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H} \mid \sum_{j=1}^{\infty} \|\psi_j\|^2 < \infty \right\},$$

$$h^s(\mathcal{H}) = \left\{ \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H} \mid \sum_{j=1}^{\infty} (j^s \|\psi_j\|)^2 < \infty \right\}.$$

We equip the space  $H^3 \cap H_0^1$  with the norm  $\|\cdot\|_{H^3 \cap H_0^1}^2 = \sum_{j=1}^3 \|\partial_x^j \cdot\|^2$  and we define the following norms for  $0 < s < 3$

$$\|\!\| \cdot \|\!\| := \|\!\| \cdot \|\!\|_{L(\mathcal{H}, \mathcal{H})}, \quad \|\!\| \cdot \|\!\|_{(s)} := \|\!\| \cdot \|\!\|_{L(H_{(0)}^s, H_{(0)}^s)},$$

$$\|\!\| \cdot \|\!\|_3 := \|\!\| \cdot \|\!\|_{L(H_{(0)}^3, H^3 \cap H_0^1)}.$$

*Assumptions* (I). Let  $B$  be a bounded symmetric operator.

1.  $\forall k \in \mathbb{N}, \exists C_k > 0$  such that  $|\langle \phi_j, B\phi_k \rangle| \geq \frac{C_k}{j^3}$  for every  $j \in \mathbb{N}$ .
2.  $\text{Ran}(B|_{D(A)}) \subseteq D(A)$  and  $\text{Ran}(B|_{H_{(0)}^3((0,1),\mathbb{C})}) \subseteq H^3 \cap H_0^1((0,1), \mathbb{C})$ .

*Remark 1.* If  $B$  satisfies Assumptions I then  $B \in L(H_{(0)}^2, H_{(0)}^2)$  thanks to [11, Remark 1]. The same argument implies that  $B \in L(H_{(0)}^3, H^3 \cap H_0^1)$ .

Let us define  $b := \|\!\| B \|\!\|_{(2)}^6 \|\!\| B \|\!\| \|\!\| B \|\!\|_3^{16} \max\{\|\!\| B \|\!\|, \|\!\| B \|\!\|_3\}$  only depending on the operator  $B$  and for every  $k, j \in \mathbb{N}, n \in \mathbb{N}$

$$E(j, k) := |k^2 - j^2|^5 C_k^{-16} k^{24} |B_{j,k}|^{-7} \max\{j, k\}^{24},$$

$$u_n(t) := \frac{\cos((k^2 - j^2)\pi^2 t)}{n}, \quad C' := \sup_{(l,m) \in \Lambda'} \left\{ \left| \sin\left(\pi \frac{|l^2 - m^2|}{|k^2 - j^2|}\right) \right|^{-1} \right\} \text{ for}$$

$$\Lambda' := \left\{ (l, m) \in \mathbb{N}^2 : \{l, m\} \cap \{j, k\} \neq \emptyset, |l^2 - m^2| \leq \frac{3}{2}|k^2 - j^2|, |l^2 - m^2| \neq |k^2 - j^2|, \langle \phi_l, B\phi_m \rangle \neq 0 \right\}.$$

Let  $N \in \mathbb{N}$  and  $B_{j,k} := \langle \phi_j, B\phi_k \rangle$  ( $\{B_{j,k}\}_{j \in \mathbb{N}} \in \ell^2$ ), we define the  $N \times N$  matrix  $M_{j,k}^N$

$$\begin{cases} (M_{j,k}^N)_{l,m} := \frac{B_{l,m}(i\mu - 2i^\mu + e^{4i\mu}(\sin(4) + i\mu \cos(4)))}{4(\mu^2 - 1)}, & \frac{|l^2 - m^2|}{|k^2 - j^2|} = \mu \in \mathbb{N} \setminus \{1\}, \\ (M_{j,k}^N)_{l,m} := \frac{B_{l,m}((-8 + 3i) + ie^{8i} + 2\pi)}{16}, & \frac{|l^2 - m^2|}{|k^2 - j^2|} = 1, \\ (M_{j,k}^N)_{l,m} := 0, & \frac{|l^2 - m^2|}{|k^2 - j^2|} \notin \mathbb{N}. \end{cases}$$

Let  $\theta \in \mathbb{R}^+$  be the smaller value so that  $e^{-i\theta} = \langle \phi_k, e^{2|B_{k,j}|^{-1}M_{j,k}^N} \phi_j \rangle$

$$T_n := \frac{n\pi}{|B_{k,j}|}, \quad \tilde{T}^N := \frac{-\theta}{(j\pi)^2}.$$

**Theorem 1.** *Let  $k, j, n, N \in \mathbb{N}$  and  $B$  satisfy Assumptions I.*

1) *If  $n \geq 6^{34}\pi^{12}b(1 + C')E(j, k)$ , then*

$$\|\Gamma_{T_n}^{u_n} \phi_j - e^{i\theta} \phi_k\|_{H_{(0)}^3} \leq C_k^2 (6k^3 \|B\|_3^2)^{-1}.$$

*Moreover there exists  $u \in L^2((0, \frac{4}{3\pi}), \mathbb{R})$  such that  $\|u\| \leq \frac{2C_k}{7\|B\|_3^2 k^3}$  and*

$$\Gamma_{\frac{4}{3\pi}}^u \Gamma_{T_n}^{u_n} \phi_j = e^{i(\theta + \frac{4k^2\pi}{3})} \phi_k.$$

2) *If  $n \geq 6^{34}10\pi^{12}b(1 + C')\|B\|E(j, k)|B_{j,k}|^{-1}$ ,  $N \geq \max\{j, k\}$  and*

$$(3) \quad \frac{2}{|B_{j,k}|} \left( \left( \sum_{l=N+1}^{\infty} |B_{l,k}|^2 \right)^{\frac{1}{2}} + \left( \sum_{l=N+1}^{\infty} |B_{l,j}|^2 \right)^{\frac{1}{2}} \right) < \frac{4\|B\|}{n\pi^2|k^2 - j^2|},$$

*then there holds*

$$\|\Gamma_{T_n}^{u_n} \Gamma_{\tilde{T}^N + \frac{4\lambda_k}{3\pi\lambda_j}}^0 \phi_j - \phi_k\|_{H_{(0)}^3} \leq C_k^2 (6k^3 \|B\|_3^2)^{-1}.$$

*Moreover there enoindexists  $u \in L^2((0, \frac{4}{3\pi}), \mathbb{R})$  such that  $\|u\| \leq \frac{2C_k}{7\|B\|_3^2 k^3}$  and*

$$\Gamma_{\frac{4}{3\pi}}^u \Gamma_{T_n}^{u_n} \Gamma_{\tilde{T}^N + \frac{4\lambda_k}{3\pi\lambda_j}}^0 \phi_j = \phi_k.$$

*Remark 2.* The results of Theorem 1 are not optimal. The aim of the work is to show how to proceed for this type of problems and we present an approach that one can use in order to establish times and controls for the global exact controllability in  $H_{(0)}^3$ .

The purpose of Theorem 1 is to exhibit readable results for generic operators  $B$  and levels  $j, k$ . For any specific choice of  $B, j$  and  $k$ , it is possible to retrace the proof by using stronger estimates and obtain sharper bounds. We briefly treat the example of  $B : \psi \rightarrow x^2\psi$ ,  $j = 2$  and  $k = 1$  in Section 5.

*Remark 3.* In the proof of Theorem 1 the choice of the control function  $u$  comes from the techniques developed in [9]. We point out that one can ensure similar results for other  $\frac{2\pi}{|\lambda_k - \lambda_j|}$ -periodic controls by using the theory exposed in [9].

