



HAL
open science

Construction of the control function for the global exact controllability and further estimates

Alessandro Duca

► **To cite this version:**

Alessandro Duca. Construction of the control function for the global exact controllability and further estimates. 2017. hal-01520173v1

HAL Id: hal-01520173

<https://hal.science/hal-01520173v1>

Preprint submitted on 10 May 2017 (v1), last revised 2 May 2019 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Construction of the control function for the global exact controllability and further estimates

Alessandro Duca

Laboratoire de Mathématiques de Besançon, Université Bourgogne Franche-Comté
16, Route de Gray, 25000 Besançon, France
`alessandro.duca@univ-fcomte.fr`

Dipartimento di Scienze Matematiche Giuseppe Luigi Lagrange, Politecnico di Torino
24, Corso Duca degli Abruzzi, 10129 Torino, Italy
`alessandro.duca@polito.it`

Dipartimento di Matematica Giuseppe Peano, Università degli Studi di Torino
10, Via Carlo Alberto, 10123 Torino, Italy
`aduca@unito.it`

SPHINX team, Inria, 54600 Villers-lès-Nancy, France
`alessandro.duca@inria.fr`

ORCID: 0000-0001-7060-1723

Abstract

We consider the one dimensional bilinear Schrödinger equations in a bounded domain. We prove the global exact controllability and we exhibit how to construct the relative control function.

AMS subject classifications: 35Q41, 93C20, 93B05, 81Q15.

Keywords: Schrödinger equation, global exact controllability, inversion problems, control estimate.

1 Introduction

In quantum mechanics any pure state of a system is mathematically represented by a wave function ψ contained in the unit sphere of a Hilbert space \mathcal{H} . If we consider a particle constrained in a one dimensional bounded region then one can choose $\mathcal{H} = L^2((0, 1), \mathbb{C})$ and the evolution of ψ is modeled by the Cauchy problem

$$(1) \quad \begin{cases} i\partial_t \psi(t, x) = A\psi(t, x) + u(t)B\psi(t, x), & x \in (0, 1), t \in (0, T), \\ \psi(0, x) = \psi^0(x) \end{cases}$$

where $A = -\Delta$ is the Laplacian with Dirichlet homogeneous boundaries ($D(A) = H^2 \cap H_0^1$), B is a bounded symmetric operator, u is a control function and $\psi^0(x)$ is the initial state of the system.

We say that Problem (1) is globally (locally) exactly controllable in $\mathcal{M} \subset \mathcal{H}$ if for any $\psi_1, \psi_2 \in \mathcal{M}$ (in a neighborhood of \mathcal{M}) there exists a control such that the related dynamics of (1) steers ψ_1 into ψ_2 .

We also say that Problem (1) is globally approximately controllable in \mathcal{M} if for any $\psi_1, \psi_2 \in \mathcal{M}$ there exists a control such that the related dynamics drives ψ_1 infinitesimally close to ψ_2 .

The global (local) exact controllability is said to be simultaneous if, given two arbitrary sequences in \mathcal{M} (in a neighborhood of \mathcal{M}), there exists a control mapping each element of the first into the corresponding of the second.

The controllability of Problem (1) has been widely studied in the literature starting by the seminal work on bilinear systems of Ball, Mardsen and Slemrod [2]. We refer the reader to [11, *Proposition 1*] for two important consequences of the results proved in [2]: well-posedness conditions and a non controllability result, both for Problem (1) in \mathcal{H} (see also Turinici [21]). However despite this non controllability feature many authors have addressed the problem for weaker notions of controllability.

For instance in [3] and [4], local exact controllability results are ensured in $H_{(0)}^s := D(|A|^{\frac{s}{2}})$ for some $s \geq 3$.

Global approximate controllability in a Hilbert space has been studied in [7], [8], [9], in Sobolev spaces in [18], [19] and in $H_{(0)}^s$ for $s > 0$ in [5], [6], [10].

Global exact controllability of one dimensional Schrödinger equation in $H_{(V)}^6 := D(|A + V|^{\frac{6}{2}})$ for $V \in H^6$ is proved in [15] while [17] proves the simultaneous local exact controllability up to phase shifts for triples in $H_{(0)}^3$.

In [16] simultaneous global exact controllability for n-tuples in $H_{(V)}^4$ is ensured and in [11] the author proves the simultaneous global exact controllability in projection for sequences in $H_{(0)}^3$.

The novelties of the present work are the following.

First, we show how to construct a neighborhood in $H_{(0)}^3$ of any eigenfunction of A in which the local exact controllability is satisfied.

Second, for any couple of eigenfunctions ϕ_j and ϕ_k , we show how to construct controls such that the relative dynamics of (1) drives ϕ_j close to ϕ_k as much desired with respect to the $H_{(0)}^3$ -norm.

Third by using together the two previous results we define a dynamics steering any eigenstate of A into any other and we provide the exact time required to get to the target state.

In more technical terms, for any ϕ_j and ϕ_k we show how to construct a

sequence of control functions u_n and a sequences of times $T_n > 0$ such that

$$\exists \theta \in \mathbb{R} \quad : \quad \lim_{n \rightarrow \infty} \|\Gamma_{T_n}^{u_n} \phi_j - e^{i\theta} \phi_k\|_{H_{(0)}^3} = 0$$

for Γ_t^u the unitary propagator of Problem (1). We also establish a neighborhood of ϕ_k of radius r where the local exact controllability is satisfied and such that there exist $n^* \in \mathbb{N}$ and $u \in L^2((0, \frac{4}{3}), \mathbb{R})$ so that

$$\|\Gamma_{T_{n^*}}^{u_{n^*}} \phi_j - e^{i\theta} \phi_k\|_{H_{(0)}^3} < r, \quad \Gamma_{\frac{4}{3\pi}}^u \Gamma_{T_{n^*}}^{u_{n^*}} \phi_j = e^{i\theta} \phi_k.$$

In conclusion we show how to get rid of the phase ambiguity and we provide a time $T_1 > 0$ such that

$$\Gamma_{\frac{4}{3\pi}}^u \Gamma_{T_{n^*}}^{u'_{n^*}} \Gamma_{T_1}^0 \phi_j = \phi_k.$$

This work represents a step for using the control theory into the experimentation of the quantum systems modeled by the bilinear Schrödinger equation. Indeed almost the entirety of the previous works focus the attention into proving the existence of controls and times such that the controllability is satisfied but none of them make the two explicit except for [9].

1.1 Scheme of the work

In Section 2 we start by exposing the main results of the work in Theorem 1. We also prove few features of Problem (1) as the well-posedness in $H_{(0)}^3$ and the time reversibility.

In Section 3, Theorem 5 ensures the global exact controllability in $H_{(0)}^3$. The result is proved by using together the global approximate controllability exposed in Proposition 3 and the local exact controllability assured in Proposition 4, both in $H_{(0)}^3$.

In Section 4 we provide the proof of Theorem 1. First for any generic eigenfunction ϕ_l of A we construct a neighborhood $B(\phi_l) \subset H_{(0)}^3$ in which the local exact controllability holds (Section 4.1). Second we consider a generic couple of eigenfunctions ϕ_j, ϕ_k and we define a sequence of control functions u_n and a sequence of times T_n such that $\Gamma_{T_n}^{u_n} \phi_j$ is close to ϕ_k up to a well known distance in the \mathcal{H} -norm depending on n (Section 4.2). Third, we refine the previous result and we provide the distance with respect to the $H_{(0)}^3$ -norm. In conclusion we establish a lower bound for n such that this distance is smaller than the radius of $B(\phi_k)$. By using the local exact controllability, the proof of the first point of Theorem 1 is accomplished (Section 4.3). The second point is achieved by acting a phase-shift in order to eliminate a phase ambiguity appearing in the previous steps (Section 4.4). In Section 5 we discuss the issues appearing during the application of the proved results and we provide few methods that one can adopt in order to

get rid of them. We also present an example in which we apply the developed techniques (Section 5.1).

2 Framework and main results

We denote $\mathcal{H} = L^2((0, 1), \mathbb{C})$, its norm $\|\cdot\|$ and its scalar product $\langle \cdot, \cdot \rangle$. The operator A is the Dirichlet Laplacian ($A = -\frac{d^2}{dx^2}$ and $D(A) = H_0^1((0, 1), \mathbb{C}) \cap H^2((0, 1), \mathbb{C})$), B is a bounded symmetric operator and u is a $L^2((0, T), \mathbb{R})$ control function.

We consider an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ composed by eigenfunctions of A associated with the eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ and

$$(2) \quad \phi_j(t, x) = e^{-iAt} \phi_j(x) = e^{-i\lambda_j t} \phi_j(x).$$

Let the spaces for $s \geq 0$

$$H_{(0)}^s = H_{(0)}^s((0, 1), \mathbb{C}) := D(A^{\frac{s}{2}}), \quad \|\cdot\|_{(s)} := \|\cdot\|_{H_{(0)}^s} = \left(\sum_{k=1}^{\infty} |k^s \langle \cdot, \phi_k \rangle|^2 \right)^{\frac{1}{2}},$$

$$\ell^\infty(\mathcal{H}) = \left\{ \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H} \mid \sup_{j \in \mathbb{N}} \|\psi_j\| < \infty \right\},$$

$$\ell^2(\mathcal{H}) = \left\{ \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H} \mid \sum_{j=1}^{\infty} \|\psi_j\|^2 < \infty \right\},$$

$$h^s(\mathcal{H}) = \left\{ \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H} \mid \sum_{j=1}^{\infty} (j^s \|\psi_j\|)^2 < \infty \right\},$$

we define the following operatorial norms for $0 < s < 3$

$$\|\!\| \cdot \|\!\| := \|\!\| \cdot \|\!\|_{L(\mathcal{H}, \mathcal{H})} = \|\!\| \cdot \|\!\|_{L(H_{(0)}^0, H_{(0)}^0)}, \quad \|\!\| \cdot \|\!\|_{(s)} := \|\!\| \cdot \|\!\|_{L(H_{(0)}^s, H_{(0)}^s)},$$

$$\|\!\| \cdot \|\!\|_3 := \|\!\| \cdot \|\!\|_{L(H_{(0)}^3, H^3 \cap H_0^1)}.$$

Assumption (I). Let B be a bounded symmetric operator.

