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# Invariance: a Theoretical Approach for Coding Sets of Words Modulo Literal (Anti)Morphisms

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**Abstract.** Let  $A$  be a finite or countable alphabet and let  $\theta$  be literal (anti)morphism onto  $A^*$  (by definition, such a correspondence is determined by a permutation of the alphabet). This paper deals with sets which are invariant under  $\theta$  ( $\theta$ -invariant for short). We establish an extension of the famous defect theorem. Moreover, we prove that for the so-called thin  $\theta$ -invariant codes, maximality and completeness are two equivalent notions. We prove that a similar property holds for some special families of  $\theta$ -invariant codes such as prefix (bifix) codes, codes with a finite (two-way) deciphering delay, uniformly synchronous codes and circular codes. For a special class of involutive antimorphisms, we prove that any regular  $\theta$ -invariant code may be embedded into a complete one.

**Keywords:** antimorphism, bifix, circular, code, complete, deciphering delay, defect, delay, embedding, equation, literal, maximal, morphism, prefix, synchronizing delay, variable length code, verbal synchronizing delay, word.

## 1 Introduction

During the last decade, in the free monoid theory, due to their powerful applications, in particular in DNA-computing, one-to-one *morphic* or *antimorphic* correspondences play a particularly important part. Given a finite or countable *alphabet*, say  $A$ , any such mapping is a substitution which is fully determined by extending a unique permutation of  $A$ , to a mapping onto  $A^*$  (the *free monoid* that is generated by  $A$ ). The resulting mapping is commonly referred to as *literal* (or *letter-to-letter*) moreover, in the case of a finite alphabet, it is well known that, with respect to the composition, some power of such a correspondence is the identity (classically, in the case where this power corresponds to the square, we say that the correspondence is *involutive*).

In that special case of involutive morphisms or antimorphisms -we write (anti)morphisms for short, lots of successful investigations have been done for extending the now classical combinatorial properties on words: we mention the study of the so-called pseudo-palindromes [3, 5], or that of pseudo-repetitions [4, 9, 13]. The framework of some peculiar families of codes [12] and equations in words [6, 7] have been also concerned. Moreover, in the larger family of one-to-one (anti)morphisms, a nice generalization of the famous theorem of Fine and Wilf [14, Proposition 1.3.5] has been recently established in [8].

Equations in words are also the starting point of the study in the present paper, where we adopt the point of view from [14, Chap. 9]. Let  $A$  be a finite or countable alphabet; a one-to-one literal (anti)morphism onto  $A^*$ , namely  $\theta$ , being fixed, consider a finite collection of unknown words, say  $Z$ . In view of making the present foreword more readable, in the first instance we take  $\theta$  as an involutive literal substitution (that is  $\theta^2 = id_{A^*}$ ). We assign that the words in  $Z$  and their images by  $\theta$  to satisfy a given equation, and we are interested in the

cardinality of any set  $T$ , whose elements allow by concatenation to compute all the words in  $Z$ . Actually, such a question might be more complex than in the classical configuration, where  $\theta$  does not interfere: it is well known that in that classical case, according to the famous defect theorem [14, Theorem 1.2.5], the words in  $Z$  may be computed as the concatenation of at most  $|Z| - 1$  words that don't satisfy any non-trivial equation. With the terminology of [14, 10],  $T$ , the set of such words is a *code*, or equivalently  $T^*$ , the submonoid that it generates, is *free*: more precisely, with respect to the inclusion of sets it is the smallest free submonoid of  $A^*$  that contains  $Z$ .

Along the way, for solving our problem, applying the defect theorem to the set  $X = Z \cup \theta(Z)$  might appear as natural. Such a methodology guarantees the existence of a code  $T$ , with  $|T| \leq |X| - 1$ , and such that  $T^*$  is the smallest free submonoid of  $A^*$  that contains  $X$ . Unfortunately, since both the words in  $Z$  and  $\theta(Z)$  are expressed as concatenations of words in  $T$ , among the elements of  $T \cup \theta(T)$  non-trivial equations can remain; in other words, by applying that methodology, the initial problem would be transferred among the words in  $T \cup \theta(T)$ . This situation is particularly illustrated by [13, Proposition 3], where the authors prove that, given an involutive antimorphism  $\theta$ , the solutions of the equation  $xy = \theta(y)x$  are  $x = (uv)^i u, y = vu$ , where the elements  $u, v$  of  $T$  satisfy the non-trivial equation  $vu = \theta(u)\theta(v)$ .

In the general case where  $\theta$  is a literal one-to-one (anti)morphism, we note that the union, say  $Y$ , of the sets  $\theta^i(T)$ , for all  $i \in \mathbb{Z}$ , is itself  $\theta$ -invariant, therefore an alternative methodology will consist in asking for some code  $Y$  which is invariant under  $\theta$ , and such that  $Y^*$  is the smallest free submonoid of  $A^*$  that contains  $X = \bigcup_{i \in \mathbb{Z}} \theta^i(Z)$ . By the way, it is straightforward to prove that the intersection of an arbitrary family of  $\theta$ -invariant free submonoids is itself a  $\theta$ -invariant free submonoid. In the present paper we prove the following result:

**Theorem 1.** *Let  $A$  be a finite or countable alphabet, let  $\theta$  be a literal (anti)morphism onto  $A^*$ , and let  $X$  be a finite  $\theta$ -invariant set. If  $X$  is not a code, then the smallest  $\theta$ -invariant free submonoid of  $A^*$  that contains  $X$  is generated by a  $\theta$ -invariant code  $Y$  which satisfies  $|Y| \leq |X| - 1$ .*

For illustrating this result in term of equations, we refer to [6, 7], where the authors considered generalizations of the famous equation in three unknowns of Lyndon-Shützenberger [14, Sect. 9.2]. They proved that, an involutive (anti)morphism  $\theta$  being fixed, given such an equation with sufficiently long members, a word  $t$  exists such that any 3-uple of “solutions” can be expressed as a concatenation of words in  $\{t\} \cup \{\theta(t)\}$ . With the notation of Theorem 1, the elements of the  $\theta$ -invariant set  $X$  are  $x, y, z, \theta(x), \theta(y), \theta(z)$  and those of  $Y$  are  $t$  and  $\theta(t)$ : we verify that  $Y$  is a  $\theta$ -invariant code with  $|Y| \leq |X| - 1$ .

