Sensitivities via Rough Paths

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SENSITIVITIES VIA ROUGH PATHS

Nicolas MARIE[1]

Abstract. Motivated by a problematic coming from mathematical finance, the paper deals with existing and additional results on the continuity and the differentiability of the Itô map associated to rough differential equations. These regularity results together with the Malliavin calculus are applied to the sensitivities analysis of stochastic differential equations driven by multidimensional Gaussian processes with continuous paths as the fractional Brownian motion. The well known results on greeks in the Itô stochastic calculus framework are extended to stochastic differential equations driven by a Gaussian process which is not a semi-martingale.

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1. Introduction

Motivated by a problematic coming from mathematical finance, the paper deals with existing and additional results on the continuity and the differentiability of the Itô map associated to rough differential equations (RDEs). These regularity results together with the Malliavin calculus are applied to the sensitivities analysis of stochastic differential equations (SDEs) driven by multidimensional Gaussian processes with continuous paths as the fractional Brownian motion.

First of all, some notions of mathematical finance are reminded.

Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a d-dimensional Brownian motion B and $\mathbb{F} := (\mathcal{A}_t; t \in [0,T])$ the filtration generated by B ($d \in \mathbb{N}^*$ and $T > 0$).
Consider the financial market consisting of $d+1$ assets (one risk-free asset and d risky assets) over the filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$. At the time $t \in [0,T]$, the deterministic price of the risk-free asset is denoted by S_t^0, and the prices of the d risky assets are given by the random vector $S_t := (S_t^1, \dots, S_t^d)$.
In a first place, assume that the process S is the solution of a stochastic differential equation, taken in the sense of Itô :

$$S_t = x + \int_0^t \mu(S_u)\, du + \int_0^t \sigma(S_u)\, dB_u \ ; \ x \in \mathbb{R}^d$$

where, $\mu : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ are some (globally) Lipschitz continuous functions.
Let $\mathbb{P}^* \sim \mathbb{P}$ be the risk-neutral probability measure of the market (i.e. such that $S^* := S/S^0$ is a $(\mathbb{F}, \mathbb{P}^*)$-martingale).

Theorem 1.1. *Consider an option of payoff $h \in L^2(\Omega, \mathcal{A}_T, \mathbb{P}^*)$. Then, there exists an admissible strategy φ such that :*

$$\forall t \in [0,T],\ V_t(\varphi) = \mathbb{E}^*\left(\frac{S_t^0}{S_0^0} h \middle| \mathcal{A}_t \right) \ \mathbb{P}^*\text{-a.s.}$$

where $V(\varphi)$ is the associated wealth process.

Theorem 1.1 is a consequence of the stochastic integral representation of the discounted claim (see N.H. Bingham and R. Kiesel [1], Lemma 6.1.2 and Theorem 6.1.5).

With the notations of Theorem 1.1, $V_T(\varphi) = \mathbb{E}^*(S_T^0/S_0^0 h)$. It is the price of the option, and when $h := F(S_T)$ with some function $F : \mathbb{R}^d \to \mathbb{R}_+$, it is possible to get the existence and an expression of the sensitivities of $V_T(\varphi)$ to perturbations of the initial condition and of the volatility function σ for instance :

$$\Delta := \partial_x \mathbb{E}^*\left[F(S_T^x)\right] \text{ and } \mathcal{V} := \partial_\sigma \mathbb{E}^*\left[F(S_T^\sigma)\right].$$

In finance, these sensitivities are called the greeks. For instance, Δ involves in the Δ-hedging which provides the admissible strategy of Theorem 1.1 (see [13], Subsection 4.3.3). However, these quantities don't involve in finance only. They could also be used in pharmacokinetics as mentioned at [19], Section 5.
The greeks have been deeply studied by several authors. In [8], E. Fournié et al. have established the existence of the greeks and have provided expressions of them via the Malliavin calculus by assuming that σ satisfied a uniform elliptic condition (see Theorem 1.2). In [11], E. Gobet and R. Münos have extended these results by assuming that σ only satisfied a hypoelliptic condition. On the computation of greeks in the Black-Scholes model, see P. Malliavin and A. Thalmaier [17], Chapter 2. On the sensitivities in models with jumps, see N. Privault et al. [23] and [7]. Finally, via the cubature formula for the Brownian motion, J. Teichmann has provided some estimators of the Malliavin weights for the computation of greeks (see J. Teichmann [26]). On

the regularity of the solution map of SDEs taken in the sense of Itô, see H. Kunita [12].
At the following theorem, δ is the divergence operator associated to the Brownian motion B (see D. Nualart [22], Section 1.3).

Theorem 1.2. *Assume that b and σ are differentiable, of bounded and Lipschitz continuous derivatives, and $F \in L^2(\mathbb{R}^d;\mathbb{R}_+)$.*

(1) *If σ satisfies the uniform elliptic condition (i.e. there exists $\varepsilon > 0$ such that for every $a, b \in \mathbb{R}^d$, $b^T\sigma^T(a)\sigma(a)b \geqslant \varepsilon\|b\|^2$), then Δ exists and*

$$\Delta = \mathbb{E}^* \left[F(S_T)\delta(h^{\Delta})\right]$$

where, h^{Δ} is an adapted d-dimensional stochastic process.

(2) *Let $\tilde{\sigma} : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ be a function such that for every ε belonging to a closed neighborhood of 0, $\sigma + \varepsilon\tilde{\sigma}$ satisfies the uniform elliptic condition. Then $\mathcal{V}$ exists and*

$$\mathcal{V} = \mathbb{E}^* \left[F(S_T)\delta(h^{\mathcal{V}})\right]$$

where, $h^{\mathcal{V}}$ is an (anticipative) d-dimensional stochastic process.

See [8], propositions 3.2 and 3.3 for a proof.

Under some technical assumptions stated at Subsection 2.3, the main purpose of this paper is to extend Theorem 1.2 to the following SDE, taken in the sense of rough paths introduced by T. Lyons in [15] :

$$X_t = x + \int_0^t \mu\left(X_s\right) ds + \int_0^t \sigma\left(X_s\right) dW_s \ ; \ x \in \mathbb{R}^d$$

where, W is a centered d-dimensional Gaussian process with continuous paths of finite p-variation ($p \geqslant 1$), and the functions μ and σ satisfy the following assumption.

Assumption 1.3. *μ and σ are $[p]+1$ times differentiable, bounded and of bounded derivatives.*

Subsections 2.1 and 2.2 deal with existing and additional results on the continuity and the differentiability of the Itô map associated to rough differential equations. In particular, the continuous differentiability of the Itô map with respect to the collection of vector fields is proved, and completes the existing results of regularity with respect to the initial condition and to the driving signal (see P. Friz and N. Victoir [10], chapters 4 and 11). In order to apply the (probabilistic) integrability results coming from T. Cass, C. Litterer and T. Lyons [3], some tailor-made upper-bounds are provided for each derivative. Subsection 2.3 reminds some definitions and results related to the good geometric rough path over a Gaussian process having a covariance function satisfying the technical Assumption 2.9, called enhanced Gaussian process by P. Friz and N. Victoir. The results of subsections 2.1 and 2.2 are applied together with the results coming from [3] in order to show the (probabilistic) integrability of the solution of a Gaussian RDE and their derivatives. The main problem is solved at Section 3 by using the results of Section 2 together with the Malliavin calculus. Some simulations of Δ and $\mathcal{V}$ are provided at Subsection 4.2.
The fractional Brownian motion (fBm) introduced in [18] by B.B. Mandelbrot and J.W. Van Ness has been studied by several authors in order to generalize the Brownian motion classically used to model the prices process of the risky assets. For instance, the regularity of the paths of the process and its memory are both controlled by the Hurst parameter H of the fBm. However, the fBm is not a semi-martingale if $H \neq 1/2$ (see. [22], Proposition 5.1.1). In [24], L.C.G. Rogers has shown the existence of arbitrages if the prices process of the assets is modeled by a fBm. In order to bypass that difficulty, several approaches have been studied. For instance, in [4], P. Cheridito assumed that the prices process was modeled by a mixed fractional Brownian motion which is a semi-martingale depending on a fBm. At Subsection 4.1, the prices of the risky assets are

modeled by a fractional SDE, in which the volatility is modeled by another one. The results of Section 3 are applied in order to show the existence and provide an expression of the sensitivity of the price of the option with respect to the collection of vector fields of the equation of the volatility.

The paper uses many results on rough paths and rough differential equations coming from [10] and, T. Lyons and Z. Qian [16]. The paper also uses results of Malliavin calculus coming from [22].
The notations, short definitions and results used throughout the paper are stated below. However, the original results of the literature are cited throughout the paper.

Notations (general) :

- $\mathbb{R}^e$ and $\mathbb{R}^d$ $(e, d \in \mathbb{N}^*)$ are equipped with their Euclidean norms, both denoted by $\|.\|$.
- The canonical basis of $\mathbb{R}^d$ is denoted by $(e_1, \dots, e_d)$. With respect to that basis, for $k = 1, \dots, d$, the k-th component of any vector $u \in \mathbb{R}^d$ is denoted by u^k.
- The closed ball of $\mathbb{R}^d$ with respect to $\|.\|$, of center $a \in \mathbb{R}^d$ and of radius $r > 0$, is denoted by $B(a, r)$.
- The usual matrix (resp. operator) norm on $\mathcal{M}_{e,d}(\mathbb{R})$ (resp. $\mathcal{L}(\mathbb{R}^e; \mathbb{R}^d)$) is denoted by $\|.\|_{\mathcal{M}}$ (resp. $\|.\|_{\mathcal{L}}$).
- Consider $0 \leqslant s < t \leqslant T$. The set of all the dissections of $[s, t]$ is denoted by $D_{s,t}$. In particular, $D_T := D_{0,T}$.
- $\Delta_T := \{(u, v) \in \mathbb{R}^2 : 0 \leqslant u < v \leqslant T\}$.
- The space of continuous (resp. continuously differentiable) functions from $[s, t]$ into $\mathbb{R}^d$ is denoted by $C^0([s, t]; \mathbb{R}^d)$ (resp. $C^1([s, t]; \mathbb{R}^d)$) and equipped with the uniform norm $\|.\|_{\infty;s,t}$.
- Differentiability means differentiability in the sense of Fréchet (see H. Cartan [2], Chapter I.2).
- Consider two Banach spaces E and F. Let $\varphi : E \to F$ be a map derivable at point $x \in E$, in the direction $h \in E$. The derivative of φ at point x, in the direction h, is denoted by :

$$D_h\varphi(x) := \lim_{\varepsilon \to 0} \frac{\varphi(x + \varepsilon h) - \varphi(x)}{\varepsilon} \text{ in } F.$$

- Consider three Banach spaces E, F and G, and a differentiable map $\varphi : E \times F \to G$. At point $(x, y) \in E \times F$, the Fréchet derivative of $\varphi(x, .)$ (resp. $\varphi(., y)$) is denoted by $\partial_y f(x, y)$ (resp. $\partial_x \varphi(x, y)$).

