On Courant's nodal domain property for linear combinations of eigenfunctions, Part I

Pierre Bérard, Bernard Helffer

ON COURANT'S NODAL DOMAIN PROPERTY FOR LINEAR COMBINATIONS OF EIGENFUNCTIONS, PART I

PIERRE BÉRARD AND BERNARD HELFFER

Abstract. According to Courant's theorem, an eigenfunction associated with the n-th eigenvalue λ_n has at most n nodal domains. A footnote in the book of Courant and Hilbert, states that the same assertion is true for any linear combination of eigenfunctions associated with eigenvalues less than or equal to λ_n. We call this assertion the *Extended Courant Property*.

In this paper, we propose new, simple and explicit examples for which the extended Courant property is false: convex domains in $\mathbb{R}^n$ (hypercube and equilateral triangle), domains with cracks in $\mathbb{R}^2$, on the round sphere $\mathbb{S}^2$, and on a flat torus $\mathbb{T}^2$. We also give numerical evidence that the extended Courant property is false for the equilateral triangle with rounded corners, and for the regular hexagon.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain or, more generally, a compact Riemannian manifold with boundary.

Consider the eigenvalue problem

$$\begin{cases} -\Delta u &= \lambda\, u \quad \text{in } \Omega\,, \\ \mathfrak{b}(u) &= 0 \qquad \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $\mathfrak{b}(u)$ is some homogeneous boundary condition on $\partial\Omega$, so that we have a self-adjoint boundary value problem (including the empty condition if Ω is a closed manifold). For example, we can choose $\mathfrak{d}(u) = u|_{\partial\Omega}$ for the Dirichlet boundary condition, or $\mathfrak{n}(u) = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ for the Neumann boundary condition.

Call $H(\Omega, \mathfrak{b})$ the associated self-adjoint extension of $-\Delta$, and list its eigenvalues in nondecreasing order, counting multiplicities, and starting with the index 1, as

$$0 \le \lambda_1(\Omega, \mathfrak{b}) < \lambda_2(\Omega, \mathfrak{b}) \le \lambda_3(\Omega, \mathfrak{b}) \le \cdots\,, \tag{1.2}$$

with an associated orthonormal basis $\{u_j, j \ge 1\}$.

Date: May 18, 2018 (berard-helffer-ext-courant-P1-1805017.tex).

2010 *Mathematics Subject Classification.* 35P99, 35Q99, 58J50.

Key words and phrases. Eigenfunction, Nodal domain, Courant nodal domain theorem.

For any eigenvalue λ of $(\Omega, \mathfrak{b})$, define the index

$$\kappa(\Omega, \mathfrak{b}, \lambda) = \min\{k \mid \lambda_k(\Omega, \mathfrak{b}) = \lambda\}. \tag{1.3}$$

Notation. If λ is an eigenvalue of $(\Omega, \mathfrak{b})$, we denote by $\mathcal{E}(\Omega, \mathfrak{b}, \lambda)$ the eigenspace associated with the eigenvalue λ.

We skip Ω or $\mathfrak{b}$ from the notations, whenever the context is clear.

Given a real continuous function v on Ω, define its *nodal set*

$$\mathcal{Z}(v) = \overline{\{x \in \Omega \mid v(x) = 0\}}\,, \tag{1.4}$$

and call $\beta_0(v)$ the number of connected components of $\Omega \setminus \mathcal{Z}(v)$ i.e., the number of *nodal domains* of v.

Theorem 1.1. *[Courant, 1923]*
For any nonzero eigenfunction u associated with $\lambda_n(\Omega, \mathfrak{b})$,

$$\beta_0(u) \leq \kappa\Big(\lambda_n(\Omega, \mathfrak{b})\Big) \leq n\,. \tag{1.5}$$

Courant's nodal domain theorem can be found in [15, Chap. V.6].

A footnote in [15, p. 454] (second footnote in the German original [14, p. 394]) indicates: *Any linear combination of the first n eigenfunctions divides the domain, by means of its nodes, into no more than n subdomains. See the Göttingen dissertation of H. Herrmann, Beiträge zur Theorie der Eigenwerte und Eigenfunktionen, 1932.*

For later reference, we write a precise statement. Given $\lambda \geq 0$, denote by $\mathcal{L}(\Omega, \mathfrak{b}, \lambda)$ the space of linear combinations of eigenfunctions of $H(\Omega, \mathfrak{b})$ associated with eigenvalues less than or equal to λ,

$$\mathcal{L}(\Omega, \mathfrak{b}, \lambda) = \left\{ \sum_{\lambda_j(\Omega,\mathfrak{b}) \leq \lambda} c_j\, u_j \mid c_j \in \mathbb{R}, u_j \in \lambda_j(\Omega, \mathfrak{b}) \right\}. \tag{1.6}$$

Statement 1.2. *[Extended Courant Property]*
Let $v \in \mathcal{L}\left(\lambda_n(\Omega, \mathfrak{b})\right)$ be any linear combination of eigenfunctions associated with the n first eigenvalues of the eigenvalue problem (1.1). Then,

$$\beta_0(v) \leq \kappa\Big(\lambda_n(\Omega, \mathfrak{b})) \leq n\,. \tag{1.7}$$

We call both Statement 1.2 and Inequality (1.7) the *Extended Courant Property*, and refer to it as the ECP(Ω), or as the ECP$(\Omega, \mathfrak{b})$ to insist on the boundary condition $\mathfrak{b}$.

The purpose of the present paper is to provide simple counterexamples to the *Extended Courant Property* for domains in $\mathbb{R}^n$, $\mathbb{T}^2$, and $\mathbb{S}^2$, including convex domains.

Some historical remarks. We begin with some historical remarks on the Extended Courant Property.

1. Statement 1.2 is true for Sturm-Liouville equations. In this context, an even more precise statement was first announced by C. Sturm in 1833, [32] and proved in [33]. Other proofs were given later on by J. Liouville and by Lord Rayleigh, who both cite Sturm explicitly, see [8] for more details.

2. Å. Pleijel mentions Statement 1.2 in his well-known paper [30] on the asymptotic behaviour of the number of nodal domains of a Dirichlet eigenfunction associated with the n-th eigenvalue of a plane domain. At the end of the paper, considering the Neumann boundary condition, he writes:
"In order to treat, for instance the case of the free three-dimensional membrane $[0,\pi]^3$*, it would be necessary to use, in a special case, the theorem quoted in* [14]*, p. 394*[1]*. This theorem which generalizes part of the Liouville-Rayleigh theorem for the string asserts that a linear combination, with constant coefficients, of the* n *first eigenfunctions can have at most* n *nodal domains. However, as far as I have been able to find there is no proof of this assertion in the literature."*

3. V. Arnold [2], see also [25], mentions that he actually discussed the footnote in [15, p. 454] with R. Courant, that the *Extended Courant Property* cannot be true in general, and that O. Viro produced counterexamples for the sphere, see [34]. More precisely, as early as 1973, Arnold [1] pointed out that the *Extended Courant Property* for the round sphere $\mathbb{S}^N$, is related to Hilbert's 16th problem. Indeed, the eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^N$ are the spherical harmonics i.e., the restrictions to the sphere of the harmonic homogeneous polynomials, so that the linear combinations of spherical harmonics of degree less than or equal to n are the restrictions to the sphere of the homogeneous polynomials of degree n in $(N+1)$ variables. The following citation is taken from [3].

Eigen oscillations of the sphere with the standard metric are described by spherical functions, i.e., polynomials. Therefore the Courant statement cited above implies the following estimate

$$\dim_{\mathbb{R}} H_0(\mathbb{R}P^N - V_n, \mathbb{R}) \leq C^N_{N+n-2} + 1 \qquad (1)$$

for the number of connected components of the complement to an algebraic hypersurface of degree n *in the* N*-dimensional projective space.*

For planar curves ($N = 2$)*, the estimation* (1) *is exact (it turns into equality on a configuration of* n *lines in general position) and can be*

[1]Pleijel refers to the German edition, this is p. 454 in the English edition [15].

proven independently of the Courant statement[2]*. For smooth surfaces of degree* 4 *in* $\mathbb{R}P^3$ *the estimation is also exact and proved (by V.M. Kharlamov).*

In the general case, the Courant statement is false (a counterexample can be constructed by a small perturbation of the standard metric on the sphere). Nonetheless the estimation (1) *seems to be plausible: for proving it one has to verify the Courant statement only for oscillation of the sphere (or the projective space) with the standard metric.*(*)

(*) *Translator's remark: the inequality* (1) *does not hold true for surfaces of even degree* ≥ 6 *in* $\mathbb{R}P^3$*. Counterexamples to* (1) *were constructed in the paper of O. Viro, "Construction of multicomponent real algebraic surfaces", Soviet Math. Dokl.* ***20****, N°. 5, 991–995 (1979).*

4. In [18], Gladwell and Zhu refer to Statement 1.2 as the *Courant-Herrmann conjecture*. They claim that this extension of Courant's theorem is not stated, let alone proved, in Herrmann's thesis or subsequent publications[3]. They consider the case in which Ω is a rectangle in $\mathbb{R}^2$, stating that they were not able to find any counterexample to the *Extended Courant Property* in this case. They also provide numerical evidence that there are counterexamples for more complicated (non convex) domains. They finally conjecture (see [18], p. 276) that the ECP(Ω) could be true for a convex domain Ω.

5. We are aware of only few papers which study linear combinations of eigenfunctions, see [24] and its bibliography. There is however a specific literature on random sums of eigenfunctions; as far as the ECP is concerned, see [31] and its bibliography. In the Neumann case, the ECP gives information on the level sets of (sums of) eigenfunctions. We point out in particular [4], in which the levels sets of a second Neumann eigenfunction are investigated.

Organization of the paper. The purpose of the present paper is to provide simple counterexamples to the *Extended Courant Property*, including *convex* domains. In Section 2, we consider the hypercube $\mathcal{C}_n$ in $\mathbb{R}^n$, with either the Dirichlet or the Neumann boundary condition. Using the strategy of Gladwell-Zhu, which fails in dimension 2, we prove that the ECP($\mathcal{C}_n, \mathfrak{d}$) is false if $n \geq 3$, and that the ECP($\mathcal{C}_n, \mathfrak{n}$) is false if $n \geq 4$. In Section 3, we prove that ECP($\mathcal{T}_e, \mathfrak{b}$) is false for the equilateral triangle, with either the Dirichlet or the Neumann boundary condition.

