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Super-linear propagation for a general, local cane toads model

Christopher Henderson*, Benoît Perthame†, Panagiotis E. Souganidis‡

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Abstract

In this article, we investigate a general local version of the cane toads equation, which models the spread of a population structured by unbounded motility. We use the thin-front limit approach of Evans and Souganidis in [*Indiana Univ. Math. J.*, 1989] to obtain a characterization of the propagation in terms of both the linearized and a geometric front equations. Our result allows for large oscillations in the motility.

1 Introduction and Main Results

The cane toads equation models the spread of a population where the motility of the individuals is not constant. Its name comes from the cane toads in Australia whose invasion has been the subject of intense biological interest in recent years; see for example Phillips et. al. [23] and Shine et. al. [24]. This phenomenon has been observed more widely, for example, the expansion of bush crickets in Great Britain, see Thomas et. al. [25]. The mathematical model presented here has its roots in the work of Arnold, Desvillettes, and Prevost [1] and Champagnat and Méléard [11] and was first introduced by Benichou et. al. in [4].

The equation that we study is the following general local version of the cane toads equation

$$\begin{cases} u_t = D(\theta)u_{xx} + u_{\theta\theta} + u(1-u) & \text{in } \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \\ u_{\theta}(x, 0, t) = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}, \end{cases} \quad (1.1)$$

where $\mathbb{R}^+ := (0, \infty)$, with the diffusion coefficient $D : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying the assumption:

Assumption 1.1. *For all $\theta \in \mathbb{R}^+ := [0, \infty)$, $D(\theta) > 0$ and $D(\theta) \rightarrow \infty$ as $\theta \rightarrow \infty$. Let $\overline{D}^\epsilon(\theta) := D(\theta/\epsilon)/D(1/\epsilon)$. There exists $\overline{D} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that, locally uniformly in \mathbb{R}^+ ,*

$$\lim_{\epsilon \rightarrow 0} \overline{D}^\epsilon(\theta) = \overline{D}(\theta).$$

In addition, $\overline{D}(\theta) = 0$ if and only if $\theta = 0$ and $\overline{D}(\theta) \rightarrow \infty$ as $\theta \rightarrow \infty$.

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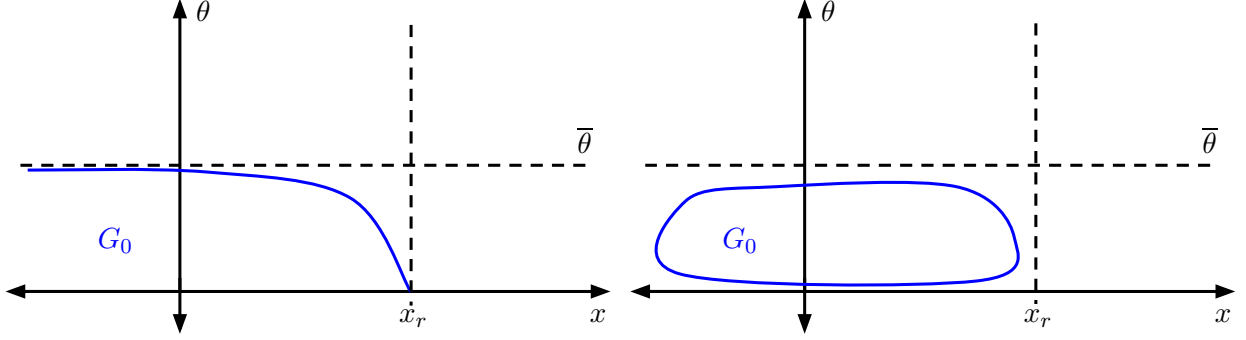


Figure 1: Two representative examples of G_0 .

The case most often considered is $D(\theta) = \theta$, whence $\overline{D}(\theta) = \theta$. A non-trivial example is

$$D(\theta) = \theta(1 + \log(\theta + 1) + \sin(\theta)/2), \quad (1.2)$$

which, although it has arbitrarily large oscillations, nevertheless satisfies $\overline{D}^\epsilon(\theta) \rightarrow \theta$.

We are interested in the long time, large space and motility limit. Hence, thinking of the time scale as ϵ^{-1} , we consider the scaled function

$$u^\epsilon(x, \theta, t) = u\left(\frac{x\sqrt{D(1/\epsilon)}}{\epsilon}, \frac{\theta}{\epsilon}, \frac{t}{\epsilon}\right),$$

which satisfies

$$\begin{cases} u_t^\epsilon = \epsilon \overline{D}^\epsilon(\theta) u_{xx}^\epsilon + \epsilon v_{\theta\theta}^\epsilon + \frac{1}{\epsilon} u^\epsilon (1 - u^\epsilon) & \text{in } \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \\ u_\theta^\epsilon(x, 0, t) = 0 & \text{on } \mathbb{R} \times \mathbb{R}^+, \end{cases} \quad (1.3)$$

which we supplement with the initial condition

$$u^\epsilon(x, \theta, 0) = u_0(x, \theta) \quad \text{with} \quad 0 \leq u_0 \leq 1, \quad (1.4)$$

where u_0 satisfies the following assumption:

Assumption 1.2. *The initial data u_0 is supported on G_0 , a smooth, non-empty, convex subset of $\mathbb{R} \times [0, \infty)$ such that $G_0 \cap (0, \infty) \times [0, \infty)$ is bounded, that is there exist $\bar{\theta} > 0$ and $x_r \in \mathbb{R}$ such that*

$$G_0 \subset (-\infty, x_r) \times [0, \bar{\theta}).$$

We note that the restriction $u_0 \leq 1$ is made for simplicity. Indeed the necessary modifications to handle the general case may be found in Lemma 1.2 and (2.5) of [14].

To study the behavior of u^ϵ as ϵ tends to zero, following Evans and Souganidis [14], we make the transformation $v^\epsilon = -\epsilon \log u^\epsilon$. Then v^ϵ satisfies

$$\begin{cases} v_t^\epsilon - \epsilon \overline{D}^\epsilon(\theta) v_{xx}^\epsilon - \epsilon v_{\theta\theta}^\epsilon + \overline{D}^\epsilon(\theta) |v_x^\epsilon|^2 + |v_\theta^\epsilon|^2 + 1 - e^{-v^\epsilon/\epsilon} = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \\ v_\theta^\epsilon(x, 0, t) = 0 & \text{on } \mathbb{R}^+ \times \mathbb{R}, \end{cases} \quad (1.5)$$

with initial conditions $v^\epsilon(x, \theta, 0) = v_0^\epsilon(x, \theta)$, where

$$v_0^\epsilon = \begin{cases} -\epsilon \log u_0 & \text{in } G_0, \\ \infty & \text{in } \overline{G_0^c}. \end{cases}$$

From the above, we see that, formally, (1.5) converges, when $v > 0$, to

$$v_t + \overline{D}|v_x|^2 + |v_\theta|^2 + 1 = 0.$$

Indeed, the following lemma shows this to be the case.

Proposition 1.3. *Assume Assumption 1.1 and Assumption 1.2. Then, as $\epsilon \rightarrow 0$ and locally uniformly in $\mathbb{R} \times [0, \infty) \times [0, \infty)$, the v^ϵ 's converge to I , which is the unique solution of*

$$\begin{cases} \min \{I_t + \overline{D}(\theta)|I_x|^2 + |I_\theta|^2 + 1, I\} = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \\ \max \{-I_\theta, \min \{I_t + |I_\theta|^2 + 1, I\}\} \geq 0 & \text{on } \mathbb{R} \times \{0\} \times \mathbb{R}^+, \\ \min \{-I_\theta, \min \{I_t + |I_\theta|^2 + 1, I\}\} \leq 0 & \text{on } \mathbb{R} \times \{0\} \times \mathbb{R}^+, \end{cases} \quad (1.6)$$

and

$$I(x, \theta, 0) = \begin{cases} \infty & \text{on } \overline{G_0^c}, \\ 0 & \text{on } G_0. \end{cases} \quad (1.7)$$

We point out that $\overline{D}(0)|I_x|^2$ does not appear in the boundary conditions because $\overline{D}(0)$ vanishes.

Recalling that $u^\epsilon = e^{-v^\epsilon/\epsilon}$, from Proposition 1.3, one might expect that the zero set of I is where u^ϵ converges to one and the set where I is positive is where u^ϵ converges to zero. This is verified by the following theorem.