1. For any $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$\forall j \in \mathbb{N}, \quad |\langle \phi_j, B\phi_k \rangle| \geq \frac{C_k}{j^3}.$$

2. $\text{Ran}(B|_{D(A)}) \subseteq D(A)$ and

$$\text{Ran}(B|_{H_{(0)}^3((0,1),\mathbb{C})}) \subseteq H^3((0, 1), \mathbb{C}) \cap H_0^1((0, 1), \mathbb{C}).$$

3. For each $j, k, l, m \in \mathbb{N}$ such that $(j, k) \neq (l, m)$ and $j^2 - k^2 - l^2 + m^2 = 0$ it is true that

$$\langle \phi_j, B\phi_j \rangle - \langle \phi_k, B\phi_k \rangle - \langle \phi_l, B\phi_l \rangle + \langle \phi_m, B\phi_m \rangle \neq 0.$$

Remark 1. If a bounded operator B satisfies Assumption (I) then $B \in L(H_{(0)}^2, H_{(0)}^2)$. Indeed B is closed in \mathcal{H} , then for every $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ such that $u_n \xrightarrow{\mathcal{H}} u$ and $Bu_n \xrightarrow{\mathcal{H}} v$ we have that $Bu = v$. Now for every $\{u_n\}_{n \in \mathbb{N}} \subset H_{(0)}^2$ such that $u_n \xrightarrow{H_{(0)}^2} u$ and $Bu_n \xrightarrow{H_{(0)}^2} v$, the convergences with respect to the \mathcal{H} -norm are implied and then $Bu = v$. Hence the operator B is closed in $H_{(0)}^2$ and $B \in L(H_{(0)}^2, H_{(0)}^2)$. Thanks to the same argument it is true that $B \in L(H_{(0)}^3, H^3 \cap H_0^1)$.

Let us introduce the quantity

$$b := \|B\|_{(2)}^6 \|B\| \|B\|_3^{16} \max\{\|B\|, \|B\|_3\}$$

depending only on the operator B and for every $k, j \in \mathbb{N}$, $n \in \mathbb{N}$

$$E(j, k) := |k^2 - j^2|^5 C_k^{-16} k^{24} |B_{j,k}|^{-7} \max\{j, k\}^{24},$$

$$u_n(t) := \frac{\cos((k^2 - j^2)\pi^2 t)}{n}, \quad C' := \sup_{(v,w) \in \Lambda'} \left\{ \left| \sin\left(\pi \frac{|v^2 - w^2|}{|k^2 - j^2|}\right) \right|^{-1} \right\},$$

where

$$\Lambda' := \{(v, w) \in \{1, \dots, N\}^2 : \{v, w\} \cap \{j, k\} \neq \emptyset, |v^2 - w^2| \leq \frac{3}{2}|k^2 - j^2|, |v^2 - w^2| \neq |k^2 - j^2|, \langle \phi_n, B\phi_m \rangle \neq 0\}.$$

Let $N \in \mathbb{N}$ and $B_{l,m} := \langle \phi_l, B\phi_m \rangle$, we define the $N \times N$ matrix $M_{j,k}^N$ with elements

$$\begin{cases} (M_{j,k}^N)_{l,m} = \frac{B_{l,m}(ik - 2i^k + e^{4ik}(\sin(4) + ik \cos(4)))}{4(k^2 - 1)}, & \frac{|l^2 - m^2|}{|k^2 - j^2|} = k \in \mathbb{N} \setminus \{1\}, \\ (M_{j,k}^N)_{l,m} = \frac{B_{l,m}((-8 + 3i) + ie^{8i} + 2\pi)}{16}, & \frac{|l^2 - m^2|}{|k^2 - j^2|} = 1, \\ (M_{j,k}^N)_{l,m} = 0, & \frac{|l^2 - m^2|}{|k^2 - j^2|} \notin \mathbb{N}, \end{cases}$$

and

$$T_n := \frac{n\pi}{|B_{j,k}|} + \frac{2}{\pi|k^2 - j^2|}, \quad \tilde{T}_n^N = -(j\pi)^{-2} \left\langle \phi_k, e^{-2M_{j,k}^N |B_{j,k}|^{-1}} \phi_j \right\rangle.$$

Theorem 1. *Let $k, j \in \mathbb{N}$ and B satisfy Assumptions (I).*

1) *Let n be an integer such that*

$$n \geq 6^{34} \pi^{12} b (1 + C') E(j, k),$$

then there exists $\theta \in \mathbb{R}$ such that

$$\|\Gamma_{T_n}^{u_n} \phi_j - e^{i\theta} \phi_k\|_{H_{(0)}^3} \leq C_k^2 (6k^3 \|B\|_3^2)^{-1}.$$

Moreover there exists $u \in L^2((0, \frac{4}{3\pi}), \mathbb{R})$ such that $\|u^{lcl}\| \leq 2C_k(7 \| B \| \frac{2}{3}k^3)^{-1}$ and

$$\Gamma_{\frac{4}{3\pi}}^u \Gamma_{T_n}^{u_n} \phi_j = e^{i\theta} \phi_k.$$

2) Let n be an integer such that

$$n \geq 6^{34} 10 \pi^{12} b (1 + C') \| B \| E(j, k) |B_{j,k}|^{-1}.$$

Let $N \geq \max\{j, k\}$ such that

$$(3) \quad \frac{2}{|B_{j,k}|} \left(\left(\sum_{l=N+1}^{\infty} |B_{l,k}|^2 \right)^{\frac{1}{2}} + \left(\sum_{l=N+1}^{\infty} |B_{l,j}|^2 \right)^{\frac{1}{2}} \right) < \frac{4 \| B \|}{n\pi^2 |k^2 - j^2|}$$

then for $v'_n(t) := u_n(t) \chi_{[\tilde{T}_n^N, \tilde{T}_n^N + T_n]}(t)$ there holds

$$\| \Gamma_{\tilde{T}_n^N + T_n}^{v'_n} \phi_j - \phi_k \|_{H_{(0)}^3} \leq C_k^2 (6k^3 \| B \| \frac{2}{3})^{-1}.$$

Moreover there exists $u \in L^2((0, \frac{4}{3\pi}), \mathbb{R})$ such that

$$\Gamma_{\frac{4}{3\pi}}^u \Gamma_{T_n}^{u_n} \Gamma_{\tilde{T}_n^N}^0 \phi_j = \phi_k.$$

Remark 2. The results of Theorem 1 are far from being optimal. Indeed the aim of the work is to show how to proceed in this kind of problems and we present an approach that one can use in order to build a control function for the global exact controllability in $H_{(0)}^3$.

Moreover the purpose of Theorem 1 is to exhibit readable results, even if its proof shows slightly stronger relations and one can surely improve them by following the approaches introduced in Section 5. We leave the optimization work to who might be interested and we briefly treat the example of $B : \psi \rightarrow x^2\psi$, $k = 1$ and $j = 2$ in Section 5.1.

Remark 3. In the proof of Theorem 1 the choice of the control function u comes from the techniques developed in [9]. We point out that one can ensure similar results for other $\frac{2\pi}{|\lambda_k - \lambda_j|}$ -periodic controls by using the theory exposed in [9].

2.1 Well-posedness of the Cauchy problem

We mention now the crucial result of well-posedness for the Cauchy problem

$$(4) \quad \begin{cases} i\partial_t \psi(t, x) = -\Delta \psi(t, x) + u(t)\mu(x)\psi(t, x) + f(t, x), \\ \psi(0, x) = \psi^0(x), \quad x \in (0, 1), \quad t \in (0, T). \end{cases}$$

Proposition 2 (Beauchard, Laurent; [3]; Lemma 1 & Proposition 2).

1) Let $T > 0$ and $\tilde{f} \in L^2((0, T), H_0^1 \cap H^3)$. The function $G : t \mapsto \int_0^t e^{iAs} \tilde{f}(s) ds$ belongs to $C^0([0, T], H_{(0)}^3)$. Moreover

$$\|G\|_{L^\infty((0, T), H_{(0)}^3)} \leq c_1(T) \|\tilde{f}\|_{L^2((0, T), H^3 \cap H_{(0)}^1)},$$

where the constant $c_1(T)$ is uniformly bounded with T lying in bounded intervals.

2) Let $\mu \in H^3((0, 1), \mathbb{R})$, $T > 0$, $\psi^0 \in H_{(0)}^3(0, 1)$, $f \in L^2((0, T), H_0^1 \cap H^3)$ and $u \in L^2((0, T), \mathbb{R})$. Then there exists a unique mild solution of (4) in $H_{(0)}^3(0, 1)$, i.e. a function $\psi \in C_0([0, T], H_{(0)}^3)$ such that for every $t \in [0, T]$

$$(5) \quad \psi(t, x) = e^{i\Delta t} \psi^0(x) - i \int_0^t e^{i\Delta(t-s)} (u(s)\mu(x)\psi(s, x) + f(s, x)) ds.$$

Moreover, for every $R > 0$, there exists $C = C(T, \mu, R) > 0$ such that, if $\|u\|_{L^2((0, T), \mathbb{R})} < R$ then the solution satisfies

$$(6) \quad \|\psi\|_{C^0([0, T], H_{(0)}^3)} \leq C(\|\psi^0\|_{H_{(0)}^3} + \|f\|_{L^2((0, T), H_0^1 \cap H^3)}), \quad \forall \psi^0 \in H_{(0)}^3.$$

If $f \equiv 0$ then

$$\|\psi\| = \|\psi^0\| \quad \forall t \in [0, T].$$

If B satisfies Assumptions (I), then Proposition 2 implies well-posedness of (1) in $H_{(0)}^3$.