Notations (rough paths) :

- Consider $p \geqslant 1$ and $\alpha \in]0, 1]$. The space of continuous functions of finite p-variation (resp. α-Hölder continuous functions) from $[s, t]$ into $\mathbb{R}^d$ is denoted by

$$C^{p\text{-var}}\left([s, t]; \mathbb{R}^d\right) := \left\{ y \in C^0\left([s, t]; \mathbb{R}^d\right) : \sup_{D=\{r_k\} \in D_{s,t}} \sum_{k=1}^{|D|-1} \|y_{r_{k+1}} - y_{r_k}\|^p < \infty \right\}$$

  (resp. $C^{\alpha\text{-höl}}([s, t]; \mathbb{R}^d)$, which is a subset of $C^{1/\alpha\text{-var}}([s, t]; \mathbb{R}^d)$) and is equipped with the p-variation distance $d_{p\text{-var};s,t}$ (resp. the α-Hölder distance $d_{\alpha\text{-höl};s,t}$). See [10], chapters 5 and 8 about these spaces.
- Consider $N \in \mathbb{N}^*$ and $y : [0, T] \to \mathbb{R}^d$ a continuous function of finite 1-variation. The step-N tensor algebra over $\mathbb{R}^d$ is denoted by

$$T^N(\mathbb{R}^d) := \mathbb{R} \oplus \mathbb{R}^d \oplus \cdots \oplus (\mathbb{R}^d)^{\otimes N},$$

  the step-N signature of y is denoted by

$$S_N(y) := \left(1, \int_0^{\cdot} dy_r, \dots, \int_{0<r_1<\cdots<r_N<\cdot} dy_{r_1} \otimes \cdots \otimes dy_{r_N}\right),$$

and the step-N free nilpotent group over $\mathbb{R}^d$ is denoted by

$$G^N(\mathbb{R}^d) := \left\{ S_N(y)_1 ; y \in C^{1\text{-var}}([0,1];\mathbb{R}^d) \right\}.$$

See [10], Chapter 7.

- For $k = 0, \dots, N$, the $(k+1)$-th component of any $X \in T^N(\mathbb{R}^d)$ is denoted by X^k.
- The space of geometric p-rough paths is denoted by

$$G\Omega_{p,T}(\mathbb{R}^d) := \overline{\left\{ S_{[p]}(y) ; y \in C^{1\text{-var}}([0,T];\mathbb{R}^d) \right\}}^{d_{p\text{-var};T}},$$

and is equipped with the p-variation distance $d_{p\text{-var};T}$, or with the uniform distance $d_{\infty;T}$, associated to the Carnot-Carathéodory distance. See [10], Chapter 9.

- The closed ball of $G\Omega_{p,T}(\mathbb{R}^d)$ with respect to $d_{p\text{-var};T}$, of center $Y \in G\Omega_{p,T}(\mathbb{R}^d)$ and of radius $r > 0$, is denoted by $B_{p,T}(Y, r)$.
- For every $Y \in G\Omega_{p,T}(\mathbb{R}^d)$, $\omega_{Y,p} : (s,t) \in \bar{\Delta}_T \mapsto \|Y\|^p_{p\text{-var};s,t}$ is a control. See [10], Chapter 1 about some properties of the controls.
- Consider $q \geqslant 1$ such that $1/p + 1/q > 1$, $Y \in G\Omega_{p,T}(\mathbb{R}^d)$ and $h \in G\Omega_{q,T}(\mathbb{R}^e)$. The geometric p-rough path over (Y^1, h^1) provided at [10], Theorem 9.26 is denoted by $S_{[p]}(Y \oplus h)$. The translation of Y by h provided at [10], Theorem 9.34 is denoted by $T_h Y$.
- Consider $\gamma > 0$. The space of collections of γ-Lipschitz (resp. locally γ-Lipschitz) vector fields on $\mathbb{R}^e$ is denoted by $\mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d)$ (resp. $\mathrm{Lip}^\gamma_{\mathrm{loc}}(\mathbb{R}^e;\mathbb{R}^d)$) (see [10], Definition 10.2). $\mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d)$ is equipped with the γ-Lipschitz norm $\|.\|_{\mathrm{lip}^\gamma}$ such that, for every $V \in \mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d)$,

$$\|V\|_{\mathrm{lip}^\gamma} := \max\left\{ \|V\|_\infty, \|DV\|_\infty, \dots, \|D^{\lfloor\gamma\rfloor}V\|_\infty, \|D^{\lfloor\gamma\rfloor}V\|_{\{\gamma\}\text{-höl}} \right\}.$$

- The closed ball of $\mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d)$ with respect to $\|.\|_{\mathrm{lip}^\gamma}$, of center $V \in \mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d)$ and of radius $r > 0$, is denoted by $B_{\mathrm{Lip}^\gamma}(V, r)$.
- Consider $\varepsilon > 0$, a compact interval I included in $[0,T]$, a control $\omega : \bar{\Delta}_T \to \mathbb{R}_+$ and $Y \in G\Omega_{p,T}(\mathbb{R}^d)$. Put

$$M_{\varepsilon,I,\omega} := \sup_{\substack{D=\{r_k\} \in D_I \\ \omega(r_k, r_{k+1}) \leqslant \varepsilon}} \sum_{k=1}^{|D|-1} \omega(r_k, r_{k+1}),$$

$M_{\varepsilon,I,p}(Y) := M_{\varepsilon,I,\omega_{Y,p}}$ and

$$N_{\varepsilon,I,p}(Y) := \sup\{n \in \mathbb{N} : \tau_n \leqslant \sup(I)\}$$

where, $\tau_0 := \inf(I)$ and for every $n \in \mathbb{N}$,

$$\tau_{n+1} := \inf\left\{ t \in I : \|Y\|^p_{p\text{-var};\tau_n,t} \geqslant \varepsilon \text{ and } t > \tau_n \right\} \wedge \sup(I).$$

In the sequel, $I := [0,T]$.

- Consider $\gamma > p$ and $V \in \mathrm{Lip}^\gamma_{\mathrm{loc}}(\mathbb{R}^e;\mathbb{R}^d)$ satisfying the p-non explosion condition (i.e. V and $D^{[p]}V$ are respectively globally Lipschitz continuous and $(\gamma - [p])$-Hölder continuous on $\mathbb{R}^e$). The unique solution of $dX = V(X)d\mathbb{W}$ with the initial condition $X_0 \in G^{[p]}(\mathbb{R}^e)$ or $X_0 \in \mathbb{R}^e$, is denoted by $\pi_V(0, X_0; \mathbb{W})$.
- By [10], Exercice 10.55, if V is a collection of affine vector fields and $\omega : \bar{\Delta}_T \to \mathbb{R}_+$ is a control satisfying $\|\mathbb{W}\|_{p\text{-var};s,t} \leqslant \omega^{1/p}(s,t)$ for every $(s,t) \in \bar{\Delta}_T$, there exists a constant $C_1 > 0$, not depending on $X_0 \in \mathbb{R}^e$ and $\mathbb{W}$, such that :

$$\|\pi_V(0, x_0; \mathbb{W})\|_{\infty;T} \leqslant C_1(1 + \|x_0\|)e^{C_1 M_{1,I,\omega}}.$$

By [10], Theorem 10.36, if $V \in \mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d)$, there exists a constant $C_2 > 0$, not depending on $X_0 \in G^{[p]}(\mathbb{R}^e)$, V and $\mathbb{W}$, such that for every $(s,t) \in \bar{\Delta}_T$,

$$\|\pi_V(0,X_0;\mathbb{W})\|_{p\text{-var};s,t} \leqslant C_2 \left(\|V\|_{\mathrm{lip}^{\gamma-1}} \|\mathbb{W}\|_{p\text{-var};s,t} \vee \|V\|^p_{\mathrm{lip}^{\gamma-1}} \|\mathbb{W}\|^p_{p\text{-var};s,t} \right).$$

By [10], Theorem 10.47, if $V \in \mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d)$, there exists a constant $C_3 > 0$, not depending on V and $\mathbb{W}$, such that for every $(s,t) \in \bar{\Delta}_T$,

$$\left\| \int V(\mathbb{W}) d\mathbb{W} \right\|_{p\text{-var};s,t} \leqslant C_3 \|V\|_{\mathrm{lip}^{\gamma-1}} \left(\|\mathbb{W}\|_{p\text{-var};s,t} \vee \|\mathbb{W}\|^p_{p\text{-var};s,t} \right).$$

Notations (Gaussian stochastic analysis) :

- For every $t \in [0,T]$, $[0,t]$ is equipped with the Borel σ-algebra $\mathcal{B}_t$ generated by the usual topology on $[0,t]$.
- $\mathbb{R}^d$ is equipped with the Borel σ-algebra generated by the usual Euclidean topology on $\mathbb{R}^d$, and $G^{[p]}(\mathbb{R}^d)$ is equipped with the Borel σ-algebra generated by the Carnot-Carathéodory topology on $G^{[p]}(\mathbb{R}^d)$. These σ-algebras are both denoted by $\mathcal{B}$.
- Let W be a d-dimensional centered Gaussian process with continuous paths. Its Cameron-Martin space is denoted by

$$H^1 := \left\{ h \in C^0([0,T];\mathbb{R}) : \exists Z \in \mathcal{W} \text{ s.t. } \forall t \in [0,T],\, h_t = \mathbb{E}(W_t Z) \right\}$$

with

$$\mathcal{W} := \overline{\mathrm{span}\,\{W_t, t \in [0,T]\}}^{L^2}$$

(see [10], Subsection 15.2.2 and Section 15.3), its reproducing kernel Hilbert space is denoted by H, and the Wiener integral with respect to W defined on H is denoted by $\mathbf{W}$ (see [22], Section 1.1).
- The Malliavin derivative associated to $\mathbf{W}$ is denoted by $\mathbf{D}$ for the $\mathbb{R}^d$-valued (resp. H-valued) random variables, and its domain in $L^2(\Omega)$ (resp. $L^2(\Omega;H)$) is denoted by $\mathbb{D}^{1,2}$ (resp. $\mathbb{D}^{1,2}(H)$) (see [22], Section 1.2).
- For the $\mathbb{R}^d$-valued random variables, the divergence operator associated to $\mathbf{D}$ is denoted by δ, and its domain in $L^2(\Omega;H)$ is denoted by $\mathrm{dom}(\delta)$ (see [22], Section 1.3).
- The vector space of the $\mathbb{R}^d$-valued (resp. H-valued) random variables locally derivable in the sense of Malliavin is denoted by $\mathbb{D}^{1,2}_{\mathrm{loc}}$ (resp. $\mathbb{D}^{1,2}_{\mathrm{loc}}(H)$) (see [22], Subsection 1.3.5).

2. Regularity of the Itô map : existing and additional results

This section deals with the regularity of the Itô map associated to RDEs. On one hand, the results on the continuity and the differentiability of the Itô map with respect to the initial condition and to the driving signal coming from [10], Chapter 11 are stated. In addition, the continuous differentiability of the Itô map with respect to the collection of vector fields is proved. On the other hand, in order to apply the integrability results coming from [3], some tailor-made upper-bounds are provided for each derivative.

First, the existing continuity results of the Itô map and of the rough integral are synthesized.

Theorem 2.1. *Consider $R > 0$:*

(1) *The Itô map $(X_0, \mathbb{W}, V) \mapsto \pi_V(0, X_0; \mathbb{W})$ is uniformly continuous from*

$$G^{[p]}(\mathbb{R}^e) \times B_{p,T}(\mathbf{1}, R) \times Lip^\gamma(\mathbb{R}^e;\mathbb{R}^d) \text{ into } G\Omega_{p,T}(\mathbb{R}^d).$$

(2) *The map*

$$\mathbb{J} : (\mathbb{W}, V) \longmapsto \int V(\mathbb{W}) d\mathbb{W}$$

is uniformly continuous from

$$B_{p,T}(\mathbf{1}, R) \times Lip^{\gamma-1}(\mathbb{R}^d; \mathbb{R}^d) \text{ into } G\Omega_{p,T}(\mathbb{R}^d).$$

In each case, the uniform continuity holds true if $B_{p,T}(\mathbf{1}, R)$ *and* $G\Omega_{p,T}(\mathbb{R}^d)$ *are equipped with the uniform distance* $d_{\infty;T}$.