In Sections 4, 5 and 6, we construct counterexamples by introducing cracks (with Neumann boundary condition on the crack), respectively

[2]We are aware of only one reference for a proof, namely J. Leydold's thesis [26], partially published in [27], using real algebraic geometry.

[3]The only relevant one seems to be [23].

on the rectangle, the rectangular torus, the Euclidean disk, and the round sphere.

In Section 7, we investigate the ECP for the equilateral triangle, with rounded corners, and for the regular hexagon, using numerical computations. In Section 8, we propose some conjectures supported by numerical simulations. In Appendix A, we give a summary of the description of the eigenvalues and eigenfunctions of the equilateral triangle, with either the Dirichlet or the Neumann boundary condition.

Acknowledgements. The authors are very much indebted to Virginie Bonnaillie-Noël who produced some simulations and pictures at an early stage of their work on this subject.

2. The hypercube

2.1. **Preparation.** Let $\mathcal{C}_n(\pi) :=]0,\pi[^n$ be the *hypercube* of dimension n, with either the Dirichlet or the Neumann boundary condition on $\partial\mathcal{C}_n(\pi)$. A point in $\mathcal{C}_n(\pi)$ is denoted by $x = (x_1, \dots, x_n)$.

A complete set of eigenfunctions of $-\Delta$ for $(\mathcal{C}_n(\pi), \mathfrak{d})$ is given by the functions

$$\prod_{j=1}^{n} \sin(k_j\, x_j)\,, \text{ with eigenvalue } \sum_{j=1}^{n} k_j^2, \quad \text{for } k_j \in \mathbb{N}\backslash\{0\}\,. \tag{2.1}$$

A complete set of eigenfunctions of $-\Delta$ for $(\mathcal{C}_n(\pi), \mathfrak{n})$ is given by the functions

$$\prod_{j=1}^{n} \cos(k_j\, x_j)\,, \text{ with eigenvalue } \sum_{j=1}^{n} k_j^2, \quad \text{for } k_j \in \mathbb{N}\,. \tag{2.2}$$

2.2. **Hypercube with Dirichlet boundary condition.** In this section, we make use of the classical Chebyshev polynomials $U_k(t), k \in \mathbb{N}$, defined by the relation,

$$\sin\left((k+1)t\right) = \sin(t)\, U_k\left(\cos(t)\right)\,,$$

and in particular,

$$U_0(t) = 1, \quad U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1\,.$$

The first Dirichlet eigenvalues of $\mathcal{C}_n(\pi)$ (as points in the spectrum) are listed in the following table, together with their multiplicities, and eigenfunctions.

For the above eigenvalues, the index defined in (1.3) is given by,

$$\kappa(n+3) = 2, \quad \kappa(n+6) = n+2, \quad \kappa(n+8) = \frac{n(n+1)}{2} + 2\,. \tag{2.3}$$

In order to study the nodal set of the above eigenfunctions or linear combinations thereof, we use the diffeomorphism

$$(x_1, \dots, x_n) \mapsto (\xi_1 = \cos(x_1), \dots, \xi_n = \cos(x_n))\,, \tag{2.4}$$

TABLE 2.1. First Dirichlet eigenvalues of $\mathcal{C}_n(\pi)$

Eigenv.	Mult.	Eigenfunctions
n	1	$\phi_1(x) := \prod_{j=1}^n \sin(x_j)$
$n+3$	n	$\phi_1(x)\, U_1(\cos(x_i))$, for $1 \leq i \leq n$
$n+6$	$\frac{n(n-1)}{2}$	$\phi_1(x)\, U_1(\cos(x_i))\ U_1(\cos(x_j))$, for $1 \leq i < j \leq n$
$n+8$	n	$\phi_1(x)\, U_2(\cos(x_i))$, for $1 \leq i \leq n$

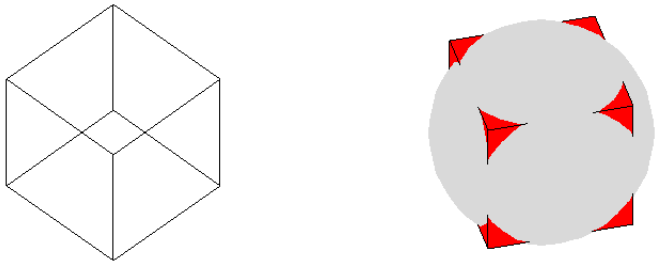

FIGURE 2.1. 3-dimensional cube

from $]0,\pi[^n$ onto $]-1,1[^n$, and factor out the function ϕ_1 which does not vanish in the open hypercube. We consider the function

$$\Xi_a(\xi_1,\dots,\xi_n) = \xi_1^2 + \cdots + \xi_n^2 - a$$

which corresponds to a linear combination Φ_a in

$$\mathcal{E}(\mathcal{C}_n(\pi),\mathfrak{d},n) \oplus \mathcal{E}(\mathcal{C}_n(\pi),\mathfrak{d},n+8).$$

Given some a, with $(n-1) < a < n$, the function Φ_a has 2^n+1 nodal domains, see Figure 2.1 in dimension 3. For $n \geq 3$, we have $2^n+1 > \kappa(n+8)$. The function Φ_a therefore provides a counterexample to the ECP for the hypercube of dimension at least 3, with Dirichlet boundary condition.

Proposition 2.1. *For $n \geq 3$, the* ECP$(\mathcal{C}_n(\pi),\mathfrak{d})$ *is false.*

Remark. An interesting feature of this example is that we get counterexamples to the ECP for linear combinations which involve eigenvalues with higher index when n increases. This is also in contrast with the fact that, in dimension 3, Courant's nodal domain theorem is sharp only for δ_1 and δ_2, [21].

2.3. **Hypercube with Neumann boundary condition.** In this section, we make use of the classical Chebyshev polynomials $T_k(t), k \in \mathbb{N}$, defined by the relation,

$$\cos(kt) = T_k(\cos(t)),$$

and in particular,

$$T_0(t) = 1\,, \quad T_1(t) = t\,, \quad T_2(t) = 2t^2 - 1\,.$$

The first Neumann eigenvalues (as points in the spectrum) are listed in the following table, together with their multiplicities, and eigenfunctions.

TABLE 2.2. First Neumann eigenvalues of $\mathcal{C}_n(\pi)$

Eigenv.	Mult.	Eigenfunctions
0	1	$\psi_1(x) := 1$
1	n	$\cos(x_i)$, for $1 \leq i \leq n$
2	$\frac{n(n-1)}{2}$	$\cos(x_i)\,\cos(x_j)$, for $1 \leq i < j \leq n$
3	$\frac{n(n-1)(n-2)}{6}$	$\cos(x_i)\,\cos(x_j)\,\cos(x_k)$, for $1 \leq i < j < k \leq n$
4	$n + \binom{n}{4}$	$T_2\,(\cos(x_i))$, for $1 \leq i \leq n$ and $\ldots$

For these Neumann eigenvalues, the index defined in (1.3) is given by,

$$\kappa(2) = n + 2\,, \kappa(3) = \frac{n(n+1)}{2} + 2\,, \kappa(4) = \frac{n(n^2+5)}{6} + 2\,. \tag{2.5}$$

In order to study the nodal set of the above eigenfunctions or linear combinations thereof, we again use the diffeomorphism (2.4) and the function Ξ_a, which here corresponds to a linear combination Ψ_a in $\mathcal{E}(\mathcal{C}_n(\pi), \mathfrak{n}, 0) \oplus \mathcal{E}(\mathcal{C}_n(\pi), \mathfrak{n}, 4)$. Given some a, with $(n-1) < a < n$, the function Ψ_a has $2^n + 1$ nodal domains. For $n \geq 4$, we have $2^n + 1 > \kappa(4)$. The function Ψ_a therefore provides a counterexample to the ECP for the hypercube of dimension at least 4, with Neumann boundary condition.

Proposition 2.2. *For $n \geq 4$, the* ECP$(\mathcal{C}_n(\pi), \mathfrak{n})$ *is false.*

2.4. **A stability result for the cube.** According to Subsection 2.2, the ECP$(\mathcal{C}_3(\pi), \mathfrak{d})$ is false. Consider the rectangular parallelepiped $\mathcal{P}_b :=]0, b_1\pi[\times]0, b_2\pi[\times]0, b_3\pi[$, with $b = (b_1, b_2, b_3)$, $b_i > 0$, and define the a_i by $\sqrt{a_i}\, b_i = 1$.

The Dirichlet eigenvalues $\delta_i(\mathcal{P}_b)$ are the numbers $a_1k_1^2 + a_2k_2^2 + a_3k_3^2$, with associated eigenfunctions

$$\prod_{i=1}^{3} \sin\left(k_i\sqrt{a_i}x_i\right), \quad k_i \in \mathbb{N}\backslash\{0\}. \tag{2.6}$$

The eigenvalues are clearly continuous in the parameters a_i. For a generic triple (a_1, a_2, a_3) close enough to $(1, 1, 1)$, the first 12 Dirichlet eigenvalues $\delta_i(\mathcal{P}_b)$ are simple, and correspond to the same type of eigenfunctions as for the ordinary cube (same choices of triples (k_1, k_2, k_3)).

TABLE 2.3. Eigenvalues for $(\mathcal{C}_3(\pi), \mathfrak{d})$ and $(\mathcal{P}_b, \mathfrak{d})$

Index	Triple	$\delta_i(\mathcal{C}_3(\pi))$	$\delta_i(\mathcal{P}_b)$
1	$(1,1,1)$	3	3.016
2	$(2,1,1)$	6	6.016
3	$(1,2,1)$	6	6.037
4	$(1,1,2)$	6	6.042
5	$(2,2,1)$	9	9.037
6	$(2,1,2)$	9	9.042
7	$(1,2,2)$	9	9.063
8	$(3,1,1)$	11	11.016
9	$(1,3,1)$	11	11.072
10	$(1,1,3)$	11	11.085
11	$(2,2,2)$	12	12.063
12	$(3,2,1)$	14	14.037

This is for example the case if we take $a_1 = 1, a_2 = 1 + \sqrt{2}/100$ and $a_3 = 1 + \sqrt{3}/100$, see the numerical values in Table 2.4, where the Dirichlet eigenvalues are denoted δ_i.