Theorem 1.4. *Assume Assumption 1.1 and Assumption 1.2 and let I be the unique solution to (1.6) and (1.7). Then*

$$\lim_{\epsilon \rightarrow 0} u^\epsilon = \begin{cases} 0 & \text{uniformly on compact subsets of } \{I > 0\}, \\ 1 & \text{uniformly on compact subsets of } \text{Int}\{I = 0\}. \end{cases}$$

Theorem 1.4 may be proved in more generality. Following the arguments of [14, Section 4], it is clear that we may replace $u(1-u)$ with $f(u)$ for any $f: \mathbb{R} \rightarrow \mathbb{R}$ that is bounded uniformly in C^2 and satisfies $f(x) > 0$ if $x \in (0, 1)$, $f(x) < 0$ if $x < 0$ or $x > 1$, and

$$f'(0) = \sup_{u \in [0, 1]} \frac{f(u)}{u}.$$

is replaced by $f'(0)$. Further, one could add periodic dependence in θ . In other words, one could replace $u(1-u)$ with $f(\theta, u)$ where f is periodic in θ and satisfies conditions like those above. In this case, one can build a corrector in θ and proceed as in the work of Majda and Souganidis [21].

In order to characterize the sets $\{I > 0\}$ and $\text{Int}\{I = 0\}$ more explicitly, we consider the geometric front equation

$$\begin{cases} w_t + 2\sqrt{\overline{D}(\theta)|w_x|^2 + |w_\theta|^2} = 0 & \text{in } \mathbb{R} \times (0, \infty) \times \mathbb{R}^+, \\ \max\{-w_\theta, w_t + 2|w_\theta|\} \geq 0 & \text{on } \mathbb{R} \times \{0\} \times \mathbb{R}^+, \\ \min\{-w_\theta, w_t + 2|w_\theta|\} \leq 0 & \text{on } \mathbb{R} \times \{0\} \times \mathbb{R}^+, \end{cases} \quad (1.8)$$

with

$$w(x, \theta, 0) = \begin{cases} 1 & \text{on } \overline{G_0^c}, \\ 0 & \text{on } G_0. \end{cases} \quad (1.9)$$

It turns out (see Section 4.2) that the zero level sets of w and I are comparable. Indeed, we have:

Proposition 1.5. *Assume that Assumption 1.1 and Assumption 1.2 hold. Then, there is a unique solution to (1.8) and (1.9) and*

$$\lim_{\epsilon \rightarrow 0} u^\epsilon = \begin{cases} 0 & \text{uniformly on compact subsets of } \text{Int}\{w = 1\}, \\ 1 & \text{uniformly on compact subsets of } \text{Int}\{w = 0\}. \end{cases}$$

It also follows from our analysis that we may compare I with the solution to the Hamilton-Jacobi equation coming from the linearized cane toads equation, that is, the equation with $u(1-u)$ replaced by u .

Indeed, consider the action

$$J(x, \theta, t) := \min_{\gamma \in C^{0,1}([0,t];(0,\infty) \times \mathbb{R}^+)} \left\{ \int_0^t \left[\frac{\dot{\gamma}_1(s)^2}{4\overline{D}(\gamma_2(s))} + \frac{\dot{\gamma}_2(s)^2}{4} - 1 \right] ds : \right. \\ \left. \gamma(0) = (x, \theta), \gamma(t) \in \overline{G_0}, \gamma_2(s) > 0 \text{ for all } s \in (0, t) \right\}. \quad (1.10)$$

Proposition 1.6. *Assume Assumption 1.1 and Assumption 1.2. Then*

$$\lim_{\epsilon \rightarrow 0} u^\epsilon = \begin{cases} 0 & \text{uniformly on compact subsets of } \{J > 0\}, \\ 1 & \text{uniformly on compact subsets of } \{J < 0\}. \end{cases}$$

We point out that this shows that the solutions are pulled. In other words, the propagation speed depends only on the linearized equation at the highest order. We also remark that this would not be the case if the original problem lead to an inhomogeneous problem in the limit; see [21] for such a discussion.

Returning to the example (1.2), we see that, if we “undo” the scaling, our result yields that

$$\text{location of the front at time } t \sim O(t\sqrt{D(t)}) \sim O(t^{3/2}\sqrt{\log(t)}).$$

Further, since $\overline{D}(\theta) = \theta$, the constant above must be the same as the case when $D(\theta) = \theta$. Hence, we find that

$$\text{location of the front at time } t \sim \frac{4}{3}t\sqrt{D(t)} \sim \frac{4}{3}t^{3/2}\sqrt{\log(t)}.$$

Further, the front is located at $(4/3)t\sqrt{D(t)}$ for any D such that $\overline{D}(\theta) = \theta$. On the other hand, when this is not the case, it is difficult to compute the exact constant in front of $t\sqrt{D(t)}$ as the Euler-Lagrange equations for the minimizers of J may not be explicitly solvable.

The approach that we follow here is based on the work of Freidlin [15, 16], [14], Barles, Evans, and Souganidis [3], and [21]. In the cane toads equation introduced by Benichou et. al. [4] $u(1-u)$ is replaced by the non-local term $u(1 - \int u d\theta)$ and $D(\theta) = \theta$. In that setting and with the additional assumption that the trait θ takes values between two fixed positive constants $[\underline{\theta}, \overline{\theta}]$, Bouin and Calvez [6] proved the existence of traveling waves, Turanova [26] showed that the speed of the traveling wave governs the spread of the population in the Cauchy problem, and Bouin, Henderson,

and Ryzhik [9] established a Bramson-type logarithmic delay. When the trait space is unbounded, as in this work, Bouin et. [7] predicted that the location of the front is of order $(4/3)t^{3/2}$. This was then verified in the local model by Berestycki, Mouhot, and Raoul [5] and by Bouin, Henderson, and Ryzhik [10] using probabilistic and analytic techniques, respectively. It was also shown in [5] that in a windowed non-local model the propagation speed is the same, while [10] obtained weak bounds of order $t^{3/2}$ for the full non-local model. A model with a trade-off term or a penalization for large of traits has been proposed and studied by Bouin, Chan, Henderson, and Kim [8]. In the present article, we investigate only the local model as the non-local model has substantial technical obstructions. We also mention related works on finite domains by Perthame and Souganidis [22] and Lam and Lou [19].

Outline of the paper

We begin by proving Theorem 1.4 in Section 2 assuming Proposition 1.3. In Section 3, we prove Proposition 1.3 using the half-relaxed limits along with uniqueness of the limiting Hamilton-Jacobi equations. New ingredients in this step are the a priori estimates, which are more difficult to obtain since the Hamiltonian is degenerate at $\theta = 0$, and the boundary conditions, since boundaries did not appear in earlier thin-front limit works. In Section 4, we prove Proposition 1.5 and Proposition 1.6, that is, the propagation of u is characterized by the solution to the geometric front equation, w , and the solution to the linearized problem, J . Again, the boundary conditions provide the main difficulties in this section. Finally, in Appendix A we end with brief comments describing why we may import the representation formulas for w and J from the boundary-less setting.

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2 The proof of Theorem 1.4

The proof hinges on the locally uniform convergence of v^ϵ to I guaranteed by Proposition 1.3. We show how to conclude Theorem 1.4 assuming this proposition, which is proved in Section 3.

Proof of Theorem 1.4. We first consider the set $\{I > 0\}$. Fix any point (x_0, θ_0, t_0) such that $J(x_0, \theta_0, t_0) > 0$ with $t_0 > 0$. Since $v^\epsilon \rightarrow I$ locally uniformly by Proposition 1.3, $v^\epsilon(x, \theta, t) > \delta$ for some $\delta, r > 0$ and any $(x, \theta, t) \in B_r(x_0, \theta_0, t_0)$. It follows that $u^\epsilon(x, \theta, t) \leq \exp\{-\delta/\epsilon\}$ for all ϵ sufficiently small and all $(x, \theta, t) \in B_r(x_0, \theta_0, t_0)$. Hence $u^\epsilon \rightarrow 0$ locally uniformly as $\epsilon \rightarrow 0$.

Now we consider the set $\text{Int}\{I = 0\}$. For $(x_0, \theta_0, t_0) \in \text{Int}\{I = 0\}$, fix a test function

$$\psi(x, \theta, t) = |t - t_0|^2 + |x - x_0|^2 + |\theta - \theta_0|^2$$

and note that, since $I(x, \theta, t) = 0$ near (x_0, θ_0, t_0) , $I - \psi$ has a strict local maximum on a small enough ball centered at (x_0, θ_0, t_0) . It follows that $v^\epsilon - \psi$ has a maximum at some point $(x_\epsilon, \theta_\epsilon, t_\epsilon)$ such that $(x_\epsilon, \theta_\epsilon, t_\epsilon) \rightarrow (x_0, \theta_0, t_0)$.