In the sequel, we will continue our investigation by studying the properties of complete  $\theta$ -invariant codes: a subset  $X$  of  $A^*$  is *complete* if any word of  $A^*$  is a factor of some words in  $X^*$ . From this point of view, a famous result from Schützenberger states that, for the wide family of the so-called *thin* codes (which contains regular codes) [10, Sect. 2.5], maximality and completeness are two equivalent notions. In the framework of invariant codes, we prove the following result:

**Theorem 2.** *Let  $A$  be a finite or countable alphabet. Given a thin  $\theta$ -invariant code  $X \subseteq A^*$ , the three following conditions are equivalent:*

- (i)  *$X$  is complete*
- (ii)  *$X$  is a maximal code*
- (iii)  *$X$  is maximal in the family of the  $\theta$ -invariant codes.*

In the proof, the main feature consists in establishing that a non-complete  $\theta$ -invariant code  $X$  cannot be maximal in the family of  $\theta$ -invariant codes: actually, the most delicate step lays upon the construction of a convenient  $\theta$ -invariant set  $Z \subseteq A^*$ , with  $X \cap Z = \emptyset$  and such that  $X \cup Z$  remains itself a  $\theta$ -invariant code.

It is well known that the preceding result from Schützenberger has been successfully extended to some famous families of thin codes, such as *prefix* (*bifix*, *uniformly synchronous*, *circular*) codes (cf [10, Proposition 3.3.8], [10, Proposition 6.2.1], [10, Theorem 10.2.11], [15, Proposition 3.6] and [11, Theorem 3.5]) and codes with a *finite deciphering delay* (f.d.d. codes, for short) [10, Theorem 5.2.2]. From this point of view, we will examine the behavior of corresponding families of  $\theta$ -invariant codes. Actually we establish a result similar to the preceding theorem 2 in the framework of the family of prefix (bifix, f.d.d., two-way f.d.d, uniformly synchronized, circular codes). In the proof, a construction very similar to the previous one may be used in the case of prefix, bifix, f.d.d., two-way f.d.d codes. At the contrary, investigating the behavior of circular codes with regards to the question necessitates the computation of a more sophisticated set; moreover the family of uniformly synchronized codes itself impose to make use of a significantly different methodology.

In the last part of our study, we address to the problem of embedding a non-complete  $\theta$ -invariant code into a complete one. For the first time, this question was stated in [2], where the author asked whether any finite code can be imbedded into a regular one. A positive answer was provided in [1], where was established a formula for embedding any regular code into a complete one. From the point of view of  $\theta$ -invariant codes, we obtain a positive answer only in the case where  $\theta$  is an involutive antimorphism which is different of the so-called mirror image; actually the general question remains open.

We now describe the contents of the paper. Section 2 contains the preliminaries: the terminology of the free monoid is settled, and the definitions of some classical families of codes are recalled. Theorem 1 is established in Section 3, where an original example of equation is studied. The proof of Theorem 2 is done in Section 3, and extensions for special families of  $\theta$ -invariant codes are studied in Section 4. The question of embedding a regular  $\theta$ -invariant code into a complete one is examined in Section 5.

## 2 Preliminaries

We adopt the notation of the free monoid theory: given an alphabet  $A$ , we denote by  $A^*$  the free monoid that it generates. Given a word  $w$ , we denote by  $|w|$  its length, the empty word, that we denote by  $\varepsilon$ , being the word with length 0. We denote by  $w_i$  the letter of position  $i$  in  $w$ : with this notation we have  $w = w_1 \cdots w_{|w|}$ . We set  $A^+ = A^* \setminus \{\varepsilon\}$ . Given  $x \in A^*$  and  $w \in A^+$ , we say that  $x$  is a *prefix* (*suffix*) of  $w$  if a word  $u$  exists such that  $w = xu$  ( $w = ux$ ). Similarly,  $x$  is a *factor* of  $w$  if a pair of words  $u, v$  exists such that  $w = uxv$ . Given a non-empty set  $X \subseteq A^*$ , we denote by  $P(X)$  ( $S(X), F(X)$ ) the set of the words that are prefix (suffix, factor) of some word in  $X$ . Clearly, we have  $X \subseteq P(X)$  ( $S(X), F(X)$ ). A set  $X \subseteq A^*$  is *complete* iff  $F(X^*) = A^*$ . Given a pair of words  $w, w'$ , we say that it *overlaps* if words  $u, v$  exist such that  $uw' = wv$  or  $w'u = vw$ , with  $1 \leq |u| < |w|$  and  $1 \leq |v| < |w'|$ ; otherwise, the pair is *overlapping-free* (in such a case, if  $w = w'$ , we simply say that  $w$  is overlapping-free).

It is assumed that the reader has a fundamental understanding with the main concepts of the theory of variable length codes: we only recall some of the main definitions and we suggest, if necessary, that he (she) report to [10]. A set  $X$  is a *variable length code* (a *code* for short) iff any equation among the words of  $X$  is trivial, that is, for any pair of sequences of words in  $X$ , namely  $(x_i)_{1 \leq i \leq m}, (y_j)_{1 \leq j \leq n}$ , the equation  $x_1 \cdots x_m = y_1 \cdots y_n$  implies  $m = n$

and  $x_i = y_i$  for each integer  $i \in [1, m]$ . By definition  $X^*$ , the submonoid of  $A^*$  which is generated by  $X$ , is *free*. Equivalently,  $X^*$  satisfies the property of *equidivisibility*, that is  $(X^*)^{-1}X^* \cap X^*(X^*)^{-1} = X^*$ .