One can then repeat the arguments of Subsection 2.2, and conclude that ECP$(\mathcal{C}_b, \mathfrak{d})$ is false, so that one has some kind of stability.

Proposition 2.3. *For $b := (b_1, b_2, b_3)$ close enough to $(1,1,1)$, the* ECP$(\mathcal{P}_b, \mathfrak{d})$ *is false.*

Clearly, the same kind of argument can be applied in higher dimension, or for the Neumann boundary condition.

3. THE EQUILATERAL TRIANGLE

Let $\mathcal{T}_e$ denote the equilateral triangle with sides equal to 1, see Figure 3.1. The eigenvalues and eigenfunctions of $\mathcal{T}_e$, with either the Dirichlet or the Neumann condition on the boundary $\partial\mathcal{T}_e$, can be completely described, see [6, 29, 28], or [7]. We provide a summary in Appendix A.

In this section, we show that the equilateral triangle provides a counterexample to the *Extended Courant Property* for both the Dirichlet and the Neumann boundary conditions.

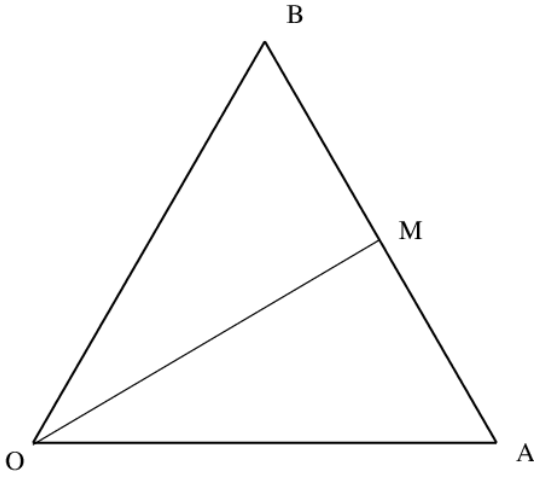

FIGURE 3.1. Equilateral triangle $\mathcal{T}_e = [OAB]$

3.1. **Neumann boundary condition.** The sequence of Neumann eigenvalues of the equilateral triangle $\mathcal{T}_e$ begins as follows,

$$0 = \lambda_1(\mathcal{T}_e, \mathfrak{n}) < \frac{16\pi^2}{9} = \lambda_2(\mathcal{T}_e, \mathfrak{n}) = \lambda_3(\mathcal{T}_e, \mathfrak{n}) < \lambda_4(\mathcal{T}_e, \mathfrak{n})\,. \tag{3.1}$$

The second eigenspace has dimension 2, and contains one eigenfunction $\varphi_2^{\mathfrak{n}}$ which is invariant under the mirror symmetry with respect to the median OM, and another eigenfunction $\varphi_3^{\mathfrak{n}}$ which is anti-invariant under the same mirror symmetry, see Appendix A.

More precisely, according to (A.21), the function $\varphi_2^{\mathfrak{n}}(x, y)$ can be chosen to be,

$$\left\{ \begin{array}{rl} \varphi_2^{\mathfrak{n}}(x,y) = & \cos(\frac{4\pi}{3}x) + \cos(\frac{2\pi}{3}(-x+\sqrt{3}y)) \\ & + \cos(\frac{2\pi}{3}(x+\sqrt{3}y))\,, \end{array} \right. \tag{3.2}$$

or, more simply,

$$\varphi_2^{\mathfrak{n}}(x,y) = 2\cos\left(\frac{2\pi x}{3}\right)\left(\cos\left(\frac{2\pi x}{3}\right) + \cos\left(\frac{2\pi y}{\sqrt{3}}\right)\right) - 1\,. \tag{3.3}$$

The set $\{\varphi_2^{\mathfrak{n}} + 1 = 0\}$ consists of the two line segments $\{x = \frac{3}{4}\} \cap \mathcal{T}_e$ and $\{x + \sqrt{3}y = \frac{3}{2}\} \cap \mathcal{T}_e$, which meet at the point $(\frac{3}{4}, \frac{\sqrt{3}}{4})$ on $\partial\mathcal{T}_e$.

The sets $\{\varphi_2^{\mathfrak{n}} + a = 0\}$, with $a \in \{0\,; 1-\varepsilon\,; 1\,; 1+\varepsilon\}$, and small positive ε, are shown in Figure 3.2. When a varies from $1-\varepsilon$ to $1+\varepsilon$, the number of nodal domains of $\varphi_2^{\mathfrak{n}} + a$ in $\mathcal{T}_e$ jumps from 2 to 3, with the jump occurring for $a = 1$.

It follows that $\varphi_2^{\mathfrak{n}} + a = 0$, for $1 \leq a \leq 1.1$, provides a counterexample to the *Extended Courant Property* for the equilateral triangle with Neumann boundary condition.

Proposition 3.1. *The* ECP$(\mathcal{T}_e, \mathfrak{n})$ *is false.*

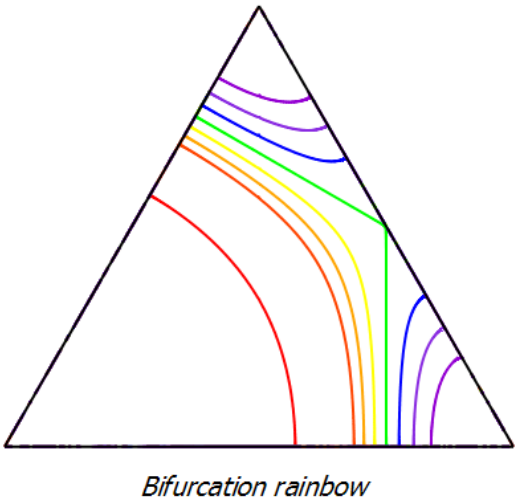

FIGURE 3.2. Level sets $\{\varphi_2^{\mathfrak{n}} + a = 0\}$ for $a \in \{0\,;\, 0.7\,;\, 0.8\,;\, 0.9\,;\, 1\,;\, 1.1\,;\, 1.2\,;\, 1.3\}$

Remark. The eigenfunction $\varphi_2^{\mathfrak{n}}$ restricted to the hemiequilateral triangle is the second Neumann eigenfunction of $\mathcal{T}_h = [OAM]$. The restriction of $\varphi_3^{\mathfrak{n}}$ to the hemiequilateral triangle is an eigenfunction of $\mathcal{T}_h$ with mixed boundary condition (Dirichlet on OM and Neumann on the other sides).

3.2. **Dirichlet boundary condition.** The sequence of Dirichlet eigenvalues of the equilateral triangle $\mathcal{T}_e$ begins as follows,

$$\lambda_1(\mathcal{T}_e, \mathfrak{d}) = \frac{16\pi^2}{3} < \lambda_2(\mathcal{T}_e, \mathfrak{d}) = \lambda_3(\mathcal{T}_e, \mathfrak{d}) = \frac{112\pi^2}{9} < \lambda_4(\mathcal{T}_e, \mathfrak{d}). \tag{3.4}$$

More precisely, according to (A.23), the function $\varphi_1^{\mathfrak{d}}(x, y)$ can be chosen to be,

$$\varphi_1^{\mathfrak{d}}(x, y) = -8 \sin \tfrac{2\pi y}{\sqrt{3}} \sin \pi(x + \tfrac{y}{\sqrt{3}}) \sin \pi(x - \tfrac{y}{\sqrt{3}})\,, \tag{3.5}$$

which shows that $\varphi_1^{\mathfrak{d}}$ does not vanish inside $\mathcal{T}_e$.

The second eigenvalue has multiplicity 2. It admits one eigenfunction, $\varphi_2^{\mathfrak{d}}$, which is symmetric with respect to the median OM, and given in (A.25), and another one, $\varphi_3^{\mathfrak{d}}$, which is anti-symmetric.

We now consider the linear combination $\varphi_2^{\mathfrak{d}} + a\,\varphi_1^{\mathfrak{d}}$, with a close to 1. The following lemma is the key for reducing the question to the previous analysis.

Lemma 3.2. *With the above notation, the following identity holds,*

$$\varphi_2^{\mathfrak{d}} = \varphi_1^{\mathfrak{d}} \varphi_2^{\mathfrak{n}}\,.$$

Proof. We express the above eigenfunctions in terms of $X := \cos\frac{2\pi}{3}x$ and $Y := \cos\frac{2\pi}{\sqrt{3}}y$.

First we observe from (3.3) that

$$\varphi_2^{\mathfrak{n}}(x,y) = 2X(X+Y) - 1\,.$$

Secondly, we have from (3.5)

$$\varphi_1^{\mathfrak{d}}(x,y) = 2\sin\frac{2\pi y}{\sqrt{3}}\,(8X^3 - 6X - 2Y)\,.$$

Finally, it remains to compute $\varphi_2^{\mathfrak{d}}$. We start from (A.25), and first factorize $\sin\frac{2\pi y}{\sqrt{3}}$ in each line. More precisely, we write,

(3.6)

$$\begin{aligned}
&\sin\tfrac{2\pi}{3}(5x+\sqrt{3}y) - \sin\tfrac{2\pi}{3}(5x-\sqrt{3}y) = 2\sin(\tfrac{2\pi y}{\sqrt{3}})\cos(5\tfrac{2\pi x}{3})\,,\\
&\sin\tfrac{2\pi}{3}(x-3\sqrt{3}y) - \sin\tfrac{2\pi}{3}(x+3\sqrt{3}y) = -2\sin(3\tfrac{2\pi y}{\sqrt{3}})\cos(\tfrac{2\pi x}{3})\,,\\
&\sin\tfrac{4\pi}{3}(2x+\sqrt{3}y) - \sin\tfrac{4\pi}{3}(2x-\sqrt{3}y) = 2\sin(2\tfrac{2\pi y}{\sqrt{3}})\cos(4\tfrac{2\pi x}{3})\,.
\end{aligned}$$

We now use the classical Chebyshev polynomials T_n, U_n, and the relations $\cos(n\theta) = T_n(\cos\theta)$ and $\sin(n+1)\theta = \sin(\theta)\,U_n(\cos\theta)$.