See [10], corollaries 10.39,40,48 for a proof.

Remark. Consider $x_0 \in \mathbb{R}^e$, $\mathbb{W} \in G\Omega_{p,T}(\mathbb{R}^d)$ and $V := (V_1, \dots, V_d)$ a collection of affine vector fields on $\mathbb{R}^e$. By [10], Theorem 10.53, $\pi_V(0, x_0; \mathbb{W})_t$ belongs to the ball $B(0; R(x_0, \mathbb{W}))$ of $\mathbb{R}^e$ for every $t \in [0, T]$, where

$$R(x_0, \mathbb{W}) := C(1 + \|x_0\|)e^{C\|\mathbb{W}\|^p_{p\text{-var};T}}$$

and $C > 0$ is a constant not depending on x_0 and $\mathbb{W}$. Moreover, for every $\tilde{x}_0 \in \mathbb{R}^e$ and every $\widetilde{\mathbb{W}} \in G\Omega_{p,T}(\mathbb{R}^d)$,

$$\|\tilde{x}_0\| \leqslant \|x_0\| \text{ and } \|\widetilde{\mathbb{W}}\|_{p\text{-var};T} \leqslant \|\mathbb{W}\|_{p\text{-var};T} \Longrightarrow R(\tilde{x}_0, \widetilde{\mathbb{W}}) \leqslant R(x_0, \mathbb{W}).$$

So, if $\hat{V} \in \mathrm{Lip}^\gamma(\mathbb{R}^e; \mathbb{R}^d)$ is the collection of vector fields satisfying $\hat{V} \equiv V$ on $B(0; R(x_0, \mathbb{W}))$, then

$$\pi_V(0, .) \equiv \pi_{\hat{V}}(0, .) \text{ on the set } B(0, \|x_0\|) \times B_{p,T}(\mathbf{1}, \|\mathbb{W}\|_{p\text{-var};T}).$$

Therefore, by Theorem 2.1, the map $\pi_V(0, .)$ is uniformly continuous from

$$B(0, \|x_0\|) \times B_{p,T}(\mathbf{1}, \|\mathbb{W}\|_{p\text{-var};T}) \text{ into } C^{p\text{-var}}([0, T]; \mathbb{R}^e).$$

The uniform continuity holds true if $B_{p,T}(\mathbf{1}, \|\mathbb{W}\|_{p\text{-var};T})$ and $C^{p\text{-var}}([0, T]; \mathbb{R}^e)$ are equipped with the uniform distance $d_{\infty;T}$.

The following technical corollary of [10], Theorem 9.26 allows to apply the integrability results of [3] to differential equations having a drift term.

Corollary 2.2. *Consider* $p > q \geqslant 1$ *such that* $1/p + 1/q > 1$, $Y \in G\Omega_{p,T}(\mathbb{R}^d)$, $h \in G\Omega_{q,T}(\mathbb{R}^e)$ *and* $\varepsilon > 0$. *There exists a constant* $C > 0$, *depending only on* p *and* q, *such that :*

$$M_{\varepsilon,I,p}[S_{[p]}(Y \oplus h)] \leqslant C[\|h\|^p_{q\text{-}var;T} + M_{\varepsilon,I,p}(Y)].$$

Proof. On one hand, for every $(s, t) \in \bar{\Delta}_T$,

$$\omega_{Y,p}(s, t) = \|Y\|_{p\text{-var};s,t} \leqslant \|S_{[p]}(Y \oplus h)\|_{p\text{-var};s,t}.$$

On the other hand, since $p/q \geqslant 1$ and, $\omega_{Y,p}$ and $\omega_{h,q}$ are two controls :

$$\omega = \|Y\|^p_{p\text{-var}} + \|h\|^p_{q\text{-var}} = \omega_{Y,p} + \omega^{p/q}_{h,q}$$

is also a control.

Then, by [10], Proposition 7.52, there exists a constant $C \geqslant 1$, depending only on p and q, such that for every $(s,t) \in \bar{\Delta}_T$,

$$\left\| S_{[p]}(Y \oplus h) \right\|^p_{p\text{-var};s,t} \leqslant C\omega(s,t).$$

In conclusion,

$$\begin{aligned} M_{\varepsilon,I,p}\left[S_{[p]}(Y \oplus h)\right] &\leqslant C \sup_{\substack{D=\{r_k\} \in D_I \\ \omega(r_k,r_{k+1}) \leqslant \varepsilon}} \sum_{k=1}^{|D|-1} \omega(r_k, r_{k+1}) \\ &\leqslant C\left[\|h\|^p_{q\text{-var};T} + M_{\varepsilon,I,p}(Y)\right] \end{aligned}$$

by the super-additivity of the control $\omega^{p/q}_{h,q}$. □

2.1. **Differentiability of the Itô map with respect to x_0 and V**

In order to prove the continuous differentiability of the Itô map of RDEs with respect to the collection of vector fields, it has to be shown for ODEs first.

Proposition 2.3. *Consider $\gamma \geqslant 1$, $x_0 \in \mathbb{R}^e$ and a continuous function $w : [0,T] \to \mathbb{R}^d$ of finite 1-variation. The map $V \mapsto \pi_V(0,x_0;w)$ is continuously differentiable from*

$$Lip^\gamma(\mathbb{R}^e;\mathbb{R}^d) \text{ into } C^{1\text{-}var}([0,T];\mathbb{R}^e).$$

Proof. In a first step, the derivability of the Itô map with respect to the collection of vector fields is established at every points and in every directions of $\mathrm{Lip}^\gamma(\mathbb{R}^d;\mathbb{R}^e)$. In a second step, via [10], Proposition B.5, the continuous differentiability of the partial Itô map is proved.

Step 1. Consider $V, \tilde{V} \in \mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d)$, $\varepsilon \in]0,1]$, $x^V := \pi_V(0,x_0;w)$ and $y^{V,\tilde{V}}$ the solution of the following ODE :

$$y^{V,\tilde{V}}_t = \int_0^t \langle DV(x^V_s), y^{V,\tilde{V}}_s \rangle dw_s + \int_0^t \tilde{V}(x^V_s) dw_s. \tag{1}$$

For every $t \in [0,T]$,

$$\begin{aligned} \frac{x^{V+\varepsilon\tilde{V}}_t - x^V_t}{\varepsilon} - y^{V,\tilde{V}}_t &= \int_0^t \left[\frac{V(x^{V+\varepsilon\tilde{V}}_s) - V(x^V_s)}{\varepsilon} - \langle DV(x^V_s), y^{V,\tilde{V}}_s\rangle\right] dw_s + \\ &\quad \int_0^t \left[\tilde{V}(x^{V+\varepsilon\tilde{V}}_s) - \tilde{V}(x^V_s)\right] dw_s \\ &= P_t(\varepsilon) + Q_t(\varepsilon) + R_t(\varepsilon) \end{aligned}$$

where,

$$\begin{aligned} P_t(\varepsilon) &:= \varepsilon^{-1} \int_0^t \left[V(x^{V+\varepsilon\tilde{V}}_s) - V(x^V_s) - \langle DV(x^V_s), x^{V+\varepsilon\tilde{V}}_s - x^V_s\rangle\right] dw_s, \\ Q_t(\varepsilon) &:= \int_0^t \left[\tilde{V}(x^{V+\varepsilon\tilde{V}}_s) - \tilde{V}(x^V_s)\right] dw_s \text{ and} \\ R_t(\varepsilon) &:= \int_0^t \langle DV(x^V_s), \varepsilon^{-1}(x^{V+\varepsilon\tilde{V}}_s - x^V_s) - y^{V,\tilde{V}}_s\rangle dw_s. \end{aligned}$$

Firstly, since V is continuously differentiable on $\mathbb{R}^e$, by [10], Lemma 4.2 :

$$\begin{aligned}\|P_t(\varepsilon)\| &\leqslant \varepsilon^{-1}\|w\|_{1\text{-var};T}\sup_{t\in[0,T]}\|V(x_t^{V+\varepsilon\tilde{V}})-V(x_t^V)-\langle DV(x_t^V),x_t^{V+\varepsilon\tilde{V}}-x_t^V\rangle\| \\ &\leqslant \eta(\varepsilon)\varepsilon^{-1}\|w\|_{1\text{-var};T}\|x^{V+\varepsilon\tilde{V}}-x^V\|_{\infty;T}\end{aligned}$$

where, $\eta(\varepsilon)\to 0$ when $\varepsilon\to 0$.

By [10], Theorem 3.18 :

$$\|P(\varepsilon)\|_{\infty;T}\leqslant M_3(\varepsilon):=2\eta(\varepsilon)e^{3M_1M_2}M_2\|\tilde{V}\|_\infty\|w\|_{1\text{-var};T} \tag{2}$$

with

$$M_1:=\|V\|_{\mathrm{lip}^\gamma}+\|\tilde{V}\|_{\mathrm{lip}^\gamma}\geqslant\|V+\varepsilon\tilde{V}\|_{\mathrm{lip}^1}\vee\|V\|_{\mathrm{lip}^1}$$

and $M_2:=\|w\|_{1\text{-var};T}$.

Secondly, since $\tilde{V}$ is continuously differentiable and of bounded derivative on $\mathbb{R}^e$, it is a collection of Lipschitz continuous vector fields. Then, by [10], Theorem 3.18 :

$$\|Q(\varepsilon)\|_{\infty;T}\leqslant M_4(\varepsilon):=2\varepsilon e^{3M_1M_2}M_2\|\tilde{V}\|_{\mathrm{lip}^\gamma}^2\|w\|_{1\text{-var};T}. \tag{3}$$

Thirdly,

$$\|R_t(\varepsilon)\|\leqslant\|V\|_{\mathrm{lip}^\gamma}\int_0^t\left\|\frac{x_s^{V+\varepsilon\tilde{V}}-x_s^V}{\varepsilon}-y_s^{V,\tilde{V}}\right\|\|dw_s\|. \tag{4}$$

Therefore, by inequalities (2), (3) and (4) :

$$\left\|\frac{x_t^{V+\varepsilon\tilde{V}}-x_t^V}{\varepsilon}-y_t^{V,\tilde{V}}\right\|\leqslant M_3(\varepsilon)+M_4(\varepsilon)+\|V\|_{\mathrm{lip}^\gamma}\int_0^t\left\|\frac{x_s^{V+\varepsilon\tilde{V}}-x_s^V}{\varepsilon}-y_s^{V,\tilde{V}}\right\|\|dw_s\|.$$

In conclusion, by the Gronwall lemma :

$$\begin{aligned}\left\|\frac{x^{V+\varepsilon\tilde{V}}-x^V}{\varepsilon}-y^{V,\tilde{V}}\right\|_{\infty;T} &\leqslant [M_3(\varepsilon)+M_4(\varepsilon)]\,e^{\|V\|_{\mathrm{lip}^\gamma}\|w\|_{1\text{-var};T}} \\ &\xrightarrow[\varepsilon\to 0]{} 0.\end{aligned}$$

Step 2. The solution of Equation (1) satisfies :

$$D_{\tilde{V}}x^V=\pi_A(0,0;.)\circ\mathbb{J}[F_{V,\tilde{V}}(.),.]\circ(\pi_V(0,x_0;.),.)(w)$$

where, $A:\mathbb{R}^e\to\mathcal{L}(\mathcal{L}(\mathbb{R}^e)\times\mathbb{R}^e;\mathbb{R}^e)$ and $F_{V,\tilde{V}}:\mathbb{R}^e\times\mathbb{R}^d\to\mathcal{L}(\mathbb{R}^e\times\mathbb{R}^d;\mathcal{L}(\mathbb{R}^e)\times\mathbb{R}^e)$ are two collections of vector fields, respectively defined by :

$$\begin{aligned}A(a)(L,b) &:= L.a+b \text{ and} \\ F_{V,\tilde{V}}(a,a')(b,b') &:= (\langle DV(a),.\rangle b';\tilde{V}(a)b')\end{aligned}$$

for every $a,b\in\mathbb{R}^e$, $a',b'\in\mathbb{R}^d$ and $L\in\mathcal{L}(\mathbb{R}^e)$.