2.2 Time reversibility

Let us now present another feature of the bilinear Schrödinger equation, the time reversibility. First we notice that

$$\begin{cases} i\partial_t \Gamma_{T-t}^u \psi^0(x) = -A \Gamma_{T-t}^u \psi^0(x) - u(T-t) B \Gamma_{T-t}^u \psi^0(x), \\ \Gamma_{T-0}^u \psi^0(x) = \Gamma_T^u \psi^0(x) = \psi^1(x). \end{cases}$$

Let us define $\tilde{u}(t) := u(T-t)$, $\tilde{\Gamma}_t^{\tilde{u}} := \Gamma_{T-t}^u$. It follows

$$\begin{cases} i\partial_t \tilde{\Gamma}_t^{\tilde{u}} \psi^1(x) = (-A - \tilde{u}(t)B) \tilde{\Gamma}_t^{\tilde{u}} \psi^1(x), \quad x \in (0, 1), \quad t \in (0, T), \\ \tilde{\Gamma}_0^{\tilde{u}} \psi^0(x) = \psi^1(x) \end{cases}$$

and $\tilde{\Gamma}_t^{\tilde{u}} \Gamma_T^u = \Gamma_{T-t}^u$. Then $\tilde{\Gamma}_T^{\tilde{u}} \Gamma_T^u = Id$ and

$$(7) \quad \tilde{\Gamma}_T^{\tilde{u}} = (\Gamma_T^u)^{-1} = (\Gamma_T^u)^*.$$

In conclusion for $t > 0$ and $\tilde{u}(\cdot) = u(t - \cdot)$ the operator $\tilde{\Gamma}_t^{\tilde{u}}$ is the propagator related to $(-A - \tilde{u}(t)B)$ and describes the reversed dynamics of Γ_t^u .

The importance of the time reversibility resides in the fact that all the controllability results that we are going to prove are still verified for the reversed problem. We will use this feature in many steps of the next proofs.

3 Global exact controllability in $H_{(0)}^3$

In this section we ensure the global exact controllability in $H_{(0)}^3$ and we start by proving the approximate controllability of Problem (1).

Definition 1. The problem $(A, B, U, \{\phi_j\}_{j \in \mathbb{N}})$ is globally approximately controllable in $H_{(0)}^s$ if for every $\psi \in H_{(0)}^s$, $\widehat{\Gamma} \in U(\mathcal{H})$ so that $\widehat{\Gamma}\psi \in H_{(0)}^s$ and $\epsilon > 0$ there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ such that

$$\|\widehat{\Gamma}\psi - \Gamma_T^u \psi\|_{H_{(0)}^s} < \epsilon.$$

Proposition 3. Let Problem (1) and B satisfy Assumptions (I). Then $(A, B, U, \{\phi_j\}_{j \in \mathbb{N}})$ is globally approximately controllable in $H_{(0)}^3$.

Proof. The result is due to [11, Theorem 6]. \square

We provide a brief proof of the local exact controllability in $H_{(0)}^3$ by rephrasing the existing local controllability results as [3], [4], [11], [15], [16] and [17]. Our purpose is to introduce the tools that we use in the proof of Theorem 1. Let

$$O_l(\epsilon) = \{\psi \in H_{(0)}^3 \mid \exists \widehat{\Gamma} \in U(\mathcal{H}) : \psi = \widehat{\Gamma}\phi_l, \|\psi - \phi_l(T)\|_{H_{(0)}^3} < \epsilon\}.$$

Proposition 4. Let B satisfy (I) and $l \in \mathbb{N}$. Then there exist $T > 0$, $\epsilon > 0$ such that for every $\psi \in O_l(\epsilon)$, there exists a control function $u \in L^2((0, T), \mathbb{R})$ so that $\psi = \Gamma_T^u \phi_l$.

Proof. Let us consider without loss of generality $l = 1$, the decomposition

$$\Gamma_t^u \phi_1 = \sum_{k=1}^{\infty} \phi_k(t) \langle \phi_k(t), \Gamma_t^u \phi_1 \rangle$$

and the map $\alpha_1(u)$, the sequence with elements

$$\alpha_{k,1}(u) = \langle \phi_k(T), \Gamma_T^u \phi_1 \rangle, \quad k \in \mathbb{N}.$$

The local existence of the control function is equivalent to prove the local right invertibility of the map α_1 for a $T > 0$ (in other words the local surjectivity). To this end, we want to use the Generalized Inverse Function Theorem ([14], p. 240) and we study the surjectivity of the Fréchet derivative of α_1 , $\gamma_1(v) := (d_u \alpha_1(0)) \cdot v$, the sequence with elements

$$\begin{aligned} \gamma_{k,1}(v) &:= \left\langle \phi_k(T), -i \int_0^T e^{-iA(T-s)} v(s) B e^{-iAs} \phi_1 ds \right\rangle \\ &= -i \int_0^T v(s) e^{i(\lambda_k - \lambda_1)s} ds B_{k,1}, \quad k \leq N, j \in \mathbb{N}, \end{aligned}$$

for $B_{k,j} = \langle \phi_k, B\phi_j \rangle = \langle B\phi_k, \phi_j \rangle = \overline{B_{j,k}}$. The right invertibility of the map γ_1 consists in proving the solvability of the moment problem

$$\frac{x_k}{B_{k,1}} = -i \int_0^T u(s) e^{i(\lambda_k - \lambda_1)s} ds.$$

for each $x \in \ell^2(\mathbb{C})$ such that $\{x_k B_{k,1}^{-1}\}_{k \leq N} \in \ell^2$. Indeed $\{x_k\}_{k \in \mathbb{N}} \in h^3$ and thanks to Assumptions (I) the claim follows.

Now the solvability of the moment problem (3) is ensured by using Ingham Theorem ([13, *Theorem 4.3*]) for $T > \frac{2\pi}{\mathcal{G}}$ and

$$\mathcal{G} := \inf_{\substack{k,j \in \mathbb{N} \\ k \neq j}} |\lambda_k - \lambda_j| = 3\pi^2 > 0.$$

Then γ_1 is surjective and the proof is achieved thanks to the Generalized Inverse Function Theorem ([14], p. 240), which ensures that the map α_1 is locally surjective. \square

Remark. We point out that one can achieve the result of Theorem 4 for any positive time $T > 0$ by using Haraux Theorem ([13, *Theorem 4.5*]), instead of Ingham Theorem ([13, *Theorem 4.3*]) as it is done in the proof of [11, *Theorem 8*].

Now we provide the global exact controllability in $H_{(0)}^3$ by using Proposition 3 and Proposition 4.

Theorem 5. *Let B satisfy (I). Then for any $\psi^1, \psi^2 \in H_{(0)}^3$ unitarily equivalent there exist $T > 0$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that*

$$\psi^2 = \Gamma_T^u \psi^1.$$

Proof. First, for any $\epsilon > 0$, Proposition 3 holds true if one substitutes A with $-A$, B with $-B$ and there exists $\tilde{T} > 0$ and a control function $u_1 \in L^2((0, \tilde{T}), \mathbb{R})$ so that

$$\left\| \phi_1(T) - \tilde{\Gamma}_{\tilde{T}}^{u_1} (\|\psi^1\|^{-1} \psi^1) \right\|_{H_{(0)}^3} < \epsilon.$$

Now $\tilde{\Gamma}_{\tilde{T}}^{u_1} (\|\psi^1\|^{-1} \psi^1) \in O_1(\epsilon)$ and thanks to Proposition 4 there exists a control function $u_2 \in L^2((0, T), \mathbb{R})$ such that

$$\tilde{\Gamma}_{\tilde{T}}^{u_1} (\|\psi^1\|^{-1} \psi^1) = \Gamma_T^{u_2} \phi_1.$$

By reversing the dynamics (see Section 2.2), for $u(t) = u_2(t)\chi_{[0,T]}(t) + u_1(T + \tilde{T} - t)\chi_{[T, T+\tilde{T}]}(t)$

$$\|\psi^1\|^{-1} \psi^1 = \Gamma_{T+\tilde{T}}^u \phi_1.$$

In conclusion by using again the time reversibility and the fact that $\|\psi^1\| = \|\psi^2\|$ there exist $\hat{T} > 0$ and a control $\hat{u} \in L^2((0, \hat{T}), \mathbb{R})$ such that

$$\psi^2 = \Gamma_{\hat{T}}^{\hat{u}} \psi^1. \quad \square$$

4 Proof of Theorem 1

4.1 Neighborhood estimate

Let us define the following terminology

$$\| \cdot \|_{L(L^2((0,T),\mathbb{R}),H_{(0)}^3)} = \| \cdot \|_{(L_t^2,H_x^3)}, \quad \| \cdot \|_{L(H_{(0)}^3,L^2((0,T),\mathbb{R}))} = \| \cdot \|_{(H_x^3,L_t^2)},$$

$$\| \cdot \|_{L^\infty((0,T),H_{(0)}^3)} = \| \cdot \|_{L_t^\infty H_x^3}, \quad \| \cdot \|_{L^2((0,T),\mathbb{R})} = \| \cdot \|_2,$$

$$\| \cdot \|_{L^1((0,T),\mathbb{R})} = \| \cdot \|_1, \quad \| \cdot \|_{BV((0,T),\mathbb{R})} = \| \cdot \|_{BV(T)},$$

for

$$\|f\|_{BV((0,T),\mathbb{R})} = \sup_{\{t_j\}_{0 \leq j \leq n}} |f(t_j) - f(t_{j-1})|$$

and $t_0 = 0 < t_1 < \dots < t_n = T$ any partition of the interval $(0, T)$.

