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# ON COURANT'S NODAL DOMAIN PROPERTY FOR LINEAR COMBINATIONS OF EIGENFUNCTIONS

PIERRE BÉRARD AND BERNARD HELFFER

ABSTRACT. We revisit Courant's nodal domain property for linear combinations of eigenfunctions, and propose new, simple and explicit counterexamples for domains in  $\mathbb{R}^2$ ,  $\mathbb{S}^2$ ,  $\mathbb{T}^2$ , or  $\mathbb{R}^3$ .

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^d$  be a bounded open domain or, more generally, a compact Riemannian manifold with boundary.

Consider the eigenvalue problem

$$(1.1) \quad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $B(u)$  is some homogeneous boundary condition on  $\partial\Omega$ , so that we have a self-adjoint boundary value problem (including the empty condition if  $\Omega$  is a closed manifold). For example, we can choose  $D(u) = u|_{\partial\Omega}$  for the Dirichlet boundary condition, or  $N(u) = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$  for the Neumann boundary condition.

Call  $H(\Omega, B)$  the associated self-adjoint extension of  $-\Delta$ , and list its eigenvalues in nondecreasing order, counting multiplicities,

$$(1.2) \quad 0 \leq \lambda_1(\Omega, B) < \lambda_2(\Omega, B) \leq \lambda_3(\Omega, B) \leq \dots$$

For any  $n \geq 1$ , define the number

$$(1.3) \quad \tau(\Omega, B, n) = \min\{k \mid \lambda_k(\Omega, B) = \lambda_n(\Omega, B)\}.$$

**Shorthand.** We use  $u \in \lambda_n(\Omega, B)$  as a shorthand for  $u$  is an eigenfunction of  $H(\Omega, B)$  associated with the eigenvalue  $\lambda_n(\Omega, B)$  i.e.,

$$H(\Omega, B)(u) = \lambda_n(\Omega, B) u.$$

Given a real continuous function  $v$  on  $\Omega$ , define its *nodal set*

$$(1.4) \quad \mathcal{Z}(v) = \overline{\{x \in \Omega \mid v(x) = 0\}},$$

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and call  $\beta_0(v)$  the number of connected components of  $\Omega \setminus \mathcal{Z}(v)$  i.e., the number of *nodal domains* of  $v$ .

**Theorem 1.1.** *[Courant (1923)]*

For any nonzero  $u \in \lambda_n(\Omega, B)$ ,

$$(1.5) \quad \beta_0(u) \leq \tau(\Omega, B, n) \leq n.$$

Courant's nodal domain theorem can be found in [9, Chap. V.6]. A footnote in [9, p. 454] indicates that this theorem also holds for any linear combination of the  $n$  first eigenfunctions, and refers to the PhD thesis of Horst Herrmann (Göttingen, 1932) [16].

For later reference, we write a precise statement. Given  $r > 0$ , denote by  $\mathcal{L}(\Omega, B, r)$  the space of linear combinations of eigenfunctions of  $H(\Omega, B)$  associated with eigenvalues less than or equal to  $r$ ,

$$(1.6) \quad \mathcal{L}(\Omega, B, r) = \left\{ \sum_{\lambda_j(\Omega, B) \leq r} c_j u_j \mid c_j \in \mathbb{R}, u_j \in \lambda_j(\Omega, B) \right\}.$$

**Statement 1.2.** *[Extended Courant Property]*

Let  $v \in \mathcal{L}(\Omega, B, \lambda_n(\Omega, B))$  be any linear combination of eigenfunctions associated with the  $n$  first eigenvalues of the eigenvalue problem (1.1). Then,

$$(1.7) \quad \beta_0(v) \leq \tau(\Omega, B, n) \leq n.$$

The footnote in [9, p. 454] actually states that the *Extended Courant Property* is true.

### Remarks on the Extended Courant Property.

1. Statement 1.2 is true for Sturm-Liouville equations. This was first announced by C. Sturm in 1833, [30] and proved in [31]. Other proofs were later on given by J. Liouville and Lord Rayleigh who both cite Sturm explicitly.

2. Å. Pleijel mentions Statement 1.2 in his well-known paper [27] on the asymptotic behaviour of the number of nodal domains of a Dirichlet eigenfunction associated with the  $n$ -th eigenvalue in a plane domain. At the end of the paper, he also considers the Neumann boundary condition.

*“In order to treat, for instance the case of the free three-dimensional membrane  $[0, \pi]^3$ , it would be necessary to use, in a special case, the theorem quoted in [8], p. 394<sup>1</sup>. This theorem which generalizes part of the Liouville-Rayleigh theorem for the string asserts that a linear combination, with constant coefficients, of the  $n$  first eigenfunctions can have at most  $n$  nodal domains. However, as far as I have been able to find there is no proof of this assertion in the literature.”*

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<sup>1</sup>Pleijel refers to the German edition, this is p. 454 in the English edition [9].

**3.** Arnold [2], see also [21, 22] and [19], mentions that he actually discussed the footnote in [9, p. 454] with R. Courant, that the *Extended Courant Property* cannot be true, and that O. Viro produced counter-examples for the 3-sphere  $\mathbb{S}^3$ , and any degree larger than or equal to 6 (see [32] and the citation below). More precisely, as early as 1973, V. Arnold [1] pointed out that, when  $\Omega$  is the round sphere  $\mathbb{S}^N$ , the *Extended Courant Property* is related to Hilbert’s 16th problem. Indeed, the eigenfunctions of the Laplace-Beltrami operator on  $\mathbb{S}^N$  are the spherical harmonics i.e., the restrictions to the sphere of the harmonic homogeneous polynomials, so that the linear combinations of spherical harmonics of degree less than or equal to  $n$  are the restrictions to the sphere of the homogeneous polynomials of degree  $n$  in  $(N + 1)$  variables. The following citation is taken from [3].

*Eigen oscillations of the sphere with the standard metric are described by spherical functions, i.e., polynomials. Therefore the Courant statement cited above implies the following estimate*

$$\dim_{\mathbb{R}} H_0(\mathbb{R}P^N - V_n, \mathbb{R}) \leq C_{N+n-2}^N + 1 \quad (1)$$

*for the number of connected components of the complement to an algebraic hypersurface of degree  $n$  in the  $N$ -dimensional projective space.*

*For planar curves ( $N = 2$ ), the estimation (1) is exact (it turns into equality on a configuration of  $n$  lines in general position) and can be proven independently of the Courant statement. For smooth surfaces of degree 4 in  $\mathbb{R}P^3$  the estimation is also exact and proved (by V.M. Kharlamov).*

*In the general case, the Courant statement is false (a counter-example can be constructed by a small perturbation of the standard metric on the sphere). Nonetheless the estimation (1) seems to be plausible: for proving it one has to verify the Courant statement only for oscillation of the sphere (or the projective space) with the standard metric.<sup>1</sup>*

<sup>1</sup> Translator’s remark: the inequality (1) does not hold true for surfaces of even degree  $\geq 6$  in  $\mathbb{R}P^3$ . Counter-examples to (1) were constructed in the paper of O. Viro, “Construction of multicomponent real algebraic surfaces”, *Soviet Math. Dokl.* **20**, N<sup>o</sup>. 5, 991–995 (1979).

**4.** In [12], Gladwell and Zhu refer to Statement 1.2 as the *Courant-Herrmann conjecture*. They claim that this extension of Courant’s theorem is not stated, let alone proved, in Herrmann’s thesis or subsequent publications<sup>2</sup>. They consider the case in which  $\Omega$  is a rectangle in  $\mathbb{R}^2$ , stating that they were not able to find a counter-example to the *Extended Courant Property* in this case. They also provide numerical evidence that there are counter-examples for more complicated (non convex) domains.

<sup>2</sup>The only relevant one seems to us [17].

The purpose of the present paper is to provide simple counter-examples to the *Extended Courant Property* for domains in  $\mathbb{R}^2$ ,  $\mathbb{T}^2$ ,  $\mathbb{S}^2$  or  $\mathbb{R}^3$ , including convex domains.

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## 2. RECTANGULAR MEMBRANE, DIRICHLET BOUNDARY CONDITION

We summarize the ideas from [12].

Consider the square  $\mathcal{S}_\pi = ]0, \pi[^2$ , with Dirichlet boundary condition. The eigenvalues are given by the numbers

$$q_2(m, n) = m^2 + n^2, \quad \text{for } m, n \in \mathbb{N}^\bullet.$$

More precisely, the Dirichlet eigenvalues of the square  $\mathcal{S}_\pi$  are

$$\begin{aligned} \delta_1 [2] < \delta_2 = \delta_3 [5] < \delta_4 [8] < \delta_5 = \delta_6 [10] < \dots \\ < \delta_7 = \delta_8 [13] < \delta_9 = \delta_{10} [17] < \delta_{11} [18] < \dots \end{aligned}$$

In this list the numbers in brackets are the actual values of the eigenvalues, for example,  $\delta_2 = \delta_3 = 5$ .