Firstly, by the second point of Theorem 2.1, the map $\mathbb{J}$ is uniformly continuous on every bounded sets of

$$C^{1\text{-var}}\left([0,T];\mathbb{R}^e\times\mathbb{R}^d\right)\times C^{1\text{-var}}\left([0,T];\mathbb{R}^e\times\mathbb{R}^d\right).$$

Secondly, the map $(V,\tilde{V},a)\mapsto F_{V,\tilde{V}}(a)$ is uniformly continuous on every bounded sets of

$$\mathrm{Lip}^\gamma\left(\mathbb{R}^e;\mathbb{R}^d\right)\times\mathrm{Lip}^\gamma\left(\mathbb{R}^e;\mathbb{R}^d\right)\times\mathbb{R}^e\times\mathbb{R}^d$$

by construction.

Thirdly, the maps $\pi_A(0,0;.)$ and $V\mapsto\pi_V(0,x_0;w)$ are respectively uniformly continuous on every bounded sets of

$$C^{1\text{-var}}\left([0,T];\mathcal{L}(\mathbb{R}^e)\times\mathbb{R}^e\right)\text{ and }\mathrm{Lip}^\gamma\left(\mathbb{R}^e;\mathbb{R}^d\right)$$

by Theorem 2.1 and its remark.

Therefore, by composition, the map $(V,\tilde{V})\mapsto D_{\tilde{V}}x^V$ is uniformly continuous on every bounded sets of

$$\mathrm{Lip}^\gamma\left(\mathbb{R}^e;\mathbb{R}^d\right)\times\mathrm{Lip}^\gamma\left(\mathbb{R}^e;\mathbb{R}^d\right).$$

In conclusion, by [10], Proposition B.5, the map $V\mapsto\pi_V(0,x_0;w)$ is continuously differentiable from

$$\mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d)\text{ into }C^{1\text{-var}}([0,T];\mathbb{R}^e).$$

□

Theorem 2.4. *Consider $\mathbb{W}\in G\Omega_{p,T}(\mathbb{R}^d)$:*

(1) *Let $V:=(V_1,\dots,V_d)$ be a collection of γ-Lipschtiz vector fields on $\mathbb{R}^e$. The map $x_0\mapsto\pi_V(0,x_0;\mathbb{W})$ is continuously differentiable from*

$$\mathbb{R}^e\text{ into }C^{p\text{-var}}\left([0,T];\mathbb{R}^e\right).$$

For every $t\in[0,T]$, the Jacobian matrix of $\pi_V(0,.;\mathbb{W})_t$ at point $x_0\in\mathbb{R}^e$ is denoted by $J^{x_0,\mathbb{W}}_{t\leftarrow 0}$.

Moreover, for every $\varepsilon>0$, there exists a constant $C_1>0$ only depending on p, γ, ε and $\|V\|_{lip^\gamma}$, such that for every $x_0\in\mathbb{R}^e$,

$$\|J^{x_0,\mathbb{W}}_{.\leftarrow 0}\|_{\infty;T}\leqslant C_1e^{C_1M_{\varepsilon,I,p}(\mathbb{W})}.$$

(2) *For every $t\in[0,T]$, $J^{x_0,\mathbb{W}}_{t\leftarrow 0}$ is an invertible matrix. Moreover, for every $\varepsilon>0$, there exists a constant $C_2>0$ only depending on p, γ, ε and $\|V\|_{lip^\gamma}$, such that for every $x_0\in\mathbb{R}^e$,*

$$\|(J^{x_0,\mathbb{W}}_{.\leftarrow 0})^{-1}\|_{\infty;T}\leqslant C_2e^{C_2M_{\varepsilon,I,p}(\mathbb{W})}.$$

(3) *Consider $x_0\in\mathbb{R}^e$. The map $V\mapsto\pi_V(0,x_0;\mathbb{W})$ is continuously differentiable from*

$$Lip^\gamma(\mathbb{R}^e;\mathbb{R}^d)\text{ into }C^{p\text{-var}}([0,T];\mathbb{R}^e).$$

Moreover, for every $R>0$ and $V,\tilde{V}\in B_{Lip^\gamma}(0,R)$, there exists two constants $\eta>0$ and $C_3>0$, depending (continuously) on R but not on $\mathbb{W}$, such that :

$$\|\partial_V\pi_V(0,x_0;\mathbb{W}).\tilde{V}\|_{\infty;T}\leqslant C_3e^{C_3M_{\eta,I,p}(\mathbb{W})}.$$

Proof. See [10], theorems 11.3-6 for a proof of the continuous differentiability of the Itô map with respect to the initial condition, and see [3], Corollary 3.4 about the upper-bound provided at the first point for $\|J^{x_0,\mathbb{W}}_{.\leftarrow 0}\|_{\infty;T}$; $x_0 \in \mathbb{R}^e$.

Let I be the identity matrix of $\mathcal{M}_e(\mathbb{R})$. The proofs of the points 1 and 2 are similar because if $w : [0,T] \to \mathbb{R}^d$ is a continuous function of finite 1-variation, then

$$\begin{aligned} J^{x_0,w}_{t\leftarrow 0} &= I + \int_0^t \langle DV[\pi_V(0,x_0;\mathbb{W})_s], J^{x_0,w}_{s\leftarrow 0}\rangle dw_s \text{ and} \\ (J^{x_0,w}_{t\leftarrow 0})^{-1} &= I - \int_0^t \langle DV[\pi_V(0,x_0;\mathbb{W})_s], (J^{x_0,w}_{s\leftarrow 0})^{-1}\rangle dw_s \end{aligned}$$

as mentioned at the proof of [10], Proposition 4.11.

The proof of the third point is detailed. In a first step, the continuous differentiability of the Itô map with respect to the collection of vector fields is proved. In a second step, in order to apply the integrability results coming from [3], a tailor-made upper-bound for the derivative of the Itô map with respect to V is provided.

Step 1. Since $\mathbb{W} \in G\Omega_{p,T}(\mathbb{R}^d)$, there exists a sequence $(w^n, n \in \mathbb{N})$ of functions belonging to $C^{1\text{-var}}([0,T];\mathbb{R}^d)$ and satisfying :

$$\lim_{n\to\infty} d_{p\text{-var};T}\left[S_{[p]}(w^n)_{0,.}, \mathbb{W}\right] = 0. \tag{5}$$

Consider $n \in \mathbb{N}$, $\mathbb{W}^n := S_{[p]}(w^n)_{0,.}$, $x_0 \in \mathbb{R}^e$, $a := (x_0, 0)$,

$$X_0 := \left(1, a, \ldots, \frac{a^{\otimes[p]}}{[p]!}\right) \in T^{[p]}\left(\mathbb{R}^{e+1}\right)$$

and $V, \tilde{V} \in \mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d)$.

By Proposition 2.3, the map $\pi_.(0,x_0;w^n)$ is continuously differentiable from

$$\mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d) \text{ into } C^{1\text{-var}}([0,T];\mathbb{R}^e).$$

In particular, $\partial_V \pi_V(0,x_0;w^n).\tilde{V} = \varphi(\mathbb{W}^n, V, \tilde{V})$ with

$$\varphi(.,V,\tilde{V}) := \pi_A(0,0;.) \circ \mathbb{J}(.;F_{V,\tilde{V}}) \circ \pi_{F_V}(0,X_0;.)$$

where,

$$\begin{aligned} A : \mathbb{R}^e &\longrightarrow \mathcal{L}(\mathcal{L}(\mathbb{R}^e)\times\mathbb{R}^e;\mathbb{R}^e), \\ F_{V,\tilde{V}} : \mathbb{R}^e\times\mathbb{R}^d &\longrightarrow \mathcal{L}(\mathbb{R}^e\times\mathbb{R}^d;\mathcal{L}(\mathbb{R}^e)\times\mathbb{R}^e) \text{ and} \\ F_V : \mathbb{R}^e &\longrightarrow \mathcal{L}(\mathbb{R}^d;\mathbb{R}^e\times\mathbb{R}^d) \end{aligned}$$

are three collections of vector fields, respectively defined by :

$$\begin{aligned} A(a)(L,b) &:= L.a + b, \\ F_{V,\tilde{V}}(a,a')(b,b') &:= (\langle DV(a),.\rangle b'; \tilde{V}(a)b') \text{ and} \\ F_V(a)b' &:= (V(a)b', b') \end{aligned}$$

for every $a, b \in \mathbb{R}^e$, $a', b' \in \mathbb{R}^d$ and $L \in \mathcal{L}(\mathbb{R}^e)$.

Consider $\varepsilon \in]0,1]$. By the Taylor formula applied to $\pi_{\cdot}(0,x_0;\mathbb{W}^n)$ between V and $V+\varepsilon\tilde{V}$, and [10], Definition 10.17 :

$$\pi_{V+\varepsilon\tilde{V}}(0,x_0;\mathbb{W}) - \pi_V(0,x_0;\mathbb{W}) = \lim_{n\to\infty} \int_0^{\varepsilon} \varphi(\mathbb{W}^n, V+\theta\tilde{V}, \tilde{V}) d\theta \tag{6}$$

uniformly.

Via the Lebesgue theorem and [10], Proposition B.1, let show that the derivative of $\pi_{\cdot}(0,x_0;\mathbb{W})$ at point V, in the direction $\tilde{V}$, exists in $C^{p\text{-var}}([0,T];\mathbb{R}^e)$ equipped with the norm $\|.\|_{p\text{-var};T}$ and coincides with $\varphi(\mathbb{W},V,\tilde{V})$.

On one hand, by the continuity results of Theorem 2.1 :

$$\forall \theta \in]0,1],\ \varphi(\mathbb{W}^n, V+\theta\tilde{V}, \tilde{V}) \xrightarrow[n\to\infty]{} \varphi(\mathbb{W}, V+\theta\tilde{V}, \tilde{V})$$

in $C^{p\text{-var}}([0,T];\mathbb{R}^e)$ equipped with $\|.\|_{\infty;T}$.