Let $T > 0$ and the space $\tilde{X} := \overline{\text{span}\{e^{i\lambda_k t}, k \in \mathbb{N}\}}^{L^2} \subset L^2((0, T), \mathbb{R})$ with the L^2 -norm. The local exact controllability is equivalent to the local surjectivity of the map

$$A_l(\cdot) := \Gamma_T^{(\cdot)} \phi_l : \tilde{X} \rightarrow H_{(0)}^3$$

defined as

$$A_l(u) = e^{-i\lambda_l T} \phi_l - i \int_0^T e^{-iA(T-s)} u(s) B \Gamma_s^u \phi_l ds.$$

In the proof of Theorem 4 we have used the Generalized Inverse Function Theorem and we have proved the surjectivity of the map

$$F_l(u) := ((d_v A_l(v=0)) \cdot u) \in H_{(0)}^3.$$

We want to estimate the radius of a neighborhood in $H_{(0)}^3$ of $\phi_l(T)$ in which the map A_l is surjective. For this reason, we use [10, Lemma 2.3; p. 42] by considering the quotient space $X := \frac{\tilde{X}}{\text{Ker}(F_l)}$ with the L^2 -norm.

1) The map $F_l : X \rightarrow H_{(0)}^3$ is bijective and we estimate a constant $M > 0$ such that

$$\|F_l(v) - F_l(w)\|_{(3)} \geq M \|v - w\|_{L^2}, \quad \forall v, w \in X.$$

Let us suppose $\|B\|_3 = 1$, we consider the map $\gamma_l : u \in X \mapsto \gamma_l(u) \in h^3$, equivalently defined for $l = 1$ in the proof of Proposition 4. We know that the surjectivity of F_l in $H_{(0)}^3$ corresponds to the surjectivity of the map γ_l in h^3 . For every $\psi \in H_{(0)}^3$, there exist $T > 0$ and $u \in X$ such that $\langle \phi_j(T), \psi \rangle = \gamma_{j,l}(u)$ and such that $F_l^{-1}(\psi) = u$.

For C_l defined in Assumptions (I), thanks to [3, *Proposition 19*; (ii)] and to Ingham Theorem ([13, *Theorem 4.3*]), there exists $\tilde{C}(T) > 0$ such that

$$\begin{aligned} \|F_l^{-1}(\psi)\|_2^2 &= \|u\|_2^2 \leq \tilde{C}(T)^2 \sum_{j=1}^{\infty} \left| \frac{\gamma_{j,l}(u)}{B_{j,l}} \right|^2 \leq \frac{\tilde{C}(T)^2}{C_l^2} \sum_{j=1}^{\infty} |j^3 \gamma_{j,l}(u)|^2 \\ &\leq \frac{\tilde{C}(T)^2}{C_l^2} \|\psi\|_{(3)}^2. \end{aligned}$$

Now for every $v, w \in X$ there exist $\psi, \varphi \in H_{(0)}^3$ such that $\psi = F_l(v)$, $\varphi = F_l(w)$ and

$$\|v - w\|_2 \leq \|F_l^{-1}(\psi - \varphi)\|_2 \leq \|F_l^{-1}\|_{(H_x^3, L_t^2)} \|\psi - \varphi\|_{(3)},$$

which implies that one can impose $M = \|F_l^{-1}\|_{(H_x^3, L_t^2)}^{-1} = \frac{C_l}{\tilde{C}(T)}$.

2) Let $u \in X$, thanks to the Duhamel's formula

$$\begin{aligned} \Gamma_T^u \phi_l &= e^{-i\lambda_l T} \phi_l - i \int_0^T e^{-iA(T-s)} u(s) B \Gamma_s^u \phi_l ds \\ &= e^{-i\lambda_l T} \phi_l - i \int_0^T e^{-iA(T-s)} u(s) B e^{-i\lambda_l s} \phi_l ds \\ &\quad - \int_0^T e^{-iA(T-s)} u(s) B \left(\int_0^s e^{-iA(s-\tau)} u(\tau) B \Gamma_\tau^u \phi_l d\tau \right) ds \\ &= e^{-i\lambda_l T} \phi_l + F_l(u) + H_l(u) \end{aligned}$$

for

$$H_l(u) := - \int_0^T e^{-iA(T-s)} u(s) B \left(\int_0^s e^{-iA(s-\tau)} u(\tau) B \Gamma_\tau^u \phi_l d\tau \right) ds.$$

We want to estimate a ball $U \subset X$ with center $u = 0$ where the map $u \mapsto \Gamma_T^u \phi_l$ is surjective by using [10, *Lemma 2.3*]. However $e^{-i\lambda_l T} \phi_l$ is constant and it is sufficient to define $U \subset X$ such that there exists a constant $0 < M_1 < M$ so that for every $v, w \in U$

$$\|H_l(v) - H_l(w)\|_{(3)} \leq M_1 \|v - w\|_{L^2}.$$

First, we notice that

$$\begin{aligned} H_l(u) - H_l(v) &= - \int_0^T e^{-iA(T-s)} u(s) B \left(\int_0^s e^{-iA(s-\tau)} u(\tau) B \Gamma_\tau^u \phi_l d\tau \right) ds \\ &\quad + \int_0^T e^{-iA(T-s)} v(s) B \left(\int_0^s e^{-iA(s-\tau)} v(\tau) B \Gamma_\tau^v \phi_l d\tau \right) ds \end{aligned}$$

$$\begin{aligned}
&= - \int_0^T e^{-iA(T-s)}(u(s) - v(s))B \left(\int_0^s e^{-iA(s-\tau)}u(\tau)B\Gamma_\tau^u\phi_l d\tau \right) ds \\
&\quad - \int_0^T e^{-iA(T-s)}v(s)B \left(\int_0^s e^{-iA(s-\tau)}(u(\tau) - v(\tau))B\Gamma_\tau^u\phi_l d\tau \right) ds \\
&\quad - \int_0^T e^{-iA(T-s)}v(s)B \left(\int_0^s e^{-iA(s-\tau)}v(\tau)B(\Gamma_\tau^u\phi_l - \Gamma_\tau^v\phi_l) d\tau \right) ds.
\end{aligned}$$

Thanks to Proposition 2, there exists a constant $C(T) > 0$ such that for every $\psi \in H^3 \cap H_0^1$, $u \in L^2((0, T), \mathbb{R})$ it holds

$$\left\| \int_0^T e^{-iA(T-s)}u(s)B\psi ds \right\|_{(3)} \leq C(T)\|u\|_2 \|B\|_3 \|\psi\|_{L_t^\infty H_x^3}$$

and then

$$\begin{aligned}
(8) \quad \|H_l(u) - H_l(v)\|_{(3)} &\leq C(T)^2 \|v - u\|_2 \|B\|_3^2 (\|u\|_2 + \|v\|_2) \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\
&\quad + C(T)^2 \|v\|_2^2 \|B\|_3^2 \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\
&\leq C(T)^2 \|v - u\|_2 (\|u\|_2 + \|v\|_2) \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\
&\quad + C(T)^2 \|v\|_2^2 \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3}.
\end{aligned}$$

One can show by using the same technique adopted in (8) that

$$\begin{aligned}
\|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} &\leq \left\| \int_0^t e^{-iA(t-s)}B(v\Gamma_s^v \phi_l - u\Gamma_s^u \phi_l) \right\|_{L_t^\infty H_x^3} \\
&\leq C(T) \|B\|_3 \|v\Gamma_t^v \phi_l - u\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \leq C(T)\|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\
&\quad + C(T)\|v\|_2 \|\Gamma_t^v - \Gamma_t^u\|_{L_t^\infty H_x^3} \leq C(T)\|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\
&\quad + C(T)^2 \|v\|_2 \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} + C(T)^2 \|v\|_2^2 \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \leq \\
&\|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \sum_{n=0}^N C(T)^{n+1} \|v\|_2^n + C(T)^N \|v\|_2^N \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3}.
\end{aligned}$$

Thanks to Remark 1, [5, Proposition 6] implies that the couple (A, B) is (2)-weakly coupled, then [6, Proposition 30; (ii)] it is satisfied and

$$(9) \quad \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} < \infty.$$

Now if $U \subseteq \{u \in X : \|u\|_2 \leq (\mu C(T))^{-1}\}$ for $\mu > 1$, it holds that for $u, v \in U$

$$\lim_{N \rightarrow \infty} C(T)^N \|v\|_2^N \|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} = 0,$$

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N C(T)^n \|v\|_2^n \leq \frac{\mu}{\mu - 1}.$$

Thus

$$\|\Gamma_t^v \phi_l - \Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \leq \frac{\mu C(T)}{\mu - 1} \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3}$$

and the relation (8) becomes

$$\begin{aligned} \|H_l(u) - H_l(v)\|_{(3)} &\leq C(T)^2 \|v - u\|_2 (\|u\|_2 + \|v\|_2) \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\ &+ \frac{\mu}{\mu - 1} C^3(T) \|v\|_2^2 \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \leq \frac{2}{\mu} C(T) \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \\ &+ \frac{1}{(\mu - 1)\mu} C(T) \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3} \leq \frac{(2\mu - 1)}{(\mu - 1)\mu} C(T) \|v - u\|_2 \|\Gamma_t^u \phi_l\|_{L_t^\infty H_x^3}. \end{aligned}$$

Thanks to Proposition 2 and to the Duhamel's formula

$$\|\Gamma_T^u \phi_l\|_{L_t^\infty H_x^3} \leq \frac{\|\phi_l\|_{(3)}}{1 - C(T)\|u\|_2 \|B\|_3} \leq \frac{\mu l^3}{\mu - 1}$$

and then

$$(10) \quad \|H_l(u) - H_l(v)\|_{(3)} \leq \frac{2\mu - 1}{(\mu - 1)^2} l^3 C(T) \|v - u\|_2.$$

Let $M_1 = \frac{2\mu - 1}{(\mu - 1)^2} l^3 C(T)$, we want to estimate μ such that $\frac{1}{2}M > M_1$ and

$$\frac{1}{2} \frac{C_l}{\tilde{C}(T)} > \frac{2\mu - 1}{(\mu - 1)^2} l^3 C(T).$$

Thus $a(2\mu - 1)(\mu - 1)^{-2} < 1$ for $a = \frac{2C(T)\tilde{C}(T)l^3}{C_l}$ and the inequality is satisfied for $\mu > a + \sqrt{a(a + 1)} + 1$.