A corresponding complete family of Dirichlet eigenfunctions for the square  $\mathcal{S}_\pi$  is given by the functions  $\phi_{m,n}(x, y) = \sin(mx) \sin(ny)$  for  $m, n \in \mathbb{N}^\bullet$ . Using the classical Chebyshev polynomials, we have

$$\phi_{m,n}(x, y) = \phi_{1,1}(x, y) U_{m-1}(\cos x) U_{n-1}(\cos y).$$

Given some positive  $r$ , denote by  $\mathcal{L}_r$  the set  $\mathcal{L}(\mathcal{S}_\pi, D, r)$ . A function  $\Phi \in \mathcal{L}_r$  is of the form

$$\Phi(x, y) := \sum_{q_2(m,n) \leq r} c_{m,n} \phi_{m,n}(x, y).$$

When looking at the nodal pattern of  $\Phi$ , we can factor out the non-vanishing factor  $\phi_{1,1}$ , and consider instead the nodal pattern of the function,

$$\Phi_1(x, y) = \sum_{q_2(m,n) \leq r} c_{m,n} U_{m-1}(\cos x) U_{n-1}(\cos y).$$

On the other-hand, using the diffeomorphism

$$F : ]0, \pi[ \ni (x, y) \mapsto (X, Y) := (\cos x, \cos y) \in ]-1, 1[,$$

we see that the nodal pattern of  $\Phi$  is diffeomorphic to the nodal pattern of the function,

$$\Psi(X, Y) := \sum_{q_2(m,n) \leq r} c_{m,n} U_{m-1}(X) U_{n-1}(Y),$$

for  $(X, Y) \in ]-1, 1[^2$ .

Choosing  $r = \delta_6 = 10$ , i.e. linear combinations involving the six first Dirichlet eigenfunctions of the square, the linear combinations  $\Psi$  generate the subspace of  $\mathbb{R}[X, Y]$  spanned by the family

$$\{1, X, Y, X^2, XY, Y^2\}$$

i.e., the polynomials in two variables, of degree less than or equal to 2. An immediate consequence of this analysis is that any  $d \in \{1, 2, 3, 4, 5\}$  is achieved as  $\beta_0(\Phi)$  for some  $\Phi \in \mathcal{L}_{10}$ . Notice that 5 is precisely Courant's bound  $\tau(\mathcal{S}_\pi, D, 6)$ , see Figure 2.1.

Choosing  $r = \delta_{10} = 17$ , i.e. linear combinations involving the ten first eigenvalues, the linear combinations generate the subspace spanned by the family

$$\{1, X, Y, X^2, XY, Y^2, XY^2, X^2Y, X^3, Y^3\},$$

which corresponds to the polynomials of degree less than or equal to 3. An immediate consequence is that any  $d \in \{1, 2, 3, 4, 5, 6, 7, 8\}$  is achieved as  $\beta_0(\Phi)$  for some  $\Phi \in \mathcal{L}_{17}$ . Notice that  $8 < 9 = \tau(\mathcal{S}_\pi, D, 10)$ .

It does not seem possible to find an example of  $\Phi \in \mathcal{L}_{17}$  with nine or more nodal domains.

**Table 1** CHC is true for the first 13 eigenfunctions on the square.









	m, n	New High Order Terms		Maximum No. of Nodal domains	
1	1, 1	1		1	
2, 3	1, 2 2, 1	X	Y	2	
4	2, 2	XY		4	
5, 6	1, 3 3, 1	X <sup>2</sup>	Y <sup>2</sup>	5	
7, 8	2, 3 3, 2	X <sup>2</sup> Y	X Y <sup>2</sup>	7	
9, 10	1, 4 4, 1	X <sup>3</sup>	Y <sup>3</sup>	8	
11	3, 3	X <sup>2</sup> Y <sup>2</sup>		10	
12, 13	2, 4 4, 2	X <sup>3</sup> Y	X Y <sup>3</sup>	12	

FIGURE 2.1. The pictures and computations of Gladwell-Zhu [12]

## 3. RECTANGLE WITH A CRACK

Let  $\mathcal{R}_0$  be the rectangle  $]0, 4\pi[ \times ]0, 2\pi[$ . For  $0 < a \leq 1$ , let  $C_a := ]0, a] \times \{\pi\}$  and  $\mathcal{R}_a := \mathcal{R} \setminus C_a$ . In this section, we only consider the Neumann boundary condition on  $C_a$ , and either the Dirichlet or Neumann boundary condition on  $\partial\mathcal{R}$ . The setting is that described in [11, Section 8].

We call

$$(3.1) \quad \begin{cases} 0 < \delta_1(0) < \delta_2(0) \leq \delta_3(0) \leq \dots \\ \text{resp.} \\ 0 = \nu_1(0) < \nu_2(0) \leq \nu_3(0) \leq \dots \end{cases}$$

the eigenvalues of  $-\Delta$  in  $\mathcal{R}$ , with Dirichlet (resp. Neumann) boundary condition on  $\partial\mathcal{R}$ . They are given by the  $\frac{m^2}{16} + \frac{n^2}{4}$  for pairs  $(m, n)$  of positive integers for the Dirichlet problem (resp. non-negative integers for the Neumann problem). Corresponding eigenfunctions are products of sines (Dirichlet) or cosines (Neumann).

The eigenvalues are arranged in non-decreasing order, counting multiplicities.

Similarly, call

$$(3.2) \quad \begin{cases} 0 < \delta_1(a) < \delta_2(a) \leq \delta_3(a) \leq \dots \\ \text{resp.} \\ 0 = \nu_1(a) < \nu_2(a) \leq \nu_3(a) \leq \dots \end{cases}$$

the eigenvalues of  $-\Delta$  in  $\mathcal{R}_a$ , with Dirichlet (resp. Neumann) boundary condition on  $\partial\mathcal{R}$ , and Neumann boundary condition on  $C_a$ .

The first three Dirichlet (resp. Neumann) eigenvalues for the rectangle  $\mathcal{R}$  are as follows.

Eigenvalue	Value	Pairs	Eigenfunctions
$\delta_1(0)$	$\frac{5}{16}$	$(1, 1)$	$\phi_1(x, y) = \sin(\frac{x}{4}) \sin(\frac{y}{2})$
$\delta_2(0)$	$\frac{1}{2}$	$(2, 1)$	$\phi_2(x, y) = \sin(\frac{x}{2}) \sin(\frac{y}{2})$
$\delta_3(0)$	$\frac{13}{16}$	$(3, 1)$	$\phi_3(x, y) = \sin(\frac{3x}{4}) \sin(\frac{y}{2})$
$\nu_1(0)$	0	$(0, 0)$	$\psi_1(x, y) = 1$
$\nu_2(0)$	$\frac{1}{16}$	$(1, 0)$	$\psi_2(x, y) = \cos(\frac{x}{4})$
$\nu_3(0)$		$(0, 1)$	$\psi_3(x, y) = \cos(\frac{y}{2})$
$\nu_4(0)$	$\frac{1}{4}$	$(2, 0)$	$\psi_4(x, y) = \cos(\frac{x}{2})$

We summarize [11], Propositions (8.5), (8.7), (9.5) and (9.9), into the following theorem.

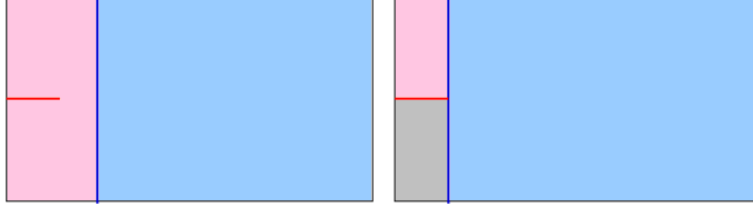


FIGURE 3.1. Rectangle with a crack (Neumann condition)

**Theorem 3.1.** *With the above notation, the following properties hold.*

- (1) *For  $i \geq 1$ , the functions  $[0, 1] \ni a \mapsto \delta_i(a)$ , resp.  $[0, 1] \ni a \mapsto \nu_i(a)$ , are non-increasing.*
- (2) *For  $i \geq 1$ , the functions  $]0, 1[ \ni a \mapsto \delta_i(a)$ , resp.  $]0, 1[ \ni a \mapsto \nu_i(a)$ , are continuous.*
- (3) *For  $i \geq 1$ ,  $\lim_{a \rightarrow 0+} \delta_i(a) = \delta_i(0)$  and  $\lim_{a \rightarrow 0+} \nu_i(a) = \nu_i(0)$ .*

It follows that for  $0 < a$ , small enough, we have

$$(3.4) \quad \begin{cases} 0 < \delta_1(a) \leq \delta_1(0) < \delta_2(a) \leq \delta_2(0) < \delta_3(a) \leq \delta_3(0), \text{ and} \\ 0 = \nu_1(a) = \nu_1(0) < \nu_2(a) \leq \nu_2(0) < \nu_3(a) \leq \nu_4(a) \leq \nu_3(0). \end{cases}$$

Observe that for  $i = 1$  and  $2$ ,  $\frac{\partial \phi_i}{\partial y}(x, \pi) = 0$  and  $\frac{\partial \psi_i}{\partial y}(x, y) = 0$ . It follows that for  $a$  small enough, the functions  $\phi_1$  and  $\phi_2$  (resp. the functions  $\psi_1$  and  $\psi_2$ ) are the first two eigenfunctions for  $\mathcal{R}_a$  with the Dirichlet (resp. Neumann) boundary condition on  $\partial \mathcal{R}$ , and the Neumann boundary condition on  $C_a$ , with associated eigenvalues  $\frac{5}{16}$  and  $\frac{1}{2}$  (resp.  $0$  and  $\frac{1}{4}$ ).