On the other hand, by applying successively [10], theorems 10.47 and 10.36, for every $\theta \in]0,1]$ and every $(s,t) \in \bar{\Delta}_T$,

$$\begin{aligned} \omega_1^{1/p}(s,t;n;\theta) &:= \left\| \int F_{V+\theta\tilde{V},\tilde{V}} \left[\pi_{F_{V+\theta\tilde{V}}}(0,X_0;\mathbb{W}^n) \right] d\pi_{F_{V+\theta\tilde{V}}}(0,X_0;\mathbb{W}^n) \right\|_{p\text{-var};s,t} \\ &\leqslant \omega_2^{1/p}(s,t;n) \end{aligned}$$

with

$$\omega_2^{1/p}(s,t;n) := \omega_3^{1/p}(s,t;n) \vee \omega_3(s,t;n) \vee \omega_3^p(s,t;n)$$

and

$$\omega_3(s,t;n) := \eta_1 \|\mathbb{W}^n\|^p_{p\text{-var};s,t}$$

where $\eta_1 > 0$ is depending on V and $\tilde{V}$, but not on $\mathbb{W}^n$ and θ.

By [10], Exercice 10.55, there exists a constant $C_4 > 0$, not depending on $\mathbb{W}^n$ and θ, such that :

$$\begin{aligned} \left\| \varphi(\mathbb{W}^n, V+\theta\tilde{V}, \tilde{V}) \right\|_{\infty;T} &\leqslant C_4 \exp\left[C_4 \sup_{\substack{D=\{r_k\} \in D_I \\ \omega_2(r_k,r_{k+1};n) \leqslant 1}} \sum_{k=1}^{|D|-1} \omega_2(r_k,r_{k+1};n) \right] \\ &= C_4 \exp\left[C_4 \sup_{\substack{D=\{r_k\} \in D_I \\ \omega_3(r_k,r_{k+1};n) \leqslant 1}} \sum_{k=1}^{|D|-1} \omega_3(r_k,r_{k+1};n) \right], \end{aligned}$$

because

$$\omega_2(.;n) \equiv \omega_3(.;n) \text{ when } \omega_2(.;n) \leqslant 1.$$

By the super-additivity of the control $\omega_3(.;n)$:

$$\left\| \varphi(\mathbb{W}^n, V+\theta\tilde{V}, \tilde{V}) \right\|_{\infty;T} \leqslant C_4 e^{\eta_1 C_4 \|\mathbb{W}^n\|^p_{p\text{-var};T}}.$$

In the right-hand side of that inequality, since η_1 and C_4 are not depending on $\mathbb{W}^n$ and θ, and since

$$\sup_{n\in\mathbb{N}^*}\|\mathbb{W}^n\|^p_{p\text{-var};T}<\infty$$

by (5) :

$$\sup_{\theta\in[0,1]}\sup_{n\in\mathbb{N}}\left\|\varphi(\mathbb{W}^n,V+\theta\tilde{V},\tilde{V})\right\|_{\infty;T}<\infty$$

in $C^{p\text{-var}}([0,T];\mathbb{R}^e)$ equipped with $\|.\|_{\infty;T}$.

Therefore, by the Lebesgue theorem and Inequality (6) :

$$\pi_{V+\varepsilon\tilde{V}}(0,x_0;\mathbb{W})-\pi_V(0,x_0;\mathbb{W})=\int_0^\varepsilon\varphi(\mathbb{W},V+\theta\tilde{V},\tilde{V})d\theta.$$

Since $\theta\mapsto\varphi(\mathbb{W},V+\theta\tilde{V},\tilde{V})$ is continuous from

$$[0,1]\text{ into }C^{p\text{-var}}([0,T];\mathbb{R}^e)\text{ (equipped with }\|.\|_{p\text{-var};T})$$

by Theorem 2.1 ; by [10], Proposition B.1, the derivative of $\pi_.(0,x_0;\mathbb{W})$ at point V, in the direction $\tilde{V}$, exists in $C^{p\text{-var}}([0,T];\mathbb{R}^e)$ equipped with $\|.\|_{p\text{-var};T}$ and coincides with $\varphi(\mathbb{W},V,\tilde{V})$.

Finally, as at the second step of the proof of Proposition 2.3, via [10], Proposition B.5 and Lemma 4.2, the map $\pi_.(0,x_0;\mathbb{W})$ is continuously differentiable from

$$\mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^d)\text{ into }C^{p\text{-var}}([0,T];\mathbb{R}^e).$$

Step 2. Consider $R>0$ and $V,\tilde{V}\in B_{\mathrm{Lip}^\gamma}(0,R)$.

By applying successively [10], theorems 10.47 and 10.36, for every $(s,t)\in\bar{\Delta}_T$,

$$\begin{aligned}\omega_4^{1/p}(s,t) &:= \left\|\int F_{V,\tilde{V}}\left[\pi_{F_V}(0,X_0;\mathbb{W})\right]d\pi_{F_V}(0,X_0;\mathbb{W})\right\|_{p\text{-var};s,t}\\ &\leqslant \omega_5^{1/p}(s,t)\end{aligned}$$

with

$$\omega_5^{1/p}(s,t):=\omega_6^{1/p}(s,t)\vee\omega_6(s,t)\vee\omega_6^p(s,t)$$

and

$$\omega_6(s,t):=\eta_2\|\mathbb{W}\|^p_{p\text{-var};s,t}$$

where $\eta_2>0$ is depending on R (continuously), but not on $\mathbb{W}$.

By [10], Exercice 10.55, there exists a constant $C_5>0$, not depending on R and $\mathbb{W}$, such that :

$$\begin{aligned}\left\|\partial_V\pi_V(0,x_0;\mathbb{W}).\tilde{V}\right\|_{\infty;T} &\leqslant C_5\exp\left[C_5\sup_{\substack{D=\{r_k\}\in D_I\\ \omega_5(r_k,r_{k+1})\leqslant 1}}\sum_{k=1}^{|D|-1}\omega_5(r_k,r_{k+1})\right]\\ &= C_5\exp\left[C_5\sup_{\substack{D=\{r_k\}\in D_I\\ \omega_6(r_k,r_{k+1})\leqslant 1}}\sum_{k=1}^{|D|-1}\omega_6(r_k,r_{k+1})\right],\end{aligned}$$

because

$$\omega_5 \equiv \omega_6 \text{ when } \omega_5 \leqslant 1.$$

However,

$$\sup_{\substack{D=\{r_k\} \in D_I \\ \omega_6(r_k,r_{k+1}) \leqslant 1}} \sum_{k=1}^{|D|-1} \omega_6(r_k,r_{k+1}) = \eta_2 M_{\eta_2^{-1},I,p}(\mathbb{W}).$$

Therefore,

$$\left\| \partial_V \pi_V(0,x_0;\mathbb{W}).\tilde{V} \right\|_{\infty;T} \leqslant C_3 e^{C_3 M_{\eta,I,p}(\mathbb{W})}$$

with $C_3 := C_5(1 \vee \eta_2)$ and $\eta := \eta_2^{-1}$. □

Notations. In the sequel, the matrices $J_{t\leftarrow 0}^{x_0,\mathbb{W}}$ and $(J_{t\leftarrow 0}^{x_0,\mathbb{W}})^{-1}$ will be respectively denoted by $J_{0\leftarrow t}^{\mathbb{W}}$ and $J_{t\leftarrow 0}^{\mathbb{W}}$ for the sake of simplicity. Moreover, for every $(s,t) \in \bar{\Delta}_T$, put

$$J_{s\leftarrow t}^{\mathbb{W}} := J_{s\leftarrow 0}^{\mathbb{W}} J_{0\leftarrow t}^{\mathbb{W}} \text{ and } J_{t\leftarrow s}^{\mathbb{W}} := J_{t\leftarrow 0}^{\mathbb{W}} J_{0\leftarrow s}^{\mathbb{W}}.$$

Then,

$$J_{s\leftarrow t}^{\mathbb{W}} J_{t\leftarrow s}^{\mathbb{W}} = J_{t\leftarrow s}^{\mathbb{W}} J_{s\leftarrow t}^{\mathbb{W}} = I.$$

At the following corollary, the upper-bounds provided at the previous theorem are extended to RDEs having a drift term.

Corollary 2.5. *Consider $m \in \mathbb{N}^*$, $p > q \geqslant 1$ such that $1/p + 1/q > 1$, $h : [0,T] \to \mathbb{R}^m$ a continuous function of finite q-variation, $\mathbb{W} \in G\Omega_{p,T}(\mathbb{R}^d)$ and $\mathbb{W}^h := S_{[p]}(\mathbb{W} \oplus h)$:*

(1) *Let $V := (V_1, \ldots, V_{d+m})$ be a collection of γ-Lipschitz vector fields on $\mathbb{R}^e$. For every $\varepsilon > 0$, there exists a constant $C_1 > 0$ depending only on p, q, γ, ε and $\|V\|_{lip^\gamma}$, such that for every $x_0 \in \mathbb{R}^e$,*

$$\|J_{\cdot\leftarrow 0}^{\mathbb{W}^h}\|_{\infty;T} \leqslant C_1 \exp\left[C_1 \left[\|h\|_{q\text{-}var;T}^p + M_{\varepsilon,I,p}(\mathbb{W})\right]\right].$$

(2) *Consider $x_0 \in \mathbb{R}^e$. For every $R > 0$ and $V, \tilde{V} \in B_{Lip^\gamma}(0,R)$, there exists two constants $\varepsilon > 0$ and $C_2 > 0$, depending on R but not on h and $\mathbb{W}$, such that :*

$$\|\partial_V \pi_V(0,x_0;\mathbb{W}^h).\tilde{V}\|_{\infty;T} \leqslant C_2 \exp\left[C_2 \left[\|h\|_{q\text{-}var;T}^p + M_{\varepsilon,I,p}(\mathbb{W})\right]\right].$$

Proof. By Corollary 2.2, there exists a constant $C_3 > 0$, depending only on p and q, such that for every $\varepsilon > 0$,

$$M_{\varepsilon,I,p}(\mathbb{W}^h) \leqslant C_3 \left[\|h\|_{q\text{-var};T}^p + M_{\varepsilon,I,p}(\mathbb{W})\right].$$

Therefore, by Theorem 2.4 :

(1) Let $V \in \mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^{d+m})$ be arbitrarily chosen. For every $\varepsilon > 0$, there exists a constant $C_4 > 0$ depending only on p, γ, ε and $\|V\|_{\mathrm{lip}^\gamma}$, such that for every $x_0 \in \mathbb{R}^e$,

$$\begin{aligned} \|J_{\cdot\leftarrow 0}^{\mathbb{W}^h}\|_{\infty;T} &\leqslant C_4 e^{C_4 M_{\varepsilon,I,p}(\mathbb{W}^h)} \\ &\leqslant C_1 \exp\left[C_1 \left[\|h\|_{q\text{-var};T}^p + M_{\varepsilon,I,p}(\mathbb{W})\right]\right] \end{aligned}$$

with $C_1 := C_4(1 \vee C_3)$.

(2) Let $x_0 \in \mathbb{R}^e$ be arbitrarily chosen. For every $R > 0$ and $V, \tilde{V} \in B_{\mathrm{Lip}^\gamma}(0, R)$, there exists two constants $\varepsilon > 0$ and $C_5 > 0$, depending on R but not on $\mathbb{W}^h$, such that :

$$\begin{aligned} \|\partial_V \pi_V(0, x_0; \mathbb{W}^h).\tilde{V}\|_{\infty;T} &\leqslant C_5 e^{C_5 M_{\varepsilon,I,p}(\mathbb{W}^h)} \\ &\leqslant C_2 \exp\left[C_2\left[\|h\|^p_{q\text{-var};T} + M_{\varepsilon,I,p}(\mathbb{W})\right]\right] \end{aligned}$$

with $C_2 := C_5(1 \vee C_3)$.