Let us establish an upper bound for $C(T)\tilde{C}(T)$. One can consider the proofs of [3, Proposition 19; (ii)] and of Ingham Theorem [13, Theorem 4.3], we know that

$$\mathcal{G} := \inf_{k \neq j} (\lambda_k - \lambda_j) = 3\pi^2.$$

Let $T = \frac{4}{3\pi}$, I' be such that $|I'| := \frac{\mathcal{G}}{\pi} T = 4$ and

$$\beta = \frac{\pi^2}{4}, \quad G(0) = \frac{\pi}{2}, \quad I_0 = [-1, +1], \quad m = (|I'| |I_0|^{-1}) = 2,$$

$$\alpha = 4R^2, \quad \hat{G}(0) = \frac{(R^2 - 1)\pi}{2}, \quad R = \frac{|I'|}{2} = 2.$$

Thanks to the proof of Ingham Theorem [13, Theorem 4.3] there holds

$$\tilde{C}(T) = \frac{2m\pi G(0)\pi}{\beta \mathcal{G}} = \frac{8}{3\pi}, \quad \hat{C}(T) = \frac{\alpha \mathcal{G}}{2\pi \hat{G}(0)\pi} = \frac{16}{\pi}.$$

and the proof of the first point of Proposition 2 (see [3]) implies

$$C(T) = 3\pi^{-3} \max \{ \sqrt{2}\widehat{C}(T), \sqrt{T} \} = 3\pi^{-3} \sqrt{2}\widehat{C}(T) = \frac{48\sqrt{2}}{\pi^4}.$$

Now $C(T)\widetilde{C}(T) \leq \frac{2}{3}$ and $a \leq \frac{4}{3}\widetilde{a}_l$ for $\widetilde{a}_l := \frac{l^3}{C_l}$. Afterwards $C_l \leq \langle \phi_1, B\phi_l \rangle \leq \|B\|$ which implies that $\widetilde{a}_l > 1$ and

$$C(T) \left(\frac{4}{3}\widetilde{a}_l + \sqrt{\frac{4}{3}\widetilde{a}_l \left(\frac{4}{3}\widetilde{a}_l + 1 \right)} + 1 \right) \leq C(T) \left(\frac{4}{3}\widetilde{a}_l + \left(\frac{4}{3}\widetilde{a}_l + 1 \right) + 1 \right) \leq \frac{7}{2}\widetilde{a}_l.$$

Let a Hilbert space Y , $y \in Y$ and $r > 0$, we define

$$B_Y(y, r) := \{ \tilde{y} \in Y \mid \|\tilde{y} - y\|_Y \leq r \}.$$

One can consider $U = B_X(0, 2(7\widetilde{a}_l)^{-1})$, we know that $M - M_1 \geq (2\widetilde{C}(T))^{-1}C_l$ and thanks to the proof of [10, Lemma 2.3]

$$\begin{aligned} A_l(B_X(0, 2(7\widetilde{a}_l)^{-1})) &\supset B_{H_{(0)}^3}(F_l(0), (M - M_1)2(7\widetilde{a}_l)^{-1}) \supset \\ &B_{H_{(0)}^3}(\phi_l(T), C_l(6\widetilde{a}_l)^{-1}) \supset B_{H_{(0)}^3}\left(\phi_l(T), \frac{C_l^2}{6l^3}\right). \end{aligned}$$

We have supposed $\|B\|_3 = 1$ but we can generalize for $\|B\|_3 \neq 1$ by considering that

$$A + uB = A + u \|B\|_3 \frac{B}{\|B\|_3}.$$

One can consider the operator $\frac{B}{\|B\|_3}$ and the control $u \|B\|_3$. Hence we substitute C_l with $C_l \|B\|_3^{-1}$ and

$$\forall \psi \in B_{H_{(0)}^3}\left(\phi_l(T), \frac{C_l^2}{6l^3 \|B\|_3^2}\right), \exists u \in B_X\left(0, \frac{2C_l}{7l^3 \|B\|_3^2}\right) \mid A_l(u) = \psi.$$

4.2 Control function for the global approximate controllability with respect to the \mathcal{H} -norm

Let ϕ_j, ϕ_k for $j \leq k$, we exhibit a control function driving the dynamics of (1) from ϕ_j to $B_{H_{(0)}^3}(\phi_k(T), C_l^2(6l^3 \|B\|_3^2)^{-1})$. For this purpose one can start by generalizing [9, Proposition 6]. Let $T = \frac{2\pi}{|\lambda_k - \lambda_j|}$ and $u(t) = \cos((\lambda_k - \lambda_j)t)$. For any $n \in \mathbb{N}$ there exists $T_n \in (nT^* - T, nT^* + T)$ such that

$$\frac{1 - |\langle \phi_k, \Gamma_{T_n}^{\frac{u}{n}} \phi_j \rangle|}{1 + 2K \|B\|} \leq \frac{(1 + C') \|(\phi_j \langle \phi_j, \cdot \rangle + \phi_k \langle \phi_k, \cdot \rangle) B\| I}{n},$$

with $T^* = \frac{\pi}{|B_{j,k}|}$, $I = \frac{4}{|\lambda_k - \lambda_j|}$, $K = \frac{2}{|B_{j,k}|}$ and

$$C' = \sup_{(v,w) \in \Lambda'} \left\{ \left| \sin \left(\pi \frac{|\lambda_v - \lambda_w|}{|\lambda_k - \lambda_j|} \right) \right|^{-1} \right\},$$

where

$$\Lambda' = \{(v, w) \in \{1, \dots, N\}^2 : \{v, w\} \cap \{j, k\} \neq \emptyset, |\lambda_v - \lambda_w| \leq \frac{3}{2}|\lambda_k - \lambda_j|, \\ |\lambda_v - \lambda_w| \neq |\lambda_k - \lambda_j|, B_{n,m} \neq 0\}.$$

Remark. As already mentioned in Remark 3 one can state similar results for other control functions $\frac{2\pi}{|\lambda_k - \lambda_j|}$ -periodics by adopting the theory from [9].

For $u_n := \frac{u}{n}$ and $R_n := (1 + 2K \lVert B \rVert)(1 + C') \lVert B \rVert In^{-1}$ there holds

$$(11) \quad \sum_{l \neq k} |\langle \phi_l, \Gamma_{T_n}^{u_n} \phi_j \rangle|^2 = 1 - |\langle \phi_k, \Gamma_{T_n}^{u_n} \phi_j \rangle|^2 \\ \leq (1 - |\langle \phi_k, \Gamma_{T_n}^{u_n} \phi_j \rangle|)(1 + |\langle \phi_k, \Gamma_{T_n}^{u_n} \phi_j \rangle|) \leq 2R_n.$$

Afterwards, there exists $\theta_n \in \mathbb{C}$ such that

$$(12) \quad |\langle \phi_k, \phi_k \rangle - \langle \phi_k, \Gamma_{T_n}^{u_n} e^{i\theta_n} \phi_j \rangle|^2 \leq R_n^2.$$

From (11), (12) it follows

$$(13) \quad R'_n := \|\phi_k - \Gamma_{T_n}^{u_n} e^{i\theta_n} \phi_j\|^2 \leq 2R_n + R_n^2,$$

hence $|B_{j,k}|^{-1} \lVert B \rVert \geq 1$ implies

$$R_n \leq \frac{(1 + C')(|B_{j,k}|^{-1} + 4|B_{j,k}|^{-1}) \lVert B \rVert^2 I}{nC_k} \\ \leq \frac{5(1 + C')|B_{j,k}|^{-1} \lVert B \rVert^2 I}{n} \leq \frac{3(1 + C')|B_{j,k}|^{-1} \lVert B \rVert^2}{n|k^2 - j^2|}.$$

and

$$(14) \quad R_n \leq \frac{3(1 + C')|B_{j,k}|^{-1} \lVert B \rVert^2}{n|k^2 - j^2|}.$$

4.3 Global approximate controllability with respect to the H^3 -norm

Let us consider the relation (14). If

$$(15) \quad n \geq \frac{3(1 + C')|B_{j,k}|^{-1} \lVert B \rVert^2}{|k^2 - j^2|},$$

then $R_n \leq 1$, $R'_n \leq R_n$ and

$$(16) \quad R'_n \leq 2R_n + R_n^2 \leq 3R_n \leq \frac{3^2|B_{j,k}|^{-1}(1 + C') \lVert B \rVert^2}{n|k^2 - j^2|}.$$

For $f_n := \phi_k - \Gamma_{T_n}^{u_n} e^{i\theta_n} \phi_j$ there holds $\|f_n\|_{(s)}^2 \leq (k^s + \|\Gamma_{T_n}^{u_n} e^{i\theta_n} \phi_j\|_{(s)})^2$ and by considering [5, relation (9)] it follows

$$(17) \quad \|f_n\|_{(3)}^4 = \|f_n\|_{\left(\frac{12}{4}\right)}^4 \leq \|f_n\|_{\left(\frac{4}{2}\right)}^2 \|f_n\|_{\left(\frac{8}{2}\right)}^2 \leq \|f_n\| \|f_n\|_{(4)}^3.$$

Let $\widehat{H}_{(0)}^4 := D(A(\lambda_\epsilon - A))$ and such as [6, Proposition 30] follows from the Kato-Rellich type arguments of [12, Section 3.10], one can ensure that for every $\lambda_\epsilon \geq 1$:

$$M := \sup_{t \in [0, T_n]} \left\| (\lambda_\epsilon - A - u_n(t)B)^{-1} \right\|_{L(H_{(0)}^2, \widehat{H}_{(0)}^4)},$$

$$N := \left\| \lambda_\epsilon - A - u_n(t)B \right\|_{BV([0, T_n], L(\widehat{H}_{(0)}^4, H_{(0)}^2))},$$

there holds:

$$\|(A + u_n(T_n)B - \lambda_\epsilon)\Gamma_{T_n}^{u_n} e^{i\theta_n} \phi_j\|_{(2)} \leq M e^{MN} \|(A - \lambda_\epsilon)\theta_n \phi_j\|_{(2)} \leq M e^{MN} (1 + \lambda_\epsilon) j^4.$$