This gives,

$$\begin{aligned}
\varphi_2^{\mathfrak{d}} &= 2\sin\tfrac{2\pi y}{\sqrt{3}}\Big(T_5(X) - XU_2(Y) + T_4(X)U_1(Y)\Big)\\
&=: 2\sin\tfrac{2\pi y}{\sqrt{3}}\,Q(X,Y)\,.
\end{aligned}$$

We find that

$$Q(X,Y) = 16X^5 - 20X^3 + 6X + 2Y(8X^4 - 8X^2 + 1) - 4XY^2\,,$$

and it turns out that the polynomial $Q(X,Y)$ can be factorized as

$$Q(X,Y) = \Big(2X(X+Y) - 1\Big)\,(8X^3 - 6X - 2Y)\,,$$

so that $\varphi_2^{\mathfrak{d}} = \varphi_1^{\mathfrak{d}}\varphi_2^{\mathfrak{n}}$.

In the above computation, we have used the relations,

$$T_4(X) = 8X^4 - 8X^2 + 1\,,\; T_5(X) = 16X^5 - 20X^3 + 5X\,,$$

and

$$U_1(Y) = 2Y\,,\; U_2(Y) = 4Y^2 - 1\,.$$

□

Observing that

$$\varphi_2^{\mathfrak{d}} + a\,\varphi_1^{\mathfrak{d}} = \varphi_1^{\mathfrak{d}}(\varphi_2^{\mathfrak{n}} + a)\,,$$

we deduce immediately from the Neumann result that the function $\varphi_2^{\mathfrak{d}} + a\,\varphi_1^{\mathfrak{d}}$, for $1 \le a \le 1.1$, provides a counterexample to the *Extended Courant Property* for the equilateral triangle with the Dirichlet boundary condition.

Proposition 3.3. *The* $\mathrm{ECP}(\mathcal{T}_e, \mathfrak{d})$ *is false.*

Remark 3.4. *Lemma 3.2 is quite puzzling. However, other such identities do exist. Indeed, consider the square $\mathcal{C}_2(\pi)$. The first eigenfunction has the form*

$$(x, y) \mapsto \alpha_0 \sin x \sin y \,,$$

with $\alpha_0 \neq 0$, and the second eigenfunctions take the form

$$(x, y) \mapsto \alpha \sin 2x \sin y + \beta \sin 2y \sin x \,,$$

with $|\alpha| + |\beta| \neq 0$. We can then observe that

$$\alpha \sin 2x \sin y + \beta \sin 2y \sin x = 2 \sin x \sin y \, (\alpha \cos x + \beta \cos y) \,,$$

and that $\alpha \cos x + \beta \cos y$ is a Neumann eigenfunction of the square. For $\mathcal{C}_2(\pi)$, more general relations between Dirichlet and Neumann eigenfunctions follow from the identity $2T_n = U_n - U_{n-2}$ between Chebyshev polynomials.
One can also prove the identity $\varphi_2^{\mathfrak{d}} = a\, \varphi_1^{\mathfrak{d}}\, \varphi_2^{\mathfrak{n}}$ between the eigenfunctions of the right isosceles triangle (for some constant a depending on the normalization of eigenfunctions).

4. Rectangle with a crack

Let $\mathcal{R}$ be the rectangle $]0, 4\pi[\times]0, 2\pi[$. For $0 < a \leq 1$, let $C_a :=]0, a] \times \{\pi\}$ and $\mathcal{R}_a := \mathcal{R} \setminus C_a$. In this section, we only consider the Neumann boundary condition on C_a, and either the Dirichlet or the Neumann boundary condition on $\partial\mathcal{R}$. The setting is the one described in [17, Section 8].

We call

$$\text{(4.1)} \qquad \begin{cases} 0 < \delta_1(0) < \delta_2(0) \leq \delta_3(0) \leq \cdots \\ \text{resp.} \\ 0 = \nu_1(0) < \nu_2(0) \leq \nu_3(0) \leq \cdots \end{cases}$$

the eigenvalues of $-\Delta$ in $\mathcal{R}$, with the Dirichlet (resp. the Neumann) boundary condition on $\partial\mathcal{R}$. They are given by the numbers $\frac{m^2}{16} + \frac{n^2}{4}$, for pairs (m, n) of positive integers for the Dirichlet problem (resp. for pairs of non-negative integers for the Neumann problem). Corresponding eigenfunctions are products of sines (Dirichlet) or cosines (Neumann).
The eigenvalues are arranged in non-decreasing order, counting multiplicities.

Similarly, call

$$\text{(4.2)} \qquad \begin{cases} 0 < \delta_1(a) < \delta_2(a) \leq \delta_3(a) \leq \cdots \\ \text{resp.} \\ 0 = \nu_1(a) < \nu_2(a) \leq \nu_3(a) \leq \cdots \end{cases}$$

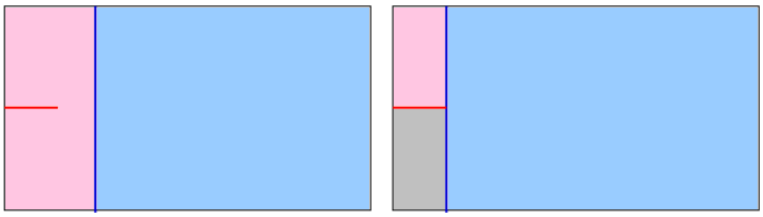

FIGURE 4.1. Rectangle with a crack (Neumann condition)

the eigenvalues of $-\Delta$ in $\mathcal{R}_a$, with the Dirichlet (resp. the Neumann) boundary condition on $\partial\mathcal{R}$, and the Neumann boundary condition on C_a.

The first three Dirichlet (resp. Neumann) eigenvalues for the rectangle $\mathcal{R}$ are as follows.

(4.3)

Eigenvalue	Value	Pairs	Dirichlet eigenfunctions
$\delta_1(0)$	$\frac{5}{16}$	$(1,1)$	$\phi_1(x,y) = \sin(\frac{x}{4})\,\sin(\frac{y}{2})$
$\delta_2(0)$	$\frac{1}{2}$	$(2,1)$	$\phi_2(x,y) = \sin(\frac{x}{2})\,\sin(\frac{y}{2})$
$\delta_3(0)$	$\frac{13}{16}$	$(3,1)$	$\phi_3(x,y) = \sin(\frac{3x}{4})\,\sin(\frac{y}{2})$

(4.4)

Eigenvalue	Value	Pairs	Neumann eigenfunctions
$\nu_1(0)$	0	$(0,0)$	$\psi_1(x,y) = 1$
$\nu_2(0)$	$\frac{1}{16}$	$(1,0)$	$\psi_2(x,y) = \cos(\frac{x}{4})$
$\nu_3(0)$		$(0,1)$	$\psi_3(x,y) = \cos(\frac{y}{2})$
$\nu_4(0)$	$\frac{1}{4}$	$(2,0)$	$\psi_4(x,y) = \cos(\frac{x}{2})$

We summarize [17], Propositions (8.5), (8.7), (9.5) and (9.9), into the following theorem.

Theorem 4.1 (Dauge-Helffer)**.**
With the above notation, the following properties hold.

1. *For $i \geq 1$, the functions $[0,1] \ni a \mapsto \delta_i(a)$, resp. $[0,1] \ni a \mapsto \nu_i(a)$, are non-increasing.*
2. *For $i \geq 1$, the functions $]0,1[\ni a \mapsto \delta_i(a)$, resp. $]0,1[\ni a \mapsto \nu_i(a)$, are continuous.*
3. *For $i \geq 1$, $\lim_{a\to 0+} \delta_i(a) = \delta_i(0)$ and $\lim_{a\to 0+} \nu_i(a) = \nu_i(0)$.*

It follows that for a positive, small enough, we have

$$(4.5)\quad \begin{cases} 0 < \delta_1(a) \leq \delta_1(0) < \delta_2(a) \leq \delta_2(0) < \delta_3(a) \leq \delta_3(0)\,, \text{and} \\ 0 = \nu_1(a) = \nu_1(0) < \nu_2(a) \leq \nu_2(0) < \nu_3(a) \leq \nu_4(a) \leq \nu_3(0)\,. \end{cases}$$

Observe that for $i = 1$ and 2, $\frac{\partial \phi_i}{\partial y}(x, \pi) = 0$ and $\frac{\partial \psi_i}{\partial y}(x, y) = 0$. It follows that for a small enough, the functions ϕ_1 and ϕ_2 (resp. the functions ψ_1 and ψ_2) are the first two eigenfunctions for $\mathcal{R}_a$ with the Dirichlet (resp. Neumann) boundary condition on $\partial\mathcal{R}$, and the Neumann boundary condition on C_a, with associated eigenvalues $\frac{5}{16}$ and $\frac{1}{2}$ (resp. 0 and $\frac{1}{16}$).

We have

$$\alpha\phi_1(x, y) + \beta\phi_2(x, y) = \sin(\frac{x}{4}) \sin(\frac{y}{2}) \left(\alpha + 2\beta \cos(\frac{x}{4})\right),$$

and

$$\alpha\psi_1(x, y) + \beta\psi_2(x, y) = \alpha + \beta \cos(\frac{x}{4}).$$

We can choose the coefficients α, β in such a way that these linear combinations of the first two eigenfunctions have two (Figure 4.1 left) or three (Figure 4.1 right) nodal domains in $\mathcal{R}_a$.