We have

$$\alpha \phi_1(x, y) + \beta \phi_2(x, y) = \sin\left(\frac{x}{4}\right) \sin\left(\frac{y}{2}\right) \left(\alpha + 2\beta \cos\left(\frac{x}{4}\right)\right),$$

and

$$\alpha \psi_1(x, y) + \beta \psi_2(x, y) = \alpha + \beta \cos\left(\frac{x}{4}\right).$$

We can choose the coefficients  $\alpha, \beta$  in such a way that these linear combinations of the first two eigenfunctions have two (Figure 3.1 left) or three nodal domains (Figure 3.1 right) in  $\mathcal{R}_a$ .

**This proves that the *Extended Courant Property* is false in  $\mathcal{R}_a$  with either Dirichlet or Neumann condition on  $\partial \mathcal{R}$  and Neumann condition on  $C_a$ .**

**Remark.** In the Neumann case, notice that we can introduce several cracks  $\{(x, b_j) \mid 0 < x < a_j\}_{j=1}^k$  in such a way that for any  $d \in \{2, 3, \dots, k+2\}$  there exists a linear combination of  $1$  and  $\cos(\frac{x}{4})$  with  $d$  nodal domains.

**Remark.** Numerical simulations, kindly provided by Virginie Bonnaille-Noël, indicate that the *Extended Courant Property* does not hold for



a rectangle with a crack, with Dirichlet boundary condition on both the boundary of the rectangle and the crack, see Section 9. Dirichlet cracks appear in another context in [13] (see also references therein).

**Remark.** It is easy to make an analogous construction for the unit disk (Neumann case) with radial cracks. As computed for example in [15] (Subsection 3.4), the second radial eigenfunction has labelling 6 ( $\lambda_6 \approx 14,68$ ) and we can introduce six radial cracks to obtain a combination of the two first radial Neumann eigenfunctions with seven nodal domains, see Figure 3.2.

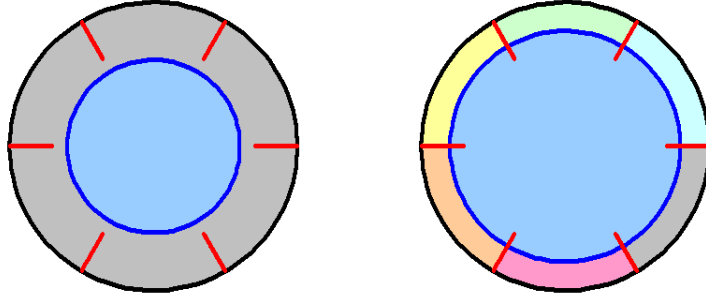


FIGURE 3.2. Disk with cracks, Neumann condition

#### 4. THE RECTANGULAR FLAT TORUS WITH CRACKS

Consider the flat torus  $\mathcal{T} := \mathbb{R}^2 / (4\pi\mathbb{Z} \oplus 2\pi\mathbb{Z})$ . Arrange the eigenvalues in nondecreasing order,

$$(4.1) \quad \lambda_1(0) < \lambda_2(0) \leq \lambda_3(0) \leq \dots$$

The eigenvalues are given by the numbers  $\frac{m^2}{4} + n^2$  for the pairs  $(m, n)$  of integers, with associated complex eigenfunctions

$$(4.2) \quad \exp(im\frac{x}{2}) \exp(iny)$$

or equivalently, the real eigenfunctions

$$(4.3) \quad \begin{aligned} &\cos(m\frac{x}{2}) \cos(ny), \cos(m\frac{x}{2}) \sin(ny), \\ &\sin(m\frac{x}{2}) \cos(ny), \sin(m\frac{x}{2}) \sin(ny), \end{aligned}$$

where  $m, n$  are non-negative integers. Accordingly, the first eigenpairs are as follows.

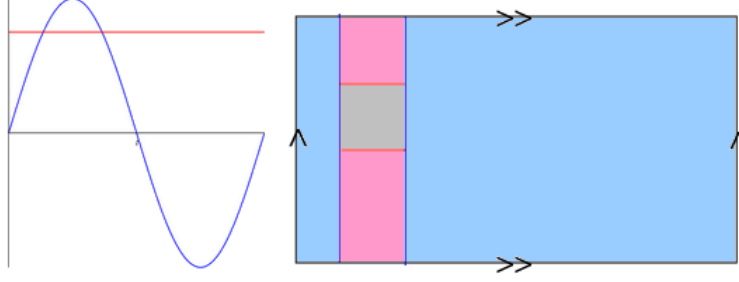


FIGURE 4.1. Flat torus with two cracks

(4.4)

Eigenvalue	Value	Pairs	Eigenfunctions
$\lambda_1(0)$	0	$(0, 0)$	$\omega_1(x, y) = 1$
$\lambda_2(0)$	$\frac{1}{4}$	$(1, 0)$	$\omega_2(x, y) = \cos(\frac{x}{2})$
$\lambda_3(0)$			$\omega_2(x, y) = \sin(\frac{x}{2})$
$\lambda_4(0)$	1	$(0, 1)$	$\omega_3(x, y) = \cos(y)$
$\lambda_5(0)$			$\omega_4(x, y) = \sin(y)$

A typical linear combination of the first three eigenfunctions is of the form  $\alpha + \beta \sin(\frac{x}{2} - \theta)$

Consider the torus  $\mathcal{T}$  and perform two (or more) cracks parallel to the  $y$  axis and with the same length  $a$ , in order to get a torus with cracks  $\mathcal{T}_a$ , Figure 4.1. For  $a$  small enough, the first three eigenfunctions of the torus  $\mathcal{T}$  remain eigenfunctions of the torus with cracks,  $\mathcal{T}_a$ , with the same  $\tau(\mathcal{T}_a, 3) = 2$ . We can choose the length  $a$  such that the nodal set of  $\alpha + \beta \sin(\frac{x}{2} - \theta)$  and the two cracks determine three nodal domains. The proof is the same as in [11].

**This proves that the *Extended Courant Property* is false on the flat torus with cracks.**

## 5. SPHERE $\mathbb{S}^2$ WITH CRACKS

On the round sphere  $\mathbb{S}^2$ , we consider the geodesic lines  $(x, y, z) \mapsto (\sqrt{1-z^2} \cos \theta_i, \sqrt{1-z^2} \sin \theta_i, z)$  through the north pole  $(0, 0, 1)$ , with distinct  $\theta_i \in [0, \pi[$ . For example, removing the geodesic segments  $\theta_0 = 0$  and  $\theta_2 = \frac{\pi}{2}$  with  $1 - z \leq a \leq 1$ , we obtain a sphere  $\mathbb{S}_a^2$  with a crack in the form of a cross. We consider the Neumann boundary condition on the crack.

We then easily produce a function in the space generated by the two first eigenspaces of the sphere with a crack having five nodal domains.

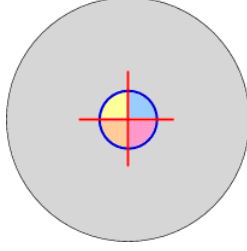


FIGURE 5.1. Sphere with crack, five nodal domains

The first eigenvalue of  $\mathbb{S}^2$  is  $\lambda_1(0) = 0$ , with corresponding eigenspace of dimension 1, generated by the function 1. The next eigenvalues of  $\mathbb{S}^2$  are  $\lambda_2(0) = \lambda_3(0) = \lambda_4(0) = 2$  with associated eigenspace of dimension 3, generated by the functions  $x, y, z$ . The following eigenvalues of  $\mathbb{S}^2$  are larger than or equal to 6.

As in [11], the eigenvalues of  $\mathbb{S}_a^2$  (with Neumann condition on the crack) are non-increasing in  $a$ , and continuous to the right at  $a = 0$ . More precisely

$$(5.1) \quad \begin{cases} 0 = \lambda_1(a) < \lambda_2(a) \leq \lambda_3(a) \leq \lambda_4(a) \leq 2 < \lambda_5(a) \leq 6, \\ \lim_{a \rightarrow 0+} \lambda_i(a) = 2 \text{ for } i = 2, 3, 4, \\ \lim_{a \rightarrow 0+} \lambda_5(a) = 6. \end{cases}$$

The function  $z$  is also an eigenfunction of  $\mathbb{S}_a^2$  with eigenvalue 2. It follows from (5.1) that for  $a$  small enough,  $\lambda_4(a) = 2$ , with eigenfunction  $z$ . For  $0 < b < a$ , the linear combination  $z - b$  has five nodal domains in  $\mathbb{S}_a^2$ , see Figure 5.1 in spherical coordinates. **It follows that the *Extended Courant Property* does not hold on the sphere with cracks.**

**Remark 1.** Removing more geodesic segments around the north pole, we can obtain a linear combination  $z - b$  with as many nodal domains as we want.

**Remark 2.** The sphere with cracks and Dirichlet condition on the cracks has been considered for another purpose in [14].

## 6. THE CUBE WITH DIRICHLET BOUNDARY CONDITION

In this section, we adapt the method used in [12] (and recalled in Section 2) to the 3D-case.