Now $N \leq \|u_n(t)\|_{BV(T_n)} \left\| B \right\|_{L(\widehat{H}_{(0)}^4, H_{(0)}^2)}$ and for every $\psi \in \widehat{H}_{(0)}^4$, $\epsilon > 0$

$$\begin{aligned} \|B\psi\|_{(2)}^2 &\leq (\epsilon \|A\psi\|_{(2)} + \left\| B \right\|_{(2)} \|\psi\|_{(2)})^2 \\ &\leq 2\epsilon^2 (\|A\psi\|_{(2)}^2 + \epsilon^{-2} \left\| B \right\|_{(2)}^2 \|\psi\|_{(2)}^2). \end{aligned}$$

If we set $\lambda_\epsilon = \left\| B \right\|_{(2)} \epsilon^{-1}$

$$\|B\psi\|_{(2)}^2 \leq 2\epsilon^2 (\|A\psi\|_{(2)}^2 + \lambda_\epsilon^2 \|\psi\|_{(2)}^2) \leq 2\epsilon^2 \|(A - \lambda_\epsilon)\psi\|_{(2)}^2$$

which leads to $N \leq \epsilon\sqrt{2}\|u_n(t)\|_{BV(T_n)}$. By following the proof of [6, Proposition 30] we can prove that

$$M \leq 2 + \epsilon \sup_{t \in [0, T_n]} (|u_n(t)| \|(\lambda_\epsilon - A)(\lambda_\epsilon - A - u_n(t)B)^{-1}\|_{(2)}).$$

If $\epsilon \leq 2^{-1}$ then $M \leq 3$ and if

$$(18) \quad n \geq \frac{2 \left\| B \right\|_{(2)}}{\pi^2} \implies \left\| A(A + u_n(T_n)B - \lambda_\epsilon)^{-1} \right\|_{(2)} \leq 2.$$

Thus

$$\begin{aligned} \|\Gamma_{T_n}^{u_n} e^{i\theta_n} \phi_j\|_{(4)} &= \|A\Gamma_{T_n}^{u_n} e^{i\theta_n} \phi_j\|_{(2)} \leq 2\|(A + u(T_n)B - \lambda_\epsilon)\Gamma_{T_n}^{u_n} e^{i\theta_n} \phi_j\|_{(2)} \\ &\leq 6e^{3\sqrt{2}\epsilon\|u_n\|_{BV(T_n)}} (1 + \lambda_\epsilon) j^4 \leq 6e^{3\sqrt{2}\epsilon\|u_n\|_{BV(T_n)}} (1 + \left\| B \right\|_{(2)} \epsilon^{-1}) j^4. \end{aligned}$$

and for a suitable value of ϵ

$$(19) \quad \|\Gamma_{T_n}^{u_n} e^{i\theta_n} \phi_j\|_{(4)} \leq 6e(1 + 3\sqrt{2}\|u_n\|_{BV(T_n)} \left\| B \right\|_{(2)}) j^4.$$

In the interval $[0, nT^* + T]$ are not contained more than d half-period of the function u for $d := 2\pi^2 n |k^2 - j^2| |B_{j,k}|^{-1} + 4$ and if

$$(20) \quad n \geq \| \| B \| \| (5\pi^{-2} |j^2 - k^2|^{-1})$$

there follows

$$(21) \quad \| u_n \|_{BV(T_n)} \leq \| u_n \|_{BV(nT^*+T)} \leq (d+1) \leq 3\pi^2 |k^2 - j^2| |B_{j,k}|^{-1}.$$

The relation (19) becomes:

$$\begin{aligned} \| \Gamma_{T_n}^{u_n} e^{i\theta n} \phi_j \|_{(4)} &\leq 6e(1 + 3^2 \sqrt{2} \pi^2 \| \| B \| \|_{(2)} |k^2 - j^2| |B_{j,k}|^{-1}) j^4 \\ &\leq 2^2 3^4 \pi^2 \| \| B \| \|_{(2)} |k^2 - j^2| |B_{j,k}|^{-1} j^4 \end{aligned}$$

and (17)

$$\begin{aligned} \| f_n \|_{(3)}^8 &\leq R'_n (2^3 3^4 \pi^2 \| \| B \| \|_{(2)} |k^2 - j^2| |B_{j,k}|^{-1} \max\{j, k\}^4)^6 \\ &\leq (2^{18} 3^{26} \pi^{12} \| \| B \| \|_{(2)}^6 |k^2 - j^2|^6 |B_{j,k}|^{-6} \max\{j, k\}^{24}) \frac{(1 + C') |B_{j,k}|^{-1} \| \| B \| \|_{(2)}^2}{n |k^2 - j^2|} \\ &\leq (2^{18} 3^{26} \pi^{12} (1 + C') \| \| B \| \|_{(2)}^6 \| \| B \| \|_{(2)}^2 |k^2 - j^2|^5 |B_{j,k}|^{-7} \max\{j, k\}^{24}) n^{-1}. \end{aligned}$$

Now we want to estimate $\| \Gamma_{nT^*+T}^{u_n} \phi_j - \phi_k \|_{(3)}$ so that one can consider $nT^* + T$ as final time.

Remark. We point out that one could use T_n , defined in [9], but its formulation is not always of easy computation. Moreover the next argument can be used for proving that the result still holds for every time in $[T_n, nT^* + T]$.

Let us consider the argument implying the relation (19) and the fact that $C(\cdot)$ introduced in Proposition 2 is increasing (see the proof of Appendix B.3, Corollary 4, [3]). It follows that $C(nT^* + T - T_n) \leq C(2T) \leq C(\frac{4}{3\pi}) = 48\sqrt{2}\pi^{-4}$. Thus one can notice that

$$\begin{aligned} \| \Gamma_{nT^*+T}^{u_n} \phi_j - \Gamma_{T_n}^{u_n} \phi_j \|_{(3)} &\leq C(nT^* + T - T_n) \| \| B \| \|_3 \int_{T_n}^{nT^*+T} |u_n(s)| ds \| \Gamma_{T_n}^{u_n} \phi_j \|_{(3)} \\ &\leq C(nT^* + T - T_n) \| \| B \| \|_3 \int_{T_n}^{nT^*+T} |u_n(s)| ds \| \Gamma_{T_n}^{u_n} \phi_j \|_{(4)} \\ &\leq C(nT^* + T - T_n) \| \| B \| \|_3 \int_{T_n}^{nT^*+T} |u_n(s)| ds 2^2 3^4 \pi^2 \| \| B \| \|_{(2)} |k^2 - j^2| |B_{j,k}|^{-1} j^4 \\ &\leq C\left(\frac{4}{3\pi}\right) \| \| B \| \|_3 \frac{2T}{n} 2^2 3^4 \pi^2 \| \| B \| \|_{(2)} |k^2 - j^2| |B_{j,k}|^{-1} j^4 \\ &\leq \frac{\| \| B \| \|_3}{n} 6^3 \pi^2 \| \| B \| \|_{(2)} |B_{j,k}|^{-1} j^4. \end{aligned}$$

By keeping in mind that $|B_{j,k}|$ is smaller than $\| \| B \| \|$, $\| \| B \| \|_{(2)}$ and $\| \| B \| \|_3$, if n is large enough so that

$$(22) \quad \frac{\| \| B \| \|_3 \pi^2}{n} 6^3 \| \| B \| \|_{(2)} |B_{j,k}|^{-1} j^4 \leq 1,$$

there holds

$$\begin{aligned}
R_n'' &:= \|\Gamma_{nT^*+T}^{u_n} \phi_j - \phi_k\|_{(3)}^8 \leq 2^7 \left(\|\Gamma_{nT^*+T}^{u_n} \phi_j - \Gamma_{T_n}^{u_n} \phi_j\|_{(3)}^8 + \|f_n\|_{(3)}^8 \right) \\
&\leq 2^7 \left(\left(\|B\|_3 6^3 \pi^2 \|B\|_{(2)} |B_{j,k}|^{-1} n^{-1} j^4 \right)^8 + \|f_n\|_{(3)}^8 \right) \\
&\leq 2^7 \left(\|B\|_3 6^3 \pi^2 \|B\|_{(2)} |B_{j,k}|^{-1} n^{-1} j^4 + \|f_n\|_{(3)}^8 \right) \leq 2^7 \frac{\pi^2 \|B\|_3 6^3 \|B\|_{(2)} j^4}{n |B_{j,k}|} \\
&\quad + \frac{2^{25} 3^{26} \pi^{12} (1 + C') \|B\|_{(2)}^6 \|B\|_{(2)}^2 |k^2 - j^2|^5 \max\{j, k\}^{24}}{|B_{j,k}|^7 n} \\
&\leq \frac{6^{26} \pi^{12} (1 + C') \|B\|_{(2)}^6 \|B\|_{(2)}^2 \max\{\|B\|, \|B\|_3\} |k^2 - j^2|^5 \max\{j, k\}^{24}}{|B_{j,k}|^7 n}.
\end{aligned}$$

Now $\lim_{n \rightarrow \infty} R_n'' = 0$, hence there exists n^* such that

$$\Gamma_{n^*T^*+T}^{u_{n^*}} \phi_j \in B_{H^3(0)} \left(\phi_k(T), C_k^2 (6k^3 \|B\|_3^2)^{-1} \right) \Rightarrow R_{n^*}'' \leq \frac{C_k^{16}}{6^8 k^{24} \|B\|_3^{16}}.$$

Thanks to the fact that for $0 \leq s < 3$ and $j, k \in \mathbb{N}$

$$\|B\|_{(s)} \geq C_k, \quad \|B\|_{(s)} \geq |B_{j,k}|$$

and defined $b := \|B\|_{(2)}^6 \|B\|_{(2)} \|B\|_3^{16} \max\{\|B\|, \|B\|_3\}$ one can assume that n^* is larger than

$$\frac{6^{34} \pi^{12} b (1 + C') |k^2 - j^2|^5 k^{24} \max\{j, k\}^{24}}{C_k^{16} |B_{j,k}|^7}.$$

In conclusion the conditions (15), (18), (20) and (22) are satisfied. Thus for

$$u_{n^*}(t) = \frac{\cos((k^2 - j^2)\pi^2 t)}{n^*}, \quad T_{n^*} = n^* \frac{\pi}{2} + \frac{2}{|k^2 - j^2| \pi},$$

there holds that $\Gamma_{T_{n^*}}^{u_{n^*}} e^{i\theta_{n^*}} \phi_j \in \tilde{O}_k$.