Proposition 4.2. *The* ECP($\mathcal{R}_a$) *is false with the Neumann condition on C_a, and either the Dirichlet or the Neumann condition on $\partial\mathcal{R}$.*

Remark 4.3. *In the Neumann case, we can introduce several cracks $\{(x, b_j) \mid 0 < x < a_j\}_{j=1}^k$ in such a way that for any $d \in \{2, 3, \dots k+2\}$ there exists a linear combination of* 1 *and* $\cos(\frac{x}{4})$ *with d nodal domains.*

Remark 4.4. *Numerical simulations, kindly provided by Virginie Bonnaillie-Noël, indicate that the* Extended Courant Property *does not hold for a rectangle with a crack, with the Dirichlet boundary condition on both the boundary of the rectangle, and the crack,* [10]. *Dirichlet cracks appear in another context in* [19] *(see also references therein)*

Remark 4.5. *It is easy to make an analogous construction for the unit disk (Neumann case) with radial cracks. As computed for example in* [22] *(Subsection 3.4), the second radial eigenfunction has labelling* 6 *($\lambda_6 \approx 14,68$), and we can introduce six radial cracks to obtain a combination of the two first radial Neumann eigenfunctions with seven nodal domains, see Figure 4.2.*

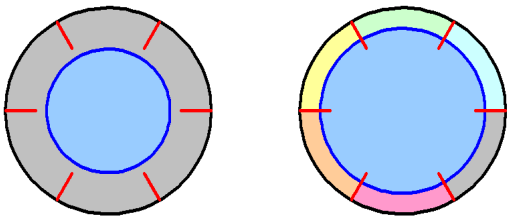

FIGURE 4.2. Disk with cracks, Neumann condition

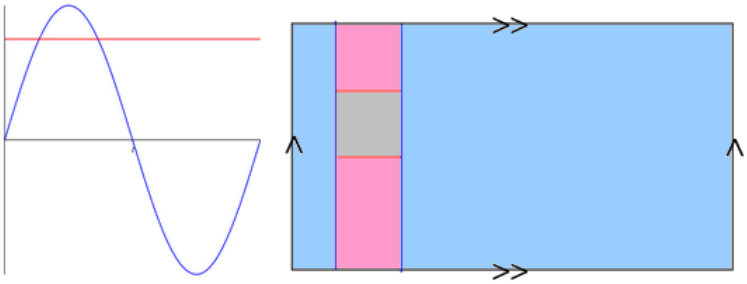

FIGURE 5.1. Flat torus with two cracks

5. The rectangular flat torus with cracks

Consider the flat torus $\mathbb{T} := \mathbb{R}^2/\left(4\pi\mathbb{Z} \oplus 2\pi Z\right)$. Arrange the eigenvalues in nondecreasing order,

$$\lambda_1(0) < \lambda_2(0) \leq \lambda_3(0) \leq \cdots \tag{5.1}$$

The eigenvalues are given by the numbers $\frac{m^2}{4} + n^2$ for (m, n) pairs of integers, with associated complex eigenfunctions

$$\exp(im\frac{x}{2})\,\exp(iny) \tag{5.2}$$

or equivalently, with real eigenfunctions

$$\begin{aligned} &\cos(m\tfrac{x}{2})\,\cos(ny),\ \cos(m\tfrac{x}{2})\,\sin(ny),\\ &\sin(m\tfrac{x}{2})\,\cos(ny),\ \sin(m\tfrac{x}{2})\,\sin(ny), \end{aligned} \tag{5.3}$$

where m, n are non-negative integers. Accordingly, the first eigenpairs of $\mathcal{T}$ are as follows.

(5.4)

Eigenvalue	Value	Pairs	Eigenfunctions
$\lambda_1(0)$	0	$(0,0)$	$\omega_1(x,y) = 1$
$\lambda_2(0)$			$\omega_2(x,y) = \cos(\frac{x}{2})$
$\lambda_3(0)$	$\frac{1}{4}$	$(1,0)$	$\omega_2(x,y) = \sin(\frac{x}{2})$
$\lambda_4(0)$			$\omega_3(x,y) = \cos(y)$
$\lambda_5(0)$	1	$(0,1)$	$\omega_4(x,y) = \sin(y)$

A typical linear combination of the first three eigenfunctions is of the form $\alpha + \beta\,\sin(\frac{x}{2} - \theta)$

Take the torus $\mathbb{T}$, and perform two (or more) cracks parallel to the x axis, and with the same length a. Call $\mathbb{T}_a$ the torus with cracks, see Figure 5.1, and choose the Neumann boundary condition on the cracks. For a small enough, the first three eigenfunctions of the torus $\mathbb{T}$ remain eigenfunctions of the torus with cracks, $\mathbb{T}_a$, with the same $\kappa(\mathbb{T}_a, 3) = 2$. The proof is the same as in [17]. We can choose the length a such that

the nodal set of $\alpha + \beta \sin(\frac{x}{2} - \theta)$ and the two cracks determine three nodal domains.

Proposition 5.1. *The* Extended Courant Property *is false for the flat torus with cracks (Neumann condition on the cracks).*

6. Sphere $\mathbb{S}^2$ with cracks

On the round sphere $\mathbb{S}^2$, we consider the geodesic lines $z \mapsto (\sqrt{1-z^2}\cos\theta_i, \sqrt{1-z^2}\sin\theta_i, z)$ through the north pole $(0,0,1)$, with distinct $\theta_i \in [0,\pi[$. For example, removing the geodesic segments $\theta_0 = 0$ and $\theta_2 = \frac{\pi}{2}$ with $1 - z \leq a \leq 1$, we obtain a sphere $\mathbb{S}^2_a$ with a crack in the form of a cross. We choose the Neumann boundary condition on the crack.
We can then easily produce a function, in the space generated by the two first eigenspaces of the sphere with a crack, having five nodal domains.

The first eigenvalue of $\mathbb{S}^2$ is $\lambda_1(0) = 0$, with corresponding eigenspace of dimension 1, generated by the function 1. The next eigenvalues of $\mathbb{S}^2$ are $\lambda_2(0) = \lambda_3(0) = \lambda_4(0) = 2$ with associated eigenspace of dimension 3, generated by the functions x, y, z. The following eigenvalues of $\mathbb{S}^2$ are larger than or equal to 6.
As in [17], the eigenvalues of $\mathbb{S}^2_a$ (with Neumann condition on the crack) are non-increasing in a, and continuous to the right at $a = 0$. More precisely

$$\text{(6.1)} \qquad \begin{cases} 0 = \lambda_1(a) < \lambda_2(a) \leq \lambda_3(a) \leq \lambda_4(a) \leq 2 < \lambda_5(a) \leq 6\,, \\ \lim_{a\to 0+} \lambda_i(a) = 2 \text{ for } i = 2,3,4, \\ \lim_{a\to 0+} \lambda_5(a) = 6\,. \end{cases}$$

The function z is also an eigenfunction of $\mathbb{S}^2_a$ with eigenvalue 2. It follows from (6.1) that for a small enough, $\lambda_4(a) = 2$, with eigenfunction z. For $0 < b < a$, the linear combination $z - b$ has five nodal domains in $\mathbb{S}^2_a$, see Figure 6.1 in spherical coordinates.

Proposition 6.1. *The* Extended Courant Property *is false for the round 2-sphere with cracks (Neumann condition on the cracks).*

Remark 6.2. *(1) Removing more geodesic segments around the north pole, we can obtain a linear combination $z - b$ with as many nodal domains as we want.*
(2) The sphere with cracks, and Dirichlet condition on the cracks, has been considered for another purpose in [20].

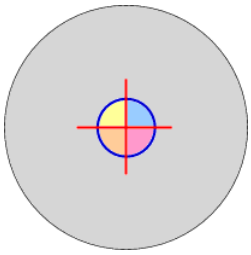

FIGURE 6.1. Sphere with crack, five nodal domains

7. NUMERICAL SIMULATIONS

7.1. **Equilateral triangle with rounded corners.** The counterexamples which we have presented so far are either domains with a non smooth boundary, or singular domains (presence of cracks). A natural question is:

Does there exist a domain Ω*, with say* C^∞ *boundary, such that the* $\mathrm{ECP}(\Omega, \mathfrak{b})$ *is false?*

A natural idea is to start from a known example, and try to smooth it up. A first step is to consider the equilateral triangle with rounded corners $\mathcal{T}_{e,a}$ (each corner is replaced by a circular sector with center on the bisector, tangent to the sides, and with radius a, small enough).

The pictures in the first row of Figure 7.1 display the level sets and nodal domains of a second Neumann eigenfunction ϕ of the equilateral triangle with rounded corners, as calculated by MATLAB. The function is almost symmetric with respect to one of the axes of symmetry of the triangle. The pictures in the second row display the nodal sets of the function $a+\phi$ for two values of a. Note that this triangle with rounded corners is C^1, not C^2. Smoother approximations of the equilateral triangle can for example be obtained by considering the super-level sets of the first Dirichlet eigenfunction of the equilateral triangle.

Claim 7.1. *Figure 7.1 gives a numerical evidence that* $\mathrm{ECP}(\mathcal{T}_{e,a}, \mathfrak{n})$ *is false.*

This numerical simulation motivated us to study the deformation of Neumann eigenfunctions along a path of C^∞ domains with the symmetries of the equilateral triangle, and to answer the above question positively. We refer to [9] for more details.

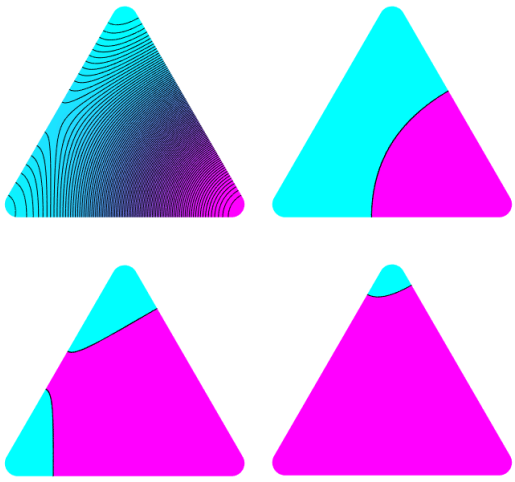

FIGURE 7.1. Level sets of one of the second Neumann eigenfunctions of the equilateral triangle with rounded corners

7.2. **The regular hexagon.** We are looking for another counterexample to the *Extended Courant Property* in a convex domain of $\mathbb{R}^2$. We have already mentioned that this quest was unsuccessful for the square, and successful for the equilateral triangle. It is natural to think of other polygons and, among them, the regular hexagon $\mathcal{H}$.

7.2.1. *Preliminaries.* Call $\mathcal{H} = [ABCDEF]$ the regular hexagon with sides of length 1, $\mathcal{T}_e = [OAB]$ the equilateral triangle, and $\mathcal{T}_h = [OAM]$ the hemiequilateral triangle. See Figure 7.2.

In this subsection, we consider both the Dirichlet, and the Neumann boundary conditions on $\partial\mathcal{H}$.