There are two cases to investigate depending on whether θ_0 is zero or not. First assume that $\theta_0 > 0$ and restrict to ϵ sufficiently small so that $\theta_\epsilon > 0$ as well. Then, using (1.5), we find

$$\psi_t - \epsilon \overline{D}^\epsilon(\theta) \psi_{xx} - \epsilon \psi_{\theta\theta} + \overline{D}^\epsilon(\theta) |\psi_x|^2 + |\psi_\theta|^2 \leq u^\epsilon - 1.$$

An explicit computation shows that the left hand side tends to zero as $\epsilon \rightarrow 0$ and, hence, $1 \leq \liminf u^\epsilon(x_\epsilon, \theta_\epsilon, t_\epsilon)$. On the other hand, recall that $(t_\epsilon, x_\epsilon, \theta_\epsilon)$ is the location of a minimum of $u^\epsilon \exp\{\psi/\epsilon\}$. Hence we have that

$$\liminf_{\epsilon \rightarrow 0} u^\epsilon(x_0, \theta_0, t_0) \geq \liminf_{\epsilon \rightarrow 0} u^\epsilon(x_\epsilon, \theta_\epsilon, t_\epsilon) \exp\left\{\frac{|t_\epsilon - t_0|^2 + |x_\epsilon - x_0|^2 + |\theta_\epsilon - \theta_0|^2}{2}\right\} \geq 1.$$

Since $u^\epsilon \leq 1$ for all ϵ , then we may replace the \liminf and \geq above with \lim and $=$, respectively.

If $\theta_0 = 0$, define

$$\psi^\epsilon(x, \theta, t) := |t - t_0|^2 + |x - x_0|^2 + |\theta - \epsilon^2|^2,$$

and let $(x_\epsilon, \theta_\epsilon, t_\epsilon)$ be a maximum of $v^\epsilon - \psi^\epsilon$. Since $\psi^\epsilon \rightarrow \psi$, $v^\epsilon \rightarrow I$, and $I - \psi$ has a strict maximum at (x_0, θ_0, t_0) , it follows that $(x_\epsilon, \theta_\epsilon, t_\epsilon) \rightarrow (t_0, x_0, 0)$.

If $\theta_\epsilon = 0$ for ϵ sufficiently small, then, since $v^\epsilon - \psi$ has a maximum at $(x_\epsilon, \theta_\epsilon, t_\epsilon)$,

$$\phi_\theta^\epsilon(x_\epsilon, 0, t_\epsilon) \leq \psi_\theta^\epsilon(x_\epsilon, 0, t_\epsilon).$$

However, by (1.3), the left hand side is 0, while the right hand side is, by construction, $-2\epsilon^2$. It follows that, for ϵ sufficiently small, we have that $\theta_\epsilon > 0$. Then (1.5) yields that, at $(x_\epsilon, \theta_\epsilon, t_\epsilon)$,

$$o(1) = \psi_t^\epsilon - \epsilon \overline{D}^\epsilon(\theta) \psi_{xx}^\epsilon - \epsilon \psi_{\theta\theta}^\epsilon + \overline{D}^\epsilon(\theta) |\psi_x^\epsilon|^2 + |\psi_\theta^\epsilon|^2 \leq u^\epsilon - 1,$$

and, hence, $\liminf u^\epsilon(x_\epsilon, \theta_\epsilon, t_\epsilon) \geq 1$, as above.

The choice of $(x_\epsilon, \theta_\epsilon, t_\epsilon)$ implies that

$$\liminf u^\epsilon(x_0, 0, t_0) \geq \liminf u^\epsilon(x_\epsilon, \theta_\epsilon, t_\epsilon) \exp\left\{\frac{|t_\epsilon - t_0|^2 + |x_\epsilon - x_0|^2 + |\theta_\epsilon - \epsilon^2|^2}{\epsilon} - \epsilon\right\} \geq 1,$$

which, together with the fact that $u^\epsilon \leq 1$, proves the claim. \square

3 The limit of the v^ϵ – the proof of Proposition 1.3

We proceed in three steps. In the first, we obtain uniform bounds on v^ϵ on compact subsets of $\{t > 0\} \cup G_0$. In the second, we take the half-relaxed limits v^ϵ to obtain v_* and v^* , and we show that they are respectively super- and sub-solutions of (3.16). Finally in the last step, we use comparison to show that $v_* = v^* = I$ and conclude that v^ϵ converges locally uniformly to I .

3.1 An upper bound for v^ϵ

By the maximum principle, $0 \leq u^\epsilon \leq 1$ and so $v^\epsilon \geq 0$. In order to take the half-relaxed limits, we need a uniform upper bound on v^ϵ .

Lemma 3.1. *Assume Assumption 1.1 and Assumption 1.2 and fix any compact subset Q of $(G_0 \times \{t = 0\}) \cup (\mathbb{R} \times \mathbb{R}^+ \times (0, \infty))$. There exists $C = C(Q) > 0$ such that, if $(x, \theta, t) \in Q$, then*

$$v^\epsilon(x, \theta, t) \leq C(Q).$$

Proof. We begin by showing that, when $\epsilon > 0$, we may ignore the boundary $\{\theta = 0\}$. Using the Neumann boundary condition, we may extend u^ϵ and v^ϵ evenly to $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$. The parabolic regularity theory yields that v^ϵ satisfies (1.5) on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ with $\overline{D}^\epsilon(\theta)$ replaced by $\overline{D}^\epsilon(|\theta|)$; for more details see [26]. For the remainder of this proof, we abuse notation by letting u^ϵ and v^ϵ refer to their even extensions.

Fix $R > 0$. We first obtain a bound on the open set $E_R(x_0, \theta_0) \times [0, \infty)$, where

$$E_R(x_0, \theta_0) := \{(x, \theta) \in \mathbb{R} \times [0, \infty) : (x - x_0)^2 + (x - x_0)(\theta - \theta_0) + (\theta - \theta_0)^2 \leq R\},$$

assuming that $\overline{E}_R \subset G_0 = \{u_0 > 0\}$.

Without loss of generality, we assume that $x_0 = \theta_0 = 0$, and we consider, for $\alpha, \beta, \rho > 0$ to be determined below, the auxiliary function

$$\psi(x, \theta, t) := \beta + \alpha t + \frac{\rho}{R^2 - (x^2 + x\theta + \theta^2)} \quad \text{in } E_R.$$

We show that ψ is a super-solution of v^ϵ . We note that this super-solution varies from the one chosen in [14]. Since $\lim_{\epsilon \rightarrow 0} \overline{D}^\epsilon(0) = 0$, the barrier in [14] will not yield a super-solution.

Straightforward calculations yield

$$\begin{aligned} & \psi_t - \epsilon \overline{D}^\epsilon \psi_{xx} - \epsilon \psi_{\theta\theta} + \overline{D}^\epsilon |\psi_x|^2 + |\psi_\theta|^2 + 1 - e^{-\psi/\epsilon} \\ & \geq \alpha - \epsilon \rho \left(\frac{\overline{D}^\epsilon(\theta) |2x + \theta|^2 + |2\theta + x|^2}{(R^2 - (x^2 + x\theta + \theta^2))^3} + \frac{2(1 + \overline{D}^\epsilon(\theta))}{(R^2 - (x^2 + x\theta + \theta^2))^2} \right) \\ & \quad + \rho^2 \overline{D}^\epsilon(\theta) \frac{|2x + \theta|^2}{|R^2 - (x^2 + x\theta + \theta^2)|^4} + \rho^2 \frac{|2\theta + x|^2}{|R^2 - (x^2 + x\theta + \theta^2)|^4}. \end{aligned} \quad (3.1)$$

We now explain how to select the parameters so that the right hand side of (3.1) is nonnegative. Before we begin, we point out that $x^2 + x\theta + \theta^2 \leq R^2$ and Young's inequality imply that $|x|, |\theta| \leq \sqrt{2}R$.

There are three regimes to consider. If $R^2 - (x^2 + x\theta + \theta^2) > R^2/10$, we choose

$$\frac{\alpha}{\rho} \geq \epsilon \left(\frac{36R^2}{R^6/10^3} + \frac{2}{R^4/10^2} \right) \max_{|\theta| \leq \sqrt{2}R} (1 + \overline{D}^\epsilon(\theta)), \quad (3.2)$$

and the right hand side of (3.1) is nonnegative.