4.4 Eliminating the phase ambiguity

In order to act a phase-shift we slightly retrace the steps of previous subsection and we adopt the theory from [9] that explains how to define precisely this phase ambiguity.

First, we refer to [9, Section 3.1] and we want to estimate $N \geq \max\{j, k\}$ such that for $C \in (0, 1)$

$$(23) \quad K \|(1 - \pi_N) B(\phi_j \langle \phi_j, \cdot \rangle + \phi_k \langle \phi_k, \cdot \rangle)\| \leq CR_n,$$

for $\pi_N(\cdot) := \sum_{k=1}^N \phi_k \langle \phi_k, \cdot \rangle$. We have that

$$\begin{aligned} & K \|(1 - \pi_N)B(\phi_j \langle \phi_j, \cdot \rangle + \phi_k \langle \phi_k, \cdot \rangle)\| \\ & \leq K \|(1 - \pi_N)B(\phi_j \langle \phi_j, \cdot \rangle)\| + K \|(1 - \pi_N)B(\phi_k \langle \phi_k, \cdot \rangle)\| \\ & \leq \frac{2}{|B_{j,k}|} \left(\left(\sum_{l=N+1}^{\infty} |B_{l,k}|^2 \right)^{\frac{1}{2}} + \left(\sum_{l=N+1}^{\infty} |B_{l,j}|^2 \right)^{\frac{1}{2}} \right) \leq CR_n \end{aligned}$$

and if $R_n \geq 4C \|B\| (n\pi^2 |k^2 - j^2|)^{-1}$, the relation (3) implies (23) by considering C small enough.

Let us take in account [9, relation (13)], [9, relation (18)] and [9, relation (19)]. Similarly to [9, relation (20)], one can consider n large enough such that

$$\begin{aligned} (24) \quad & 1 - |\langle \phi_k, \Gamma_{T_n}^{\frac{u}{n}} \phi_j \rangle| \leq K \|(1 - \pi_N)B(\phi_j \langle \phi_j, \cdot \rangle + \phi_k \langle \phi_k, \cdot \rangle)\| \\ & + 4KR_n \|(1 - \pi_N)B\pi_N\| + R_n \leq CR_n + 8|B_{j,k}|^{-1} \|B\| R_n + R_n \\ & \leq CR_n + 9|B_{j,k}|^{-1} \|B\| R_n \leq 10|B_{j,k}|^{-1} \|B\| R_n =: \tilde{R}_n. \end{aligned}$$

Thus one can substitute R_n with \tilde{R}_n in the relation (16) and by repeating the argument of the previous section one can use \tilde{n} equal to

$$\frac{6^{34} 10\pi^{12} b(1 + C') \|B\| |k^2 - j^2|^5 k^{24} \max\{j, k\}^{24}}{C_k^{16} |B_{j,k}|^8}.$$

By referring to the proofs of [9, Proposition 2] and [9, Corollary 3], we introduce the $N \times N$ matrix M such that for $l, m \in \mathbb{N}$ if $|\lambda_l - \lambda_m| |\lambda_k - \lambda_j|^{-1} \in \mathbb{N}$

$$M_{l,m} = \langle \phi_l, M\phi_m \rangle = \frac{B_{l,m}}{I} \int_0^I e^{i(\lambda_l - \lambda_m)s} |u(s)| ds,$$

otherwise $M_{l,m} = 0$. Now for every l, m such that $|\lambda_l - \lambda_m| |\lambda_k - \lambda_j|^{-1} = \mu \in \mathbb{N} \setminus \{1\}$ it follows

$$M_{l,m} = \frac{B_{l,m}}{4} \int_0^4 e^{i\mu x} |\cos(x)| dx = \frac{B_{l,m}(i\mu - 2i^\mu + e^{4i\mu}(\sin(4) + i\mu \cos(4)))}{4(\mu^2 - 1)},$$

while if $|\lambda_l - \lambda_m| = |\lambda_k - \lambda_j|$ there holds that

$$M_{l,m} = \frac{B_{l,m}}{4} \int_0^4 e^{ix} |\cos(x)| dx = \frac{B_{l,m}((-8 + 3i) + ie^{8i} + 2\pi)}{16}.$$

Thank to [9], one can ensure that $e^{i\theta_{n^*}} = \langle \phi_k, e^{-KM} \phi_j \rangle$. In conclusion for

$$\tilde{T}_{n^*} = -\lambda_j^{-1} \theta_{n^*}, \quad \tilde{w}(t) := u_{n^*} \chi_{[\tilde{T}_{n^*}, \tilde{T}_{n^*} + T_{n^*}]},$$

there holds that $\Gamma_{\tilde{T}_{n^*}}^{\tilde{w}} \phi_j \in \tilde{O}_k$ and the proof it is achieved thanks to the local exact controllability.

5 Optimization methods, example and other approaches

As mentioned in Remark 2 the results provided in Theorem 1 are far from being optimal. We are sure of the existence of many techniques that one can use in order to improve the estimates and here we present few of them.

1. One can retrace the proof of Theorem 1 fixing B and $j, k \in \mathbb{N}$ in order to use sharper estimates. We briefly show an example in the next subsection.
2. By using the techniques adopted in Section 4.1, one can look for a larger neighborhood of validity of the local exact controllability. Indeed the larger it is, the smaller n has to be. We suggest to change the time from the one chosen of $\frac{4}{3\pi}$ and study the variation of the radius as function of the time. Moreover, one can use the Haraux Theorem ([13, *Theorem 4.5*]), instead of Ingham Theorem ([13, *Theorem 4.3*]) in order to prove the local exact controllability. By retracing the steps of Section 4.1, one can establish the new constants and study how the neighborhood changes as a function of the time.
3. Another try is to look for more precise upper bounds of the kind

$$\|\Gamma_t^u f\|_k \leq C_1 \|f\|_k, \quad f \in H_{(0)}^k, \quad 3 \leq k < 5, \quad C_1 > 0,$$

$$\|f\|_3 \leq C_2 \|f\|, \quad f \in H_{(0)}^3, \quad C_2 > 0,$$

than the ones used in Section 4.3 (with our approach C_2 was function of $\|f\|_{(4)}$).

4. In conclusion one can obtain the same result of Theorem 1 with T_n as final time instead of using $nT + T^*$ as stated in Remark 4.3, Section 4.2.

5.1 Example: dipolar moment

Let $B : \psi \mapsto x^2 \psi$, we want to define a control function and a time so that the dynamics of (1) drives the second eigenstate ϕ_2 into the first ϕ_1 .

First, Assumptions (I) are satisfied because

$$|\langle \phi_j, x^2 \phi_k \rangle| = \left| \frac{(-1)^{j-k}}{(j-k)^2 \pi^2} - \frac{(-1)^{j+k}}{(j+k)^2 \pi^2} \right| = \frac{4jk}{(j^2 - k^2)^2 \pi^2}, \quad j \neq k,$$

$$|\langle \phi_k, x^2 \phi_k \rangle| = \left| \frac{1}{3} - \frac{1}{2k^2 \pi^2} \right|, \quad k \in \mathbb{N}$$

which imply the validity of the conditions 1) and 2). The condition 3) is verified as

$$j^2 - k^2 - l^2 + m^2 = 0 \quad \implies \quad j^{-2} - k^{-2} - l^{-2} + m^{-2} \neq 0.$$

Now there holds

$$\begin{aligned}\|\partial_x(x^2\psi)\| &\leq \|2x\psi\| + \|x^2\partial_x\psi\|, \\ \|\partial_x^2(x^2\psi)\| &\leq \|2\psi\| + \|4x\partial_x\psi\| + \|x^2\partial_x^2\psi\|, \\ \|\partial_x^3(x^2\psi)\| &\leq \|6\partial_x\psi\| + \|6x\partial_x^2\psi\| + \|x^2\partial_x^3\psi\|.\end{aligned}$$

We know that for every $\psi \in H_{(0)}^3$ it follows that $\|\psi\| \leq \pi^{-2}\|A\psi\|$ and the Poincaré inequality implies that $\|\psi\| \leq \pi^{-1}\|A^{\frac{1}{2}}\psi\|$, $\|A\psi\| \leq \pi^{-1}\|A^{\frac{3}{2}}\psi\|$. Thus

$$\begin{aligned}\|B\|_3 &= \sup_{\substack{\psi \in H_{(0)}^3 \\ \|\psi\|_{(3)} \leq 1}} \|x^2\psi\|_{H^3} \leq \sup_{\substack{\psi \in H_{(0)}^3 \\ \|\psi\|_{(3)} \leq 1}} (\|6\partial_x\psi\| + \|6\partial_x^2\psi\| + \|\partial_x^3\psi\| \\ &\quad + \|2\psi\| + \|4\partial_x\psi\| + \|\partial_x^2\psi\| + \|2\psi\| + \|\partial_x\psi\|) \\ &\leq \sup_{\substack{\psi \in H_{(0)}^3 \\ \|\psi\|_{(3)} \leq 1}} (1 + 7\pi^{-1} + \pi^{-2}(11 + 5\pi^{-1}))\|\partial_x^3\psi\| \leq 4, 6.\end{aligned}$$

Thanks to density arguments and to the Poincaré inequality

$$\sup_{\substack{\psi \in H_{(0)}^2 \\ \|\psi\|_{(2)} = 1}} \|\partial_x\psi\| = \sup_{\substack{\psi \in H_0^2 \\ \|\psi\|_{(2)} = 1}} \|\partial_x\psi\| \leq \sup_{\substack{\psi \in H_0^2 \\ \|\psi\|_{(2)} = 1}} \pi^{-1}\|\partial_x^2\psi\| = \pi^{-1}$$

and then $\|B\|_{(2)} \leq (2\pi^{-2} + 4\pi^{-1} + 1) \leq 2, 5$, $\|B\| = 1$, $C' = 0$ and there holds

$$|B_{1,1}| = C_1 = \frac{2\pi - 3}{6\pi^2}, \quad |B_{1,2}| = C_2 = \frac{8}{9\pi^2}, \quad I = \int_0^{\frac{2}{3\pi}} |u(s)| ds = \frac{4}{3\pi^2},$$

Then we retrace the steps of the proof of the first point of Theorem 1 in order to produce more precise estimates.