### 3 Local exact controllability in $H_{(0)}^3$

In this section we provide a brief proof of the local exact controllability in  $H_{(0)}^3$  by rephrasing the existing results of local controllability as [3], [4], [11], [15] and [16]. Our purpose is to introduce the tools that we use in the proof of Theorem 1. Let  $\psi \in H_{(0)}^3$  and  $U(\mathcal{H})$  the space of the unitary operators on  $\mathcal{H}$ , we define

$$B_{H_{(0)}^3}(\psi, \epsilon) := \{\tilde{\psi} \in H_{(0)}^3 \mid \exists \widehat{\Gamma} \in U(\mathcal{H}) : \tilde{\psi} = \widehat{\Gamma}\psi, \|\tilde{\psi} - \psi\|_{H_{(0)}^3} < \epsilon\}.$$

**Theorem 2.** *Let  $B$  satisfy Assumptions I. For every  $l \in \mathbb{N}$  there exist  $T > 0$  and  $\epsilon > 0$  such that for every  $\psi \in B_{H_{(0)}^3}(\phi_l(T), \epsilon)$  there exists a control function  $u \in L^2((0, T), \mathbb{R})$  so that  $\psi = \Gamma_T^u \phi_l$ .*

*Proof.* Let the decomposition  $\Gamma_t^u \phi_l = \sum_{k=1}^{\infty} \phi_k(t) \langle \phi_k(t), \Gamma_t^u \phi_l \rangle$  and the map  $\alpha_l(u)$ , the sequence with elements

$$\alpha_{k,l}(u) = \langle \phi_k(T), \Gamma_T^u \phi_l \rangle, \quad k \in \mathbb{N}.$$

Ensuring the local existence of the control function is equivalent to prove the local right invertibility of the map  $\alpha_l$  for a  $T > 0$  (in other words the local surjectivity). To this end, we want to use the Generalized Inverse Function Theorem ([14], p. 240) and we study the surjectivity of the Fréchet derivative of  $\alpha_l$ ,  $\gamma_l(v) := (d_u \alpha_l(0)) \cdot v$ , the sequence with elements

$$\begin{aligned} \gamma_{k,l}(v) &:= \left\langle \phi_k(T), -i \int_0^T e^{-iA(T-s)} v(s) B e^{-iAs} \phi_l ds \right\rangle \\ &= -i \int_0^T v(s) e^{i(\lambda_k - \lambda_l)s} ds B_{k,l}, \quad k \in \mathbb{N}, \end{aligned}$$

for  $B_{k,j} = \langle \phi_k, B \phi_j \rangle = \langle B \phi_k, \phi_j \rangle = \overline{B_{j,k}}$ . The right invertibility of the map  $\gamma_l$  consists in proving the resolvability of the moment problem

$$(4) \quad \frac{x_k}{B_{k,l}} = -i \int_0^T u(s) e^{i(\lambda_k - \lambda_l)s} ds.$$

for each  $x \in \ell^2(\mathbb{C})$  such that  $\{x_k B_{k,l}^{-1}\}_{k \in \mathbb{N}} \in \ell^2$ . Now thanks to Assumptions I and to the fact that  $\{x_k\}_{k \in \mathbb{N}} \in h^3$ , the resolvability of the moment problem

(4) is due to Ingham Theorem ([13, *Theorem 4.3*]) for  $T > \frac{2\pi}{\mathcal{G}}$  and

$$\mathcal{G} := \inf_{\substack{k, j \in \mathbb{N} \\ k \neq j}} |\lambda_k - \lambda_j| = 3\pi^2 > 0.$$

Then  $\gamma_l$  is surjective for  $T$  large enough and the proof is achieved thanks to the Generalized Inverse Function Theorem ([14], p. 240), which provides the local surjectivity of the map  $\alpha_l$  at the same time  $T$ .  $\square$

*Remark.* We point out that one can achieve the result of Theorem 2 for any positive time  $T > 0$  by using Haraux Theorem ([13, *Theorem 4.5*]), instead of Ingham Theorem ([13, *Theorem 4.3*]) as made in the proof of [11, *Theorem 8*].

## 4 Proof of Theorem 1

The proof follows by gathering a local exact controllability result with a global approximate controllability one. In particular it consists in the following steps.

- For any generic eigenfunction  $\phi_l$  of  $A$  we construct a neighborhood  $B_{H_{(0)}^3}(\phi_l, r)$  of radius  $r$  in which the local exact controllability is verified.
- We consider a generic couple of eigenfunctions  $\phi_j, \phi_k$ . We define a sequence of control functions  $u_n$  and a sequence of times  $T_n$  such that  $\Gamma_{T_n}^{u_n} \phi_j$  is close to  $\phi_k$  up to a known distance in the  $\mathcal{H}$ -norm depending on  $n$ .
- By establishing this distance with respect to the  $H_{(0)}^3$ -norm, we provide a lower bound for  $n$  so that it is smaller than the radius of  $B_{H_{(0)}^3}(\phi_k, r)$ .
- The local exact controllability in  $B_{H_{(0)}^3}(\phi_k, r)$  and the time reversibility (see [11, *Section 1.3*]) allow to define a dynamics steering  $\phi_j$  in  $B_{H_{(0)}^3}(\phi_k, r)$  and after in  $\phi_k$ .

### 4.1 Neighborhood estimate

Let us define the following terminology

$$\begin{aligned} \|\cdot\|_{L(L^2((0,T),\mathbb{R}),H_{(0)}^3)} &= \|\cdot\|_{(L_t^2, H_x^3)}, & \|\cdot\|_{L(H_{(0)}^3, L^2((0,T),\mathbb{R}))} &= \|\cdot\|_{(H_x^3, L_t^2)}, \\ \|\cdot\|_{L^\infty((0,T),H_{(0)}^3)} &= \|\cdot\|_{L_t^\infty H_x^3}, & \|\cdot\|_{L^2((0,T),\mathbb{R})} &= \|\cdot\|_2, \\ \|\cdot\|_{L^1((0,T),\mathbb{R})} &= \|\cdot\|_1, & \|\cdot\|_{BV((0,T),\mathbb{R})} &= \|\cdot\|_{BV(T)}, \end{aligned}$$



$$\text{for } \|f\|_{BV((0,T),\mathbb{R})} = \sup_{\{t_j\}_{j \leq n} \in P} \sum_{j=1}^n |f(t_j) - f(t_{j-1})|$$

and  $P$  the set of the partitions of  $(0, T)$  so that  $t_0 = 0 < t_1 < \dots < t_n = T$ .

Let  $T > 0$  and the space  $\tilde{X} := \overline{\text{span}\{e^{i\lambda_k t}, k \in \mathbb{N}\}}^{L^2} \subset L^2((0, T), \mathbb{R})$  equipped with the  $L^2$ -norm. The local exact controllability is equivalent to the local surjectivity of the map  $A_l(\cdot) := \Gamma_T^{(\cdot)} \phi_l : \tilde{X} \rightarrow H_{(0)}^3$  so that

$$A_l(u) = e^{-i\lambda_l T} \phi_l - i \int_0^T e^{-iA(T-s)} u(s) B \Gamma_s^u \phi_l ds.$$

The proof of Theorem 2 is equivalent to the surjectivity of the Fréchet derivative of  $A_l(u)$ , the map  $F_l(u) := ((d_v A_l(v=0)) \cdot u) \in H_{(0)}^3$  which implies the local surjectivity of  $A_l$  thanks to the Generalized Inverse Function Theorem ([14], p. 240). We want to estimate the radius of a neighborhood in which the map  $A_l$  is surjective. For this reason, we use [10, Lemma 2.3; p. 42] by considering the quotient space  $X := \tilde{X}/\text{Ker}(F_l)$  with the  $L^2$ -norm.