If $R^2 - (x^2 + x\theta + \theta^2) \leq R^2/10$ and $|\theta| < R/10$, it is easy to check that $|x| \geq 3R/4$. Hence, $|2\theta + x| \geq R/2$ and the last term on the right hand side of (3.1) dominates. Indeed,

$$\begin{aligned} & \psi_t - \epsilon \overline{D}^\epsilon(\theta) \psi_{xx} - \epsilon \psi_{\theta\theta} + \overline{D}^\epsilon(\theta) |\psi_x|^2 + |\psi_\theta|^2 + 1 - e^{-\psi/\epsilon} \\ & \geq \alpha - \epsilon \rho \left(\frac{(\max_{E_R} \overline{D}^\epsilon(\theta) + 1) 18R^2}{(R^2 - (x^2 + x\theta + \theta^2))^3} + \frac{2(\max_{E_R} \overline{D}^\epsilon(\theta) + 1)}{(R^2 - (x^2 + x\theta + \theta^2))^2} \right) + \rho^2 \frac{R^2}{4|R^2 - (x^2 + x\theta + \theta^2)|^4} \\ & \geq \frac{\rho}{|R^2 - (x^2 + x\theta + \theta^2)|^4} \left[\frac{\rho R^2}{4} - \epsilon 3R^4 (\max_{E_R} \overline{D}^\epsilon(\theta) + 1) \right], \end{aligned}$$

and the right hand side of the above inequality is nonnegative if

$$\rho \geq 6\epsilon R^2 \left(\max_{E_R} \overline{D}^\epsilon(\theta) + 1 \right), \quad (3.3)$$

which may be chosen uniformly in ϵ , since, as $\epsilon \rightarrow 0$, $\overline{D}^\epsilon(\theta) \rightarrow \overline{D}$ locally uniformly.

Finally, we need to consider what happens if $x^2 + x\theta + \theta^2 \leq R^2/10$ and $\theta \geq R/10$. In this case, there exists $C_R > 0$ that depends only on R , such that $\overline{D}^\epsilon(\theta) \geq C_R^{-1}$. It follows that, since $|2x + \theta|^2 + |2\theta + x|^2 \geq x^2 + \theta^2$, $\overline{D}^\epsilon(\theta)|2x + \theta|^2 + |2\theta + x|^2 \geq C_R^{-1}(x^2 + \theta^2) \geq C_R^{-1}R^2/100$. Hence, the last two terms on the right hand side of (3.1) dominate. Indeed,

$$\begin{aligned} & \psi_t - \epsilon \overline{D}^\epsilon(\theta) \psi_{xx} - \epsilon \psi_{\theta\theta} + \overline{D}^\epsilon(\theta) |\psi_x|^2 + |\psi_\theta|^2 + 1 - e^{-\psi/\epsilon} \\ & \geq \alpha - \epsilon \rho \left(\frac{(\max_{E_R} \overline{D}^\epsilon(\theta) + 1) 18R^2}{(R^2 - (x^2 + x\theta + \theta^2))^3} + \frac{2(\max_{E_R} \overline{D}^\epsilon(\theta) + 1)}{(R^2 - (x^2 + x\theta + \theta^2))^2} \right) + \rho^2 \frac{R^2}{100C_R |R^2 - (x^2 + x\theta + \theta^2)|^4} \\ & \geq \frac{\rho}{|R^2 - (x^2 + x\theta + \theta^2)|^4} \left[\frac{\rho R^2}{100C_R} - \epsilon 3R^4 (\max_{E_R} \overline{D}^\epsilon(\theta) + 1) \right]. \end{aligned}$$

The right hand side of the above inequality is nonnegative if

$$\rho \geq 300\epsilon R^2 C_R \left(\max_{E_R} \overline{D}^\epsilon(\theta) + 1 \right). \quad (3.4)$$

Choosing first ρ to satisfy (3.3)-(3.4) and then α to satisfy (3.2), we obtain

$$\psi_t - \epsilon \overline{D}^\epsilon(\theta) \psi_{xx} - \epsilon \psi_{\theta\theta} + \overline{D}^\epsilon(\theta) |\psi_x|^2 + |\psi_\theta|^2 + 1 - e^{-\psi/\epsilon} \geq 0.$$

Choose β so that $\psi(\cdot, \cdot, t=0) \geq \beta \geq v_0^\epsilon$ in E_R , and finally, notice that, since $\psi = +\infty$ on ∂E_R , $\psi \geq v^\epsilon$ on ∂E_R . The maximum principle implies that $0 \leq v^\epsilon \leq \psi$ on $E_r \times \mathbb{R}^+$. In particular, there exists some $C_R > 0$, which depend only on R and u_0 , such that, on $E_{R/2} \times [0, \infty)$ and for all ϵ sufficiently small,

$$0 \leq v^\epsilon \leq \psi \leq C_R(1+t). \quad (3.5)$$

We also note that, as $\epsilon \rightarrow 0$, we may choose ρ as small as we like, and, moreover, since $v_0^\epsilon = O(\epsilon)$, we may also select β as small as we need.

Now fix $T > 0$, choose $L \geq R$ and β sufficiently large so that $\zeta \geq v^\epsilon$ on $\partial D_{R/2} \times [1/T, T]$, which can be done in view of (3.5), and define, for α and ρ to be determined below,

$$\zeta(x, \theta, 1/T + t) := \alpha t + \beta + \frac{\rho}{t(L^2 - (x^2 + x\theta + \theta^2))} \quad \text{in } (E_L \cap E_{R/2}^c) \times [0, T].$$

Since $\zeta \rightarrow \infty$ on ∂D_L , then $\zeta \geq v^\epsilon$ on $\partial(E_L \cap E_{R/2}^c) \times [1/T, T]$. Moreover, $\zeta \rightarrow \infty$ when $t \rightarrow 0$, so that $\zeta(\cdot, \cdot, 1/T) \geq v^\epsilon(\cdot, \cdot, 1/T)$.

Thus, to show that $\zeta \geq v^\epsilon$ in $(E_L \cap E_{R/2}) \times [1/T, T]$, we need only show that ζ is a super-solution to (1.5). We again point out that our super-solution differs from the one in [14], in order to deal with the fact that $\overline{D}^\epsilon(\theta) \rightarrow 0$ and $\overline{D}^\epsilon(\theta) \rightarrow \infty$ as $\theta \rightarrow \infty$.

We claim that ζ is a super-solution of (1.5). To this end, we compute that

$$\begin{aligned} & \zeta_t - \epsilon \overline{D}^\epsilon \zeta_{xx} - \epsilon \zeta_{\theta\theta} + \overline{D}^\epsilon |\zeta_x|^2 + |\zeta_\theta|^2 + \zeta - e^{-\zeta/\epsilon} \\ & \geq \alpha - \frac{\rho}{t^2(L^2 - x^2 - x\theta - \theta^2)} - \epsilon \rho \left(\frac{\overline{D}^\epsilon(\theta) |2x + \theta|^2 + |2\theta + x|^2}{t(L^2 - (x^2 + x\theta + \theta^2))^3} + \frac{2(1 + \overline{D}^\epsilon(\theta))}{t(L^2 - (x^2 + x\theta + \theta^2))^2} \right) \\ & \quad + \frac{\rho^2 \overline{D}^\epsilon(\theta)}{t^2} \frac{|2x + \theta|^2}{|L^2 - x^2 - x\theta - \theta^2|^4} + \frac{\rho^2}{t^2} \frac{|2\theta + x|^2}{|L^2 - x^2 - x\theta - \theta^2|^4} \end{aligned}$$

When $x^2 + x\theta + \theta^2 \rightarrow L^2$, we argue as before, choosing ρ and α sufficiently large, depending on T and L , so that the above is non-negative. When $t \rightarrow 0$, we may, if necessary, enlarge ρ so that the last two terms dominate and the above is non-negative. Hence, in all cases, we conclude that

$$\zeta_t - \epsilon \overline{D}^\epsilon(\theta) \zeta_{xx} - \epsilon \zeta_{\theta\theta} + \overline{D}^\epsilon(\theta) |\zeta_x|^2 + |\zeta_\theta|^2 + \zeta - e^{-\zeta/\epsilon} \geq 0. \quad (3.6)$$

Since we have controlled ζ on the parabolic boundary of $D_L \cap D_{R/2}^c \times [1/T, T)$, we know that $v^\epsilon \leq \zeta$ on $D_L \cap D_{R/2}^c \times [1/T, T)$. Hence, for any $T_0 > 0$, this provides a bound on $D_{L/2} \cap D_{R/2}^c \times (T_0, T)$ depending only on T_0 , L , and R .

Combining (3.5) and (3.6) finishes the proof. \square

3.2 The half-relaxed limits

We recall next the definition of the classical half-relaxed limits v^* and v_* :

$$v^*(x, \theta, t) = \limsup_{\substack{(y, \eta, s) \rightarrow (x, \theta, t), \\ \epsilon \rightarrow 0}} v^\epsilon(y, \eta, s) \quad \text{and} \quad v_*(x, \theta, t) = \liminf_{\substack{(y, \eta, s) \rightarrow (x, \theta, t), \\ \epsilon \rightarrow 0}} v^\epsilon(y, \eta, s). \quad (3.7)$$

The existence of these limits is guaranteed by Lemma 3.1 along with the fact that $v^\epsilon \leq 0$. We point out that v_* is lower semi-continuous while v^* is upper semi-continuous.