Some famous families of codes that have been studied in the literature:  $X$  is a *prefix (suffix, bifix) code* iff  $X \neq \{\varepsilon\}$  and  $X \cap XA^+ = \emptyset$  ( $X \cap A^+X = \emptyset$ ,  $X \cap XA^+ = X \cap A^+X = \emptyset$ ).  $X$  is a code with a *finite deciphering delay (f.d.d. code for short)* if it is a code and if a non-negative integer  $d$  exists such that  $X^{-1}X^* \cap X^dA^+ \subseteq X^+$ . With this condition, if another integer  $d'$  exists such that we have  $X^*X^{-1} \cap A^+X^{d'} \subseteq X^+$ , we say that  $X$  is a *two-way f.d.d. code*.  $X$  is a *uniformly synchronized code* if it is a code and if a positive integer  $k$  exists such that, for all  $x, y \in X^k$ ,  $u, v \in A^+$ :  $uxyv \in X^* \implies ux, xv \in X^*$ .  $X$  is a *circular code* if for any pair of sequences of words in  $X$ , namely  $(x_i)_{1 \leq i \leq m}$ ,  $(y_j)_{1 \leq j \leq n}$ , and any pair of words  $s, p$ , with  $s \neq \varepsilon$ , the equation  $x_1 \cdots x_m = sy_2 \cdots y_np$ , with  $y_1 = ps$ , implies  $m = n$ ,  $p = \varepsilon$  and  $x_i = y_i$  for each  $i \in [1, m]$ .

In the whole paper, we consider a *finite* or *countable* alphabet  $A$  and a mapping  $\theta$  which satisfies each of the three following conditions:

- (a)  $\theta$  is a one-to-one correspondence onto  $A^*$
- (b)  $\theta$  is *literal*, that is  $\theta(A) \subseteq A$
- (c) either  $\theta$  is a *morphism* or it is an *antimorphism* (it is an antimorphism if  $\theta(\varepsilon) = \varepsilon$  and  $\theta(xy) = \theta(y)\theta(x)$ , for any pair of words  $x, y$ ); for short in any case we write that  $\theta$  is an *(anti)morphism*.

In the case where  $A$  is a finite set, it is well known that a positive integer  $n$  exists such that  $\theta^n = id_{A^*}$ . In the whole paper, we are interested in the family of sets  $X \subseteq A^*$  that are invariant under the mapping  $\theta$  ( $\theta$ -invariant for short), that is  $\theta(X) = X$ .

### 3 A Defect Effect for Invariant Sets

Informally, the famous defect theorem says that if some words of a set  $X$  satisfy a non-trivial equation, then these words may be written upon an alphabet of smaller size. In this section, we examine whether a corresponding result may be stated in the framework of  $\theta$ -invariant sets. The following property comes from the definition:

**Proposition 1.** *Let  $M$  be a submonoid of  $A^*$  and let  $S \subseteq A^*$  be such that  $M = S^*$ . Then  $M$  is  $\theta$ -invariant if and only if  $S$  is  $\theta$ -invariant.*

Clearly the intersection of a non-empty family of  $\theta$ -invariant free submonoids of  $A^*$  is itself a  $\theta$ -invariant free submonoid. Given a submonoid  $M$  of  $A^*$ , recall that its *minimal generating set* is  $(M \setminus \{\varepsilon\}) \setminus (M \setminus \{\varepsilon\})^2$ .

**Theorem 2.** *Let  $A$  be a finite or countable alphabet, let  $X \subseteq A^*$  be a  $\theta$ -invariant set and let  $Y$  be the minimal generating set of the smallest  $\theta$ -invariant free submonoid of  $A^*$  which contains  $X$ . If  $X$  is not a code, then we have  $|Y| \leq |X| - 1$ .*

*Proof.* With the notation of Theorem 2, since  $Y$  is a code, each word  $x \in X$  has a unique factorization upon the words of  $Y$ , namely  $x = y_1 \cdots y_n$ , with  $y_i \in Y$  ( $1 \leq i \leq n$ ). In a classical way, we say that  $y_1$  ( $y_n$ ) is the *initial (terminal) factor* of  $x$  (with respect to such a factorization). At first, we shall establish the following lemma:

**Lemma 3.** *With the preceding notation, each word in  $Y$  is the initial (terminal) factor of a word in  $X$ .*

*Proof.* By contradiction, assume that a word  $y \in Y$  that is never initial of any word in  $X$  exists. Set  $Y_0 = (Y \setminus \{y\})\{y\}^*$  and  $Y_i = \theta^i(Y_0)$ , for each integer  $i \in \mathbb{Z}$ . In a classical way (cf e.g. [14, p. 7]), since  $Y$  is a code,  $Y_0$  itself is a code. Since  $\theta^i$  is a one-to-one correspondence, for each integer  $i \in \mathbb{Z}$ ,  $Y_i$  is a code, that is  $Y_i^*$  is a free submonoid of  $A^*$ . Consequently, the intersection, namely  $M$ , of the family  $(Y_i^*)_{i \in \mathbb{Z}}$  is itself a free submonoid of  $A^*$ . Moreover we have  $\theta(M) \subseteq M$  (indeed, given a word  $w \in M$ ,  $\theta(w) \notin Y_i$  implies  $w \notin Y_{i-1}$ ) therefore, since  $\theta$  is onto, we obtain  $\theta(M) = M$ . Let  $x$  be an arbitrary word in  $X$ . Since  $X \subseteq Y^*$ , and according to the definition of  $y$ , we have  $x = (y_1 y^{k_1})(y_2 y^{k_2}) \cdots (y_n y^{k_n})$ , with  $y_1, \dots, y_n \in Y \setminus \{y\}$  and  $k_1, \dots, k_n \geq 0$ . Consequently  $x$  belongs to  $Y_0^*$ , therefore we have  $X \subseteq Y_0^*$ . Since  $X$  is  $\theta$ -invariant, this implies  $X = \theta(X) \subseteq Y_i^*$  for each  $i \in \mathbb{Z}$ , thus  $X \subseteq M$ .