1) The map  $F_l : X \rightarrow H_{(0)}^3$  is an homeomorphism and we want to estimate a constant  $M > 0$  such that

$$\|F_l(v) - F_l(w)\|_{(3)} \geq M \|v - w\|_{L^2}, \quad \forall v, w \in X.$$

Let us suppose  $\|B\|_3 = 1$ . By recalling the proof of Theorem 2 we know that the surjectivity of  $F_l$  in  $H_{(0)}^3$  is equivalent to the surjectivity of  $\gamma_l$  in  $h^3$ . For every  $\psi \in H_{(0)}^3$ , there exist  $T > 0$  and  $u \in X$  such that  $\langle \phi_j(T), \psi \rangle = \gamma_{j,l}(u)$  and such that  $F_l^{-1}(\psi) = u$ .

For  $C_l$  defined in Assumptions I, thanks to [3, Proposition 19; (ii)] and to Ingham Theorem ([13, Theorem 4.3]), there exists  $\tilde{C}(T) > 0$  such that

$$\begin{aligned} \|F_l^{-1}(\psi)\|_2^2 &= \|u\|_2^2 \leq \tilde{C}(T)^2 \sum_{j=1}^{\infty} \left| \frac{\gamma_{j,l}(u)}{B_{j,l}} \right|^2 \leq \frac{\tilde{C}(T)^2}{C_l^2} \sum_{j=1}^{\infty} |j^3 \gamma_{j,l}(u)|^2 \\ &\leq \frac{\tilde{C}(T)^2}{C_l^2} \|\psi\|_{(3)}^2. \end{aligned}$$

For each  $v, w \in X$ , there exist  $\psi, \varphi \in H_{(0)}^3$  so that  $\psi = F_l(v)$ ,  $\varphi = F_l(w)$  and

$$\|v - w\|_2 \leq \|F_l^{-1}(\psi - \varphi)\|_2 \leq \|F_l^{-1}\|_{(H_x^3, L_t^2)} \|\psi - \varphi\|_{(3)},$$

which imply that we can choose  $M = \|F_l^{-1}\|_{(H_x^3, L_t^2)}^{-1} = \frac{C_l}{\tilde{C}(T)}$ .

2) Let  $u \in X$ , thanks to the Duhamel's formula

$$\begin{aligned}\Gamma_T^u \phi_l &= e^{-i\lambda_l T} \phi_l - i \int_0^T e^{-iA(T-s)} u(s) B e^{-i\lambda_l s} \phi_l ds \\ &\quad - \int_0^T e^{-iA(T-s)} u(s) B \left( \int_0^s e^{-iA(s-\tau)} u(\tau) B \Gamma_\tau^u \phi_l d\tau \right) ds \\ &= e^{-i\lambda_l T} \phi_l + F_l(u) + H_l(u)\end{aligned}$$

$$\text{for } H_l(u) := - \int_0^T e^{-iA(T-s)} u(s) B \left( \int_0^s e^{-iA(s-\tau)} u(\tau) B \Gamma_\tau^u \phi_l d\tau \right) ds.$$

We exhibit a ball  $U \subset X$  with center  $u = 0$  where the map  $A_l : u \mapsto \Gamma_T^u \phi_l$  is surjective thanks to [10, Lemma 2.3]. However  $e^{-i\lambda_l T} \phi_l$  is constant and it is sufficient to study the surjectivity of  $F_l + H_l$ . We define the neighborhood  $U$  so that there exists  $M_1 \leq M/2$  such that for every  $v, w \in U$

$$\|H_l(v) - H_l(w)\|_{(3)} \leq M_1 \|v - w\|_{L^2}.$$

First, we notice

$$\begin{aligned}H_l(u) - H_l(v) &= - \int_0^T e^{-iA(T-s)} u(s) B \left( \int_0^s e^{-iA(s-\tau)} u(\tau) B \Gamma_\tau^u \phi_l d\tau \right) ds \\ &\quad + \int_0^T e^{-iA(T-s)} v(s) B \left( \int_0^s e^{-iA(s-\tau)} v(\tau) B \Gamma_\tau^v \phi_l d\tau \right) ds \\ &= - \int_0^T e^{-iA(T-s)} (u(s) - v(s)) B \left( \int_0^s e^{-iA(s-\tau)} u(\tau) B \Gamma_\tau^u \phi_l d\tau \right) ds \\ &\quad - \int_0^T e^{-iA(T-s)} v(s) B \left( \int_0^s e^{-iA(s-\tau)} (u(\tau) - v(\tau)) B \Gamma_\tau^u \phi_l d\tau \right) ds \\ &\quad - \int_0^T e^{-iA(T-s)} v(s) B \left( \int_0^s e^{-iA(s-\tau)} v(\tau) B (\Gamma_\tau^u \phi_l - \Gamma_\tau^v \phi_l) d\tau \right) ds.\end{aligned}$$

Thanks to [11, Proposition 5], there exists a constant  $C(T) > 0$  such that for every  $\psi \in H^3 \cap H_0^1$  and  $u \in L^2((0, T), \mathbb{R})$

$$\left\| \int_0^T e^{-iA(T-s)} u(s) B \psi ds \right\|_{(3)} \leq C(T) \|u\|_2 \|B\|_3 \|\psi\|_{L_t^\infty H_x^3}.$$

Then

$$\begin{aligned}(5) \quad \|H_l(u) - H_l(v)\|_{(3)} &\leq C(T)^2 \|v - u\|_2 \|B\|_3^2 (\|u\|_2 + \|v\|_2) \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\ &\quad + C(T)^2 \|v\|_2^2 \|B\|_3^2 \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\ &\leq C(T)^2 \|v - u\|_2 (\|u\|_2 + \|v\|_2) \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\ &\quad + C(T)^2 \|v\|_2^2 \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3}.\end{aligned}$$

One can show by using the same technique adopted in (5) that

$$\begin{aligned}
& \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \leq \left\| \int_0^t e^{-iA(t-s)} B(v\Gamma_t^v \phi_l - u\Gamma_t^u \phi_l) \right\|_{L_t^\infty H_x^3} \\
& \leq C(T) \| \| B \| \|_3 \| v\Gamma_t^v \phi_l - u\Gamma_t^u \phi_l \|_{L_t^\infty H_x^3} \leq C(T) \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\
& + C(T) \|v\|_2 \|\Gamma_t^v - \Gamma_t^u\|_{L_t^\infty H_x^3} \leq C(T) \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\
& + C(T)^2 \|v\|_2 \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} + C(T)^2 \|v\|_2^2 \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \leq \\
& \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \left( \sum_{n=0}^N C(T)^{n+1} \|v\|_2^n \right) + C(T)^N \|v\|_2^N \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3}.
\end{aligned}$$

In addition, [5, *Proposition 6*] implies that the couple  $(A, B)$  is (2)-weakly coupled thanks to Remark 1. Then [6, *Proposition 30; (ii)*] is satisfied and

$$(6) \quad \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} < \infty.$$

Let  $\mu > 1$ , if  $U \subseteq \{u \in X : \|u\|_2 \leq (\mu C(T))^{-1}\}$ , then for  $u, v \in U$

$$\lim_{N \rightarrow \infty} C(T)^N \|v\|_2^N \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} = 0,$$

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N C(T)^n \|v\|_2^n \leq \frac{\mu}{\mu - 1}$$

$$\implies \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \leq \frac{\mu C(T)}{\mu - 1} \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3}.$$

The relation (5) becomes

$$\begin{aligned}
& \|H_l(u) - H_l(v)\|_{(3)} \leq C(T)^2 \|v - u\|_2 (\|u\|_2 + \|v\|_2) \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\
& + \frac{\mu}{\mu - 1} C^3(T) \|v\|_2^2 \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \leq \frac{2}{\mu} C(T) \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\
& + \frac{C(T)}{(\mu - 1)\mu} \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \leq \frac{(2\mu - 1)}{(\mu - 1)\mu} C(T) \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3}.
\end{aligned}$$