Equations for v_* and v^*

Our first step is to prove that v_* and v^* satisfy the limits that the theory of viscosity solutions suggest. The issues here are the boundary behavior and the infinite initial data.

Lemma 3.2. *The relaxed lower limit v_* satisfies in the viscosity sense*

$$\begin{cases} \min \{ (v_*)_t + \overline{D}(\theta) |(v_*)_x|^2 + |(v_*)_\theta|^2 + 1, v_* \} \geq 0 & \text{in } \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \\ \max \{ -(v_*)_\theta, \min \{ (v_*)_t + |(v_*)_\theta|^2 + 1, v_* \} \} \geq 0 & \text{on } \mathbb{R} \times \{0\} \times \mathbb{R}^+, \end{cases} \quad (3.8)$$

and

$$v_*(\cdot, \cdot, 0) = \begin{cases} 0 & \text{in } G_0, \\ \infty & \text{in } \overline{G}_0^c. \end{cases} \quad (3.9)$$

Proof. Assume that, for some smooth function ψ , $v_* - \psi$ has a strict local minimum at $(x_0, \theta_0, t_0) \in \mathbb{R} \times [0, \infty) \times [0, \infty)$. We may then choose $\epsilon_k \rightarrow 0$ and $(y_k, \eta_k, s_k) \rightarrow (x_0, \theta_0, t_0)$ be such that (y_k, η_k, s_k) is a minimum of $\phi^{\epsilon_k} - \psi$ in $\mathbb{R} \times [0, \infty) \times [0, \infty)$ and

$$v_*(x_0, t_0, \theta_0) = \lim_{k \rightarrow \infty} \phi^{\epsilon_k}(y_k, \eta_k, s_k).$$

If $(x_0, \theta_0, t_0) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$, then, for sufficiently large k , $(y_k, \eta_k, s_k) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$. Since v^ϵ solves (1.5), at (y_k, η_k, s_k) , we have

$$0 \leq \psi_t - \epsilon_k \overline{D}^{\epsilon_k} \psi_{xx} - \epsilon_k \psi_{\theta\theta} + \overline{D}^{\epsilon_k} |\psi_x|^2 + |\psi_\theta|^2 + 1 - e^{-\phi^{\epsilon_k}/\epsilon_k} \leq \psi_t - \overline{D} |\psi_x|^2 + |\psi_\theta|^2 + 1 + o(1).$$

Taking the limit $k \rightarrow \infty$ yields the claim.

Assume next that $\theta_0 = 0$. If $\eta_k > 0$ infinitely often, the fact that ϕ^{ϵ_k} solves (1.5) yields, at (y_k, η_k, s_k) ,

$$0 \leq \psi_t - \epsilon_k \overline{D}^{\epsilon_k} \psi_{xx} - \epsilon_k \psi_{\theta\theta} + \overline{D}^{\epsilon_k} \psi_x^2 + \psi_\theta^2 + 1 - e^{-\phi^{\epsilon_k}/\epsilon_k} \leq \psi_t + \overline{D}^{\epsilon_k} \psi_x^2 + \psi_\theta^2 + 1 + o(1).$$

If $\eta_k = 0$ for all k sufficiently large, then, since ϕ^{ϵ_k} satisfies Neumann boundary conditions, we have

$$0 \leq -\psi_\theta(y_k, \eta_k, s_k).$$

In either case letting $k \rightarrow \infty$ yields the claim.

Finally we need to consider the case $t_0 = 0$. Fix $\mu > 0$ and smooth function $\zeta \in C^\infty(\mathbb{R} \times [0, \infty); [0, 1])$ such that $\zeta|_{\overline{G_0}} \equiv 0$ and $\zeta|_{\mathbb{R} \times (0, \infty) \setminus \overline{G_0}} > 0$. Then

$$\begin{cases} \max \{ (v_*)_t + \overline{D} |(v_*)_x|^2 + |(v_*)_\theta|^2 + 1, v_* - \mu\zeta \} \geq 0 & \text{in } \mathbb{R} \times [0, \infty) \times \{0\}, \\ \max \{ -(v_*)_\theta, (v_*)_t + |(v_*)_\theta|^2 + 1, v_* - \mu\zeta \} \geq 0 & \text{in } \times \mathbb{R} \times \{0\} \times \{0\}. \end{cases} \quad (3.10)$$

Indeed, if $(x_0, \theta_0) \in \overline{G_0}$, (3.10) holds since $v_* \geq 0$ and $\zeta \equiv 0$ on $\overline{G_0}$. If $(x_0, \theta_0) \in \mathbb{R} \times (0, \infty) \setminus \overline{G_0}$ and $v_*(0, x_0, \theta_0) < \mu\zeta(x_0, \zeta_0)$ then, since v_* is finite at (x_0, θ_0) , we argue as above to obtain that, in the viscosity sense,

$$(v_*)_t + \overline{D} |(v_*)_x|^2 + |(v_*)_\theta|^2 + 1 \geq 0$$

We proceed similarly if $\theta_0 = 0$. Hence, we conclude that (3.10) holds.

It is straightforward to check from the definition of \liminf that $v_* = 0$ on $\{0\} \times G_0$. If $(x_0, \theta_0) \in \mathbb{R} \times (0, \infty) \setminus \overline{G_0}$, then we assume, by contradiction, that $v_*(x_0, \theta_0, 0) < \infty$.

Choose μ sufficiently large so that $v_*(x_0, \theta_0, 0) < \mu\zeta(x_0, \theta_0, 0)$, fix $\lambda_\delta > 0$ to be selected below and define

$$\psi_\delta(x, \theta, t) := -\frac{|x - x_0|^2 + |\theta - \theta_0|^2}{\delta} - \lambda_\delta t.$$

Since v_* is lower semi-continuous, $v_* - \psi_\delta$ attains minimum at some $(x_\delta, \theta_\delta, t_\delta) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ and $(x_\delta, \theta_\delta, t_\delta) \rightarrow (x_0, \theta_0, 0)$. It follows that

$$v_*(x_\delta, \theta_\delta, t_\delta) + \lambda_\delta t_\delta + \frac{|x_\delta - x_0|^2 + |\theta_\delta - \theta_0|^2}{\delta} \leq v_*(x_0, \theta_0, 0)., \quad (3.11)$$

The above, along with the fact that $v_*(x_0, \theta_0, 0) < \mu\zeta(x_0, \theta_0, 0)$, implies that, for all δ small enough, $v_*(x_\delta, \theta_\delta, t_\delta) < \mu\zeta(x_\delta, \theta_\delta, t_\delta)$. In addition, since $(x_\delta, \theta_\delta, t_\delta) \rightarrow (x_0, \theta_0, 0)$ and $\theta_0 > 0$, then we may also assume that $\theta_\delta > 0$.

If $t_\delta > 0$, using (3.11), we have that $v_*(x_0, \theta_0, 0) > 0$ and, in view of lower semi-continuity of the v_* , for δ small enough, $v_*(x_\delta, \theta_\delta, t_\delta) > 0$. Hence, using that v_* satisfies (3.8) for $t_\delta > 0$, we have

$$0 \leq -\lambda_\delta + \frac{4(\overline{D}(\theta_\delta)|x_\delta - x_0|^2 + |\theta_\delta - \theta_0|^2)}{\delta^2} + 1. \quad (3.12)$$

Then (3.11) and the fact that $\theta_\delta \rightarrow \theta_0$ implies $\overline{D}(\theta_\delta) \leq 2D(\theta_0)$ for all δ sufficiently small, give

$$0 \leq -\lambda_\delta + \frac{4(\overline{D}(\theta_\delta)|x_\delta - x_0|^2 + |\theta_\delta - \theta_0|^2)}{\delta^2} + 1 \leq -\lambda_\delta + \frac{8\overline{D}(\theta_0)v_*(0, x_0, \theta_0)}{\delta} + 1. \quad (3.13)$$

Choosing $\lambda_\delta = 2 + 8\overline{D}(\theta_0)v_*(0, x_0, \theta_0)/\delta$ yields a contradiction. Hence either $t_\delta = 0$ or $\theta_\delta = 0$.

If $t_\delta = 0$, by assumption, we have that $v_* - \mu\zeta < 0$ at $(x_0, \theta_0, 0)$. Using (3.11), we have that the same is true at $(x_\delta, \theta_\delta)$. Hence we apply (3.10) to obtain the same contradiction as in (3.13).