But the word  $y$  belongs to  $Y^*$  and doesn't belong to  $Y_0^*$  thus it doesn't belong to  $M$ . This implies  $X \subseteq M \subsetneq Y^*$ : a contradiction with the minimality of  $Y^*$ . ■

*Proof of Theorem 2.* Let  $\alpha$  be the mapping from  $X$  onto  $Y$  which, with every word  $x \in X$ , associates the initial factor of  $x$  in its (unique) factorization over  $Y^*$ . According to Lemma 3,  $\alpha$  is onto. We will prove that it is not one-to-one. Classically, since  $X$  is not a code, a non-trivial equation may be written among its words, say:

$x_1 \cdots x_n = x'_1 \cdots x'_m$ , with  $x_i, x'_j \in X$   $x_1 \neq x'_1$  ( $1 \leq i \leq n, 1 \leq j \leq m$ ). Since  $Y$  is a code, a unique sequence of words in  $Y$ , namely  $y_1, \dots, y_p$  exists such that:

$x_1 \cdots x_n = x'_1 \cdots x'_m = y_1 \cdots y_p$ . This implies  $y_1 = \alpha(x_1) = \alpha(x'_1)$  and completes the proof. ■

In what follows we discuss some interpretation of Theorem 2 with regards to equations in words. For this purpose, we assume that  $A$  is finite, thus a positive integer  $n$  exists such that  $\theta^n = id_{A^*}$ . Consider a finite set of words, say  $Z$ , and denote by  $X$  the union of the sets  $\theta^i(Z)$ , for  $i \in [1, n]$ ; assume that a non-trivial equation holds among the words of  $X$ , namely  $x_1 \cdots x_m = y_1 \cdots y_p$ . By construction  $X$  is  $\theta$ -invariant therefore, according to Theorem 2, a  $\theta$ -invariant code  $Y$  exists such that  $X \subseteq Y^*$ , with  $|Y| \leq |X| - 1$ . This means that each of the words in  $X$  can be expressed by making use of at most  $|X| - 1$  words of type  $\theta^i(u)$ , with  $u \in Y$  and  $1 \leq i \leq n$ . It will be easily verified that the examples from [6, 13, 7] corroborate this fact, moreover below we mention an original one:

*Example 4.* Let  $\theta$  be a literal antimorphism such that  $\theta^3 = id_{A^*}$ . Consider two different words  $x, y$ , with  $|x| > |y|$ , which satisfy the following equation:

$$x\theta(y) = \theta^2(y)\theta(x).$$

With these conditions, a pair of words  $u, v$  exists such that  $x = uv$ ,  $\theta^2(y) = u$ , thus  $y = \theta(u)$ , moreover we have  $v = \theta(v)$  and  $u = \theta(u) = \theta^2(u)$ . With the preceding notation, we have  $Z = \{x, y\}$ ,  $X = Z \cup \theta(Z) \cup \theta^2(Z)$ ,  $Y = \{u\} \cup \{v\} \cup \{\theta(u)\} \cup \{\theta(v)\} \cup \{\theta^2(u)\} \cup \{\theta^2(v)\}$ . It follows from  $y = \theta(y) = \theta^2(y)$  that  $X = \{x\} \cup \{\theta(x)\} \cup \{\theta^2(x)\} \cup \{y\}$ .

- At first, assume that no word  $t$  exists such that  $u, v \in t^+$ . In a classical way, we have  $uv \neq vu$ , thus  $X = \{x, \theta(x), \theta^2(x), y\}$  and  $Y = \{u, v\}$ . We verify that  $|Y| \leq |X| - 1$ .

- Now, assume that we have  $u, v \in t^+$ . We obtain  $X = Z = \{x, y\}$  and  $Y = \{t\}$ . Once more we have  $|Y| \leq |X| - 1$ .

## 4 Maximal $\theta$ -Invariant Codes

Given set  $X \subseteq A^*$ , we say that it is *thin* if  $A^* \neq F(X)$ . Regular codes are well known examples of thin codes. From the point of view of maximal codes, below we recall one of the famous result stated by Schützenberger:

**Theorem 5.** [10, Theorem 2.5.16] *Let  $X$  be an thin code. Then the following conditions are equivalent:*

- (i)  $X$  is complete
- (ii)  $X$  is a maximal code.

The aim of this section is to examine whether a corresponding result may be stated in the family of thin  $\theta$ -invariant codes.

In the case where  $|A| = 1$ , we have  $\theta = id_{A^*}$ , moreover the codes are all the singletons in  $A^+$ . Therefore any code is  $\theta$ -invariant, maximal and complete. In the rest of the paper, we assume that  $|A| \geq 2$ .

*Some notations.* Let  $X$  be a non-complete  $\theta$ -invariant code, and let  $y \notin F(X^*)$ . Without loss of generality, we may assume that the initial and the terminal letters of  $y$  are different (otherwise, substitute to  $y$  the word  $ay\bar{a}$ , with  $a, \bar{a} \in A$  and  $a \neq \bar{a}$ ), we may also assume that  $|y| \geq 2$ . Set:

$$y = ax\bar{a}, \quad z = \bar{a}^{|y|}y\bar{a}^{|y|} = \bar{a}^{|y|}ax\bar{a}^{|y|}. \quad (1)$$

Since  $\theta$  is a literal (anti)morphism, for each integer  $i \in \mathbb{Z}$ , a pair of different letters  $b, \bar{b}$  and a word  $x'$  exist such that  $|x'| = |x| = |y| - 2$ , and:

$$\theta^i(z) = \bar{b}^{|y|}\theta^i(y)b^{|y|} = \bar{b}^{|y|}bx'\bar{b}b^{|y|}. \quad (2)$$

Given two (not necessarily different) integers  $i, j \in \mathbb{Z}$ , we will accurately study how the two words  $\theta^i(z), \theta^j(z)$  may overlap.