Thanks to [11, *Proposition 5*] and to the Duhamel's formula

$$\|\Gamma_T^u \phi_l\|_{L_t^\infty H_x^3} \leq \frac{\|\phi_l\|_{(3)}}{1 - C(T)\|u\|_2 \| \| B \| \|_3} \leq \frac{\mu l^3}{\mu - 1},$$

$$(7) \quad \implies \|H_l(u) - H_l(v)\|_{(3)} \leq \frac{2\mu - 1}{(\mu - 1)^2} l^3 C(T) \|v - u\|_2.$$

Let  $M_1 = \frac{2\mu - 1}{(\mu - 1)^2} l^3 C(T)$ . We estimate  $\mu$  such that  $M_1 \leq \frac{1}{2} M$ , in other words

$$\frac{2\mu - 1}{(\mu - 1)^2} l^3 C(T) \leq \frac{1}{2} \frac{C_l}{\tilde{C}(T)},$$

and for  $a = \frac{2C(T)\tilde{C}(T)l^3}{C_l}$ , the inequality is satisfied when

$$\mu \geq a + \sqrt{a(a+1)} + 1.$$

Let us establish an upper bound for  $C(T)\tilde{C}(T)$ . By keeping in mind the proof of [3, *Proposition 19; (ii)*], we set  $T = \frac{4}{3\pi}$  and thanks to the proof of Ingham Theorem [13, *Theorem 4.3*]

$$\tilde{C}(T) = \frac{8}{3\pi}, \quad \hat{C}(T) = \frac{16}{\pi}.$$

The proof of the first point of [11, *Proposition 5*] (see [3]) implies

$$C(T) = 3\pi^{-3} \max \{ \sqrt{2\hat{C}(T)}, \sqrt{T} \} = 3\pi^{-3} \sqrt{2\hat{C}(T)} = \frac{48\sqrt{2}}{\pi^4},$$

then  $C(T)\tilde{C}(T) \leq \frac{2}{3}$  and  $a \leq \frac{4}{3}\tilde{a}_l$  for  $\tilde{a}_l := \frac{l^3}{C_l}$ . Afterwards by recalling the definition of  $C_l$  provided in Assumption I, we have  $C_l \leq \langle \phi_1, B\phi_l \rangle \leq \| \| B \| \|$  which ensures that  $\tilde{a}_l > 1$  and

$$C(T) \left( \frac{4}{3}\tilde{a}_l + \sqrt{\frac{4}{3}\tilde{a}_l \left( \frac{4}{3}\tilde{a}_l + 1 \right)} + 1 \right) \leq C(T) \left( \frac{4}{3}\tilde{a}_l + \left( \frac{4}{3}\tilde{a}_l + 1 \right) + 1 \right) \leq \frac{7}{2}\tilde{a}_l.$$

Let  $B_X(x, r) := \{ \tilde{x} \in X \mid \|\tilde{x} - x\|_X \leq r \}$ , one can consider  $U = B_X(0, 2(7\tilde{a}_l)^{-1})$ , we know that  $M - M_1 \geq (2\tilde{C}(T))^{-1}C_l$  and thanks to the proof of [10, *Lemma 2.3*]

$$\begin{aligned} A_l(B_X(0, 2(7\tilde{a}_l)^{-1})) &\supset B_{H^3_{(0)}}(F_l(0), (M - M_1)2(7\tilde{a}_l)^{-1}) \supset \\ &B_{H^3_{(0)}}(\phi_l(T), C_l(6\tilde{a}_l)^{-1}) \supset B_{H^3_{(0)}}\left(\phi_l(T), \frac{C_l^2}{6l^3}\right). \end{aligned}$$

We supposed  $\| \| B \| \|_3 = 1$  but we can generalize for  $\| \| B \| \|_3 \neq 1$  thanks to

$$A + uB = A + u \| \| B \| \|_3 \frac{B}{\| \| B \| \|_3}.$$

One can consider the operator  $\frac{B}{\| \| B \| \|_3}$  and the control  $u \| \| B \| \|_3$ . By recalling that  $T = \frac{4}{3\pi}$  we substitute  $C_l$  with  $C_l \| \| B \| \|_3^{-1}$  and

$$\forall \psi \in B_{H^3_{(0)}}\left(\phi_l\left(\frac{4}{3\pi}\right), \frac{C_l^2}{6l^3 \| \| B \| \|_3^2}\right), \exists u \in B_X\left(0, \frac{2C_l}{7l^3 \| \| B \| \|_3^2}\right) : A_l(u) = \psi.$$

## 4.2 Control function for the global approximate controllability in the $\mathcal{H}$ -norm

Let  $\phi_j, \phi_k$  for  $j, k \in \mathbb{N}$ , we exhibit a control function driving the dynamics of (1) from  $\phi_j$  close to  $\phi_k$  by using [9, *Proposition 6*]. Let  $T^* = \frac{\pi}{|B_{j,k}|}$ ,

$T = \frac{2\pi}{|\lambda_k - \lambda_j|}$  and  $u(t) = \cos((\lambda_k - \lambda_j)t)$ . For any  $n \in \mathbb{N}$  there exists  $T_n \in (nT^* - T, nT^* + T)$  such that

$$\frac{1 - |\langle \phi_k, \Gamma_{T_n}^{u/n} \phi_j \rangle|}{1 + 2K \lll B \rrl} \leq \frac{(1 + C') \lll (\phi_j \langle \phi_j, \cdot \rangle + \phi_k \langle \phi_k, \cdot \rangle) B \rrl I}{n},$$

for  $I = \frac{4}{|\lambda_k - \lambda_j|}$ ,  $K = \frac{2}{|B_{j,k}|}$ ,  $C' = \sup_{(l,m) \in \Lambda'} \left\{ \left| \sin \left( \pi \frac{|\lambda_l - \lambda_m|}{|\lambda_k - \lambda_j|} \right) \right|^{-1} \right\}$  and

$$\Lambda' = \left\{ (l, m) \in \mathbb{N}^2 : \{l, m\} \cap \{j, k\} \neq \emptyset, |\lambda_l - \lambda_m| \leq \frac{3}{2} |\lambda_k - \lambda_j|, |\lambda_l - \lambda_m| \neq |\lambda_k - \lambda_j|, B_{l,m} \neq 0 \right\}.$$

We point out that the definition of  $T^*$  provided in [9, *Proposition 6*] is incorrect and its formulation is provided above.

*Remark.* As mentioned in Remark 3, one can state similar results for other control functions  $\frac{2\pi}{|\lambda_k - \lambda_j|}$ -periodics by adopting the theory from [9].

For  $u_n := \frac{u}{n}$  and  $R_n := (1 + 2K \lll B \rrl)(1 + C') \lll B \rrl I n^{-1}$  there holds

$$(8) \quad \sum_{l \neq k} |\langle \phi_l, \Gamma_{T_n}^{u_n} \phi_j \rangle - \langle \phi_l, \phi_k \rangle|^2 = \sum_{l \neq k} |\langle \phi_l, \Gamma_{T_n}^{u_n} \phi_j \rangle|^2 = 1 - |\langle \phi_k, \Gamma_{T_n}^{u_n} \phi_j \rangle|^2 \\ \leq (1 - |\langle \phi_k, \Gamma_{T_n}^{u_n} \phi_j \rangle|)(1 + |\langle \phi_k, \Gamma_{T_n}^{u_n} \phi_j \rangle|) \leq 2R_n.$$

Afterwards, there exists  $\theta_n \in \mathbb{C}$  such that

$$(9) \quad |\langle \phi_k, e^{i\theta_n} \phi_k \rangle - \langle \phi_k, \Gamma_{T_n}^{u_n} \phi_j \rangle|^2 \leq R_n^2.$$