Having reached a contradiction in all cases, we conclude that $v_*(0, \cdot, \cdot) \geq \mu\zeta$. However, $\zeta > 0$ on $\mathbb{R} \times (0, \infty) \setminus \overline{G_0}$ and μ is arbitrary. Letting $\mu \rightarrow \infty$ finishes the proof. \square

We now obtain the equation for v^* . The argument is slightly more complicated since $v^* \geq 0$ and, hence, for the first equation must consider the cases where v^* is zero or positive.

Lemma 3.3. *The upper relaxed half limit v^* is a viscosity solution to*

$$\begin{cases} \min \{ (v^*)_t + \overline{D}|(v^*)_x|^2 + |(v^*)_\theta|^2 + 1, v^* \} \leq 0 & \text{in } \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \\ \min \{ -(v^*)_\theta, \min \{ (v^*)_t + |(v^*)_\theta|^2 + 1, v^* \} \} \leq 0, & \text{on } \mathbb{R} \times \{0\} \times \mathbb{R}^+, \end{cases} \quad (3.14)$$

and

$$v^*(\cdot, \cdot, 0) = \begin{cases} 0 & \text{in } G_0, \\ \infty & \text{in } \overline{G}_0^c. \end{cases} \quad (3.15)$$

Proof. The proof of Lemma 3.3 is similar to that of Lemma 3.2, thus we omit some details and provide only a sketch of the proof.

Assume that, for some smooth function ψ , $v^* - \psi$ has a strict local maximum at $(x_0, \theta_0, t_0) \in \mathbb{R} \times [0, \infty) \times [0, \infty)$. We may then choose $\epsilon_k \rightarrow 0$ and $(y_k, \eta_k, s_k) \rightarrow (x_0, \theta_0, t_0)$ be such that (y_k, η_k, s_k) is a maximum of $\phi^{\epsilon_k} - \psi$ in $\mathbb{R} \times [0, \infty) \times [0, \infty)$ and

$$v^*(x_0, t_0, \theta_0) = \lim_{k \rightarrow \infty} \phi^{\epsilon_k}(y_k, \eta_k, s_k).$$

To check (3.14), we need only consider the set $\{v^* > 0\}$ since (3.14) is satisfied whenever $v^* = 0$. If $t_0 > 0$ and $\theta_0 > 0$, then for sufficiently large k , $t_k, \theta_k > 0$ and, at (y_k, η_k, s_k) ,

$$0 \geq \psi_t - \epsilon_k \overline{D}^{\epsilon_k} \psi_{xx} - \epsilon_k \psi_{\theta\theta} + \overline{D}^{\epsilon_k} |\psi_x|^2 + |\psi_\theta|^2 + 1 - e^{-\phi^{\epsilon_k}/\epsilon_k}.$$

Since $\phi^{\epsilon_k}(y_k, \eta_k, s_k) \rightarrow \phi^*(x_0, \theta_0, t_0) > 0$ as $k \rightarrow \infty$, the last term tends uniformly to zero. In addition, the regularity of ψ implies that we make take the limit as $k \rightarrow \infty$ to obtain, as desired, at (x_0, θ_0, t_0) ,

$$0 \geq \psi_t + \overline{D} |\psi_x|^2 + |\psi_\theta|^2 + 1.$$

If $\theta_0 = 0$ we argue similarly to the arguments in Lemma 3.2.

We now consider the case $t_0 = 0$. Fix any point (x_0, θ_0) and observe that there is $R > 0$ such that $(x_0, \theta_0) \in E_R$ and $\overline{E}_R \subset G_0$. Using the comments appearing after (3.5), for any $\epsilon_0 > 0$, we find $C > 0$, depending only on (x_0, θ_0) , R , D , and \overline{D} , and α_{ϵ_0} such that, for all $\epsilon < \epsilon_0$ and $(x, \theta) \in E_R$,

$$0 \leq v^\epsilon(x, \theta, t) \leq C\epsilon_0 + \alpha_{\epsilon_0} t + \frac{C\epsilon_0}{R^2 - (x - x_0)^2 + (x - x_0)(\theta - \theta_0) + (\theta - \theta_0)^2}.$$

Taking the lim sup of this inequality as $(x, \theta, t) \rightarrow (x_0, \theta_0, 0)$ yields

$$0 \leq v^*(x_0, \theta_0) \leq C\epsilon_0.$$

Since ϵ_0 is arbitrary, then we find $v^*(x_0, \theta_0) = 0$.

On the other hand, fix any point $(x_0, \theta_0) \in \overline{G}_0^c$ and it follows from the definition of lim sup that $v^*(x_0, \theta_0, 0) = \infty$. This concludes the proof. \square

3.3 The equality of v_* and v^*

As noted above, by construction, $v_* \leq v^*$. In addition, v_* and v^* are a super- and a sub-solution to the same equation with the same initial conditions except on the small set ∂G_0 . In this section, we show that $v_* = v^*$.

Existence and uniqueness of I

We outline the argument developed in Crandall, Lions and Souganidis [13] that yields that there exists a unique solution to

$$\begin{cases} \min \{I_t + \overline{D}|I_x|^2 + |I_\theta|^2 + 1, I\} = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \\ \max \{-I_\theta, \min \{I_t + |I_\theta|^2 + 1, I\}\} \geq 0 & \text{in } \mathbb{R} \times \{0\} \times \mathbb{R}^+, \\ \min \{-I_\theta, \min \{I_t + |I_\theta|^2 + 1, I\}\} \leq 0 & \text{in } \mathbb{R} \times \{0\} \times \mathbb{R}^+, \end{cases} \quad (3.16)$$

with the initial condition

$$I(\cdot, \cdot, 0) = \begin{cases} \infty & \text{on } G_0, \\ 0 & \text{on } \overline{G_0}^c. \end{cases}$$

Fix $\mathcal{C} := \{\zeta \in C^0(\mathbb{R} \times [0, \infty)) : \zeta|_{G_0^c} \equiv 0\}$ and denote by $S(t)\zeta$ the solution to (3.16) with the initial data ζ . The existence and uniqueness of $S(t)\zeta$ are well-understood; see, for example, Crandall, Ishii, and Lions's "User's Guide" [12]. In addition, arguments as in Section 3.1 give uniform bounds on $S(t)\zeta$ for $t > 0$.

Let

$$I(x, \theta, t) := \sup_{\zeta \in \mathcal{C}^0} S(t)\zeta.$$

Following [13], we observe that I is the unique maximal solution of (3.16). We note that, due to the Neumann boundary conditions, this does not follow directly from [13]. The extension is, however, straightforward and we leave the details to the reader.

The equality of v_* and v^*

First, we show that $v_* \geq I$. To this end, fix any $\zeta \in C^0$ and let I_ζ be the solution of (3.16) with initial data ζ . By construction, $I \geq I_\zeta$. In addition, $v_* \geq I_\zeta$ on $\mathbb{R} \times (0, \infty) \times \{0\}$. The standard comparison argument, along with Lemma 3.2, gives that $v_* \geq I_\zeta$ on $[0, \infty) \times \mathbb{R} \times (0, \infty)$ since, by construction, v_* is lower-semicontinuous. Since this is true for all ζ , we find

$$I = \sup_{\zeta} I_\zeta \leq v_*,$$

and, hence, $I \leq v^*$.

Next, we show that $v^* \leq I$. Fix $\sigma, \delta > 0$ and define

$$G_\delta := \{(x, \theta) \in G_0 : \text{dist}((x, \theta), G_0^c) > \delta\} \quad \text{and} \quad \Lambda_{\sigma, \delta} := \sup_{(x, \theta) \in G_\delta} v^*(x, \theta, \sigma).$$

The arguments in Section 3.1 yield that $\Lambda_{\sigma, \delta} < \infty$. In addition, a close look at the estimates there (more specifically, the paragraph below (3.5)) yields that, for fixed δ , $\Lambda_{\sigma, \delta} \rightarrow 0$ as $\sigma \rightarrow 0$.

Let $I_{\sigma, \delta}$ to be the solution of (3.16) with initial data at $t = \sigma$ which is $\Lambda_{\sigma, \delta}$ on G_0 and is ∞ on $\overline{G_\delta}^c$, which is defined using the theory of maximal functions in [13]. Then Lemma 3.3 and the comparison principle for time $t \geq \sigma$ imply that, on $\mathbb{R} \times [0, \infty) \times [0, \infty)$,

$$v^* \leq I_{\sigma, \delta}$$

Letting first $\sigma \rightarrow 0$ and then δ to zero and using that $I_{\sigma, \delta} \rightarrow I$, we find

$$v^* \leq I.$$

Hence we have that $v_* \leq v^* \leq I \leq v_* \leq v^*$, which implies that all three functions must be equal. In particular, we have that v^ϵ converges locally uniformly to I .