**Lemma 6.** *With the notation in (2), let  $u, v \in A^+$  and  $i, j \in \mathbb{Z}$  such that  $|u| \leq |z| - 1$  and  $\theta^i(z)v = u\theta^j(z)$ . Then we have  $|u| = |v| \geq 2|y|$ , moreover a letter  $b$  and a unique positive integer  $k$  (depending of  $|u|$ ) exist such that we have  $\theta^i(z) = ub^k, \theta^j(z) = b^k v$ , with  $k \leq |y|$ .*

*Proof.* According to (2), we set  $\theta^i(z) = \bar{b}^{|y|}bx'\bar{b}b^{|y|}$  and  $\theta^j(z) = \bar{c}^{|y|}cx''\bar{c}c^{|y|}$ , with  $b, \bar{b}, c, \bar{c} \in A$  and  $b \neq \bar{b}, c \neq \bar{c}$ . Since  $\theta$  is a literal (anti)morphism, we have  $|\theta^i(z)| = |\theta^j(z)|$  thus  $|u| = |v|$ ; since we have  $1 \leq |u| \leq 3|y| - 1$ , exactly one of the following cases occurs:

*Case 1:*  $1 \leq |u| \leq |y| - 1$ . With this condition, we have  $(\theta^i(z))_{|u|+1} = \bar{b} = \bar{c} = (u\theta^j(z))_{|u|+1}$  and  $(\theta^i(z))_{|y|+1} = b = \bar{c} = (u\theta^j(z))_{|y|+1}$ , which contradicts  $b \neq \bar{b}$ .

*Case 2:*  $|u| = |y|$ . This condition implies  $(\theta^i(z))_{|u|+1} = b = \bar{c} = (u\theta^j(z))_{|u|+1}$  and  $(\theta^i(z))_{2|y|} = \bar{b} = \bar{c} = (u\theta^j(z))_{2|y|}$ , which contradicts  $b \neq \bar{b}$ .

*Case 3:*  $|y|+1 \leq |u| \leq 2|y|-1$ . We obtain  $(\theta^i(z))_{2|y|} = \bar{b} = \bar{c} = (u\theta^j(z))_{2|y|}$  and  $(\theta^i(z))_{2|y|+1} = b = \bar{c} = (u\theta^j(z))_{2|y|+1}$  which contradicts  $b \neq \bar{b}$ .

*Case 4:*  $2|y| \leq |u| \leq 3|y| - 1$ . With this condition, necessarily we have  $b = \bar{c}$ , therefore an integer  $k \in [1, |y|]$  exists such that  $\theta^i(z) = ub^k$  and  $\theta^j(z) = b^k v$ . ■

Set  $Z = \{\theta^i(z) | i \in \mathbb{Z}\}$ . Since  $y \notin F(X^*)$  and since  $X$  is  $\theta$ -invariant, for any integer  $i \in \mathbb{Z}$  we have  $\theta^i(z) \notin F(X^*)$ , hence we obtain  $Z \cap F(X^*) = \emptyset$ . By construction, all the words in  $Z$  have length  $|z|$  moreover, as a consequence of Lemma 6:

**Lemma 7.** *With the preceding notation, we have  $A^+ZA^+ \cap ZX^*Z = \emptyset$ .*



*Proof.* By contradiction, assume that  $z_1, z_2, z_3 \in Z$ ,  $x \in X^*$  and  $u, v \in A^+$  exist such that  $uz_1v = z_2xz_3$ . By comparing the lengths of the words  $u$  and  $v$  with  $|z|$ , exactly one of the three following cases occurs:

*Case 1:*  $|z| \leq |u|$  and  $|z| \leq |v|$ . With this condition, we have  $z_2 \in P(u)$  and  $z_3 \in S(v)$ , therefore the word  $z_1$  is a factor of  $x$ : this contradicts  $Z \cap F(X^*) = \emptyset$ .

*Case 2:*  $|u| < |z| \leq |v|$ . We have in fact  $u \in P(z_2)$  and  $z_3 \in S(v)$ . We are in the condition of Lemma 6: the words  $z_2, z_1$  overlap. Consequently,  $u \in A^+$  and  $b \in A$  exist such that  $z_2 = ub^k$  and  $z_1 = b^kz'_1$ , with  $1 \leq k \leq |y|$ . But by construction we have  $|uz_1| = |z_2xz_3| - |v|$ : since we assume  $|v| \geq |z|$ , this implies  $|uz_1| \leq |z_2xz_3| - |z| = |z_2x|$ , therefore we obtain  $uz_1 = ub^kz'_1 \in P(z_2x)$ . It follows from  $z_2 = ub^k$  that  $z'_1 \in P(x)$ . Since  $z_1 \in Z$  and according to (2),  $i \in \mathbb{Z}$  and  $\bar{b} \in A$  exist such that we have  $z_1 = b^kz'_1 = b^{|y|}\theta^i(y)\bar{b}^{|y|}$ . Since by Lemma 6 we have  $|z'_1| = |u| \geq 2|y|$ , we obtain  $\theta^i(y) \in F(z'_1)$ , which contradicts  $y \notin F(X^*)$ .

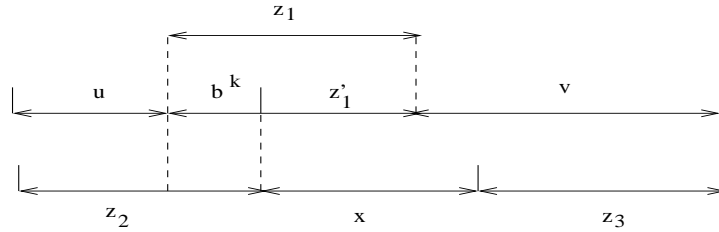
*Case 3:*  $|v| < |z| \leq |u|$ . Same arguments on the reversed words lead to a conclusion similar to that of Case 2.