Consider the cube  $\mathcal{C}_\pi = ]0, \pi[^3$ . The eigenvalues are the numbers

$$q_3(k, m, n) = k^2 + m^2 + n^2, \quad k, m, n \in \mathbb{N}^\bullet.$$

A corresponding complete set of eigenfunctions is given by the functions

$$\phi_{k,m,n}(x, y, z) = \sin(kx) \sin(my) \sin(nz), \quad k, m, n \in \mathbb{N}^\bullet.$$

The first Dirichlet eigenvalues of the cube are given by

$$\delta_1 [3] < \delta_2 = \delta_3 = \delta_4 [6] < \delta_5 = \delta_6 = \delta_7 [9] < \dots \\ \delta_8 = \delta_9 = \delta_{10} [11] < \delta_{11} \dots$$

Using Chebyshev polynomials, for  $k, m, n \in \mathbb{N}^\bullet$  we have

$$\phi_{k,m,n}(x, y, z) = \phi_{1,1,1}(x, y, z) U_{k-1}(\cos x) U_{m-1}(\cos y) U_{n-1}(\cos z).$$

The factor  $\phi_{1,1,1}$  does not vanish in the cube  $\mathcal{C}_\pi$ . The map

$$\mathcal{C}_\pi \ni (x, y, z) \mapsto (X, Y, Z) := (\cos(x), \cos(y), \cos(z)) \in ]-1, 1[^3$$

is a diffeomorphism from  $\mathcal{C}_\pi$  to the cube  $] - 1, 1[^3$ .

Let  $\mathcal{L}_r$  now denote the set  $\mathcal{L}(\mathcal{C}_\pi, D, r)$ .

In view of the preceding remarks, in order to count the nodal domains of a linear combination  $\Phi \in \mathcal{L}_r$ ,

$$\Phi = \sum_{q_3(k,m,n) \leq r} c_{k,m,n} \phi_{k,m,n}$$

in the cube  $\mathcal{C}_\pi$ , it suffices to count the nodal domains of the corresponding linear combination,

$$\Psi = \sum_{q_3(k,m,n) \leq r} c_{k,m,n} U_{k-1}(X) U_{m-1}(Y) U_{n-1}(Z)$$

in the cube  $] - 1, 1[^3$ .

Using the formulas for the Chebyshev polynomials, it is easy to see that the linear combinations  $\Psi$  for  $k^2 + m^2 + n^2 \leq 11 = \delta_{10}$  correspond to the polynomials of degree less than or equal to 2 in the variables  $X, Y$  and  $Z$ .

In particular, the polynomial  $f_a(X, Y, Z) := X^2 + Y^2 + Z^2 - a$  can be represented as such a linear combination  $\Psi$  with  $k^2 + m^2 + n^2 \leq 11$ . We can consider the corresponding linear combination  $\phi_a(x, y, z) \in \mathcal{L}_{11} = \mathcal{L}_{\delta_{10}}$ .

When  $a < 0$ , the polynomial  $f_a$  is positive in  $] - 1, 1[^3$  and, correspondingly, the function  $\phi_a \in \mathcal{L}_{11}$  does not vanish in  $\mathcal{C}_\pi$ , so that it has one nodal domain.

When  $0 < a < \sqrt{2}$ , the polynomial  $f_a$  has two nodal domains in  $] - 1, 1[^3$  and, correspondingly, the function  $\phi_a \in \mathcal{L}_{11}$  has two nodal domains in  $\mathcal{C}_\pi$ .

When  $\sqrt{2} < a < \sqrt{3}$ , the polynomial  $f_a$  has nine nodal domains in  $] - 1, 1[^3$  and, correspondingly, the function  $\phi_a \in \mathcal{L}_{11}$  has nine nodal domains in  $\mathcal{C}_\pi$ .

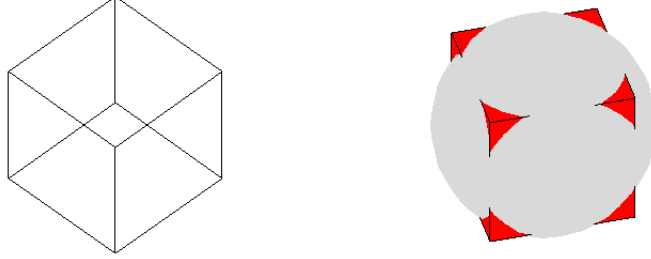


FIGURE 6.1. Cube with Dirichlet boundary condition

On the other-hand,  $\mathcal{L}_{11}$  involves eigenfunctions associated with eigenvalues less than or equal to 11 i.e., the ten first eigenfunctions. Since  $11 = \delta_8 = \delta_9 = \delta_{10}$ , Courant's upper bound is  $8 = \tau(\mathcal{C}_\pi, D, 10)$ .

**It follows that  $\phi_a$ ,  $\sqrt{2} < a < \sqrt{3}$ , provides a counter-example to the *Extended Courant Property* for the 3D-cube with Dirichlet boundary condition.**

**Remark.** The same method can be applied to the cube with Neumann boundary condition, but does apparently not provide counter-examples in this case.

## 7. AROUND HEXAGONS

We are looking for a convex counterexample in  $\mathbb{R}^2$ . We have already mentioned that this quest was unsuccessful for the square. It is natural to think of other polygons and between them the regular hexagon.

**7.1. Symmetries.** The Dirichlet and the Neumann boundary condition can both be considered.

Call  $\mathcal{H} = [ABCDEF]$  the regular hexagon with sides of length 1,  $\mathcal{T} = [OAB]$  the equilateral triangle, and  $\mathcal{T}_2 = [OAM]$  the hemiequilateral triangle. See Figure 7.1.

The Dirichlet and Neumann eigenvalues of  $\mathcal{T}$  and  $\mathcal{T}_2$  can be described explicitly, together with complete sets of eigenfunctions, see [5, 25, 24, 6].

In particular, the eigenvalues of  $\mathcal{T}$  are the numbers  $\frac{16\pi^2}{9}(m^2 + mn + n^2)$ , where  $m, n \in \mathbb{N}$  for the Neumann boundary condition, and  $m, n \in \mathbb{N}^\bullet$  for the Dirichlet boundary condition. Complete sets of eigenfunctions are given by functions  $C_{m,n}^*$  and  $S_{m,n}^*$  with  $*$   $\in \{D, N\}$  which are linear combinations of cosines and sines respectively. We provide a summary in Appendix A.

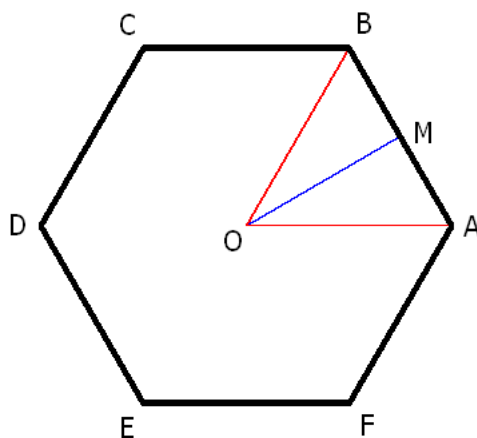


FIGURE 7.1. The hexagon

The Dirichlet (resp. Neumann) eigenfunctions of  $\mathcal{T}$  and  $\mathcal{T}_2$  yield special Dirichlet (resp. Neumann) eigenfunctions of  $\mathcal{H}$ . Although the eigenvalues / eigenfunctions of the hexagon are not all known explicitly, numerical methods provide very accurate results for the low lying eigenvalues, see for example [4, 10] for the Dirichlet boundary condition. This is sufficient to identify low lying simple eigenvalues and the corresponding eigenfunctions. Indeed, if an eigenvalue of  $\mathcal{H}$  is simple, a corresponding eigenfunction must be invariant or anti-invariant with respect to the mirror symmetries of the hexagon, and therefore arises from an eigenfunction of  $\mathcal{T}$  or  $\mathcal{T}_2$ , possibly with mixed boundary condition.

For example, the first Dirichlet eigenfunction  $u_{1,D}$  of  $\mathcal{H}$  arises from the first eigenfunction  $f_1$  of  $\mathcal{T}_2$ , with mixed boundary condition DNN (Dirichlet on the smaller side, Neumann on the other sides).

**7.2. Dirichlet boundary condition.** Numerical computations of the Dirichlet eigenvalues of the regular hexagon, see Figures 7.2 and 7.3, show that the 6-th eigenvalue is simple, and that the nodal set of the corresponding eigenfunction  $u_{6,D}$  is a closed simple line. It follows that this eigenfunction is invariant with respect to all mirror symmetries of the hexagon (symmetries with respect to the diagonals through the point 0, and with respect to the mediators of the sides). It therefore arises from an eigenfunction of  $\mathcal{T}_2$  with mixed boundary condition. The approximate values of the eigenvalues show that this must be the second eigenfunction of  $\mathcal{T}_2$  with mixed DNN boundary condition.