From (8), (9)

$$(10) \quad R'_n := \|e^{i\theta_n} \phi_k - \Gamma_{T_n}^{u_n} \phi_j\|^2 \leq 2R_n + R_n^2,$$

hence  $|B_{j,k}|^{-1} \lll B \rrl \geq 1$  implies

$$R_n \leq \frac{(1 + C')(|B_{j,k}|^{-1} + 4|B_{j,k}|^{-1}) \lll B \rrl^2 I}{nC_k} \leq \frac{5(1 + C')|B_{j,k}|^{-1} \lll B \rrl^2 I}{n},$$

$$(11) \quad R_n \leq \frac{3(1 + C')|B_{j,k}|^{-1} \lll B \rrl^2}{n|k^2 - j^2|}.$$

### 4.3 Global approximate controllability with respect to the $H^3$ -norm

Now, we exhibit a control function driving the dynamics of (1) from  $\phi_j$  to  $B_{H^3_{(0)}}(\phi_k, C_l^2(6l^3 \lll B \rrl_3^2)^{-1})$  by using the previous subsection. Let us consider the relation (11), if

$$(12) \quad n \geq \frac{3(1 + C')|B_{j,k}|^{-1} \lll B \rrl^2}{|k^2 - j^2|}, \quad j \neq k,$$

then  $R_n \leq 1$ ,  $R_n^2 \leq R_n$  and

(13)

$$R'_n = \|e^{i\theta_n} \phi_k - \Gamma_{T_n}^{u_n} \phi_j\|^2 \leq 2R_n + R_n^2 \leq 3R_n \leq \frac{3^2 |B_{j,k}|^{-1} (1 + C') \|B\|^2}{n|k^2 - j^2|}.$$

For  $f_n := e^{i\theta_n} \phi_k - \Gamma_{T_n}^{u_n} \phi_j$  there holds  $\|f_n\|_{(s)}^2 \leq (k^s + \|\Gamma_{T_n}^{u_n} \phi_j\|_{(s)})^2$  and thanks [5, relation (9)] it follows

$$(14) \quad \|f_n\|_{(3)}^4 = \|f_n\|_{\left(\frac{12}{4}\right)}^4 \leq \|f_n\|_{\left(\frac{4}{2}\right)}^2 \|f_n\|_{\left(\frac{8}{2}\right)}^2 \leq \|f_n\| \|f_n\|_{(4)}^3.$$

If we establish an upper bound for  $\|f_n\|_{(4)}$  independent from  $n$ , then  $\|f_n\| \xrightarrow{n \rightarrow +\infty} 0$  implies that  $\|f_n\|_{(3)} \xrightarrow{n \rightarrow +\infty} 0$  up to a rest that we can bound from above. Let  $\epsilon > 0$ ,  $\lambda_\epsilon = \|B\|_{(2)} \epsilon^{-1}$  and  $\widehat{H}_{(0)}^4 := D(A(i\lambda_\epsilon - A))$ . We proceed as in the proof of [6, Proposition 30] that follows from [12, Section 3.10]. Let  $n \geq 3\epsilon$  and

$$\begin{aligned} M &:= \sup_{t \in [0, T_n]} \|(i\lambda_\epsilon - A - u_n(t)B)^{-1}\|_{L(H_{(0)}^2, \widehat{H}_{(0)}^4)} \\ &= \sup_{t \in [0, T_n]} \|(i\lambda_\epsilon - A)(i\lambda_\epsilon - A - u_n(t)B)^{-1}\|_{(2)} \\ &= \sup_{t \in [0, T_n]} \|(I - u_n(t)B(i\lambda_\epsilon - A)^{-1})^{-1}\|_{(2)}. \end{aligned}$$

Now  $\|u_n(t)B(i\lambda_\epsilon - A)^{-1}\|_{(2)} \leq \frac{\|B\|_{(2)}}{n\lambda_\epsilon} = \frac{\epsilon}{n} < 1$  and

$$(15) \quad \begin{aligned} M &= \sup_{t \in [0, T_n]} \left\| \sum_{l=1}^{+\infty} (u_n(t)(i\lambda_\epsilon - A)^{-1}B)^l \right\|_{(2)} \\ &\leq \sum_{l=1}^{+\infty} \|n^{-1}(i\lambda_\epsilon - A)^{-1}B\|_{(2)}^l = \frac{1}{1 - \|B\|_{(2)} n^{-1} \lambda_\epsilon^{-1}} = \frac{n}{n - \epsilon} \leq 2. \end{aligned}$$

Let us consider

$$N := \|(i\lambda_\epsilon - A - u_n(\cdot)B)\|_{BV([0, T_n], L(\widehat{H}_{(0)}^4, H_{(0)}^2))} \leq \|u_n\|_{BV(T_n)} \|B\|_{L(\widehat{H}_{(0)}^4, H_{(0)}^2)},$$

there holds

$$\|(A + u_n(T_n)B - i\lambda_\epsilon)\Gamma_{T_n}^{u_n} \phi_j\|_{(2)} \leq Me^{MN} \|(A - i\lambda_\epsilon)\phi_j\|_{(2)} \leq Me^{MN} (1 + \lambda_\epsilon) j^4.$$

Now for every  $\psi \in \widehat{H}_{(0)}^4$

$$\|B\psi\|_{(2)}^2 \leq (\epsilon \|A\psi\|_{(2)} + \|B\|_{(2)} \|\psi\|_{(2)})^2 \leq 2\epsilon^2 (\|A\psi\|_{(2)}^2 + \lambda_\epsilon^2 \|\psi\|_{(2)}^2).$$

As  $\|(A - i\lambda_\epsilon)\psi\|_{(2)}^2 = \|A\psi\|_{(2)}^2 + \lambda_\epsilon^2\|\psi\|_{(2)}^2$ , it follows  $\|B\psi\|_{(2)}^2 \leq 2\epsilon^2(\|(A - i\lambda_\epsilon)\psi\|_{(2)}^2)$  and  $N \leq \epsilon\sqrt{2}\|u_n\|_{BV(T_n)}$ . In addition, thanks to the techniques of relation (15), it is verified

$$\| \| A(A + u_n(T_n)B - i\lambda_\epsilon)^{-1} \| \|_{(2)} \leq 4.$$

Thus

$$\begin{aligned} \|\Gamma_{T_n}^{u_n}\phi_j\|_{(4)} &= \|A\Gamma_{T_n}^{u_n}\phi_j\|_{(2)} \leq 4\|(A + u(T_n)B - i\lambda_\epsilon)\Gamma_{T_n}^{u_n}\phi_j\|_{(2)} \\ &\leq 8e^{2\sqrt{2}\epsilon\|u_n\|_{BV(T_n)}}(1 + \lambda_\epsilon)j^4 \leq 8e^{2\sqrt{2}\epsilon\|u_n\|_{BV(T_n)}}(1 + \| \| B \| \|_{(2)}\epsilon^{-1})j^4 \end{aligned}$$

and for  $\epsilon = (2\sqrt{2}\|u_n\|_{BV(T_n)})^{-1}$

$$(16) \quad \|\Gamma_{T_n}^{u_n}\phi_j\|_{(4)} \leq 8e(1 + 2\sqrt{2}\|u_n\|_{BV(T_n)} \| \| B \| \|_{(2)})j^4.$$

The interval  $[0, nT^* + T]$  contains less than  $d$  half-periods of the function  $u$  for  $d := 2\pi^2n|k^2 - j^2||B_{j,k}|^{-1} + 4$  and if

$$(17) \quad n \geq \| \| B \| \| (5\pi^{-2}|j^2 - k^2|^{-1})$$

$$(18) \quad \implies \|u_n\|_{BV(T_n)} \leq \|u_n\|_{BV(nT^*+T)} \leq (d+1)/n \leq 3\pi^2|k^2 - j^2||B_{j,k}|^{-1}.$$