*Case 4:*  $|z| > |u|$  and  $|z| > |v|$ . With this condition, both the pairs of words  $z_2, z_1$  and  $z_1, z_3$  overlap. Once more we are in the condition of Lemma 6: letters  $c, d$ , words  $u, v, s, t$ , and integers  $h, k$  exist such that the two following properties hold:

$$z_2 = uc^h, \quad z_1 = c^hs, \quad |u| = |s| \geq 2|y|, \quad h \leq |y|, \quad (3)$$

$$z_1 = td^k, \quad z_3 = d^kv, \quad |v| = |t| \geq 2|y|, \quad k \leq |y|. \quad (4)$$

It follows from  $uz_1v = z_2xz_3$  that  $uz_1v = (uc^h)x(d^kv)$ , thus  $z_1 = c^hxd^k$ . Once more according to (2),  $i \in \mathbb{Z}$  and  $\bar{c} \in A$  exist such that we have  $z_1 = c^{|y|}\theta^i(y)\bar{c}^{|y|}$ . Since we have  $h, k \leq |y|$ , this implies  $d = \bar{c}$  moreover  $\theta^i(y)$  is a factor of  $x$ . Once more, this contradicts  $y \notin F(X^*)$ . ■



**Fig. 1.** Proof of Lemma 7: Case 2

Thanks to Lemma 7 we will prove some meaningful results in Section 5. Presently, we will apply it in a special context:

**Corollary 8.** *With the preceding notation,  $X^*Z$  is a prefix code.*

*Proof.* Let  $z_1, z_2 \in Z$ ,  $x_1, x_2 \in X^*$ ,  $u \in A^+$ , such that  $x_1z_1u = x_2z_2$ . For any word  $z_3 \in Z$ , we have  $(z_3x_1)z_1(u) = z_3x_2z_2$ , a contradiction with Lemma 7. ■

We are now ready to prove the main result of the section:



**Theorem 9.** *Let  $A$  be a finite or countable alphabet and let  $X \subseteq A^*$  be a thin  $\theta$ -invariant code. Then the following conditions are equivalent:*

- (i)  $X$  is complete
- (ii)  $X$  is a maximal code
- (iii)  $X$  is maximal in the family  $\theta$ -invariant codes.

*Proof.* Let  $X$  be a  $\theta$ -invariant code. According to Theorem 5, if  $X$  is thin and complete, then it is a maximal code, therefore  $X$  is maximal in the family of  $\theta$ -invariant codes. For proving the converse, we consider a set  $X$  which is maximal in the family of  $\theta$ -invariant codes.

Assume that  $X$  is not complete and let  $y \notin F(X^*)$ . Define the word  $z$  as in (1) and consider the set  $Z = \{\theta^i(z) | i \in \mathbb{Z}\}$ . At first, we will prove that  $X \cup Z$  remains a code. In view of that, we consider an arbitrary equation between the words in  $X \cup Z$ . Since  $X$  is a code, without loss of generality, we may assume that at least one element of  $Z$  has at least one occurrence in one of the two sides of this equation. As a matter of fact, with such a condition and since  $Z \cap F(X^*) = \emptyset$ , two sequences of words in  $X^*$ , namely  $(x_i)_{1 \leq i \leq n}$ ,  $(x'_j)_{1 \leq j \leq p}$  and two sequences of words in  $Z$ , namely  $(z_i)_{1 \leq i \leq n-1}$ ,  $(z'_j)_{1 \leq j \leq p-1}$  exist such that the equation takes the following form:

$$x_1 z_1 x_2 z_2 \cdots x_{n-1} z_{n-1} x_n = x'_1 z'_1 x'_2 z'_2 \cdots x'_{p-1} z'_{p-1} x'_p. \quad (5)$$

Without loss of generality, we assume  $n \geq p$ . At first, according to Corollary 8, necessarily, we have  $x_1 = x'_1$ , therefore Equation (5) is equivalent to:  $z_1 x_2 z_2 \cdots x_{n-1} z_{n-1} x_n = z'_1 x'_2 z'_2 \cdots x'_{p-1} z'_{p-1} x'_p$ , however, since all the words in  $Z$  have a common length, we have  $z_1 = z'_1$  hence our equation is equivalent to  $x_2 z_2 \cdots x_{n-1} z_{n-1} x_n = x'_2 z'_2 \cdots x'_{p-1} z'_{p-1} x'_p$ . Consequently, by applying iteratively the result of Corollary 8, we obtain:  $x_2 = x'_2, \dots, x_p = x'_p$ , which implies  $x_{p+1} z_{p+1} \cdots z_{n-1} x_n = \varepsilon$ , thus  $n = p$ . In other words Equation (5) is trivial, thus  $X \cup Z$  is a code.

Next, since  $\theta$  is one-to-one and since we have  $\theta(X \cup Z) \subseteq \theta(X) \cup \theta(Z) = X \cup Z$ , the code  $X \cup Z$  is  $\theta$ -invariant. It follows from  $z \in Z \setminus X$  that  $X$  is strictly included in  $X \cup Z$ : this contradicts the maximality of  $X$  in the whole family of  $\theta$ -invariant codes, and completes the proof of Theorem 9. ■

*Example 10.* Let  $A = \{a, b, c\}$ . Consider the antimorphism  $\theta$  which is generated by the permutation  $\sigma(a) = b, \sigma(b) = c, \sigma(c) = a$  and let  $X = \{ab, cb, ca, ba, bc, ac\}$ ; it can be easily verified that  $X$  is a  $\theta$ -invariant code. Since we have  $a^3 \notin F(X^*)$ , by setting  $y = a^3 b$  and  $z = b^4 \cdot a^3 b \cdot a^4$  we are in Condition (1). The corresponding set  $Z$  is  $\{\theta^i(z) | i \in \mathbb{Z}\} = \{b^4 c b^3 c^4, a^4 c^3 a c^4, a^4 b a^3 b^4, c^4 b^3 c b^4, c^4 a c^3 a^4, b^4 a^3 b a^4\}$ . Since  $X \cup Z$  is a prefix set, this guarantees that  $X \cup Z$  remains a  $\theta$ -invariant code.

## 5 Maximality in Some Families of $\theta$ -Invariant Codes

In the literature, statements similar to Theorem 5 were established in the framework of some special families of thin codes. In this section we will draw similar investigations with regards to  $\theta$ -invariant codes. We will establish the following result:

**Theorem 11.** *Let  $A$  be a finite or countable alphabet and let  $X \subseteq A^*$  be a thin  $\theta$ -invariant prefix (resp. bifix, f.d.d., two-way f.d.d, uniformly synchronized, circular) code. Then the following conditions are equivalent:*

- (i)  $X$  is complete

(ii)  $X$  is a maximal code

(iii)  $X$  is maximal in the family of prefix (bifix, f.d.d., two-way f.d.d, uniformly synchronized, circular) codes

(iv)  $X$  is maximal in the family  $\theta$ -invariant codes

(v)  $X$  is maximal in the family of  $\theta$  invariant prefix (bifix, f.d.d., two-way f.d.d, uniformly synchronized, circular) codes.