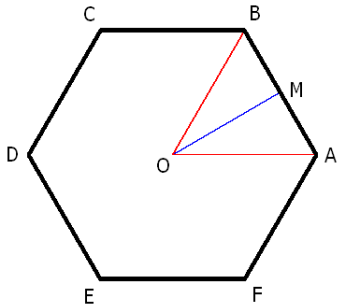

FIGURE 7.2. The hexagon

Only a small portion (asymptotically one-sixth) of the eigenvalues, and of the eigenfunctions, of the regular hexagon $\mathcal{H}$ are known explicitly, namely those which arise from the equilateral triangle (with the Dirichlet or the Neumann boundary condition).

7.2.2. *Numerical computations.* Numerical computations of the Dirichlet eigenvalues, and of the nodal patterns of Dirichlet eigenfunctions, are available in the literature, see for example [5, 16]. They strongly rely on the symmetries of the hexagon, Figure 7.3.

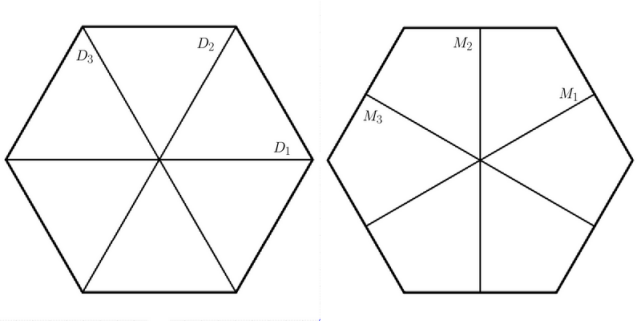

FIGURE 7.3. The lines of symmetry of the hexagon

We did not find similar computations for the Neumann eigenvalues and eigenfunctions of $\mathcal{H}$ in the literature. We performed numerical computations for this case with MATLAB, making use of the symmetries as in [16], and computing the eigenvalues of the fundamental domain $\mathcal{R}$ for the action of the symmetries D_1 and M_2, see Figure 7.4.

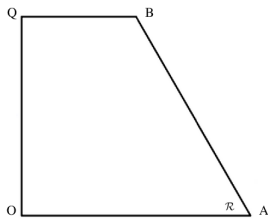

FIGURE 7.4. The domain $\mathcal{R}$

More precisely, the strategy is as follows. Fix a boundary condition $\mathfrak{b}$ on $\partial\mathcal{H}$, either the Dirichlet or the Neumann boundary condition. Numerically compute the eigenvalues of $\mathcal{R}$, with the boundary condition $\mathfrak{b}$ on $\partial\mathcal{R} \cap \partial\mathcal{H} = [ABQ]$, and with mixed boundary conditions, $\mathfrak{d}$ or $\mathfrak{n}$ on the other sides, $[OA]$ and $[OQ]$. Merge the four sets of numerical eigenvalues, and re-order the result to obtain the numerical eigenvalues of $\mathcal{H}$, with boundary condition $\mathfrak{b}$. In order to identify eigenfunctions invariant or anti-invariant under all the symmetries with respect to the lines M_i and D_j, compute the eigenvalues of $\mathcal{T}_h$, with boundary condition $\mathfrak{b}$ on $\partial\mathcal{T}_h \cap \partial\mathcal{H}$, and with mixed boundary conditions on the other sides.

It turns out that the low lying eigenvalues of $\mathcal{R}$ and $\mathcal{T}_h$ are simple. It follows that one can identify the low lying eigenvalues of $\mathcal{H}$ corresponding to the symmetries of the eigenfunctions. Using the rotation with

center O and angle $\frac{2\pi}{3}$, one can also identify certain double eigenvalues of $\mathcal{H}$.

Assuming that the computed eigenvalues are close enough to the true eigenvalues, using symmetries, and Courant's nodal domain theorem, it is possible to identify the first six Dirichlet or Neumann eigenfunctions of $\mathcal{H}$, and their nodal patterns.

7.2.3. *Dirichlet boundary condition.* Figure 7.5 displays the nodal patterns of the first six Dirichlet eigenfunctions of $\mathcal{H}$ (see the figure in [5, p. 512]).

The computations indicate that the 6th eigenvalue is simple, and that the nodal set of the corresponding Dirichlet eigenfunction $u_6^{\mathfrak{d}}$ is a simple closed curve. Taking for granted that $\delta_6(\mathcal{H})$ is simple, using the symmetries of the hexagon, and making use of Courant's nodal domain theorem, one can show that $u_6^{\mathfrak{d}}$ is invariant under all the symmetries with respect to D_i, M_j. It follows that $u_6^{\mathfrak{d}}$ arises from the second eigenfunction of $\mathcal{T}_h$, with mixed boundary condition $\mathfrak{nnd}$ (Dirichlet on the smaller side of $\mathcal{T}_h$, Neumann on the other sides).

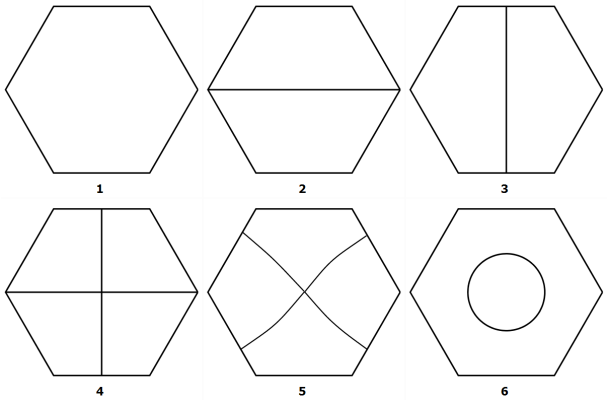

FIGURE 7.5. Nodal structure for the six first eigenfunctions of the Dirichlet Laplacian in the hexagon

Plotting the nodal set of the linear combination $u_6^{\mathfrak{d}} + a\, u_1^{\mathfrak{d}}$ for several values of a, one finds some values of a for which this function has 7 nodal domains, see Figure 7.6.

Claim 7.2. *Figure 7.6 provides a numerical evidence that the* $\mathrm{ECP}(\mathcal{H}, \mathfrak{d})$ *is false.*

Remark 7.3. *Figure 7.6 also suggests that the* $\mathrm{ECP}(\mathcal{T}_e, \mathfrak{nnd})$ *is false (equilateral triangle with the Dirichlet boundary condition on one side, and the Neumann boundary condition on the other sides).*

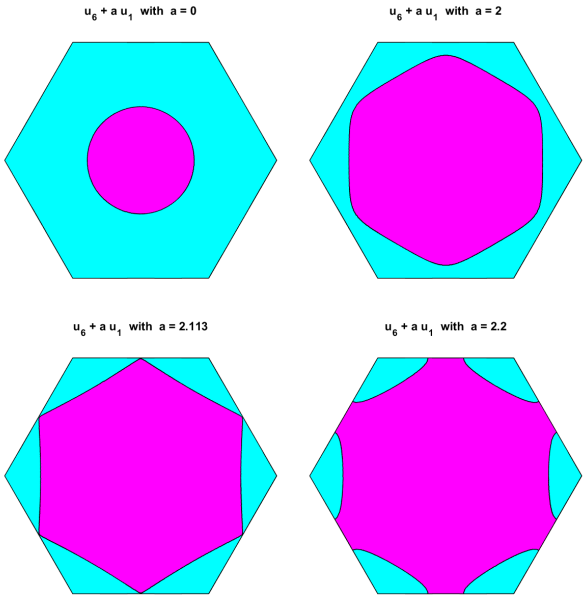

FIGURE 7.6. Nodal set of $u_6^{\mathfrak{d}} + a\, u_1^{\mathfrak{d}}$ for $a \in \{0\,;2\,;2.113\,;2.2\}$

7.2.4. *Neumann boundary condition.* The first Neumann eigenvalue of the hexagon, $\nu_1(\mathcal{H})$, is 0, with associated eigenfunction $u_1^{\mathfrak{n}} \equiv 1$. Figure 7.7 displays the (computed) eigenvalues, and corresponding nodal patterns, for the next six eigenvalues $\nu_i(\mathcal{H}), 2 \le i \le 7$. It turns out that the 8th Neumann eigenvalue of the hexagon satisfies $\nu_8(\mathcal{H}) \approx 24.8989$.

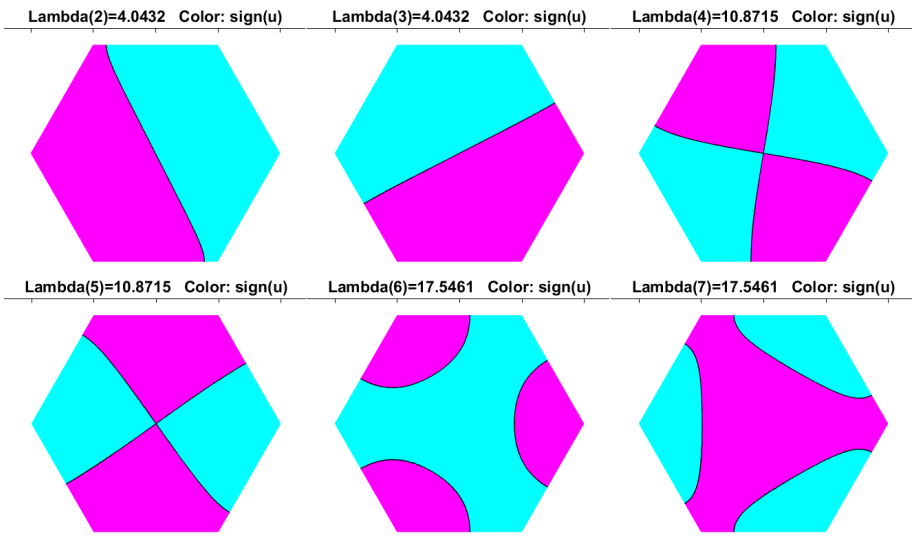

FIGURE 7.7. Eigenpairs 2–7 for the hexagon with Neumann condition using MATLAB

Figure 7.7 shows that the (numerical) eigenvalues $\nu_i(\mathcal{H}), 2 \leq i \leq 7$ come in pairs. The common value for $\nu_6(\mathcal{H})$ and $\nu_7(\mathcal{H})$ is ≈ 17.5461. This suggests that these eigenvalues are equal to $\nu_2(\mathcal{T}_e) = \nu_2(\mathcal{T}_h) = \frac{16\pi^2}{9} \approx 17.545963$. Where do the eigenfunctions associated to ν_6 and ν_7 come from?