Thanks to  $\| \| B \| \|_{(2)} \geq |B_{j,k}|$ , the relation (16) becomes

$$\begin{aligned} \|\Gamma_{T_n}^{u_n}\phi_j\|_{(4)} &\leq 8e(1 + 3 \cdot 2\sqrt{2}\pi^2 \| \| B \| \|_{(2)}|k^2 - j^2||B_{j,k}|^{-1})j^4 \\ &\leq 2^23^4\pi^2 \| \| B \| \|_{(2)}|k^2 - j^2||B_{j,k}|^{-1}j^4 \end{aligned}$$

and (14) changes into

$$\begin{aligned} \|f_n\|_{(3)}^8 &\leq R'_n(2^33^4\pi^2 \| \| B \| \|_{(2)}|k^2 - j^2||B_{j,k}|^{-1} \max\{j, k\}^4)^6 \\ &\leq (2^{18}3^{26}\pi^{12} \| \| B \| \|_{(2)}^6|k^2 - j^2|^6|B_{j,k}|^{-6} \max\{j, k\}^{24}) \frac{(1 + C')|B_{j,k}|^{-1} \| \| B \| \|_{(2)}^2}{n|k^2 - j^2|} \\ &\leq (2^{18}3^{26}\pi^{12}(1 + C') \| \| B \| \|_{(2)}^6 \| \| B \| \|_{(2)}^2|k^2 - j^2|^5|B_{j,k}|^{-7} \max\{j, k\}^{24})n^{-1}. \end{aligned}$$

After, for  $I := [nT^* - T, nT^* + T]$  we estimate  $\sup_{t \in I} \|\Gamma_t^{u_n}\phi_j - \Gamma_{T_n}^{u_n}\phi_j\|_{(3)}$  so that one can use any time in  $I$  as final time for the dynamics, since  $T_n$  (defined in [9]) is not always easy to compute.

Let us consider the argument used in (16). The function  $C(\cdot)$  introduced in [11, Proposition 5] is increasing (see the proof of Appendix B.3, Corollary 4, [3]). It follows  $\sup_{t \in I} C(t - T_n) \leq C(nT^* + T - T_n) \leq C(2T) \leq C(\frac{4}{3\pi}) =$

$48\sqrt{2}\pi^{-4}$ . Thus one can notice that

$$\begin{aligned}
& \sup_{t \in I} \|\Gamma_t^{u_n} \phi_j - \Gamma_{T_n}^{u_n} \phi_j\|_{(3)} \leq \sup_{t \in I} C(t - T_n) \|B\|_3 \int_{T_n}^t |u_n(s)| ds \|\Gamma_{T_n}^{u_n} \phi_j\|_{(3)} \\
& \leq C(nT^* + T - T_n) \|B\|_3 \int_{nT^* - T}^{nT^* + T} |u_n(s)| ds \|\Gamma_{T_n}^{u_n} \phi_j\|_{(4)} \\
& \leq C(nT^* + T - T_n) \|B\|_3 \frac{2T}{n} 2^2 3^4 \pi^2 \|B\|_{(2)} |k^2 - j^2| |B_{j,k}|^{-1} j^4 \\
& \leq C\left(\frac{4}{3\pi}\right) \|B\|_3 \frac{2T}{n} 2^2 3^4 \pi^2 \|B\|_{(2)} |k^2 - j^2| |B_{j,k}|^{-1} j^4 \\
& \leq \frac{\|B\|_3}{n} 6^3 \pi^2 \|B\|_{(2)} |B_{j,k}|^{-1} j^4.
\end{aligned}$$

Then we use  $nT^*$  as final time for the dynamics and we keep in mind that  $|B_{j,k}|$  is smaller than  $\|B\|$ ,  $\|B\|_{(2)}$  and  $\|B\|_3$ . If

$$(19) \quad n \geq \|B\|_3 \pi^2 6^3 \|B\|_{(2)} |B_{j,k}|^{-1} j^4,$$

there holds

$$\begin{aligned}
(20) \quad R_n'' & := \|\Gamma_{nT^*}^{u_n} \phi_j - e^{i\theta_n} \phi_k\|_{(3)}^8 \leq 2^7 \left( \|\Gamma_{nT^*}^{u_n} \phi_j - \Gamma_{T_n}^{u_n} \phi_j\|_{(3)}^8 + \|f_n\|_{(3)}^8 \right) \\
& \leq 2^7 \left( \left( \|B\|_3 6^3 \pi^2 \|B\|_{(2)} |B_{j,k}|^{-1} n^{-1} j^4 \right)^8 + \|f_n\|_{(3)}^8 \right) \\
& \leq 2^7 \left( \|B\|_3 6^3 \pi^2 \|B\|_{(2)} |B_{j,k}|^{-1} n^{-1} j^4 + \|f_n\|_{(3)}^8 \right) \leq 2^7 \frac{\pi^2 \|B\|_3 6^3 \|B\|_{(2)} j^4}{n |B_{j,k}|} \\
& + \frac{2^{25} 3^{26} \pi^{12} (1 + C') \|B\|_{(2)}^6 \|B\|^2 |k^2 - j^2|^5 \max\{j, k\}^{24}}{|B_{j,k}|^7 n} \\
& \leq \frac{6^{26} \pi^{12} (1 + C') \|B\|_{(2)}^6 \|B\|^2 \max\{\|B\|, \|B\|_3\} |k^2 - j^2|^5 \max\{j, k\}^{24}}{|B_{j,k}|^7 n}.
\end{aligned}$$

Now  $\lim_{n \rightarrow \infty} R_n'' = 0$ , hence there exists  $n^*$  such that

$$\Gamma_{n^* T^*}^{u_{n^*}} \phi_j \in B_{H_{(0)}^3} \left( e^{i\theta_{n^*}} \phi_k, C_k^2 (6k^3 \|B\|_3^2)^{-1} \right) \implies R_{n^*}'' \leq \frac{C_k^{16}}{6^8 k^{24} \|B\|_3^{16}}.$$

For  $0 \leq s < 3$  and  $j, k \in \mathbb{N}$  it follows  $\|B\|_{(s)} \geq C_k$  and  $\|B\|_{(s)} \geq |B_{j,k}|$ . Defined  $b := \|B\|_{(2)}^6 \|B\| \|B\|_3^{16} \max\{\|B\|, \|B\|_3\}$ , one can assume that  $n^*$  is an integer number larger than

$$\frac{6^{34} \pi^{12} b (1 + C') |k^2 - j^2|^5 k^{24} \max\{j, k\}^{24}}{C_k^{16} |B_{j,k}|^7}.$$

In conclusion the conditions (12), (17) and (19) are satisfied. We point out that the local exact controllability is verified in a neighborhood of  $\phi_k(4/3\pi)$



while our dynamics is pointing  $e^{i\theta_{n^*}}\phi_k$ . For this reason we have to pay attention to the phase of the target state. For

$$u_{n^*}(t) = \frac{\cos((k^2 - j^2)\pi^2 t)}{n^*}, \quad T_{n^*}^* = n^*T = n^* \frac{\pi}{|B_{j,k}|},$$

thanks to the first point of the proof and to the time reversibility of the system (1) (see [11, Section 1.3]), there exists  $u \in L^2((0, \frac{4}{3\pi}), \mathbb{R})$  such that

$$(21) \quad \Gamma_{\frac{4}{3\pi}}^u \Gamma_{T_{n^*}^*}^{u_{n^*}} \phi_j = e^{i\theta_{n^*}} \phi_k \left( -\frac{4}{3\pi} \right) = e^{i\theta_{n^*}} e^{i\lambda_k \frac{4}{3\pi}} \phi_k.$$

#### 4.4 Eliminating the phase ambiguity

In order to obtain a phase-shift  $e^{i\theta_n}$  we retrace the steps of previous subsection and we adopt the theory from [9] that explains how to define it.