4 The relationship between I , J and w

We now characterize the location of the front in a more tractable manner. We do not follow the approach of [15, 16], which relied on a condition on the minimizing paths of J . Instead, we opt for a PDE proof based on [21].

Recalling (1.10) and (1.8) from Section 1, we now show that the movement of $\partial\{I > 0\}$ can be understood in terms of $\{J = 0\}$ and $\partial\{w > 0\}$. To this end, we first show that J is a sub-solution to I . Then we discuss the existence and uniqueness of w . Finally, we use these to prove Proposition 1.5 and Proposition 1.6.

4.1 The equation for J

We first show that J solves

$$\begin{cases} J_t + \overline{D}(\theta)|J_x|^2 + |J_\theta|^2 + 1 = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \\ \min\{-J_\theta, J_t + |J_\theta|^2 + 1\} \leq 0 & \text{on } \mathbb{R} \times \{0\} \times \mathbb{R}^+, \end{cases} \quad (4.1)$$

from which it follows that

$$\begin{cases} \min\{J_t + \overline{D}(\theta)|J_x|^2 + |J_\theta|^2 + 1, J\} \leq 0 & \text{in } \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \\ \min\{-J_\theta, \min\{J_t + |J_\theta|^2 + 1, J\}\} \leq 0 & \text{on } \mathbb{R} \times \{0\} \times \mathbb{R}^+. \end{cases}$$

Recalling the initial conditions for I , we then find that $J \leq I$. We note that J actually satisfies the Neumann boundary condition in θ , but this is not necessary for our purposes so we do not show it.

Proof of (4.1). In Appendix A, we show that the classical arguments may be easily adapted to give that, on $\mathbb{R} \times (0, \infty) \times (0, \infty)$, J solves

$$J_t + \overline{D}|J_x|^2 + |J_\theta|^2 + 1 = 0. \quad (4.2)$$

The main point is that optimal and almost-optimal trajectories in the definition of J remain bounded away from the boundary $\theta = 0$. As such, one may show that the dynamic programming principle is verified and argue as usual.

We now show that $\min\{-J_\theta, J_t + |J_\theta|^2 + 1\} \leq 0$ on $\mathbb{R} \times \{0\} \times (0, \infty)$. Assume that for a test function ϕ , $(x, 0, t)$ with $t > 0$ is a strict maximum $J - \phi$ in a ball $\overline{B}_r(x_0, 0, t_0)$, and, without loss of generality, assume that $(J - \phi)(x, 0, t) = 0$.

Since $J - \phi$ has a strict maximum of zero at $(x_0, 0, t_0)$, we consider, for any $\lambda \in \mathbb{R}$,

$$\phi_\lambda(x, \theta, t) := \phi(x, \theta, t) + \lambda\theta^2(r^2 - |x - x_0|^2 - |\theta|^2 - |t - t_0|^2).$$

If $\lambda \geq 0$, then 0 is a strict maximum of $J - \phi_\lambda$ on $\overline{B}_r(x_0, 0, t_0)$. Define

$$\lambda_0 = \inf\{\lambda \in \mathbb{R} : J(x, \theta, t) - \phi_\lambda(x, \theta, t) \geq 0 \text{ for all } (x, \theta, t) \in B_r(x_0, 0, t_0)\}.$$

The observation above shows that $\lambda_0 \leq 0$.

Fix $\epsilon > 0$ and let $(x_\epsilon, \theta_\epsilon, t_\epsilon)$ be a positive maximum of $J - \phi_{\lambda_0 - \epsilon}$ on $\overline{B}_r(x_0, 0, t_0)$. By the construction of ϕ_λ , $(x_\epsilon, \theta_\epsilon, t_\epsilon)$ is in the interior of $\overline{B}_r(x_0, 0, t_0)$.

It is now a standard argument in the theory of viscosity solutions that, as $\epsilon \rightarrow 0$, $(x_\epsilon, \theta_\epsilon, t_\epsilon) \rightarrow (x_0, 0, t_0)$.

Since $(x_\epsilon, \theta_\epsilon, t_\epsilon)$ is a local maximum with $\theta_\epsilon > 0$ and $t_\epsilon > 0$ and we have already shown that J solves (4.1) in $\mathbb{R} \times (0, \infty) \times (0, \infty)$, at $(x_\epsilon, \theta_\epsilon, t_\epsilon)$, we have

$$0 \geq \phi_t - 2(\lambda_0 - \epsilon)\theta_\epsilon^2(t_\epsilon - t_0) + D(\theta_\epsilon)|\phi_x - 2(\lambda_0 - \epsilon)\theta_\epsilon^2(x_\epsilon - x_0)|^2 \\ + |\phi_\theta + 2\theta_\epsilon(\lambda - \epsilon)(r^2 - |x_\epsilon - x_0|^2 - |\theta_\epsilon|^2 - |t_\epsilon - t_0|^2) - 2\theta_\epsilon^3(\lambda_0 - \epsilon)|^2 + 1.$$

Letting $\epsilon \rightarrow 0$ we find that, at $(x_0, 0, t_0)$, $0 \geq \phi_t + |\phi_\theta|^2 + 1$, and the proof is now complete. \square

4.2 A representation formula for w

Recall that w satisfies (1.8) and (1.9). Following work of Lions [20], we define, for any $p \in \mathbb{R}^2$,

$$N(x, \theta, p) := \frac{1}{2} \sqrt{\frac{p_x^2}{D(\theta)} + p_\theta^2},$$

and, for any $(x, \theta), (y, \eta) \in \mathbb{R} \times [0, \infty)$,

$$d((x, \theta), (y, \eta)) := \\ \inf \left\{ \int_0^t N(\gamma(s), \dot{\gamma}(s)) ds : \gamma \in W^{1,\infty}, \gamma(0) = (x, \theta), \gamma(t) = (y, \eta), \gamma_2(s) > 0 \text{ for all } s \in (0, t) \right\}.$$

Without the boundary, it follows from [20, Section 3.4] that

$$w(x, \theta, t) = \inf \{ w(y, \eta, 0) : d((x, \theta), (y, \eta)) \leq t \}.$$

We check that the same formula holds in our setting.

Lemma 4.1. *The solution w to (1.8) and (1.9) is given by*

$$w(x, \theta, t) = \inf \{ w(y, \eta, 0) : d((x, \theta), (y, \eta)) \leq t \}.$$

Proof. Let

$$\tilde{w}(x, \theta, t) := \inf \{ w(y, \eta, 0) : d((x, \theta), (y, \eta)) \leq t \}.$$

First we show that \tilde{w} satisfies the equation (1.8). We may then conclude that $\tilde{w} = w$ by uniqueness; recall that, by definition, $\tilde{w}(\cdot, \cdot, 0) = w(\cdot, \cdot, 0)$ in $\mathbb{R} \times (0, \infty)$. Uniqueness follows since (1.8) and (1.9) has a maximum principle, which can be found, for example in Barles [2], Giga and Sato [17] and Ishii and Sato [18].

As above, the classical arguments may be easily adapted to show that \tilde{w} satisfies (1.8) in $\mathbb{R} \times (0, \infty)$. Thus, we need only show that (1.8) is satisfied on $\mathbb{R} \times \{0\} \times (0, \infty)$. Arguments as in Section 4.1 show that $\min\{-\tilde{w}_\theta, \tilde{w}_t + 2|\tilde{w}_\theta|\} \leq 0$.