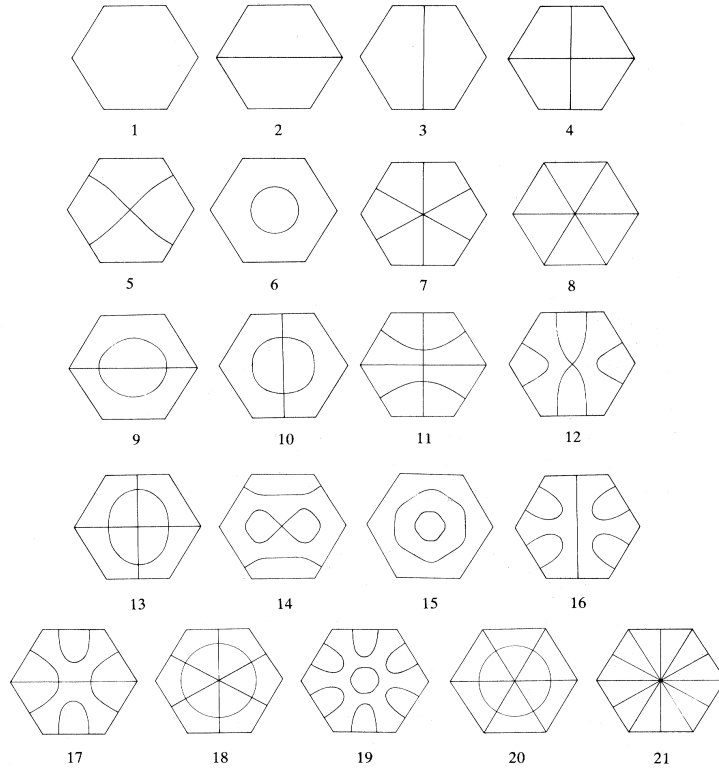


FIGURE 7.2. Nodal structure for the Dirichlet Laplacian in the hexagon [4, Fig. 2]

The corresponding pictures in Figure 7.2 and Figure 7.3 appear as slightly different. This is normal because we have 2-dimensional eigenspaces associated with  $\lambda_2 = \lambda_3$  and  $\lambda_4 = \lambda_5$ . The computations in [4] (Figure 7.2) take the symmetries into account from the beginning. Figure 7.3 was obtained using the MATLAB programme (PDE Toolbox) directly on the hexagon, the computed eigenfunctions are not always the symmetric ones.

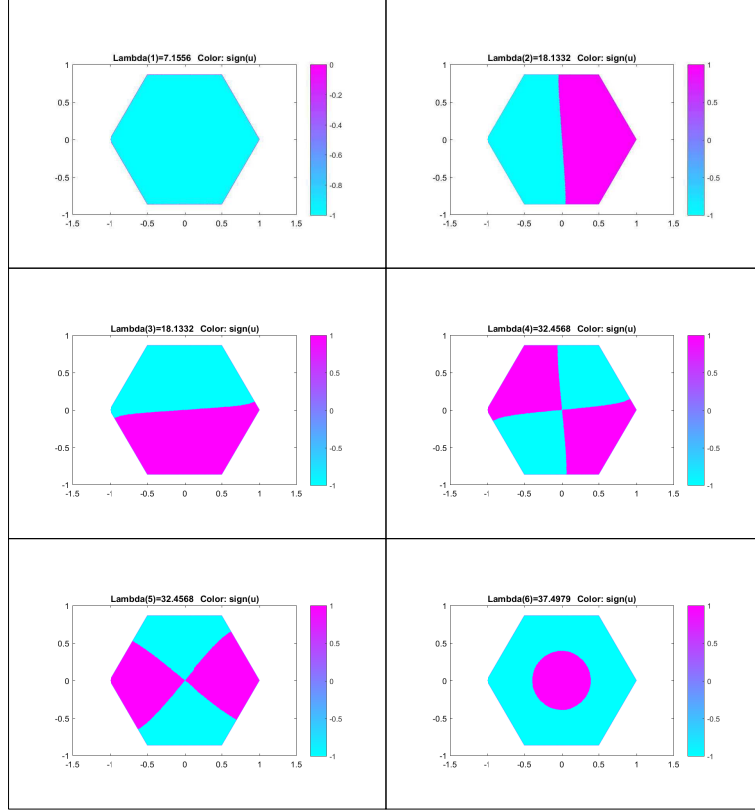


FIGURE 7.3. Eigenpairs for the hexagon with Dirichlet condition using MATLAB

The natural question is:

**Question 1:** Does there exist a linear combination of  $u_{1,D}$  and  $u_{6,D}$  with at least 7 nodal domains?

To answer Question 1, it would be sufficient to prove that one of the level lines  $f_2/f_1$  cuts the equilateral triangle into three parts.

As far as we know, the eigenfunctions  $f_2$  and  $f_1$  are not trigonometric polynomials. This is at least claimed in [26] (Section 3), without detailed proof.

This question can at least be attacked numerically (the main difficulty being that both  $f_1$  and  $f_2$  vanish on one side).



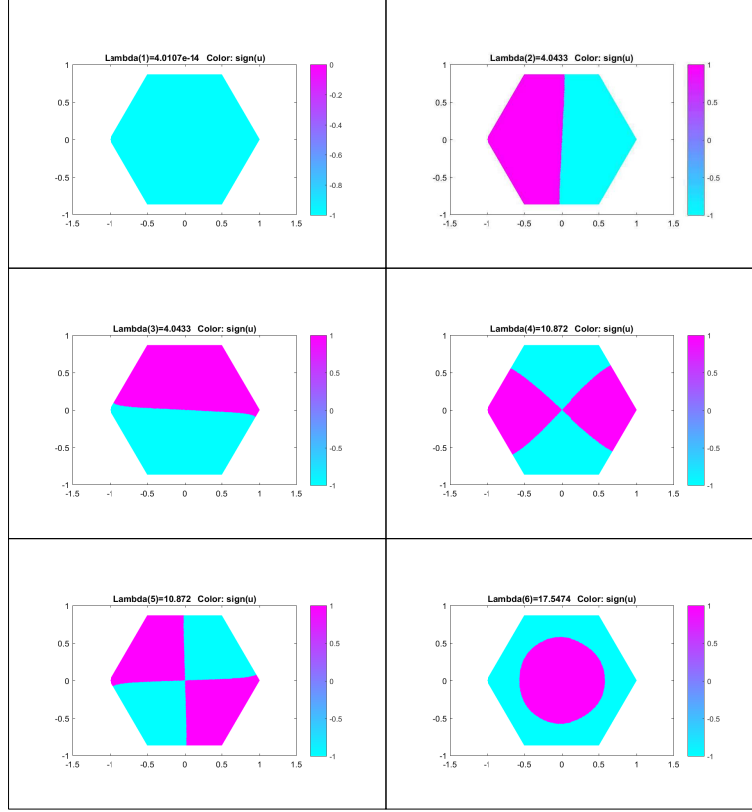


FIGURE 7.4. Eigenpairs for the hexagon with Neumann condition using MATLAB

Numerical simulations, kindly provided by Virginie Bonnaillie-Noël, suggest that the answer to Question 1 is positive. See Section 9.

**7.3. Neumann boundary condition.** The first eigenpairs for the hexagon with sides 1 and Neumann boundary conditions, as computed with MATLAB (PDE Toolbox), are shown in Figure 7.4.

The first Neumann eigenfunction of the hexagon is 0, with associated eigenfunction  $u_{1,N} \equiv 1$ . Figure 7.4 indicates that the 6-th Neumann eigenvalue of the hexagon is simple, and that the nodal set of the corresponding eigenfunction is a simple closed curve. It follows that the corresponding eigenfunction  $u_{6,N}$  is invariant under all mirror symmetries of the hexagon. It is obtained from an eigenfunction of the

hemiequilateral triangle  $\mathcal{T}_2 = [OAM]$  with Neumann boundary condition. The computed value  $\approx 17.5474$  shows that this corresponds to the second Neumann eigenfunction of the hemiequilateral triangle, associated with the eigenvalue  $\frac{16\pi^2}{9}$ .

The natural question is:

**Question 2:** Does there exist a linear combination of  $u_{1,N}$  and  $u_{6,N}$  with at least 7 nodal domains?

In the case of the Neumann boundary condition, we can give a complete proof, and not only, as under the Dirichlet boundary condition, a numerical proof. Indeed, the Neumann eigenvalues of the equilateral triangle  $\mathcal{T} = [OAB]$  can be described explicitly. The sequence begins as follows,

$$(7.1) \quad 0 = \lambda_1(\mathcal{T}, N) < \frac{16\pi^2}{9} = \lambda_2(\mathcal{T}, N) = \lambda_3(\mathcal{T}, N) < \lambda_4(\mathcal{T}, N).$$

Furthermore, the second eigenspace has dimension 2, and contains one eigenfunction  $\varphi_2$  which is invariant under the mirror symmetry with respect to the median  $OM$ , and another eigenfunction  $\varphi_3$  which is anti-invariant under the same mirror symmetry. The eigenfunction  $\varphi_2$  restricted to the hemiequilateral triangle is the second Neumann eigenfunction of  $\mathcal{T}_2$ . The restriction of  $\varphi_3$  to the hemiequilateral triangle is an eigenfunction of  $\mathcal{T}_2$  with mixed boundary condition (Dirichlet on  $OM$  and Neumann on the other sides).

More precisely, up to a scaling factor, we have

$$(7.2) \quad \begin{cases} \varphi_2(x, y) = \cos(\frac{4\pi}{3}x) + \cos(\frac{2\pi}{3}(-x + \sqrt{3}y)) \\ \quad + \cos(\frac{2\pi}{3}(x + \sqrt{3}y)), \\ \varphi_3(x, y) = \sin(\frac{4\pi}{3}x) + \sin(\frac{2\pi}{3}(-x + \sqrt{3}y)) \\ \quad - \sin(\frac{2\pi}{3}(x + \sqrt{3}y)). \end{cases}$$

The expression for  $\varphi_2$  can be simplified. More precisely,

$$(7.3) \quad \varphi_2(x, y) = 2 \cos\left(\frac{2\pi x}{3}\right) \left( \cos\left(\frac{2\pi x}{3}\right) + \cos\left(\frac{2\pi y}{\sqrt{3}}\right) \right) - 1.$$

The set  $\{\varphi_2 + 1 = 0\}$  consists of the two line segments  $\{x = \frac{3}{4}\} \cap \mathcal{T}$  and  $\{x + \sqrt{3}y = \frac{3}{2}\} \cap \mathcal{T}$ , which meet at the point  $(\frac{3}{4}, \frac{\sqrt{3}}{4})$  on  $\partial\mathcal{T}$ . The sets  $\{\varphi_2 + a = 0\}$ , with  $a \in \{0, 1 - \varepsilon, 1, 1 + \varepsilon\}$ , and small positive  $\varepsilon$ , are shown in Figure 7.5. When  $a$  varies from 0 to  $1 + \varepsilon$  the number of nodal domains of  $\varphi_2 + a$  in  $\mathcal{T}$  jumps from 2 to 3, with the jump occurring for  $a = 1$ .