□

2.2. **Differentiability of the Itô map with respect to the driving signal**

First of all, the notion of differentiability introduced by P. Friz and N. Victoir on $G\Omega_{p,T}(\mathbb{R}^d)$ is reminded.

Definition 2.6. *Consider a Banach space F, $p > q \geqslant 1$ such that $1/p + 1/q > 1$, and an open set U of $G\Omega_{p,T}(\mathbb{R}^d)$. The map $\varphi : G\Omega_{p,T}(\mathbb{R}^d) \to F$ is continuously differentiable in the sense of Friz-Victoir on U if and only if, for every $Y \in U$, the map*

$$h \in C^{q\text{-}var}([0,T];\mathbb{R}^d) \longmapsto \varphi(T_h Y) \in F$$

is continuously differentiable.

With the notations of Definition 2.6, if φ is continuously differentiable from U into F in the sense of Friz-Victoir, then

$$\forall Y \in U,\ \psi^Y : h \in C^{q\text{-var}}([0,T];\mathbb{R}^d) \longmapsto \psi^Y(h) = \varphi(T_h Y)$$

is derivable at every points and in every directions of $C^{q\text{-var}}([0,T];\mathbb{R}^d)$.

Notation. For every continuous function $h : [0,T] \to \mathbb{R}^d$ of finite q-variation,

$$\begin{aligned} D^{\mathrm{FV}}_h \varphi(Y) &:= D_h \psi^Y(0) \\ &= \lim_{\varepsilon \to 0} \frac{\varphi(T_{\varepsilon h} Y) - \varphi(T_0 Y)}{\varepsilon}. \end{aligned}$$

In the sequel, D^{FV} is called the Friz-Victoir (directional) derivative operator.

Theorem 2.7. *Consider a collection $V := (V_1, \ldots, V_d)$ of γ-Lipschitz vector fields on $\mathbb{R}^e$ and $x_0 \in \mathbb{R}^e$. The map $\mathbb{W} \mapsto \pi_V(0, x_0; \mathbb{W})$ is continuously differentiable from*

$$G\Omega_{p,T}(\mathbb{R}^d) \text{ into } C^{p\text{-}var}([0,T];\mathbb{R}^e)$$

in the sense of Friz-Victoir.

Moreover, for every $\mathbb{W} \in G\Omega_{p,T}(\mathbb{R}^d)$ and every continous function $h : [0,T] \to \mathbb{R}^d$ of finite q-variation,

$$D^{FV}_h \pi_V(0, x_0; \mathbb{W}) = \int_0^{\cdot} J^{\mathbb{W}}_{\cdot \leftarrow s} V\left[\pi_V(0, x_0; \mathbb{W})_s\right] dh_s.$$

(Duhamel principle).

Consider $\mathbb{W} \in G\Omega_{p,T}(\mathbb{R}^d)$ and a control $\omega : \bar{\Delta}_T \to \mathbb{R}_+$ satisfying :

$$\forall (s,t) \in \bar{\Delta}_T,\ \|\mathbb{W}\|_{p\text{-}var;s,t} \leqslant \omega^{1/p}(s,t).$$

(1) *There exists a constant $C_1 > 0$, not depending on $\mathbb{W}$ and ω, such that for every continous function $h : [0,T] \to \mathbb{R}^d$ of finite q-variation,*

$$\|D_h^{FV}\pi_V(0,x_0;\mathbb{W})\|_{\infty;T} \leqslant C_1 \exp\left[C_1(\|h\|_{q\text{-}var;T}^p + M_{1,I,\omega})\right].$$

(2) *There exists a constant $C_2 > 0$, not depending on $\mathbb{W}$ and ω, such that for every continous function $h : [0,T] \to \mathbb{R}^d$ of finite q-variation,*

$$\|D_h^{FV}\pi_V(0,x_0;\mathbb{W})\|_{p\text{-}var;T} \leqslant C_2 \exp\left[C_2\left[\|h\|_{q\text{-}var;T}^p + \omega(0,T)\right]\right].$$

Proof. See [10], theorems 11.3-6 and Exercice 11.9 for a proof of the first part.

Consider a continuous function $h : [0,T] \to \mathbb{R}^d$ of finite q-variation, $\mathbb{W}^h := S_{[p]}(\mathbb{W} \oplus h)$, $a := (x_0, 0, 0)$ and

$$X_0 := \left(1, a, \ldots, \frac{a^{\otimes[p]}}{[p]!}\right) \in T^{[p]}\left(\mathbb{R}^{e+2}\right).$$

By [10], Theorem 11.3 :

$$D_h^{\mathrm{FV}}\pi_V(0,x_0;\mathbb{W}) = \pi_A(0,0;.) \circ \mathbb{J}(.,F) \circ \pi_G(0,X_0;.)(\mathbb{W}^h)$$

where,

$$\begin{array}{rcl} A : \mathbb{R}^e & \longrightarrow & \mathcal{L}(\mathcal{L}(\mathbb{R}^e) \times \mathbb{R}^e; \mathbb{R}^e), \\ F : \mathbb{R}^e \times \mathbb{R}^d \times \mathbb{R}^d & \longrightarrow & \mathcal{L}(\mathbb{R}^e \times \mathbb{R}^d \times \mathbb{R}^d; \mathcal{L}(\mathbb{R}^e) \times \mathbb{R}^e) \text{ and} \\ G : \mathbb{R}^e & \longrightarrow & \mathcal{L}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^e \times \mathbb{R}^d \times \mathbb{R}^d) \end{array}$$

are three collections of vector fields, respectively defined by :

$$\begin{array}{rcl} A(a)(L,b) & := & L.a + b, \\ F(a,a',a'')(b,b',b'') & := & (\langle DV(a), .\rangle b'; V(a)b'') \text{ and} \\ G(a)(b',b'') & := & (V(a)b', b', b'') \end{array}$$

for every $a, b \in \mathbb{R}^e$, $a', b', a'', b'' \in \mathbb{R}^d$ and $L \in \mathcal{L}(\mathbb{R}^e)$.

By applying successively [10], theorems 10.47 and 10.36, for every $(s,t) \in \bar{\Delta}_T$,

$$\begin{array}{rcl} \omega_1^{1/p}(s,t) & := & \left\| \int F\left[\pi_G(0,X_0;\mathbb{W}^h)\right] d\pi_G(0,X_0;\mathbb{W}^h) \right\|_{p\text{-var};s,t} \\ & \leqslant & \omega_2^{1/p}(s,t) \end{array}$$

with

$$\omega_2^{1/p}(s,t) := \omega_3^{1/p}(s,t) \vee \omega_3(s,t) \vee \omega_3^p(s,t)$$

and, by [10], Proposition 7.52 :

$$\omega_3(s,t) := \varepsilon_1\left[\|h\|_{q\text{-var};s,t}^p + \omega(s,t)\right] \geqslant \varepsilon_2 \|\mathbb{W}^h\|_{p\text{-var};s,t}^p$$

where, $\varepsilon_1, \varepsilon_2 \geqslant 1$ are two constants not depending on $\mathbb{W}$, ω and h.

On one hand, by [10], Exercice 10.55, there exists a constant $C_3 > 0$, not depending on $\mathbb{W}$, ω and h, such that :

$$\begin{aligned}\left\|D_h^{\mathrm{FV}}\pi_V(0,x_0;\mathbb{W})\right\|_{\infty;T} &\leqslant C_3\exp\left[C_3\sup_{\substack{D=\{r_k\}\in D_I\\ \omega_2(r_k,r_{k+1})\leqslant 1}}\sum_{k=1}^{|D|-1}\omega_2(r_k,r_{k+1})\right]\\ &= C_3\exp\left[C_3\sup_{\substack{D=\{r_k\}\in D_I\\ \omega_3(r_k,r_{k+1})\leqslant 1}}\sum_{k=1}^{|D|-1}\omega_3(r_k,r_{k+1})\right]\\ &\leqslant C_1\exp\left[C_1\left(\|h\|_{q\text{-var};T}^p+M_{1,I,\omega}\right)\right]\end{aligned}$$

with $C_1 := C_3\varepsilon_1$, because

$$\omega_2 \equiv \omega_3 \text{ when } \omega_2 \leqslant 1 \tag{7}$$

and

$$\forall (s,t)\in\bar{\Delta}_T,\ \omega(s,t)\leqslant\omega_3(s,t).$$

On the other hand, by [10], Theorem 10.53, there exists a constant $C_4 > 0$, not depending on $\mathbb{W}$, ω and h, such that for every $(s,t)\in\bar{\Delta}_T$ satisfying $\omega_2(s,t)\leqslant 1$,

$$\begin{aligned}\|D_h^{\mathrm{FV}}\pi_V(0,x_0;\mathbb{W})_{s,t}\| &\leqslant C_4\left[1+\|D_h^{\mathrm{FV}}\pi_V(0,x_0;\mathbb{W})_s\|\right]\omega_2^{1/p}(s,t)e^{C_4\omega_2(s,t)}\\ &\leqslant C_4\left[1+\|D_h^{\mathrm{FV}}\pi_V(0,x_0;\mathbb{W})\|_{\infty;T}\right]\omega_3^{1/p}(s,t)e^{C_4\omega_3(0,T)}\end{aligned}$$

by (7).

Therefore, by the super-additivity of the control ω_3, there exists a constant $C_2 > 0$, not depending on $\mathbb{W}$, ω and h, such that :

$$\|D_h^{\mathrm{FV}}\pi_V(0,x_0;\mathbb{W})\|_{p\text{-var};T}\leqslant C_2\exp\left[C_2\left[\|h\|_{q\text{-var;T}}^p+\omega(0,T)\right]\right].$$

□

At the following corollary, the upper-bounds provided at the previous theorem are extended to RDEs having a drift term.

Corollary 2.8. *Consider $m\in\mathbb{N}^*$, $p>q\geqslant 1$ such that $1/p+1/q>1$, $r\in[1,p[$ such that $1/p+1/r>1$, $g:[0,T]\to\mathbb{R}^m$ a continuous function of finite r-variation, $\mathbb{W}\in G\Omega_{p,T}(\mathbb{R}^d)$, $\mathbb{W}^g := S_{[p]}(\mathbb{W}\oplus g)$, $V := (V_1,\dots,V_{d+m})$ a collection of γ-Lipschitz vector fields on $\mathbb{R}^e$ and $x_0\in\mathbb{R}^e$. There exists a constant $C>0$, not depending on g and $\mathbb{W}$, such that for every continuous function $h:[0,T]\to\mathbb{R}^{d+m}$ of finite q-variation,*

$$\|D_h^{FV}\pi_V(0,x_0;\mathbb{W}^g)\|_{\infty;T}\leqslant C\exp\left[C\left[\|h\|_{q\text{-var};T}^p+\|g\|_{r\text{-var};T}^p+M_{1,I,p}(\mathbb{W})\right]\right].$$

Proof. Let $h:[0,T]\to\mathbb{R}^{d+m}$ be a continuous function of finite q-variation. By Corollary 2.2, there exists a constant $C_1>0$, depending only on p and r, such that :

$$M_{1,I,p}(\mathbb{W}^g)\leqslant C_1\left[\|g\|_{r\text{-var};T}^p+M_{1,I,p}(\mathbb{W})\right].$$

Then, by Theorem 2.7, there exists a constant $C_2 > 0$, not depending on $\mathbb{W}^g$ and h, such that :

$$\begin{aligned}\|D_h^{\mathrm{FV}}\pi_V(0,x_0;\mathbb{W}^g)\|_{\infty;T} &\leqslant C_2\exp\left[C_2\left[\|h\|^p_{q\text{-var};T}+M_{1,I,p}(\mathbb{W}^g)\right]\right]\\ &\leqslant C\exp\left[C\left[\|h\|^p_{q\text{-var};T}+\|g\|^p_{r\text{-var};T}+M_{1,I,p}(\mathbb{W})\right]\right]\end{aligned}$$

with $C := C_2(1 \vee C_1)$. □

2.3. **Application to the Gaussian stochastic analysis**

Consider a d-dimensional stochastic process W and the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where Ω is the vector space of continuous functions from $[0,T]$ into $\mathbb{R}^d$, $\mathcal{A}$ is the σ-algebra generated by cylinder sets of Ω, and $\mathbb{P}$ is the probability measure induced by the process W on $(\Omega, \mathcal{A})$.