By referring to [9, Section 3.1] we estimate  $N \geq \max\{j, k\}$  so that

$$(22) \quad K \|(1 - \pi_N)B(\phi_j \langle \phi_j, \cdot \rangle + \phi_k \langle \phi_k, \cdot \rangle)\| \leq CR_n$$

for  $C \in (0, 1)$ ,  $\pi_N(\cdot) := \sum_{k=1}^N \phi_k \langle \phi_k, \cdot \rangle$ . We have

$$\begin{aligned} & K \|(1 - \pi_N)B(\phi_j \langle \phi_j, \cdot \rangle + \phi_k \langle \phi_k, \cdot \rangle)\| \\ & \leq K \|(1 - \pi_N)B(\phi_j \langle \phi_j, \cdot \rangle)\| + K \|(1 - \pi_N)B(\phi_k \langle \phi_k, \cdot \rangle)\| \\ & \leq \frac{2}{|B_{j,k}|} \left( \left( \sum_{l=N+1}^{\infty} |B_{l,k}|^2 \right)^{\frac{1}{2}} + \left( \sum_{l=N+1}^{\infty} |B_{l,j}|^2 \right)^{\frac{1}{2}} \right) \leq CR_n. \end{aligned}$$

If  $R_n \geq 4C \|| B \|| (n\pi^2|k^2 - j^2|)^{-1}$ , then t

he relation (3) implies (22) if  $C$  is small enough. By using [9, relations (13), (18), (19)] as made for [9, relation (20)], we consider  $n$  large enough so that

$$(23) \quad \begin{aligned} & 1 - |\langle \phi_k, \Gamma_{T_n}^{\frac{u}{n}} \phi_j \rangle| \leq K \|(1 - \pi_N)B(\phi_j \langle \phi_j, \cdot \rangle + \phi_k \langle \phi_k, \cdot \rangle)\| \\ & + 4KR_n \|(1 - \pi_N)B\pi_N\| + R_n \leq CR_n + 8|B_{j,k}|^{-1} \|| B \|| R_n + R_n \\ & \leq CR_n + 9|B_{j,k}|^{-1} \|| B \|| R_n \leq 10|B_{j,k}|^{-1} \|| B \|| R_n =: \tilde{R}_n. \end{aligned}$$

Thus we substitute  $R_n$  with  $\tilde{R}_n$  in the relation (13). The argument of the previous section leads to ensure that  $n^* \in \mathbb{N}$  has to be larger than

$$\frac{6^{34} 10\pi^{12} b(1 + C') \|| B \|| |k^2 - j^2|^5 k^{24} \max\{j, k\}^{24}}{C_k^{16} |B_{j,k}|^8}.$$

By referring to the proofs of [9, Proposition 2] and [9, Corollary 3], we introduce the  $N \times N$  matrix  $M$  such that for  $l, m \in \mathbb{N}$

$$M_{l,m} = \langle \phi_l, M\phi_m \rangle = \frac{B_{l,m}}{I} \int_0^I e^{i(\lambda_l - \lambda_m)x} |u(x)| dx \quad \text{if} \quad \frac{|\lambda_l - \lambda_m|}{|\lambda_k - \lambda_j|} \in \mathbb{N}$$

otherwise  $M_{l,m} = 0$ . Now for every  $l, m$  such that  $|\lambda_l - \lambda_m||\lambda_k - \lambda_j|^{-1} = \mu \in \mathbb{N} \setminus \{1\}$

$$M_{l,m} = \frac{B_{l,m}}{4} \int_0^4 e^{i\mu x} |\cos(x)| dx = \frac{B_{l,m}(i\mu - 2i^\mu + e^{4i\mu}(\sin(4) + i\mu \cos(4)))}{4(\mu^2 - 1)},$$

while if  $|\lambda_l - \lambda_m| = |\lambda_k - \lambda_j|$

$$M_{l,m} = \frac{B_{l,m}}{4} \int_0^4 e^{ix} |\cos(x)| dx = \frac{B_{l,m}((-8 + 3i) + ie^{8i} + 2\pi)}{16}.$$

Thanks to [9, p. 5] there holds  $|\langle \phi_k, e^{KM} \phi_j \rangle| = 1$  and  $\theta_{n^*}$  is the real value so that  $e^{-i\theta_{n^*}} = \langle \phi_k, e^{KM} \phi_j \rangle$ . In conclusion thanks to [9] and to the relation (21) for  $\tilde{T}_{n^*} = -\lambda_j^{-1} \theta_{n^*}$ , there follows

$$\Gamma_{T_{n^*}^*}^{u_{n^*}} \Gamma_{\tilde{T}_{n^*}}^0 \frac{\Gamma_{\frac{\lambda_k}{\lambda_j} \frac{4}{3\pi}}^0}{\lambda_j \frac{4}{3\pi}} \phi_j \in B_{H_{(0)}^3}(\phi_k, C_k^2(6k^3 \|B\|_3^2)^{-1}).$$

The proof is achieved thanks to the local exact controllability and the time reversibility (see [11, Section 1.3]) since  $\exists u \in L^2((0, \frac{4}{3\pi}), \mathbb{R})$  so that

$$\Gamma_{\frac{4}{3\pi}}^u \Gamma_{T_{n^*}^*}^{u_{n^*}} \Gamma_{\tilde{T}_{n^*}}^0 \frac{\Gamma_{\frac{\lambda_k}{\lambda_j} \frac{4}{3\pi}}^0}{\lambda_j \frac{4}{3\pi}} \phi_j = \phi_k.$$

## 5 Example: dipolar moment

In the current section we retrace the proof of the first point of Theorem 1 by fixing  $B$  and  $j, k \in \mathbb{N}$ . Let  $B : \psi \mapsto x^2 \psi$ , we define a control function and a time so that the dynamics of (1) drives the second eigenstate  $\phi_2$  into the first  $\phi_1$ .

First, Assumptions I are satisfied since

$$\begin{aligned} |\langle \phi_j, x^2 \phi_k \rangle| &= \left| \frac{(-1)^{j-k}}{(j-k)^2 \pi^2} - \frac{(-1)^{j+k}}{(j+k)^2 \pi^2} \right| = \frac{4jk}{(j^2 - k^2)^2 \pi^2}, \quad j \neq k, \\ |\langle \phi_k, x^2 \phi_k \rangle| &= \left| \frac{1}{3} - \frac{1}{2k^2 \pi^2} \right|, \quad k \in \mathbb{N}. \end{aligned}$$

Now for every  $\psi \in H_{(0)}^3$  we know that  $x^2 \psi \in H^3 \cap H_0^1$ ,  $\|\partial_x \psi\| \leq \|\partial_x^2 \psi\|$  and thanks to the Poincaré inequality  $\|\psi\| \leq \pi^{-1} \|\partial_x \psi\|$ ,  $\|\partial_x^2 \psi\| \leq \pi^{-1} \|\partial_x^3 \psi\|$ . In addition  $\|x\psi\| \leq \frac{1}{\sqrt{3}} \|\psi\|$ ,  $\|x^2 \psi\| \leq \frac{1}{\sqrt{5}} \|\psi\|$  and

$$\begin{aligned} \|\partial_x(x^2 \psi)\| &\leq \|2x\psi\| + \|x^2 \partial_x \psi\| \leq \frac{2}{\sqrt{3}} \|\psi\| + \frac{1}{\sqrt{5}} \|\partial_x \psi\| \\ &\leq \left( \frac{2}{\sqrt{3}\pi} + \frac{1}{\sqrt{5}} \right) \|\partial_x \psi\| \leq \left( \frac{2\sqrt{5} + \sqrt{3}\pi}{\sqrt{15}\pi^2} \right) \|\partial_x^3 \psi\|, \\ \|\partial_x^2(x^2 \psi)\| &\leq \|2\psi\| + \|4x\partial_x \psi\| + \|x^2 \partial_x^2 \psi\| \leq \left( \frac{2\sqrt{15} + 4\sqrt{5}\pi + \sqrt{3}\pi}{\sqrt{15}\pi^2} \right) \|\partial_x^3 \psi\|, \\ \|\partial_x^3(x^2 \psi)\| &\leq \|6\partial_x \psi\| + \|6x\partial_x^2 \psi\| + \|x^2 \partial_x^3 \psi\| \leq \left( \frac{6\sqrt{15} + 6\sqrt{5}\pi + \sqrt{3}\pi}{\sqrt{15}\pi} \right) \|\partial_x^3 \psi\|. \end{aligned}$$