The function  $u_{6,N} + a$  in the hexagon  $\mathcal{H}$  is obtained from the function  $\varphi_2 + a$  in the equilateral triangle  $\mathcal{T}$  by reflections with respect to the diagonals of the hexagon. When  $a$  varies from 0 to  $1 + \varepsilon$  the number of nodal domains of  $u_{6,N} + a$  in  $\mathcal{H}$  jumps from 2 to 7, with the jump

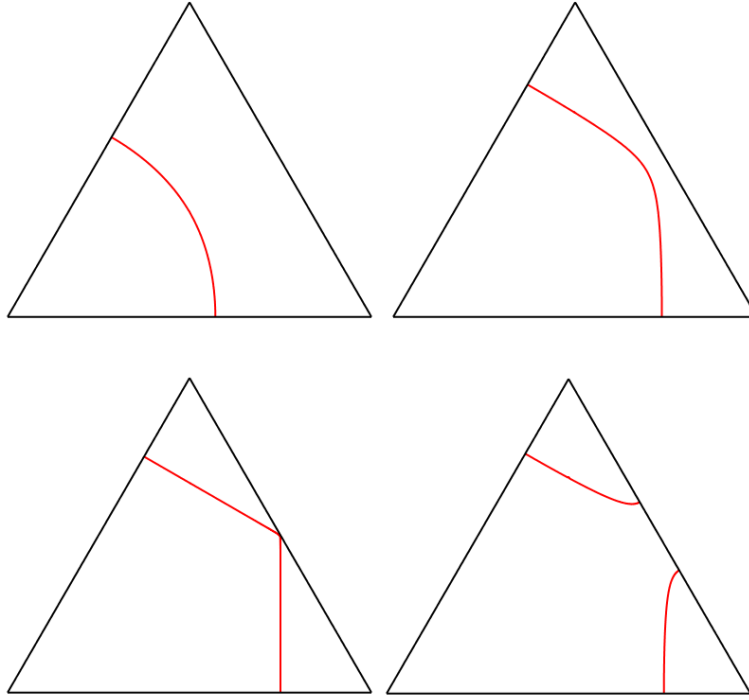


FIGURE 7.5. Levels sets  $\{\varphi_2 + a = 0\}$  for  $a \in \{0; 0.85; 1; 1.15\}$

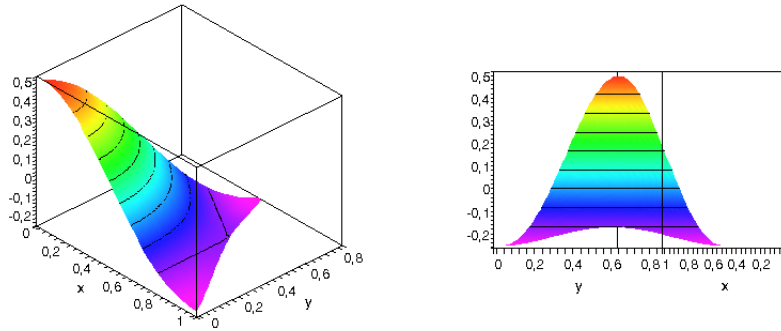


FIGURE 7.6. Equilateral triangle: 2nd Neumann eigenfunction (symmetric one)

occurring for  $a = 1$ . Figure 7.6 displays the function  $\varphi_2$  conveniently scaled.

It follows that the *Extended Courant property* does not hold on the regular hexagon with Neumann boundary condition.

## 8. EXTENSIONS

In the case of a Euclidean domain  $\Omega \subset \mathbb{R}^d$ , with the Dirichlet boundary condition, another natural upper bound for the number of nodal domains is provided by Pleijel's method [27] which uses the Faber-Krahn isoperimetric inequality. Namely, for  $u \in \lambda_n(\Omega, D)$  we have

$$(8.1) \quad \beta_0(u) \leq |\Omega| \left( \frac{\lambda_n(\Omega, D)}{\lambda_1(\mathbb{B}_1^d, D)} \right)^{\frac{d}{2}},$$

where  $|\Omega|$  is the Euclidean volume of  $\Omega$ , and where  $\mathbb{B}_1^d$  is the Euclidean ball with volume 1. When  $n$  tend to infinity the right-hand side of (8.1) is asymptotically smaller than or equal to  $\gamma(d) n$  for some positive constant  $\gamma(d) < 1$ .

It would be natural to investigate the *Extended Courant Property* with Courant's bound replaced by Pleijel's bound (8.1).

## 9. NUMERICAL SIMULATIONS

The numerical simulations in this section were kindly performed by Virginie BONNAILLIE-NOËL. The eigenvalues and eigenfunctions are computed using a finite element method [23]. Similar simulations are done in [7] to determine the minimal 3-partition of a square.

**Rectangle with a crack, and Dirichlet boundary condition.**

Figure 9.1 shows (from top to bottom) the level lines of  $v_1$ ,  $v_2$ , the first and second eigenfunctions of the rectangle with a horizontal crack, with Dirichlet boundary condition on both the crack and the boundary of the rectangle. In this example, the length of the rectangle is equal to twice the width. The crack begins on the left side, at half the width; its length is equal to  $\frac{1}{10}$  of the length of the rectangle.

The third picture shows the level lines of the quotient  $\frac{v_2}{v_1}$ , and suggests that the linear combination  $2v_1 - v_2$  has three nodal domains, two of them on each side of the crack. One can therefore **conjecture that the *Extended Courant Property* does not hold on a rectangle with a crack and Dirichlet boundary condition.**

**Hexagon with Dirichlet boundary condition.** Figure 9.2 shows (from top to bottom) the level lines of  $f_1$ ,  $f_2$ , the first and second eigenfunctions of the equilateral triangle, with Dirichlet boundary condition on one side, Neumann condition on the other sides. Here,  $f_1$  and  $f_2$  are normalized by the conditions  $\sup_{\mathcal{T}} f_1 = 1$  and  $\sup_{\mathcal{T}} f_2 = 1$ .

The third picture shows the level lines of the quotient  $\frac{f_2}{f_1}$ , and suggests that the linear combination  $\frac{3}{2}f_1 + f_2$  has three nodal domains, two of them in the neighborhood of the vertices of the side with Dirichlet boundary condition.

The first and sixth Dirichlet eigenfunctions of the hexagon are obtained from the functions  $f_1$  and  $f_2$  by reflection with respect to the diagonals of the hexagon. Figure 9.3 shows (from top to bottom) the level lines of  $u_{1,D}$ ,  $u_{6,D}$ , the first and sixth eigenfunctions of the hexagon with Dirichlet boundary condition. The third picture shows the level lines of the quotient  $\frac{u_{6,D}}{u_{1,D}}$ .

The pictures suggest that  $\frac{3}{2}u_{1,D} + u_{2,D}$  has seven nodal domains, six near the vertices of the hexagon and one containing the center  $O$ .

One can therefore **conjecture that the *Courant Extended Property* does not hold on the regular hexagon with Dirichlet boundary condition.**

**Regular polygons.** Figure 9.4 shows the nodal lines of the ratio  $\frac{w_{6,D}}{w_{1,D}}$  where  $w_{1,D}$  (resp.  $w_{6,D}$ ) is the first (resp. sixth) eigenfunction of a regular polygon with 7 sides and Dirichlet boundary condition.

One can therefore **conjecture that the *Extended Courant Property* does not hold for the regular heptagon.** When  $n$  tends to infinity, the eigenvalues of the regular polygon with  $n$  sides tend to the eigenvalues of the disk. Observing that the second radial eigenfunction of the Dirichlet or Neumann problem for the disk have labelling 6, one may **more generally conjecture that the *Extended Courant Property* does not hold for the regular polygon with  $n \geq 6$  sides, and Dirichlet or Neumann boundary condition.**

This type of example does not seem to yield a counter-example for the regular pentagon.