*Sketch proof.* According to Theorem 9, and thanks to [10, Proposition 3.3.8], [10, Proposition 6.2.1], [10, Theorem 5.2.2], [15, Proposition 3.6] and [11, Theorem 3.5], if  $X$  is complete then it is maximal in the family of  $\theta$ -invariant codes and maximal in the family of  $\theta$ -invariant prefix (bifix, f.d.d., two-way f.d.d, uniformly synchronized, circular) codes. Consequently, the proof of Proposition 11 comes down to establish that if  $X$  is not complete, then it cannot be maximal in the family of  $\theta$ -invariant prefix (bifix, f.d.d., wo-way f.d.d, uniformly synchronized, circular) codes.

1) We begin by  $\theta$ -invariant prefix codes. At first, we assume that  $\theta$  is an antimorphism. Since  $X \cap XA^+ = \emptyset$ , and since  $\theta$  is injective, we have  $\theta(X) \cap \theta(XA^+) = \emptyset$ , thus  $X \cap A^+X = \emptyset$ , hence  $X$  is also a suffix code. Assume that  $X$  is not complete. According to [10, Proposition 3.3.8], it is non-maximal in both the families of prefix codes and suffix codes. Therefore a pair of words  $y, y' \in A^+ \setminus X$  exists such  $X \cup \{y\}$  ( $X \cup \{y'\}$ ) remains a prefix (suffix) code. By construction  $X \cup \{yy'\}$  remains a code which is both prefix and suffix.

Set  $Y = \{\theta^i(yy') \mid i \in \mathbb{Z}\}$ : since all the words in  $Y$  have same positive length,  $Y$  is a prefix code. From the fact that  $\theta$  is one-to-one, for any integer  $i \in \mathbb{Z}$  we obtain  $\theta^i(\{yy'\}) \cap \theta^i(P(X)) = \theta^i(X) \cap P(\theta^i(yy')) = \emptyset$ , consequently  $X \cup Y$  remains a prefix code. By construction,  $Y$  is  $\theta$ -invariant and it is not included in  $X$ , thus  $X$  is not a maximal prefix code.

In the case where  $\theta$  is a morphism, the preceding arguments may be simplified. Actually, a word  $y \in A^+ \setminus X$  exists such that  $X \cup \{y\}$  remains a prefix code, thereferore by setting  $Y = \{\theta^i(y) \mid i \in \mathbb{Z}\}$ ,  $X \cup Y$  remains a prefix code.

2) (sketch) The preceding arguments may be applied for proving that in any case, if  $X$  is a non-complete bifix code, then it is maximal.

3,4) (sketch) In the case where  $X$  is a (two-way) f.d.d.-code, according to [10, Proposition 5.2.1], similar arguments leads to a similar conclusion.

5) In the case where  $X$  is a  $\theta$ -invariant uniformly synchronized code with *verbal delay*  $k$  ([10, Section 10.2]), we must make use of different arguments. Actually, according to [15, Theorem 3.10], a complete uniformly synchronized code  $X'$  exists, with synchronizing delay  $k$ , and such that  $X \subsetneq X'$ . More precisely,  $X'$  is the minimal generating set of the submonoid  $M$  of  $A^*$  which is defined by  $M = (X^{2k}A^* \cap A^*X^{2k}) \cup X^*$ . According to Proposition 1 in the present paper,  $X'$  is  $\theta$ -invariant. Since  $X$  is stictly included in  $X'$ , it cannot be maximal in the family of  $\theta$ -invariant uniformly synchronized codes with delay  $k$ .

6) It remains to study the case where  $X$  is a non-complete  $\theta$ -invariant circular code. Let  $y \notin F(X^*)$  and let  $z$  and  $Z$  be computed as in Section 3: this guarantees that  $X \cup Z$  is a  $\theta$ -invariant set. For proving that  $X \cup Z$  is a circular code, by contradiction we assume that some words  $y_1, \dots, y_n, y'_1, \dots, y'_m \in X \cup Z$  (with  $m + n$  minimal),  $p \in A^*$  and  $s \in A^+$ , exist such that the following equation holds:

$$y_1 y_2 \cdots y_n = s y'_2 y'_3 \cdots y'_m p \quad \text{and} \quad y'_1 = ps. \quad (6)$$

Once more since  $X$  is a code, and since  $Z \cap F(X^*) = \emptyset$ , without loss of generality we assume that at least one integer  $i \in \mathbb{Z}$  exists such that  $y_i \in Z$ ; similarly, at least one integer  $j \in [1, m]$  exists such that  $y'_j \in Z$ . By construction, we have  $y_i \in F(y'_j \cdots y'_m y'_1 \cdots y'_j \cdots y'_m y'_1 \cdots y'_j)$ ; consequently, since all the words in  $Z$  have the same length, a pair of integers  $h, k \in [1, m]$  and a pair of words  $u, v$  exist such that  $u y_i v \in y'_h X^* y'_k$ . According to Lemma 7, necessarily

we have either  $u = \varepsilon$  or  $v = \varepsilon$ ; this implies  $y_i = y'_h$  or  $y_i = y'_k$ , which contradicts the minimality of  $m + n$ , therefore  $X \cup Z$  is a circular code. ■

## 6 Embedding a Regular Invariant Code into a Complete One

In this section, we consider a non-complete regular  $\theta$ -invariant code  $X$  and we are interested in the problem of computing a complete one, namely  $Y$ , such that  $X \subseteq Y$ . Historically, such a question appears for the first time in [2], where the author asked for the possibility of embedding a finite code into a regular complete one. With regards to  $\theta$ -invariant codes, it seems natural to generalize the formula from [1] by making use of the code  $Z$  that was introduced in Section 4. More precisely we would consider the set  $X' = X \cup (ZU)^*Z$ , with  $U = A^* \setminus (X^* \cup A^*ZA^*)$ . Unfortunately, with such a construction we observe that some pairs of words in  $Z$  may overlap, therefore a non-trivial equation could hold among the words of  $X'$ .