To better understand the situation, we have to go back to the domain $\mathcal{R}$, and look at its eigenvalues with mixed boundary conditions. For this purpose, we decompose the boundary $\partial\mathcal{R}$ into three pieces, $\Gamma_1 = [OQ]$, $\Gamma_2 = [OA]$, and $\Gamma_3 = [ABQ]$. We consider the eigenvalue problem for $\mathcal{R}$, with mixed boundary conditions $\mathfrak{abn}$, i.e., with boundary condition $\mathfrak{a}$ on Γ_1, $\mathfrak{b}$ on Γ_2, and $\mathfrak{n}$ on Γ_3, with $a, b \in \{d, n\}$.

TABLE 7.1. First Neumann eigenvalues of $\mathcal{H}$

Eigenvalue of $\mathcal{H}$	Approximation	Eigenvalue of $\mathcal{R}$
$\nu_2(\mathcal{H})$	≈ 4.0432	$\mu_1(\mathcal{R}, \mathfrak{dnn})$
$\nu_3(\mathcal{H})$	≈ 4.0432	$\mu_1(\mathcal{R}, \mathfrak{ndn})$
$\nu_4(\mathcal{H})$	≈ 10.8715	$\mu_1(\mathcal{R}, \mathfrak{ddn})$
$\nu_5(\mathcal{H})$	≈ 10.8715	$\mu_2(\mathcal{R}, \mathfrak{nnn})$
$\nu_6(\mathcal{H})$	≈ 17.5461	$\mu_2(\mathcal{R}, \mathfrak{dnn})$
$\nu_7(\mathcal{H})$	≈ 17.5461	$\mu_3(\mathcal{R}, \mathfrak{nnn})$

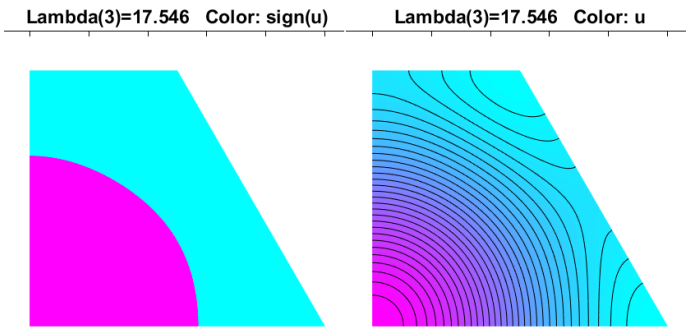

FIGURE 7.8. Nodal pattern for $\mu_3(\mathcal{R}, \mathfrak{nnn})$

Figure 7.8 displays the nodal pattern and the level lines of a third eigenfunction of $(\mathcal{R}, \mathfrak{nnn})$. By reflection with respect to the lines D_1 and M_2, one obtains a Neumann eigenfunction $u_{\mathcal{H}}$ of $\mathcal{H}$, associated with $\nu_6(\mathcal{H}) = \nu_7(\mathcal{H})$, whose nodal set is a closed simple curve around O, and whose level lines are displayed in Figure 7.9; some level lines of $u_{\mathcal{H}}$ have six connected components, one component near each vertex of the hexagon.

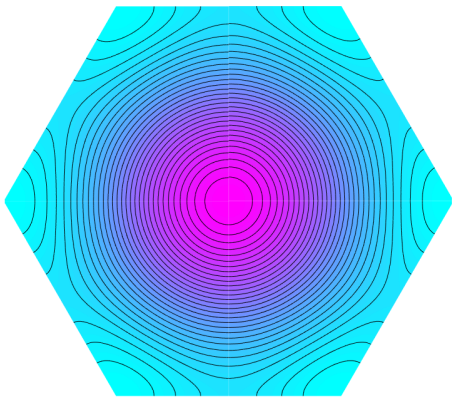

FIGURE 7.9. Level lines of $u_{\mathcal{H}}$

Claim 7.4. *Figures 7.8 and 7.9 provide a numerical evidence that the* $\mathrm{ECP}(\mathcal{H}, \mathfrak{n})$ *is false.*

Remark 7.5. *The above counterexample actually arises from the counterexample to* $\mathrm{ECP}(\mathcal{T}_e, \mathfrak{n})$ *given in Section 3.*

8. Conjectures

Conjectures 8.1. *Numerical simulations performed by Virginie Bonnaillie-Noël [13] suggest,*

(1) *The* Extended Courant Property *does not hold on a rectangle with a crack, and Dirichlet boundary condition on both the crack and the boundary of the rectangle,* [10, Figure 9.1].
(2) *The* Extended Courant Property *does not hold for the regular heptagon, with the Dirichlet boundary condition,* [10, Figure 9.4].
(3) *The* Extended Courant Property *does not hold for the regular polygon with* n *sides,* $n \geq 6$*, with the Dirichlet or the Neumann boundary condition.*

Conjectures 8.2. *The numerical computations in Section 7 suggest,*

(1) *The* Courant Extended Property *does not hold for the regular hexagon with either the Dirichlet boundary or the Neumann boundary conditions.*
(2) *The* Courant Extended Property *does not hold for the equilateral triangle, with the Dirichlet boundary condition on one side, and the Neumann condition on the other sides.*

Appendix A. Eigenvalues of the equilateral triangle

In this appendix, we recall the description of the eigenvalues of the equilateral triangle. For the reader's convenience, we retain the notation of [7, Section 2].

A.1. **General formulas.** Let $\mathbb{E}^2$ be the Euclidean plane with the canonical orthonormal basis $\{e_1 = (1,0), e_2 = (0,1)\}$, scalar product $\langle \cdot, \cdot \rangle$ and associated norm $|\cdot|$.

Consider the vectors

$$\alpha_1 = (1, -\frac{1}{\sqrt{3}}), \alpha_2 = (0, \frac{2}{\sqrt{3}}), \alpha_3 = (1, \frac{1}{\sqrt{3}}) = \alpha_1 + \alpha_2\,, \tag{A.1}$$

and

$$\alpha_1^\vee = (\frac{3}{2}, -\frac{\sqrt{3}}{2}), \alpha_2^\vee = (0, \sqrt{3}), \alpha_3^\vee = (\frac{3}{2}, \frac{\sqrt{3}}{2}) = \alpha_1^\vee + \alpha_2^\vee\,. \tag{A.2}$$

Then

$$\alpha_i^\vee = \frac{3}{2}\alpha_i, |\alpha_i|^2 = \frac{4}{3}, |\alpha_i^\vee|^2 = 3. \tag{A.3}$$

Define the mirror symmetries

$$s_i(x) = x - 2\frac{\langle x, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i = x - \frac{2}{3}\langle x, \alpha_i^\vee\rangle \alpha_i^\vee\,, \tag{A.4}$$

whose axes are the lines

$$L_i = \{x \in \mathbb{E}^2 \mid \langle x, \alpha_i\rangle = 0\}. \tag{A.5}$$

Let W be the group generated by these mirror symmetries. Then,

$$W = \{1, s_1, s_2, s_3, s_1 \circ s_2, s_1 \circ s_1\}\,, \tag{A.6}$$

where $s_1 \circ s_2$ (resp. $s_2 \circ s_1$) is the rotation with center the origin and angle $\frac{2\pi}{3}$ (resp. $-\frac{2\pi}{3}$).

Remark. The above vectors are related to the root system A_2 and W is the Weyl group of this root system.

Let

$$\Gamma = \mathbb{Z}\alpha_1^\vee \oplus \mathbb{Z}\alpha_2^\vee \tag{A.7}$$

be the (equilateral) lattice. The set

$$\mathcal{D}_\Gamma = \{s\alpha_1^\vee + t\alpha_2^\vee \mid 0 \le s, t \le 1\} \tag{A.8}$$

is a fundamental domain for the action of Γ on $\mathbb{E}^2$. Another fundamental domain is the closure of the open hexagon (see Figure 7.2)

$$\mathcal{H} = [A, B, C, D, E, F]\,, \tag{A.9}$$

whose vertices are given by

$$\begin{cases} A = (1,0); B = (\frac{1}{2}, \frac{\sqrt{3}}{2}); (-\frac{1}{2}, \frac{\sqrt{3}}{2}); \\ D = (-1,0); E = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}); F = (\frac{1}{2}, -\frac{\sqrt{3}}{2})\,. \end{cases} \tag{A.10}$$

Call $\mathcal{T}_e$ the equilateral triangle

$$\mathcal{T}_e = [O, A, B]\,, \tag{A.11}$$

where $O = (0,0)$.

Let Γ^* be the dual lattice of the lattice Γ, defined by

$$\Gamma^* = \{x \in \mathbb{E}^2 \mid \forall \gamma \in \Gamma, \langle x, \gamma \rangle \in \mathbb{Z}\}\,. \tag{A.12}$$

Then,

$$\begin{cases} \Gamma^* = \mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2\,, \\ \text{where } \varpi_1 = (\frac{2}{3}, 0) \text{ and } \varpi_2 = (\frac{1}{3}, \frac{1}{\sqrt{3}})\,. \end{cases} \tag{A.13}$$

Define the set C (an open Weyl chamber of the root system A_2),

$$C = \{x\varpi_1 + y\varpi_2 \mid x, y > 0\}\,, \tag{A.14}$$

and let $\mathbb{T}_e$ denote the equilateral torus $\mathbb{E}^2/\Gamma$.

A complete set of orthogonal (not normalized) eigenfunctions of $-\Delta$ on $\mathbb{T}_e$ is given (in complex form) by the exponentials

$$\phi_p(x) = \exp(2i\pi\langle x, p\rangle) \text{ where } x \in \mathbb{E}^2 \text{ and } p \in \Gamma^*\,. \tag{A.15}$$

Furthermore, for $p = m\varpi_1 + n\varpi_2$, with $m, n \in \mathbb{Z}$, the multiplicity of the eigenvalue $\hat{\lambda}(m,n) = 4\pi^2|p|^2 = \frac{16\pi^2}{9}(m^2 + mn + n^2)$ is equal to the number of points (k, ℓ) in $\mathbb{Z}^2$ such that $k^2 + k\ell + \ell^2 = m^2 + mn + n^2$.

The closure of the equilateral triangle $\mathcal{T}_e$ is a fundamental domain of the action of the semi-direct product $\Gamma \rtimes W$ on $\mathbb{E}^2$ or equivalently, a fundamental domain of the action of W on $\mathbb{T}^2_e$.