In order to prove Corollary 2.15 which is crucial at Section 3, the existing results on Gaussian rough paths proved by P. Friz and N. Victoir in [9], and by T. Cass, C. Litterer and T. Lyons in [3] have to be stated first.

Consider the two following technical assumptions on the stochastic process W.

Assumption 2.9. *W is a d-dimensionnel centered Gaussian process with continuous paths. Moreover, its covariance function c_W is of finite 2D ρ-variation with $\rho \in [1,2[$ (see [10], Definition 5.50).*

Assumption 2.10. *There exists $p > q \geqslant 1$ such that :*

$$\frac{1}{p}+\frac{1}{q} > 1 \text{ and } H^1 \hookrightarrow C^{q\text{-}var}([0,T];\mathbb{R}^d).$$

Example. By [10], Proposition 15.5, Proposition 15.7 and Exercice 20.2, the fractional Brownian motion of Hurst parameter $H \in]1/4, 1/2]$ satisfies assumptions 2.9 and 2.10.

Theorem 2.11. *Consider a stochastic process W satisfying Assumption 2.9, and $p > 2\rho$. For almost every $\omega \in \Omega$, there exists a geometric p-rough path $\mathbb{W}(\omega)$ over $W(\omega)$ satisfying :*

1. *There exists a deterministic constant $C > 0$, only depending on ρ, p and $\|c_W\|_{\rho\text{-}var;[0,T]^2}$, such that :*

$$\mathbb{E}\left(e^{C\|\mathbb{W}\|^2_{p\text{-}var;T}}\right) < \infty.$$

   (generalized Fernique theorem).
2. *Let $(W^n, n \in \mathbb{N})$ be a sequence of linear approximations, or of mollifier approximations, of the process W. $\mathbb{W}$ is the limit in p-variation, in $L^r(\Omega)$ for every $r \geqslant 1$, of the sequence $(S_3(W^n), n \in \mathbb{N})$ (universality).*

$\mathbb{W}$ is the enhanced Gaussian process over W.

See [10], Theorem 15.33 for a proof.

Proposition 2.12. *Consider a stochastic process W satisfying assumptions 2.9 and 2.10, $\mathbb{W}$ the enhanced Gaussian process over W, and the Cameron-Martin's space $H^1 \subset \Omega$ of the process W. Then,*

$$\forall \omega \in \Omega,\ \forall h \in H^1,\ \mathbb{W}(\omega + h) = T_h\mathbb{W}(\omega).$$

See [10], Lemma 15.58 for a proof.

Proposition 2.13. *For every geometric p-rough path Y and every $\varepsilon > 0$,*

$$M_{\varepsilon,I,p}(Y) \leqslant \varepsilon\left[2N_{\varepsilon,I,p}(Y)+1\right].$$

See [3], Proposition 4.6 for a proof.

Theorem 2.14. *Consider a stochastic process W satisfying assumptions 2.9 and 2.10, and $\mathbb{W}$ the enhanced Gaussian process over W. Then,*

$$\forall C, \varepsilon, r > 0,\ Ce^{CN_{\varepsilon,I,p}(\mathbb{W})} \in L^r(\Omega).$$

See [3], Theorem 6.4 and Remark 6.5 for a proof.

Corollary 2.15. *Consider $x_0 \in \mathbb{R}^e$, $V := (V_1, \dots, V_{d+1})$ and $\tilde{V} := (\tilde{V}_1, \dots, \tilde{V}_{d+1})$ two collections of γ-Lipschitz vector fields on $\mathbb{R}^e$, a stochastic process W satisfying assumptions 2.9 and 2.10, $\mathbb{W}$ the enhanced Gaussian process over W, $\mathbb{W}^g := S_{[p]}(\mathbb{W} \oplus g)$ with $g := Id_{[0,T]}$, and a continuous function $h : [0,T] \to \mathbb{R}^{d+1}$ of finite q-variation.*

$$\|J^{\mathbb{W}^g}_{\cdot\leftarrow 0}\|_{\infty;T},\ \|\partial_V \pi_V(0, x_0; \mathbb{W}^g).\tilde{V}\|_{\infty;T} \text{ and } \|D_h^{FV}\pi_V(0, x_0; \mathbb{W}^g)\|_{\infty;T}$$

belong to $L^r(\Omega)$ for every $r > 0$.

Proof. It is a straightforward consequence of corollaries 2.5 and 2.8, of Proposition 2.13 (deterministic results), and of Theorem 2.14 (probabilistic result). □

3. Sensitivity analysis of Gaussian rough differential equations

This section solves the problem stated in the introduction of the paper by using the deterministic results on RDEs of subsections 2.1 and 2.2, the probabilistic results on Gaussian RDEs of Subsection 2.3 and the Malliavin calculus.

Assume that W, μ and σ defined in the introduction satisfy the following assumption.

Assumption 3.1. *The process W satisfies assumptions 2.9 and 2.10, and*

$$C_0^1\left([0,T];\mathbb{R}^d\right) \subset H^1.$$

Moreover, there exists a constant $C > 0$ such that :

$$\forall h \in C_0^1\left([0,T];\mathbb{R}^d\right),\ \|h\|_{H^1} \leqslant C\|\dot{h}\|_{\infty;T}.$$

The functions μ and σ satisfy Assumption 1.3 and, for every $a \in \mathbb{R}^d$, $\sigma(a)$ is an invertible matrix. Moreover, the function $\sigma^{-1} : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ is bounded.

Example. The fractional Brownian motion B^H of Hurst parameter $H \in]1/4, 1[$ satisfies Assumption 3.1. Indeed, it has been stated at Subsection 2.3 that B^H satisfies assumptions 2.9 and 2.10. Moreover, by the first point of L. Decreusefond and S. Ustunel [5], Theorem 3.3 :

$$C_0^1\left([0,T];\mathbb{R}^d\right) \subset H^1.$$

Consider $h \in C_0^1([0,T];\mathbb{R}^d)$. By the second point of [5], Theorem 3.3 :

$$\begin{aligned}\|h\|_{H^1} &= \|J_H(\dot{h})\|_{H^1} = \|\dot{h}\|_{L^2([0,T])} \\ &\leqslant T^{1/2}\|\dot{h}\|_{\infty;T}.\end{aligned}$$

Assume also that the function $F : \mathbb{R}^d \to \mathbb{R}$ satisfies the following assumption.

Assumption 3.2. *The function F is continuously differentiable from $\mathbb{R}^d$ into $\mathbb{R}$. Moreover, there exists two constants $C > 0$ and $N \in \mathbb{N}^*$ such that, for every $a \in \mathbb{R}^d$,*

$$|F(a)| \leqslant C(1 + \|a\|)^N \text{ and } \|DF(a)\|_{\mathcal{L}} \leqslant C(1 + \|a\|)^N.$$

The following results are solving, at least partially, the problem stated in the introduction of the paper.

Notations :

- Under Assumption 3.1, the enhanced Gaussian process over W is denoted by $\mathbb{W}$, $\mathbb{W}^g := S_{[p]}(\mathbb{W} \oplus g)$ with $g := \mathrm{Id}_{[0,T]}$, and $V := (V_1, \dots, V_{d+1})$ is the collection of vector fields defined by :

$$V(a)(b,c) := \mu(a)c + \sigma(a)b$$

  for every $a, b \in \mathbb{R}^d$ and $c \in \mathbb{R}$.
- Let $\mathbb{S}_p \subset \mathrm{Lip}^\gamma(\mathbb{R}^d; \mathbb{R}^d)$ be the space of functions from $\mathbb{R}^d$ into $\mathcal{M}_d(\mathbb{R})$, $[p]+1$ times differentiable, bounded and of bounded derivatives.
- For every $x \in \mathbb{R}^d$, $\mathbb{E}[F(X_T)]$ is denoted by $f_T(x, \sigma)$.

Lemma 3.3. *Let $I = (I^1, \dots, I^d)$ be the map from H into H^1 such that :*

$$I^i(h) := \mathbb{E}\left[\mathbf{W}^i(h^i) W^i\right] \in H^1$$

for every $h \in H := H_1 \oplus \cdots \oplus H_d$ and $i = 1, \dots, d$. I is an isometry from H into H^1.

Proof. On one hand, the linearity of $I : H \to H^1$ is a straightforward consequence of the linearity of $\mathbf{W} : H \to L^2(\Omega)$.

On the other hand, by construction of $\mathbf{W}$ and of the scalar products on H and H^1 :

$$\begin{aligned} \langle I(h), I(g) \rangle_{H^1} &= \sum_{i=1}^d \langle \mathbb{E}\left[\mathbf{W}^i(h^i) W^i\right]; \mathbb{E}\left[\mathbf{W}^i(g^i) W^i\right] \rangle_{H_i^1} \\ &= \sum_{i=1}^d \mathbb{E}\left[\mathbf{W}^i(h^i) \mathbf{W}^i(g^i)\right] = \langle h, g \rangle_H \end{aligned}$$

for every functions $h, g \in H$. □

The following lemma extends [10], Proposition 20.5 to Gaussian RDEs having a drift term.

Lemma 3.4. *For every $x_0 \in \mathbb{R}^d$ and almost every $\omega \in \Omega$, the map $h \mapsto \pi_V[0, x_0; \mathbb{W}^g(\omega + h)]$ is continuously differentiable from*

$$H^1 \text{ into } C^{p\text{-}var}\left([0,T]; \mathbb{R}^d\right).$$

For every $t \in [0,T]$, $\pi_V(0, x_0; \mathbb{W}^g)_t \in \mathbb{D}^{1,2}_{loc}$ and for every $h \in H^1$,

$$\begin{aligned} \langle \mathbf{D}\pi_V(0, x_0; \mathbb{W}^g)_t, I^{-1}(h) \rangle_H &= D^{FV}_{(h,0)} \pi_V(0, x_0; \mathbb{W}^g)_t \\ &= \int_0^t J^{\mathbb{W}^g}_{t \leftarrow s} \sigma\left[\pi_V(0, x_0; \mathbb{W}^g)_s\right] dh_s. \end{aligned}$$

Proof. By Proposition 2.12, for almost every $\omega \in \Omega$ and every $h \in H^1$,

$$\begin{aligned} \mathbb{W}^g(\omega + h) &= S_{[p]}[\mathbb{W}(\omega + h) \oplus g] \\ &= S_{[p]}[T_h \mathbb{W}(\omega) \oplus g] \\ &= T_{(h,0)} S_{[p]}[\mathbb{W}(\omega) \oplus g] \\ &= T_{(h,0)} \mathbb{W}^g(\omega). \end{aligned}$$

Then, almost surely :

$$\pi_V[0,x_0;\mathbb{W}^g(.+h)] = \pi_V[0,x_0;T_{(h,0)}\mathbb{W}^g]. \tag{8}$$

By Assumption 2.10 and Corollary 2.8, $h \mapsto \pi_V[0,x_0;T_{(h,0)}\mathbb{W}^g]$ is continuously differentiable from

$$H^1 \subset C^{q\text{-var}}\left([0,T];\mathbb{R}^d\right) \text{ into } C^{p\text{-var}}\left([0,T];\mathbb{R}^d\right).$$

Then, by Equality (8), the map $h \mapsto \pi_V[0,x_0;\mathbb{W}^g(.+h)]$ is also continuously differentiable from

$$H^1 \text{ into } C^{p\text{-var}}\left([0,T];\mathbb{R}^d\right),$$

and for almost every $\omega \in \Omega$ and every $h \in H^1$,

$$D^{\mathrm{FV}}_{(h,0)}\pi_V\left[0,x_0;\mathbb{W}^g(\omega)\right] = D_hF^\omega(0)$$

with $F^\omega := \pi_V[0,x_0;\mathbb{W}^g(\omega+.)]$.