Let $T = \frac{2}{3\pi}$, $u(t) = \cos(3\pi^2 t)$, $T^* = \frac{9\pi^3}{8}$, $K = \frac{9\pi^2}{4}$, for $u_n := \frac{u}{n}$ there exists $\theta_n \in \mathbb{C}$ such that

$$R'_n = \|\phi_1 - \Gamma_{T_n}^{u_n} e^{i\theta_n} \phi_2\|^2 \leq \frac{27\pi^2}{8n}.$$

Afterwards for n large enough thanks to (21)

$$\|u_n\|_{BV(0, nT^* + T)} \leq 3^4 2^{-3} \pi^4.$$

Thus

$$\begin{aligned}\|\phi_1 - \Gamma_{nT^* + T}^{u_n} e^{i\theta_n} \phi_2\|_3^8 &\leq 2^7 (27\pi^2 (8n)^{-1} (6e(1 + 3\sqrt{2}3^4 2^{-3} (2, 5)\pi^4) 2^4 + 1)^6 \\ &\quad + (4, 6) 6^3 2^3 9\pi^2 n^{-1}) \leq 1, 77 \cdot 10^{42} n^{-1}\end{aligned}$$

and we can ensure that in the neighborhood $B_{H_{(0)}^3}(\phi_1(T), 1.5 \cdot 10^{-3})$ the local exact controllability is verified. Thus the first point of Theorem 1 is satisfied for $n = (1.3 \cdot 10^{67}) \frac{8}{9\pi^2}$ and by referring to Remark 4.3 there exists $\theta \in \mathbb{R}$ such that for:

$$u(t) = (1.3 \cdot 10^{67})^{-1} \frac{9\pi^2}{8} \cos(3\pi^2 t), \quad T = 1.3 \cdot 10^{68}$$

it follows

$$\|\Gamma_T^u \phi_2 - e^{i\theta} \phi_1\|_{H_{(0)}^3} \leq 1.5 \cdot 10^{-3}.$$

Moreover there exists $u^{lcl} \in L^2((0, \frac{4}{3\pi}), \mathbb{R})$ so that

$$\Gamma_T^u \Gamma_{\frac{4}{3\pi}}^{u^{lcl}} \phi_2 = e^{i\theta} \phi_1.$$

We point out that by considering $\tilde{B} : \psi \mapsto \frac{x^2}{E_0} \psi$ instead of $B : \psi \mapsto x^2 \psi$ for $E_0 > 0$, one can obtain the same result for $n \geq (1.3 \cdot 10^{67}) \frac{8}{9\pi^2 E_0}$ even if the rescale does not affect the time controllability.

Regarding the application of the second point of Theorem 1 we refer the reader to the next section.

5.2 Computing the phase ambiguity and other techniques

Establishing the matrix M and then acting a phase-shift are in general difficult tasks. The more n grows, the more the size of the matrix M does, making more difficult the computation of e^{KM} .

First, we suggest to optimize of the results of Theorem 1 as explained in the beginning of the section. If this approach would not be enough one can adopt one of the following methods.

1) We can numerically construct a matrix \tilde{E} that approximates e^{KM} up to an error ϵ and such that $|\langle \phi_k, \tilde{E} \phi_j \rangle| = 1$. By using the theory from [9] as we have done for in (24), we can modify the relation (24) as follows

$$\begin{aligned} |\langle \phi_k, \tilde{E} \phi_j \rangle - \langle \phi_k, \Gamma_{T_n}^{\frac{u}{n}} \phi_j \rangle| &\leq K \|(1 - \pi_N) B(\phi_j \langle \phi_j, \cdot \rangle + \phi_k \langle \phi_k, \cdot \rangle)\| \\ &\quad + 4K(R_n + \epsilon) \|(1 - \pi_N) B \pi_N\| + R_n. \end{aligned}$$

If ϵ is small enough and n large enough, then one can obtain the same result of the second point of Theorem 1.

2) Another approach is to refer to the perturbation theory technique developed in [11]. By decomposing $A + u(t)B = (A + u_0 B) + u_1(t)B$, we can consider $u_0 B$ as a perturbative term of the operator A so that the perturbed eigenvalues $\{\lambda_j^{u_0}\}_{j \in \mathbb{N}}$ of the operator $A + u_0 B$ fulfill

$$|\lambda_k^{u_0} - \lambda_j^{u_0}| \neq K |\lambda_n^{u_0} - \lambda_m^{u_0}|, \quad \forall k, j, n, m \in \mathbb{N}, K \in \mathbb{N}.$$

Hence if we consider the dynamics between perturbed eigenstates $\{\phi_j^{u_0}\}_{j \in \mathbb{N}}$ of the operator $A + u_0 B$, one can choose a perturbation so that the matrix M has just two non-diagonal elements different from 0, so that M and e^{KM} are easier to compute. In this framework one can define only dynamics between two generic perturbed eigenfunctions $\phi_j^{u_0}, \phi_k^{u_0}$. However for u_0 small enough

$$\phi_j^{u_0} \in B_{H^3(0)} \left(\phi_j, \frac{C_j^2}{6j^3 \|B\|_3^2} \right), \quad \phi_k^{u_0} \in B_{H^3(0)} \left(\phi_k, \frac{C_k^2}{6k^3 \|B\|_3^2} \right)$$

and Proposition 4 implies that one can respectively move from ϕ_j to $\phi_j^{u_0}$ and from ϕ_k to $\phi_k^{u_0}$ with two suitable control functions.

Acknowledgments. The author thanks Thomas Chambrion for suggesting him the problem and for the explanation about the work [9]. He is also grateful to the colleagues Nabile Boussaid, Lorenzo Tentarelli and Riccardo Adami for the fruitful discussions.

References

- [1] C. Baiocchi, V. Komornik, P. Loreti, Ingham-Beurling type theorems with weakened gap conditions, *Acta Mathematica Hungarica*, **97**, (2002), 55-95.
- [2] J. M. Ball, J. E. Marsden, M. Slemrod, Controllability for distributed bilinear systems, *SIAM J. Control and Optimization*, **20**, (1993), 576-597.
- [3] K. Beauchard, C. Laurent, Local controllability of 1D linear and non linear Schrödinger equations with bilinear control, *J. Math. Pures Appl.*, **94**, 720-554, (2010).
- [4] K. Beauchard, Local controllability of a 1D Schrödinger equation, *J. Math. Pures et Appl.*, 84, 851-956, (2005).
- [5] N. Boussaid, M. Caponigro, T. Chambrion, Weakly-coupled systems in quantum control, *IEEE Trans. Automat. Control*, **58(9)**, 2205-2216, (2013).
- [6] N. Boussaid, M. Caponigro, T. Chambrion, Regular propagators of bilinear quantum systems. 2014. [jhal-01016299](https://arxiv.org/abs/1401.0162)
- [7] T. Chambrion, P. Mason, M. Sigalotti and U. Boscain, Controllability of the discrete-spectrum Schrödinger equation driven by an external field. *Annales de l'IHP, Non Linear Analysis*, **26(1)**, 329-349, (2009).

- [8] U. Boscain, M. Caponigro, T. Chambrion, M. Sigalotti, A weak spectral condition for the controllability of the bilinear Schrödinger equation with application to the control of a rotating planar molecule, *Communications on Mathematical Physics*, **311**, 423-455, (2012).
- [9] T. Chambrion, Periodic excitation of bilinear quantum systems, *Automatica*, Elsevier, **48(9)**, 2040-2046, (2012).
- [10] G. Christol, A. Cot, C.-M. Marle, *Calcul différentiel*, Paris, Ellipses, (1997).
- [11] A. Duca, Simultaneous global exact controllability in projection, submitted: <https://arxiv.org/abs/1703.00966>.
- [12] T. Kato, Integration of the equation of evolution in a Banach space, *J. M. S. J.*, **2(5)**, 208-234, (1953).
- [13] V. Kolornik, P. Loreti, *Fourier Series in Control Theory*, New York, Springer, (2005).
- [14] D. G Luenberger, *Optimization by Vector Space Methods*, New York, John Wiley, (1969).
- [15] M. Morancey, V. Nersesyan, Global exact controllability of 1D Schrödinger equations with a polarizability term, *C. R. Math. Acad. Sci. Paris*, **372(5)**, 425-429, (2014).
- [16] M. Morancey, V. Nersesyan, Simultaneous global exact controllability of an arbitrary number of 1D bilinear Schrödinger equations, *J. Math. Pures et Appl.*, **103(1)**:228-254, (2015).
- [17] M. Morancey, Simultaneous local exact controllability of 1D bilinear Schrödinger equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **31(3)**, (2014).
- [18] V. Nersesyan, Growth of Sobolev norms and controllability of Schrödinger equation, *Comm. Math. Phys.*, 290:371-387, (2009).
- [19] V. Nersesyan, Global approximate controllability for Schrödinger equation in higher Sobolev norms and applications, *Ann. I. H. Poincaré*, **27**, 901-915, (2010).
- [20] F. Rellich, *Perturbation Theory of Eigenvalue Problems*, New York, CRC Press, (1969).
- [21] G. Turinici, On the controllability of bilinear quantum systems, In M. Defranceschi and C. Le Bris, editors, *Mathematical models and methods for ab initio Quantum Chemistry*, volume 74 of Lecture Notes in Chemistry. Springer, (2000).