For the following proposition, we refer to [6].

Proposition A.1. *Complete orthogonal (not normalized) sets of eigenfunctions of the equilateral triangle $\mathcal{T}_e$ in complex form are given, respectively for the Dirichlet (resp. Neumann) boundary condition on $\partial\mathcal{T}_e$, as follows.*

(1) *Dirichlet boundary condition on $\partial\mathcal{T}_e$. The family is*

$$\Phi^{\mathfrak{d}}_p(x) = \sum_{w \in W} det(w) exp(2i\pi\langle x, w(p)\rangle) \tag{A.16}$$

with $p \in C \cap \Gamma^*$. Furthermore, for $p = m\varpi_1 + n\varpi_2$, with m, n positive integers, the multiplicity of the eigenvalue $4\pi^2|p|^2$ is equal to the number of solutions $q \in C \cap \Gamma^*$ of the equation $|q|^2 = |p|^2$.

(2) *Neumann boundary condition on $\partial\mathcal{T}_e$. The family is*

$$\Phi^{\mathfrak{n}}_p(x) = \sum_{w \in W} exp(2i\pi\langle x, w(p)\rangle) \tag{A.17}$$

with $p \in \overline{C} \cap \Gamma^*$. Furthermore, for $p = m\varpi_1 + n\varpi_2$, with m, n non-negative integers, the multiplicity of the eigenvalue $4\pi^2|p|^2$ is equal to the number of solutions $q \in \overline{C} \cap \Gamma^*$ of the equation $|q|^2 = |p|^2$.

Remark. To obtain corresponding complete orthogonal sets of real eigenfunctions, it suffices to consider the functions

$$C_p = \Re(\Phi_p) \text{ and } S_p = \Im(\Phi_p)\,.$$

For $p = m\varpi_1 + n\varpi_2$, with $m, n \in \mathbb{N} \setminus \{0\}$ for the Dirichlet boundary condition (resp. $m, n \in \mathbb{N}$ for the Neumann boundary condition), we denote these functions by $C_{m,n}$ and $S_{m,n}$.

In order to give explicit formulas for the first eigenfunctions, we have to examine the action of the group W on the lattice Γ^*. A simple calculation yields the following table in which we simply denote $m\varpi_1 + n\varpi_2$ by (m, n).

(A.18)

w	(m,n)	$w(m,n)$	$\det(w)$
1	(m,n)	(m,n)	1
s_1	(m,n)	$(-m, m+n)$	-1
s_2	(m,n)	$(m+n, -n)$	-1
s_3	(m,n)	$(-n, -m)$	-1
$s_1 \circ s_2$	(m,n)	$(-m-n, m)$	1
$s_2 \circ s_1$	(m,n)	$(n, -m-n)$	1

Remark. The above table should be compared with [7, Table], in which there is a slight unimportant error (the lines $s_1 \circ s_2$ and $s_2 \circ s_1$ are interchanged).

Remark. Using the above chart, one can easily prove the following relations.

(A.19)
$$\begin{cases} C^{\mathfrak{d}}_{n,m} = -C^{\mathfrak{d}}_{m,n} & \text{and} \quad S^{\mathfrak{d}}_{n,m} = S^{\mathfrak{d}}_{m,n}\,, \\ C^{\mathfrak{n}}_{n,m} = C^{\mathfrak{n}}_{m,n} & \text{and} \quad S^{\mathfrak{n}}_{n,m} = -S^{\mathfrak{n}}_{m,n}\,. \end{cases}$$

A.2. **Neumann boundary condition, first three eigenfunctions.** The first Neumann eigenvalue of $\mathcal{T}_e$ is 0, corresponding to the point $0 = (0,0) \in \Gamma^*$, with first eigenfunction $\varphi_1 \equiv 1$ up to scaling.

The second Neumann eigenvalue corresponds to the pairs $(1,0)$ and $(0,1)$. According to the preceding remark, it suffices to consider $C^{\mathfrak{n}}_{1,0}$ and $S^{\mathfrak{n}}_{1,0}$. Using Proposition A.1, and the table (A.18), we find that, at the point $[s,t] = s\alpha_1^\vee + t\alpha_2^\vee$,

(A.20)
$$\begin{cases} C^{\mathfrak{n}}_{1,0}([s,t]) = 2\left(\cos(2\pi s) + \cos(2\pi(-s+t)) + \cos(2\pi t)\right)\,, \\ S^{\mathfrak{n}}_{1,0}([s,t]) = 2\left(\sin(2\pi s) + \sin(2\pi(-s+t)) - \sin(2\pi t)\right)\,. \end{cases}$$

Up to a factor 2, this gives the following two independent eigenfunctions for the Neumann eigenvalue $\frac{16\pi^2}{9}$, in the (x,y) variables, with $(x,y) = \left(\frac{3}{2}s, -\frac{\sqrt{3}}{2}s + \sqrt{3}t\right)$ or $(s,t) = \left(\frac{2}{3}x, \frac{1}{3}x + \frac{1}{\sqrt{3}}y\right)$,

$$\text{(A.21)} \qquad \begin{cases} \varphi_2^{\mathfrak{n}}(x,y) = & \cos(\frac{4\pi}{3}x) + \cos(\frac{2\pi}{3}(-x+\sqrt{3}y)) \\ & + \cos(\frac{2\pi}{3}(x+\sqrt{3}y))\,, \\ \varphi_3^{\mathfrak{n}}(x,y) = & \sin(\frac{4\pi}{3}x) + \sin(\frac{2\pi}{3}(-x+\sqrt{3}y)) \\ & - \sin(\frac{2\pi}{3}(x+\sqrt{3}y))\,. \end{cases}$$

The first eigenfunction is invariant under the mirror symmetry with respect to the median OM of the equilateral triangle, see Figure 3.1. The second eigenfunction is anti-invariant under the mirror symmetry with respect to this median. Its nodal set is equal to the median itself.

A.3. **Dirichlet boundary condition, first three eigenfunctions.** The first Dirichlet eigenvalue of $\mathcal{T}_e$ is $\delta_1(\mathcal{T}_e) = \frac{16\pi^2}{3}$. A first eigenfunction is given by $S^{\mathfrak{d}}_{1,1}$. Using Proposition A.1 and Table A.18, we find that this eigenfunction is given, at the point $[s,t] = s\alpha_1^\vee + t\alpha_2^\vee$, by the formula

$$\text{(A.22)} \qquad \begin{cases} \varphi_1^{\mathfrak{d}}([s,t]) = & 2\sin 2\pi(s+t) + 2\sin 2\pi(s-2t) \\ & + 2\sin 2\pi(t-2s)\,. \end{cases}$$

Substituting the expressions of s and t in terms of x and y, one obtains the formula,

$$\text{(A.23)} \qquad \begin{array}{rl} \varphi_1^{\mathfrak{d}}(x,y) = & 2\sin\left(2\pi(x+\frac{y}{\sqrt{3}})\right) - 2\sin\left(4\pi\frac{y}{\sqrt{3}}\right) \\ & -2\sin\left(2\pi(x-\frac{y}{\sqrt{3}})\right)\,, \end{array}$$

The second Dirichlet eigenvalue has multiplicity 2,

$$\delta_2(\mathcal{T}_e) = \delta_3(\mathcal{T}_e) = \frac{112\pi^2}{9}\,.$$

The eigenfunctions $C^{\mathfrak{d}}_{2,1}$ and $S^{\mathfrak{d}}_{2,1}$ are respectively anti-invariant and invariant under the mirror symmetry with respect to $[OM]$, with values at the point $[(s,t)]$ given by the formulas,

$$\text{(A.24)} \qquad \begin{cases} \varphi_2^{\mathfrak{d}}([s,t]) = & \sin 2\pi(2s+t) + \sin 2\pi(s+2t) \\ & + \sin 2\pi(2s-3t) - \sin 2\pi(3s-2t) \\ & + \sin 2\pi(s-3t) - \sin 2\pi(3s-t)\,, \\ \varphi_3^{\mathfrak{d}}([s,t]) = & \cos 2\pi(2s+t) - \cos 2\pi(s+2t) \\ & - \cos 2\pi(2s-3t) + \cos 2\pi(3s-2t) \\ & + \cos 2\pi(s-3t) - \cos 2\pi(3s-t)\,. \end{cases}$$

Substituting the expressions of s and t in terms of x and y, one obtains the formulas,

$$\begin{aligned}\varphi_2^{\mathfrak{d}}(x,y) = \ & \sin\left(\tfrac{2\pi}{3}(5x+\sqrt{3}y)\right) - \sin\left(\tfrac{2\pi}{3}(5x-\sqrt{3}y)\right)\\ & + \sin\left(\tfrac{2\pi}{3}(x-3\sqrt{3}y)\right) - \sin\left(\tfrac{2\pi}{3}(x+3\sqrt{3}y)\right)\\ & + \sin\left(\tfrac{4\pi}{3}(2x+\sqrt{3}y)\right) - \sin\left(\tfrac{4\pi}{3}(2x-\sqrt{3}y)\right)\ .\end{aligned} \tag{A.25}$$

and

$$\begin{aligned}\varphi_3^{\mathfrak{d}}(x,y) = \ & \cos\left(\tfrac{2\pi}{3}(5x+\sqrt{3}y)\right) - \cos\left(\tfrac{2\pi}{3}(5x-\sqrt{3}y)\right)\\ & + \cos\left(\tfrac{2\pi}{3}(x-3\sqrt{3}y)\right) - \cos\left(\tfrac{2\pi}{3}(x+3\sqrt{3}y)\right)\\ & + \cos\left(\tfrac{4\pi}{3}(2x+\sqrt{3}y)\right) - \cos\left(\tfrac{4\pi}{3}(2x-\sqrt{3}y)\right)\ .\end{aligned} \tag{A.26}$$

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PB: Institut Fourier, Université Grenoble Alpes and CNRS, B.P.74, F38402 Saint Martin d'Hères Cedex, France.
E-mail address: pierrehberard@gmail.com

BH: Laboratoire Jean Leray, Université de Nantes and CNRS, F44322 Nantes Cedex, France, and LMO, Université Paris-Sud.
E-mail address: Bernard.Helffer@univ-nantes.fr