We now consider the case where, for a test function ψ , $\tilde{w} - \psi$ has a minimum at $(x_0, 0, t_0)$ and, in addition, $\tilde{w} - \psi(x_0, 0, t_0) = 0$ and assume that

$$-\psi_\theta(x_0, 0, t_0) < 0.$$

As discussed in the Appendix, d satisfies the dynamic programming principle. If γ be the optimal path from $(x_0, 0)$ to \overline{G}_0 , for any $\delta > 0$,

$$d((x_0, 0), \overline{G}_0) = \int_0^\delta N(\gamma, \dot{\gamma}) ds + d(\gamma(\delta), G_0).$$

It follows that $d(\gamma(\delta), \overline{G}_0) \leq d((x_0, 0), \overline{G}_0) \leq t_0$, which, in particular, implies that $\tilde{w}(\gamma(\delta), t_0) \leq \tilde{w}(x_0, 0, t_0)$. Since the minimum value of $\tilde{w} - \psi$ is zero, then, for δ sufficiently small, $\tilde{w}(\gamma(\delta), t_0) \leq \psi(\gamma(\delta), t_0)$ and, hence,

$$\begin{aligned} \tilde{w}(x_0, 0, t_0) + \gamma_2(\delta)\psi_\theta(x_0, 0, t_0) + O(\gamma(\delta)^2) &= \psi(x_0, 0, t_0) + \gamma_2(\delta)\psi_\theta(x_0, 0, t_0) + O(\gamma(\delta)^2) \\ &= \psi(\gamma(\delta), t_0) \leq \tilde{w}(\gamma(\delta), t_0) \leq \tilde{w}(x_0, 0, t_0). \end{aligned}$$

Using that $\psi_\theta(x_0, 0, t_0) > 0$ and $\gamma_2(\delta) > 0$, we obtain a contradiction. \square

4.3 The proofs of Proposition 1.5 and Proposition 1.6

First, we claim that $\{I > 0\} \subset \{w > 0\}$. To begin, we note that w is a super-solution to I because

$$2\sqrt{\overline{D}(\theta)p_x^2 + p_\theta^2} \leq \overline{D}(\theta)p_x^2 + p_\theta^2 + 1.$$

Following [21], we let $\overline{I} := \tanh(I)$ and observe that \overline{I} and w satisfy the same initial data. The maximum principle implies that $\overline{I} \leq w$, which, in turn, gives $\{\overline{I} > 0\} \subset \{w > 0\}$. Since \tanh is increasing, we have that $\{\overline{I} > 0\} = \{I > 0\}$, and thus $\{I > 0\} \subset \{w > 0\}$. On the other hand, we note that J is a sub-solution to I . This implies that $\{J > 0\} \subset \{I > 0\} \subset \{w > 0\}$.

Now we show that $\{I = 0\} \subset \{J \leq 0\} \subset \{w = 0\}$. We remark that it is known that this is not true, in general, for all propagation problems [21].

That J is a sub-solution to I yields $J \leq I$. Next, fix $(x, \theta, t) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$ such that $I(x, \theta, t) = 0$. It follows that $J(x, \theta, t) \leq 0$. Then any minimizer in the formula for J starting at (x, θ) must end at some $(y, \eta) \in \overline{G}_0$. The formulae for J and d along with the Cauchy-Schwarz inequality give

$$\sqrt{t} \geq \sqrt{J(x, \theta, t) + t} = \left(\int_0^t \left[\frac{\dot{\gamma}_1^2}{4\overline{D}(\gamma_2)} + \frac{\dot{\gamma}_2^2}{4} \right] ds \right)^{1/2} \geq \left(\int_0^t N(\gamma, \dot{\gamma}) ds \right) / \left(\int_0^t ds \right)^{1/2},$$

which implies that $d((x, \theta), \overline{G}_0) \leq t$, and, hence, that $w(x, \theta, t) = 0$, finishing the claim.

Having shown that $\{I > 0\} = \text{Int}\{w = 1\} = \{J > 0\}$ and that $\text{Int}\{I = 0\} = \text{Int}\{w = 0\} = \{J < 0\}$ finishes the proofs of Proposition 1.5 and Proposition 1.6.

A Appendix: Brief comments about J and w as a solutions of (4.2), (1.8)

Due to the degeneracy of (4.2) at $\theta = 0$ and the loss of coercivity of the quadratic form in the equation as $\theta \rightarrow \infty$, (4.2) falls outside the classical theory of Hamilton-Jacobi equations. In view of this, we include here some remarks which are meant to convince the reader that J and w have the usual properties, that is they satisfy the dynamic programming principle, solve respectively (4.2) and (1.8), and their extremal paths are given by the Euler-Lagrange equations. Since the arguments are similar, in the remainder of the Appendix we only discuss J .

The main observation is that extremal paths are bounded away from ∞ and 0. The first claim is the following.

Lemma A.1. *Assume Assumption 1.1 and Assumption 1.2, and fix $(x, \theta, t) \in \mathbb{R} \times [0, \infty) \times (0, \infty)$. Let $\gamma \in (W^{1,\infty})^2$ be a trajectory such that*

$$\int_0^t \left[\frac{\dot{\gamma}_1(s)^2}{4\overline{D}(\gamma_2(s))} + \frac{\dot{\gamma}_2(s)^2}{4} - 1 \right] ds \leq J(x, \theta, t) + 1.$$

There exists $C_{x,\theta,t}$, depending only on (x, θ, t) and \overline{D} , such that, for all $s \in (0, t)$, $\gamma_2(s) \leq C_{x,\theta,t}$.

Proof. Firstly, we point out that by choosing any trajectory connecting (x, θ, t) and a point in \overline{G}_0 , like, for example, a linear one, yields $|J(x, \theta, t)| \leq C_{x,\theta,t}$. Secondly, we integrate $\dot{\gamma}_2$ from 0 to s , for any $s \in (0, t)$, we obtain

$$\gamma_2(s) - \theta = \int_0^s \dot{\gamma}_2(r) dr \leq 2\sqrt{r} \sqrt{\int_0^s \frac{\dot{\gamma}_2(r)^2}{4} dr} \leq 2\sqrt{t} \sqrt{J(x, \theta, t) + t} \leq C_{x,\theta,t}.$$

□

It follows that any approximately extremal trajectory γ_2 is bounded. As a result, $\overline{D}(\gamma_2)$ is bounded as well from above and the quadratic form in the integrand of J is uniformly coercive. Hence any approximately extremal trajectory will be bounded in H^1 . Using compactness we obtain a single extremal trajectory.

The second claim:

Lemma A.2. *Assume that Assumption 1.1 and Assumption 1.2, fix $(x, \theta, t) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$ and let $\gamma \in W^{1,\infty}$ be a trajectory such that*

$$J(x, \theta, t) = \int_0^t \left[\frac{\dot{\gamma}_1(s)^2}{4\overline{D}(\gamma_2(s))} + \frac{\dot{\gamma}_2(s)^2}{4} - 1 \right] ds.$$

Fix θ_0 such that $\overline{D}(\theta_0) = \min\{\overline{D}(\theta), \overline{D}(\underline{\theta})\}$ and $\theta < \theta_0$ implies that $\overline{D}(\theta) < \overline{D}(\theta_0)$. Then, for all $s \in [0, t]$, $\gamma(s) \geq \theta_0$.

Proof. We argue by contradiction. If there exists s such that $\gamma(s) < \theta_0$, then, by continuity, we may find $s_0 < s_1$ such that $\gamma(s_0) = \theta_0$, $\gamma(s) < \theta_0$ for all $s \in (s_0, s_1)$, and either $\gamma(s_1) = \theta_0$ or $s_1 = t$.

Define a new trajectory, α , by

$$\alpha(s) = \begin{cases} \gamma(s) & \text{if } s \in [0, s_0] \cup [s_1, t], \\ (\gamma_1(s), \theta_0) & \text{if } s \in [s_0, s_1], \end{cases}$$

Since $\alpha(t) \in \overline{G}_0$, α is an admissible trajectory in the definition of J . Using this and the fact that $\overline{D}(\gamma_2(s)) < \overline{D}(\theta_0)$ for $s \in (s_0, s_1)$, we obtain

$$\begin{aligned} J(x, \theta, t) &\leq \int_0^t \left[\frac{\dot{\alpha}_1(s)^2}{4\overline{D}(\alpha_2(s))} + \frac{\dot{\alpha}_2(s)^2}{4} - 1 \right] ds \\ &= \int_{[0, s_0] \cup [s_1, t]} \left[\frac{\dot{\gamma}_1(s)^2}{4\overline{D}(\gamma_2(s))} + \frac{\dot{\gamma}_2(s)^2}{4} - 1 \right] ds + \int_{s_0}^{s_1} \left[\frac{\dot{\gamma}_1(s)^2}{4\overline{D}(\theta_0)} - 1 \right] ds \\ &< \int_{[0, s_0] \cup [s_1, t]} \left[\frac{\dot{\gamma}_1(s)^2}{4\overline{D}(\gamma_2(s))} + \frac{\dot{\gamma}_2(s)^2}{4} - 1 \right] ds + \int_{s_0}^{s_1} \left[\frac{\dot{\gamma}_1(s)^2}{4\overline{D}(\gamma_2(s))} + \frac{\dot{\gamma}_2(s)^2}{4} - 1 \right] ds \\ &= \int_0^t \left[\frac{\dot{\gamma}_1(s)^2}{4\overline{D}(\gamma_2(s))} + \frac{\dot{\gamma}_2(s)^2}{4} - 1 \right] ds = J(x, \theta, t), \end{aligned}$$

which is a contradiction. □

Since extremal trajectories remain bounded away from zero, they do not “see” the boundary. Hence the standard theory of Hamilton-Jacobi equation applies showing that J solves (4.2) and has all the expected properties.

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