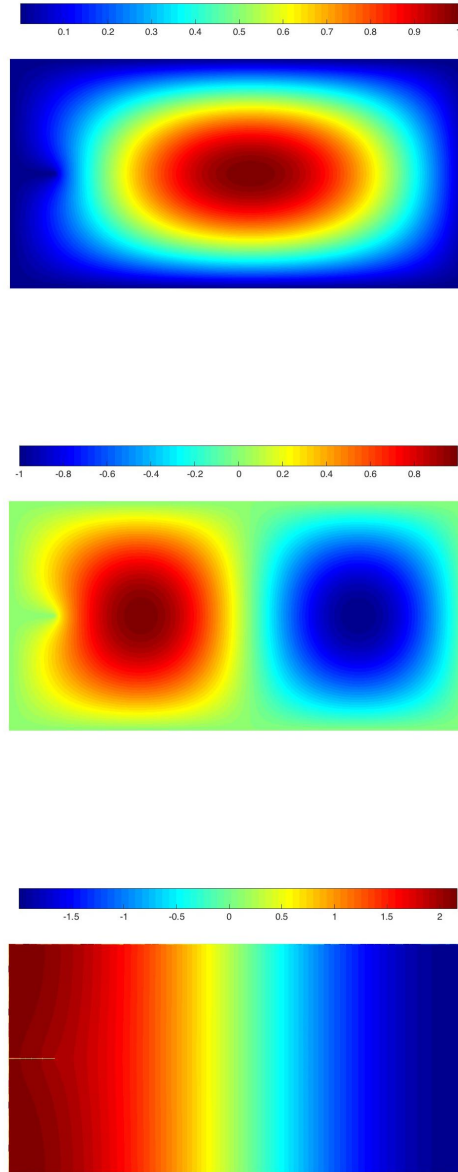


FIGURE 9.1. Level lines of  $v_1, v_2$  and  $\frac{v_2}{v_1}$  in a rectangle with an horizontal crack coming from the left

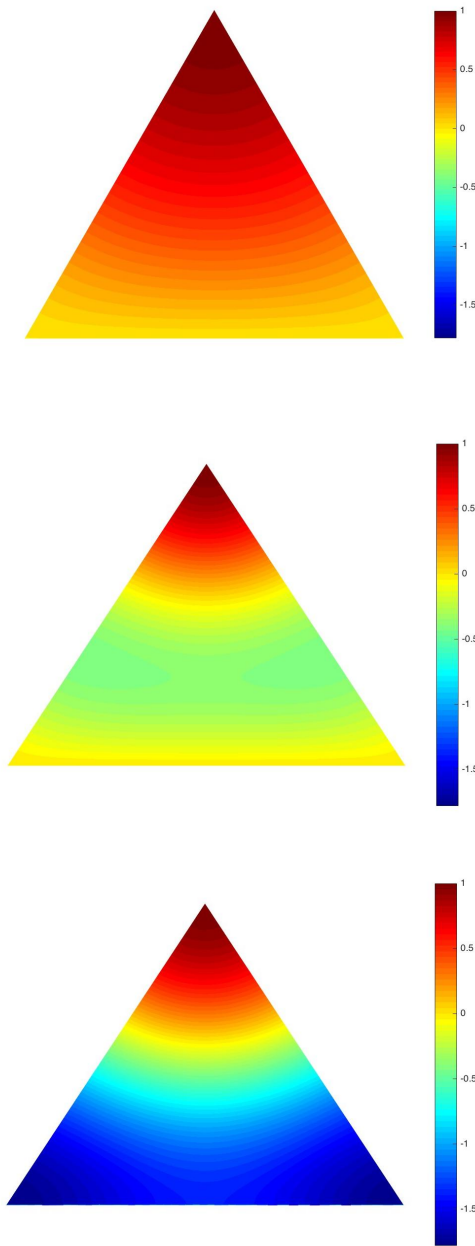


FIGURE 9.2. Level lines of  $f_1, f_2$  and  $\frac{f_2}{f_1}$  for the NND problem in the equilateral triangle

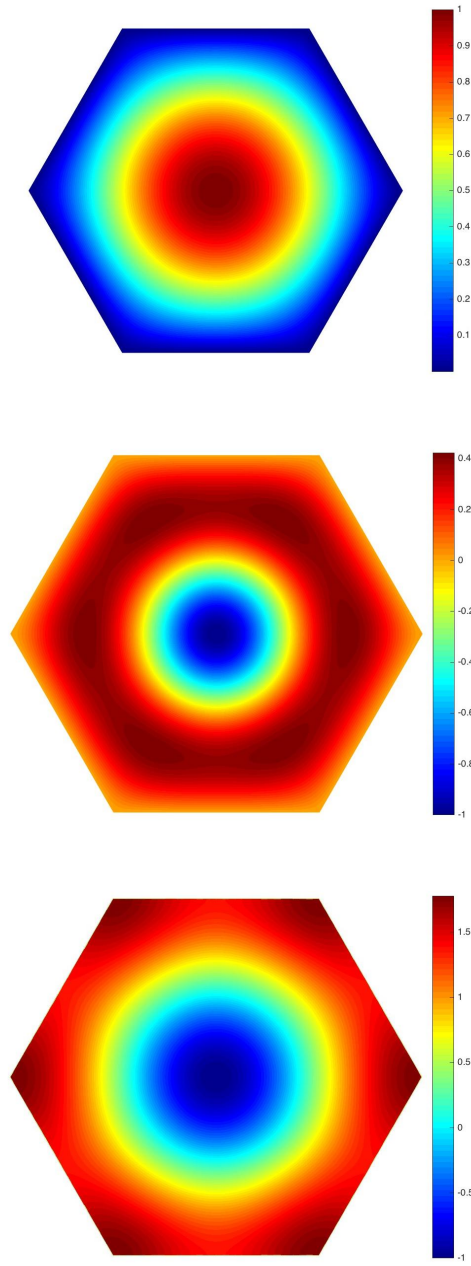


FIGURE 9.3. Level lines of  $u_{1,D}$ ,  $u_{6,D}$  and  $\frac{u_{6,D}}{u_{1,D}}$  for the Dirichlet problem in the regular hexagon



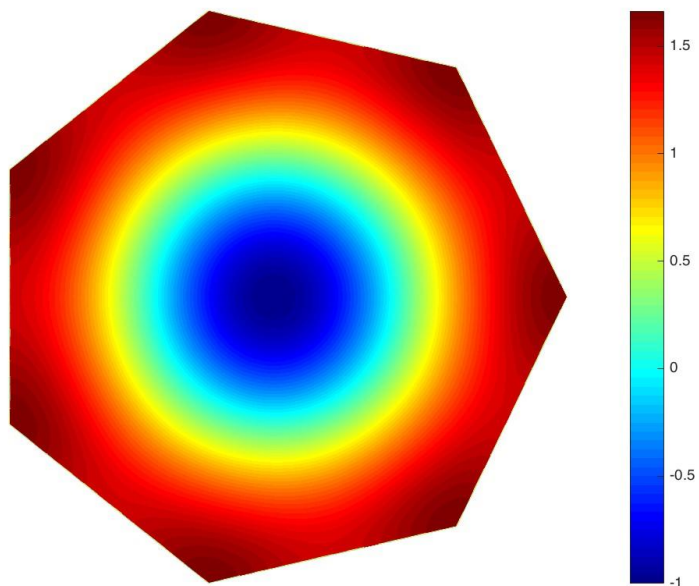


FIGURE 9.4. Level lines of  $\frac{w_{6,D}}{w_{1,D}}$  for the Dirichlet problem in the regular heptagon

## APPENDIX A. EIGENVALUES OF THE EQUILATERAL TRIANGLE

In this appendix, we recall the description of the eigenvalues of the equilateral triangle. For the reader's convenience, we retain the notation of [6, Section 2].

Let  $\mathbb{E}^2$  be the Euclidean plane with the canonical orthonormal basis  $\{e_1 = (1, 0), e_2 = (0, 1)\}$ , scalar product  $\langle \cdot, \cdot \rangle$  and associated norm  $|\cdot|$ .

Consider the vectors

$$(A.1) \quad \alpha_1 = (1, -\frac{1}{\sqrt{3}}), \alpha_2 = (0, \frac{2}{\sqrt{3}}), \alpha_3 = (1, \frac{1}{\sqrt{3}}) = \alpha_1 + \alpha_2,$$

and

$$(A.2) \quad \alpha_1^\vee = (\frac{3}{2}, -\frac{\sqrt{3}}{2}), \alpha_2^\vee = (0, \sqrt{3}), \alpha_3^\vee = (\frac{3}{2}, \frac{\sqrt{3}}{2}) = \alpha_1^\vee + \alpha_2^\vee.$$

Then

$$(A.3) \quad \alpha_i^\vee = \frac{3}{2}\alpha_i, |\alpha_i|^2 = \frac{4}{3}, |\alpha_i^\vee|^2 = 3.$$

Define the mirror symmetries

$$(A.4) \quad s_i(x) = x - 2 \frac{\langle x, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i = x - \frac{2}{3} \langle x, \alpha_i^\vee \rangle \alpha_i^\vee,$$

whose axes are the lines

$$(A.5) \quad L_i = \{x \in \mathbb{E}^2 \mid \langle x, \alpha_i \rangle = 0\}.$$

Let  $W$  be the group generated by these mirror symmetries. Then,

$$(A.6) \quad W = \{1, s_1, s_2, s_3, s_1 \circ s_2, s_1 \circ s_3\},$$

where  $s_1 \circ s_2$  (resp.  $s_2 \circ s_1$ ) is the rotation with center the origin and angle  $\frac{2\pi}{3}$  (resp.  $-\frac{2\pi}{3}$ ).