$$\begin{aligned}
\implies \quad & \| \| B \| \|_3^2 = \sup_{\substack{\psi \in H_{(0)}^3 \\ \| \psi \|_{(3)} \leq 1}} (\| \partial_x x^2 \psi \|^2 + \| \partial_x^2 x^2 \psi \|^2 + \| \partial_x^3 x^2 \psi \|^2) \\
& \leq \sup_{\substack{\psi \in H_{(0)}^3 \\ \| \psi \|_{(3)} \leq 1}} \left( \frac{2\sqrt{5} + \sqrt{3}\pi}{\sqrt{15}\pi^2} \right)^2 \| \partial_x^3 \psi \|^2 + \left( \frac{2\sqrt{15} + 4\sqrt{5}\pi + \sqrt{3}\pi}{\sqrt{15}\pi^2} \right)^2 \| \partial_x^3 \psi \|^2 \\
& + \left( \frac{6\sqrt{15} + 6\sqrt{5}\pi + \sqrt{3}\pi}{\sqrt{15}\pi} \right)^2 \| \partial_x^3 \psi \|^2 \leq \left( \frac{2\sqrt{15} + 4\sqrt{5}\pi + \sqrt{3}\pi}{\sqrt{15}\pi^2} \right)^2 \\
& + \left( \frac{2\sqrt{5} + \sqrt{3}\pi}{\sqrt{15}\pi^2} \right)^2 + \left( \frac{6\sqrt{15} + 6\sqrt{5}\pi + \sqrt{3}\pi}{\sqrt{15}\pi} \right)^2
\end{aligned}$$

and  $\| \| B \| \|_3 \leq 5,93$ . Equivalently  $\| \| B \| \|_{(2)} \leq 3,4$ ,  $\| \| B \| \| = 1/\sqrt{5}$ ,  $C' = 0$ . Then there holds

$$|B_{1,1}| = C_1 = \frac{2\pi - 3}{6\pi^2}, \quad |B_{1,2}| = C_2 = \frac{8}{9\pi^2}, \quad I = \int_0^{\frac{2}{3\pi}} |u(s)| ds = \frac{4}{3\pi^2}.$$

We retrace the proof of the first point of Theorem 1. Let  $T = \frac{2}{3\pi}$ ,  $u(t) = \cos(3\pi^2 t)$ ,  $T^* = \frac{9\pi^3}{8}$ ,  $K = \frac{9\pi^2}{4}$ , for  $u_n := \frac{u}{n}$  there exists  $\theta_n \in \mathbb{C}$  such that

$$\| e^{i\theta_n} \phi_1 - \Gamma_{T_n}^{u_n} \phi_2 \|^2 \leq \frac{3^2 |B_{1,2}^{-1}| \| \| B \| \|^2}{n |2^2 - 1^2|} = \frac{27\pi^2}{40n}.$$

Afterwards, for  $n$  large enough thanks to (18)

$$\| u_n \|_{BV(0, nT^* + T)} \leq 3\pi^2 |k^2 - j^2| |B_{j,k}|^{-1} \leq 3^4 2^{-3} \pi^4.$$

By following the proof of Theorem 1 for  $I := [nT^* - T, nT^* + T]$  we have

$$\begin{aligned}
& \| e^{i\theta_n} \phi_1 - \Gamma_{T_n}^{u_n} \phi_2 \|_{(3)}^8 \leq 2^7 \left( \| e^{i\theta_n} \phi_1 - \Gamma_{T_n}^{u_n} \phi_2 \|_{(4)}^6 \| e^{i\theta_n} \phi_1 - \Gamma_{T_n}^{u_n} \phi_2 \|^2 \right) \\
& + \sup_{t \in I} \left( 2^7 \| \Gamma_{T_n}^{u_n} \phi_2 - \Gamma_t^{u_n} \phi_2 \|_{(3)}^8 \right) \leq 2^7 \left( \frac{27\pi^2}{40n} (8e(1 + 3^4 \sqrt{2} \cdot 2^{-2} \cdot 3,4 \cdot \pi^4) 2^4 + 1)^6 \right. \\
& \left. + 5,93 \cdot 3,4 \cdot 6^3 \cdot 2 \cdot 9\pi^4 n^{-1} \right) \leq 1,11 \cdot 10^{42} n^{-1}.
\end{aligned}$$

In the neighborhood  $B_{H_{(0)}^3}(\phi_1(T), 1,5 \cdot 10^{-3})$  the local exact controllability is verified and the first point of Theorem 1 is satisfied for  $n = 4,31 \cdot 10^{64}$ . In conclusion, there exists  $\theta \in \mathbb{R}$  such that for

$$u_n(t) = (4,31 \cdot 10^{64})^{-1} \cos(3\pi^2 t), \quad T = (4,31 \cdot 10^{64}) \frac{9\pi^3}{8},$$

$$\implies \quad \| e^{i\theta} \phi_1 - \Gamma_T^{u_n} \phi_2 \|_{H_{(0)}^3} \leq 1,5 \cdot 10^{-3}.$$

Moreover there exists  $u^{lcl} \in L^2((0, \frac{4}{3\pi}), \mathbb{R})$  so that

$$\Gamma_T^{u_n} \Gamma_{\frac{4}{3\pi}}^{u^{lcl}} \phi_2 = e^{i\theta} \phi_1.$$

## 6 Moving forward

The nature of the work opens several questions, first and foremost, if the techniques developed may be adopted in the simultaneous global exact controllability together with the approaches of the works [11] and [15].

Moreover, the results provided in Theorem 1 are far from being optimal and one might be interested in optimizing them.

1. As already mentioned in Remark 3, Theorem 1 can be stated for other  $\frac{2\pi}{|\lambda_k - \lambda_j|}$ -periodic controls by using the theory exposed in [9]. A natural question is when we can retrace the theory of the work with different controls and obtain sharper estimates for  $n$ .
2. By using the techniques adopted in Section 4.1, one can look for a larger neighborhood of validity of the local exact controllability. A try is to change the time  $\frac{4}{3\pi}$  and study the variation of the radius as a time depending function.
3. One can adopt Haraux Theorem ([13, *Theorem 4.5*]) instead of Ingham Theorem ([13, *Theorem 4.3*]) in order to prove the local exact controllability. By retracing the steps of Section 4.1, one can establish the new constants and study how the neighborhood changes according to the time.
4. By referring to Section 4.3, the result of Theorem 1 is also valid if we change the dynamics time from  $nT$  to  $T_n$  as explained in its proof. In this framework the lower bound required for  $n$  decreases.

Another interesting question is how to simplify the matrix  $M$  and then the estimate of  $\theta$ , since the more  $n$  grows, the more the size of the matrix  $M$  does, making the computation of  $e^{KM}$  more difficult.

One can numerically approximate the phase ambiguity  $\theta_n$  by defining a value  $\tilde{\theta}$  such that  $\|e^{i\tilde{\theta}}\phi_k - e^{i\theta_n}\phi_k\|_{H^3_{(0)}} \leq \epsilon$ . Then for  $R''_n$  introduced in (20)

$$\|\Gamma_{T_n^*}^{u_n}\phi_j - e^{i\tilde{\theta}}\phi_k\|_{(3)} \leq \epsilon + (R''_n)^{1/8}.$$

Now for  $\epsilon$  small enough one can establish  $n^*$  large enough so that

$$\Gamma_{T_{n^*}^*}^{u_{n^*}}\phi_j \in B_{H^3_{(0)}}\left(e^{i\tilde{\theta}}\phi_k, C_k^2(6k^3 \|B\|_3^2)^{-1}\right)$$

and proceed as in the proof of Theorem 1.

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