Nevertheless, we shall see that in the very special case where  $\theta$  is an involutive antimorphism, convenient invariant overlapping-free words can be computed. Denote by  $\theta_0$  the antimorphism which is generated by the identity onto  $A$ ; in other words, with every word  $w = w_1 \cdots w_n \in A^*$  (with  $w_i \in A$ , for  $1 \leq i \leq n$ ), it associates  $\theta_0(w) = w_n \cdots w_1$ .

**Proposition 12.** *Let  $A$  be a finite alphabet and let  $\theta$  be an antimorphism onto  $A^*$ , with  $\theta \neq \theta_0$ . If  $\theta$  is involutive, then any non-complete regular  $\theta$ -invariant code can be embedded into a complete one.*

*Proof.* Let  $X$  be such that  $\theta(X) = X$ . Assume that  $X$  is not complete. We will construct an overlapping-free word  $t \notin F(X^*)$  such that  $\theta(t) = t$ . At first, we consider a word  $x$  such that  $x \notin F(X^*)$  and  $|x| \geq 2$ . Without loss of generality, we assume that  $x$  is overlapping-free (otherwise, as in [10, Proposition 1.3.6], a word  $s$  exists such that  $xs$  is overlapping-free). If  $\theta(x) = x$ , then we set  $t = x$ , otherwise let  $y = cx$ , where  $c$  stands for the initial letter of  $x$ . Once more, without loss of generality we assume that  $y$  is overlapping-free. By construction we have  $y \in ccA^+$ , thus  $|y| \geq 3$  and  $y_1 = y_2 = c$ . If  $\theta(y) = y$ , then we set  $t = y$ . Now assume  $\theta(y) \neq y$ ; according to the condition of Proposition 12, we have  $\theta|_A \neq id_A$ , therefore a pair of letters  $a, b$  exists such that the following property holds:

$$a \neq b, \quad b \neq c, \quad \theta(a) = b, \quad \theta(b) = a. \quad (7)$$

Set  $t = a^{|y|}b\theta(y)yab^{|y|}$ . By construction, we have  $\theta(t) = t$ , moreover the following property holds:

**Claim.**  $t$  is an overlapping-free word.

*Proof.* Let  $u, v \in A^*$  such that  $ut = tv$ , with  $1 \leq |u| \leq |t| - 1$ . According to the length of  $u$ , exactly one of the following cases occurs:

*Case 1:*  $1 \leq |u| \leq |y|$ . With this condition, we obtain  $t_{|y|+1} = b = (ut)_{|y|+1} = a$ : a contradiction with  $a \neq b$ .

*Case 2:*  $|y| + 1 \leq |u| \leq 2|y|$ . This condition implies  $\theta(y_1) = t_{2|y|+1} = a$ , therefore we obtain  $c = y_1 = \theta(a) = b$ : a contradiction with (7).

*Case 3:*  $|u| = 2|y| + 1$ . We have  $y = a^{|y|}$ : since we have  $|y| \geq 3$ , this contradicts the fact that  $y$  is overlapping-free.

*Case 4:*  $|u| = 2|y| + 2$ . We have  $t_{2|y|+3} = y_2 = c = (ut)_{2|y|+3} = a$ . It follows from  $y_1 = y_2 = c$

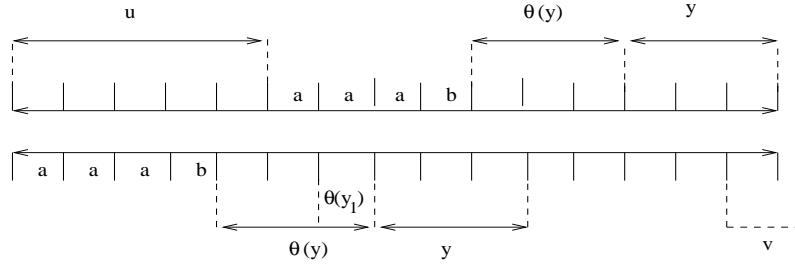
that  $y = a^{|y|}$ : once more this contradicts the fact that  $y$  is overlapping-free.

*Case 5:*  $2|y| + 3 \leq |u| \leq 3|y| + 2$ . By construction, we have  $t_{|u|+|y|} = b = (ut)_{|uy|} = a$ , a contradiction with (7).

*Case 6:*  $3|y| + 3 \leq |u| \leq |t| - 1 = 4|y| + 1$ . We obtain  $t_{|u|+1} = b = (ut)_{|u|+1} = a$ : once more this contradicts (7).

In any case we obtain a contradiction: this establishes the claim.

Since we have  $t \notin F(X^*)$ , and since  $t$  is overlapping-free, the classical method from [1] may be applied without any modification to ensure that  $X$  may be embedded into a complete code, say  $X'$ . Recall that it computes in fact a code  $X'$  as  $X \cup V$ , with  $V = t(Ut)^*$  and  $U = A^* \setminus (X^* \cup A^*tA^*)$ . Moreover, since  $\theta(t) = t$ , it is straightforward to verify that  $\theta(X') = X'$ . ■



**Fig. 2.** Proof of Proposition 12: Case 2 with  $|y| = 3$  and  $|u| = 5$

With regards to the antimorphism  $\theta_0$ , necessarily the words  $w, \theta_0(w)$  overlap, therefore the preceding methodology seems to be unreliable in the most general case. We finish our paper by stating the following open problem:

*Problem.* Let  $A$  be a finite alphabet and let  $\theta$  be an (anti)morphism onto  $A^*$ . Given a non-complete regular  $\theta$ -invariant code  $X \subset A^*$ , can we compute a complete regular  $\theta$ -invariant code  $Y$  such that  $X \subseteq Y$ ?

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