**Remark.** The above vectors are related to the root system  $A_2$  and  $W$  is the Weyl group of this root system.

Let

$$(A.7) \quad \Gamma = \mathbb{Z}\alpha_1^\vee \oplus \mathbb{Z}\alpha_2^\vee$$

be the (equilateral) lattice. The set

$$(A.8) \quad \mathcal{D}_\Gamma = \{s\alpha_1^\vee + t\alpha_2^\vee \mid 0 \leq s, t \leq 1\}$$

is a fundamental domain for the action of  $\Gamma$  on  $\mathbb{E}^2$ . Another fundamental domain is the closure of the open hexagon (see Figure 7.1)

$$(A.9) \quad \mathcal{H} = [A, B, C, D, E, F],$$

whose vertices are given by

$$(A.10) \quad \begin{cases} A = (1, 0); B = (\frac{1}{2}, \frac{\sqrt{3}}{2}); (-\frac{1}{2}, \frac{\sqrt{3}}{2}); \\ D = (-1, 0); E = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}); F = (\frac{1}{2}, -\frac{\sqrt{3}}{2}). \end{cases}$$

Call  $\mathcal{T}$  the equilateral triangle

$$(A.11) \quad \mathcal{T} = [O, A, B],$$

where  $O = (0, 0)$ .

Let  $\Gamma^*$  be the dual lattice of the lattice  $\Gamma$ , defined by

$$(A.12) \quad \Gamma^* = \{x \in \mathbb{E}^2 \mid \forall \gamma \in \Gamma, \langle x, \gamma \rangle \in \mathbb{Z}\}.$$

Then,

$$(A.13) \quad \begin{cases} \Gamma^* = \mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2, \\ \text{where } \varpi_1 = (\frac{2}{3}, 0) \text{ and } \varpi_2 = (\frac{1}{3}, \frac{1}{\sqrt{3}}). \end{cases}$$

Define the set  $C$  (an open Weyl chamber of the root system  $A_2$ ),

$$(A.14) \quad C = \{x\varpi_1 + y\varpi_2 \mid x, y > 0\},$$

and let  $\mathbb{T}$  denote the equilateral torus  $\mathbb{E}^2/\Gamma$ .

A complete set of orthogonal (not normalized) eigenfunctions of  $-\Delta$  on  $\mathbb{T}$  is given (in complex form) by the exponentials

$$(A.15) \quad \phi_p(x) = \exp(2i\pi\langle x, p \rangle) \text{ where } x \in \mathbb{E}^2 \text{ and } p \in \Gamma^*.$$

Furthermore, for  $p = m\varpi_1 + n\varpi_2$ , with  $m, n \in \mathbb{Z}$ , the multiplicity of the eigenvalue  $\hat{\lambda}(m, n) = 4\pi^2|p|^2 = \frac{16\pi^2}{9}(m^2 + mn + n^2)$  is equal to the number of points  $(k, \ell)$  in  $\mathbb{Z}^2$  such that  $k^2 + k\ell + \ell^2 = m^2 + mn + n^2$ .

The closure of the equilateral triangle  $\mathcal{T}$  is the fundamental domain of the action of the semi-direct product  $\Gamma \rtimes W$  on  $\mathbb{E}^2$  or equivalently, the fundamental domain of the action of  $W$  on  $\mathbb{T}^2$ .

For the following proposition, we refer to [5].

**Proposition A.1.** *Complete orthogonal (not normalized) sets of eigenfunctions of the equilateral triangle  $\mathcal{T}$  in complex form are given, respectively for the Dirichlet (resp. Neumann) boundary condition on  $\partial\mathcal{T}$ , as follows.*

(1) *Dirichlet boundary condition on  $\partial\mathcal{T}$ . The family is*

$$(A.16) \quad \Phi_p^D(x) = \sum_{w \in W} \det(w) \exp(2i\pi\langle x, w(p) \rangle)$$

*with  $p \in C \cap \Gamma^*$ . Furthermore, for  $p = m\varpi_1 + n\varpi_2$ , with  $m, n$  positive integers, the multiplicity of the eigenvalue  $4\pi^2|p|^2$  is equal to the number of solutions  $q \in C \cap \Gamma^*$  of the equation  $|q|^2 = |p|^2$ .*

(2) *Neumann boundary condition on  $\partial\mathcal{T}$ . The family is*

$$(A.17) \quad \Phi_p^N(x) = \sum_{w \in W} \exp(2i\pi\langle x, w(p) \rangle)$$

*with  $p \in \overline{C} \cap \Gamma^*$ . Furthermore, for  $p = m\varpi_1 + n\varpi_2$ , with  $m, n$  non-negative integers, the multiplicity of the eigenvalue  $4\pi^2|p|^2$*

is equal to the number of solutions  $q \in \overline{C} \cap \Gamma^*$  of the equation  $|q|^2 = |p|^2$ .

**Remark.** To obtain corresponding complete orthogonal sets of real eigenfunctions, it suffices to consider the functions

$$C_p = \Re(\Phi_p) \text{ and } S_p = \Im(\Phi_p).$$

For  $p = m\varpi_1 + n\varpi_2$  with  $m, n \in \mathbb{N}^\bullet$  for the Dirichlet boundary condition (resp.  $m, n \in \mathbb{N}$  for the Neumann boundary condition) we denote these functions by  $C_{m,n}$  and  $S_{m,n}$ .

In order to give explicit formulas for the first eigenfunctions, we have to examine the action of the group  $W$  on the lattice  $\Gamma^*$ . A simple calculation yields the following table in which we simply denote  $m\varpi_1 + n\varpi_2$  by  $(m, n)$ .

(A.18)

$w$	$(m, n)$	$w(m, n)$
1	$(m, n)$	$(m, n)$
$s_1$	$(m, n)$	$(-m, m + n)$
$s_2$	$(m, n)$	$(m + n, -n)$
$s_3$	$(m, n)$	$(-n, -m)$
$s_1 \circ s_2$	$(m, n)$	$(-m - n, m)$
$s_2 \circ s_1$	$(m, n)$	$(n, -m - n)$

**Remark.** The above table should be compared with [6, Table], where this is a slight unimportant error (the lines  $s_1 \circ s_2$  and  $s_2 \circ s_1$  are interchanged).

**Remark.** Using the above chart, one can easily prove the following relations.

$$(A.19) \quad \begin{cases} C_{n,m}^D = -C_{m,n}^D & \text{and} & S_{n,m}^D = S_{m,n}^D, \\ C_{n,m}^N = C_{m,n}^N & \text{and} & S_{n,m}^N = -S_{m,n}^N. \end{cases}$$

**Neumann boundary condition.** The first Neumann eigenvalue of  $\mathcal{T}$  is 0, corresponding to the point  $0 = (0, 0) \in \Gamma^*$ , with first eigenfunction  $\varphi_1 \equiv 1$  up to scaling. The second Neumann eigenvalue corresponds to the pairs  $(1, 0)$  and  $(0, 1)$ . According to the preceding remark, it suffices to consider  $C_{1,0}$  and  $S_{1,0}$ . Using Proposition A.1 and the table (A.18), we find that for the point  $[s, t] = s\varpi_1 + t\varpi_2$

$$(A.20) \quad \begin{cases} C_{1,0}([s, t]) = 2(\cos(2\pi s) + \cos(2\pi(-s + t)) + \cos(2\pi t)) , \\ S_{1,0}([s, t]) = 2(\sin(2\pi s) + \sin(2\pi(-s + t)) - \sin(2\pi t)) . \end{cases}$$

Up to a factor 2, this gives the following two independent eigenfunctions for the Neumann eigenvalue  $\frac{16\pi^2}{9}$  in the  $(x, y)$  variables with  $(x, y) = (\frac{3}{2}s, -\frac{\sqrt{3}}{2}s + \sqrt{3}t)$ .

$$(A.21) \quad \begin{cases} \varphi_2(x, y) = \cos(\frac{4\pi}{3}x) + \cos(\frac{2\pi}{3}(-x + \sqrt{3}y)) \\ \quad \quad \quad + \cos(\frac{2\pi}{3}(x + \sqrt{3}y)), \\ \varphi_3(x, y) = \sin(\frac{4\pi}{3}x) + \sin(\frac{2\pi}{3}(-x + \sqrt{3}y)) \\ \quad \quad \quad - \sin(\frac{2\pi}{3}(x + \sqrt{3}y)). \end{cases}$$

The first eigenfunction is invariant under the mirror symmetry with respect to the median  $OM$  of the equilateral triangle, see Figure 7.1. The second eigenfunction is anti-invariant under the mirror symmetry with respect to this median. Its nodal set is equal to the median itself. We can rewrite  $\varphi_2(x, y)$  as

$$(A.22) \quad \varphi_2(x, y) = 2 \cos(\frac{2\pi}{3}x) \left( \cos(\frac{2\pi}{3}x) + \cos(\frac{2\pi}{3}\sqrt{3}y) \right) - 1.$$

It is clear that the nodal set of  $\varphi_2 + 1$  consists of two line segments  $\{x - \frac{3}{4} = 0\} \cap \mathcal{T}$  and  $\{x + \sqrt{4}y - \frac{3}{2} = 0\} \cap \mathcal{T}$ .

For  $\varepsilon > 0$  small enough, the nodal set of  $\varphi_1(x, y) - 1 - \varepsilon$  consists of a single arc from the side  $OA$  to the side  $OB$  which does not meet the side  $AB$ . Correspondingly the function has two nodal domains inside  $\mathcal{T}$ . The nodal set of  $\varphi_1(x, y) - 1 + \varepsilon$  consists of two disjoint arcs, one from the side  $OA$  to the side  $AB$  and one from the side  $OB$  to the side  $AB$ . Correspondingly, the function has three nodal domains inside  $\mathcal{T}$ .

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