By the Duhamel principle (see Theorem 2.7), for every $t \in [0,T]$ and every $h \in H^1$,

$$\begin{aligned} D^{\mathrm{FV}}_{(h,0)}\pi_V(0,x_0;\mathbb{W}^g)_t &= \int_0^t J^{\mathbb{W}^g}_{t\leftarrow s}V\left[\pi_V(0,x_0;\mathbb{W}^g)_s\right](dh_s,0) \\ &= \int_0^t J^{\mathbb{W}^g}_{t\leftarrow s}\sigma\left[\pi_V(0,x_0;\mathbb{W}^g)_s\right]dh_s. \end{aligned}$$

In conclusion, by [22], Proposition 4.1.3 and Lemma 4.1.2, $\pi_V(0,x_0;\mathbb{W}^g)_t$ is continuously H^1-differentiable and then locally derivable in the sense of Malliavin, with

$$\langle \mathbf{D}\pi_V(0,x_0;\mathbb{W}^g)_t, I^{-1}(h)\rangle_H = D_hF^\omega(0).$$

□

Theorem 3.5. *Under assumptions 3.1 and 3.2 :*

(1) *The function $f_T(.,\sigma)$ is differentiable from $\mathbb{R}^d$ into $\mathbb{R}$ and, for every $x,v \in \mathbb{R}^d$,*

$$\partial_x f_T(x,\sigma).v = \mathbb{E}\left[\langle \mathbf{D}(F\circ X^x_T), I^{-1}(h^{x,v})\rangle_H\right]$$

where

$$h^{x,v} := \frac{1}{T}\int_0^\cdot \sigma^{-1}\left(X^x_s\right) J^{\mathbb{W}^g}_{s\leftarrow 0}v ds.$$

(2) *For every $x \in \mathbb{R}^d$, the function $f_T(x,.)$ is differentiable from $\mathbb{S}_p$ into $\mathbb{R}$ and, for every $\sigma,\tilde{\sigma} \in \mathbb{S}_p$ satisfying Assumption 3.1,*

$$\partial_\sigma f_T(x,\sigma).\tilde{\sigma} = \mathbb{E}\left[\langle \mathbf{D}(F\circ X^\sigma_T), I^{-1}(h^{\sigma,\tilde{\sigma}})\rangle_H\right]$$

where

$$h^{\sigma,\tilde{\sigma}} := \frac{1}{T}\int_0^\cdot \sigma^{-1}\left(X^\sigma_s\right) J^{\mathbb{W}^g}_{s\leftarrow T}\partial_\sigma X^\sigma_T.\tilde{\sigma} ds.$$

Proof. (1) On one hand, for every $\varepsilon \in]0,1]$, $\eta > 0$ and $x,v \in \mathbb{R}^d$, by the Taylor formula, and the first point of Corollary 2.5 ; there exists a constant $C_1 > 0$, depending only on p, γ, η and $\|V\|_{\mathrm{lip}^\gamma}$, such that :

$$\begin{aligned} \frac{\left|F(X^{x+\varepsilon v}_T) - F(X^x_T)\right|}{\varepsilon} &= \left|\int_0^1 \langle DF(X^{x+\theta\varepsilon v}_T), DX^{x+\theta\varepsilon v}_T.v\rangle d\theta\right| \\ &\leqslant C_1\|v\|e^{C_1M_{\eta,I,p}(\mathbb{W})}\int_0^1 \|DF(X^{x+\theta\varepsilon v}_T)\|_{\mathcal{L}}d\theta. \end{aligned}$$

Moreover, since F satisfies Assumption 3.2, there exists two constants $C_2 > 0$ and $N \in \mathbb{N}^*$, depending only on F, such that for every $\theta \in [0,1]$,

$$\left\|DF\left(X_T^{x+\theta\varepsilon v}\right)\right\|_{\mathcal{L}} \leqslant C_2\left(1+\left\|X_T^{x+\theta\varepsilon v}\right\|\right)^N.$$

Then, by the triangle inequality together with [10], Theorem 10.36, there exists a constant $C_3 > 0$, not depending on x, v, θ, ε, V and $\mathbb{W}$, such that :

$$\begin{aligned}\|DF(X_T^{x+\theta\varepsilon v})\|_{\mathcal{L}} \quad \leqslant \quad & C_2[1+\|x\|+\|v\|+ \\ & C_3[\|V\|_{\mathrm{lip}^{\gamma-1}}\|\mathbb{W}^g\|_{p\text{-var};T} \vee \|V\|^p_{\mathrm{lip}^{\gamma-1}}\|\mathbb{W}^g\|^p_{p\text{-var};T}]]^N.\end{aligned}$$

Since W satisfies assumptions 2.9 and 2.10, by Corollary 2.15, the generalized Fernique theorem (see Theorem 2.11) and the Cauchy-Schwarz inequality :

$$\varepsilon \in]0,1] \longmapsto \frac{\left|F(X_T^{x+\varepsilon v}) - F(X_T^x)\right|}{\varepsilon}$$

is bounded by an integrable random variable not depending on ε. Therefore, by the Lebesgue theorem, $f_T(.,\sigma)$ is differentiable on $\mathbb{R}^d$ and

$$\forall x, v \in \mathbb{R}^d,\ \partial_x f_T(x,\sigma).v = \mathbb{E}\left[\langle DF(X_T^x), \partial_x X_T^x.v\rangle\right]. \tag{9}$$

On the other hand, consider $x, v \in \mathbb{R}^d$. By construction, the paths of the process $h^{x,v}$ are continuously differentiable from $[0,T]$ into $\mathbb{R}^d$ and $h_0^{x,v} = 0$. Then, since W satisfies Assumption 3.1, $h^{x,v}$ is a H^1-valued random variable. By the Duhamel principle (see Theorem 2.7) :

$$\begin{aligned}D^{\mathrm{FV}}_{(h^{x,v},0)}X_T^x \quad &= \quad \int_0^T J^{\mathbb{W}^g}_{T\leftarrow s}\sigma\left(X_s^x\right)dh_s^{x,v} \\ &= \quad \partial_x X_T^x.v.\end{aligned}$$

Therefore, by Equality (9), Lemma 3.4 and [22], Proposition 1.2.3 (the chain rule of the Malliavin derivative) :

$$\begin{aligned}\partial_x f_T(x,\sigma).v \quad &= \quad \mathbb{E}[DF(X_T^x).D^{\mathrm{FV}}_{(h^{x,v},0)}X_T^x] \\ &= \quad \mathbb{E}[DF(X_T^x).\langle \mathbf{D}X_T^x, I^{-1}(h^{x,v})\rangle_H] \\ &= \quad \mathbb{E}[\langle \mathbf{D}(F\circ X_T^x), I^{-1}(h^{x,v})\rangle_H].\end{aligned}$$

(2) Let $x \in \mathbb{R}^d$ be arbitrarily fixed. On one hand, for every $\varepsilon \in]0,1]$ and $\sigma, \tilde{\sigma} \in \mathbb{S}_p$, by the Taylor formula :

$$\begin{aligned}\frac{\left|F(X_T^{\sigma+\varepsilon\tilde{\sigma}}) - F(X_T^{\sigma})\right|}{\varepsilon} \quad &= \quad \left|\int_0^1 \langle DF(X_T^{\sigma+\theta\varepsilon\tilde{\sigma}}), DX_T^{\sigma+\theta\varepsilon\tilde{\sigma}}.\tilde{\sigma}\rangle d\theta\right| \\ &\leqslant \quad C_2\int_0^1\left(1+\left\|X_T^{\sigma+\theta\varepsilon\tilde{\sigma}}\right\|\right)^N\left\|DX^{\sigma+\theta\varepsilon\tilde{\sigma}}.\tilde{\sigma}\right\|_{\infty;T}d\theta.\end{aligned}$$

At [10], Theorem 10.36, the constant involving in the upper-bound does not depend on the signal and on the collection of vector fields. At the second point of Corollary 2.5, the two constants involving in the upper-bound depend continuously on the γ-Lipschitz norm of the collection of vector fields. Then, there exists a constant $C_4 > 0$, depending on σ and $\tilde{\sigma}$ but not on ε and $\mathbb{W}$, such that for every $\theta \in [0,1]$,

$$\left\|DX^{\sigma+\theta\varepsilon\tilde{\sigma}}.\tilde{\sigma}\right\|_{\infty;T} \leqslant C_4 e^{C_4 M_{C_4,I,p}(\mathbb{W})}$$

and

$$\|X_T^{\sigma+\theta\varepsilon\tilde{\sigma}}\| \leqslant C_4 \left(\|\mathbb{W}^g\|_{p\text{-var};T} \vee \|\mathbb{W}^g\|_{p\text{-var};T}^p \right).$$

Since W satisfies assumptions 2.9 and 2.10, by Proposition 2.13, Theorem 2.14, the generalized Fernique theorem (see Theorem 2.11) and the Cauchy-Schwarz inequality :

$$\varepsilon \in]0,1] \longmapsto \frac{\left|F(X_T^{\sigma+\varepsilon\tilde{\sigma}}) - F(X_T^{\sigma})\right|}{\varepsilon}$$

is bounded by an integrable random variable not depending on ε. Therefore, by the Lebesgue theorem, $f_T(x,.)$ is differentiable on $\mathbb{S}_p$ and

$$\forall \sigma, \tilde{\sigma} \in \mathbb{S}_p,\ \partial_\sigma f_T(x,\sigma).\tilde{\sigma} = \mathbb{E}\left[\langle DF(X_T^\sigma), \partial_\sigma X_T^\sigma.\tilde{\sigma}\rangle\right]. \tag{10}$$

On the other hand, consider $\sigma, \tilde{\sigma} \in \mathbb{S}_p$ satisfying Assumption 3.1. By construction, the paths of the process $h^{\sigma,\tilde{\sigma}}$ are continuously differentiable from $[0,T]$ into $\mathbb{R}^d$ and $h_0^{\sigma,\tilde{\sigma}} = 0$. Then, $h^{\sigma,\tilde{\sigma}}$ is a H^1-valued random variable. By the Duhamel principle (see Theorem 2.7) :

$$\begin{aligned} D^{\mathrm{FV}}_{(h^{\sigma,\tilde{\sigma}},0)} X_T^\sigma &= \int_0^T J^{\mathbb{W}^g}_{T\leftarrow s}\sigma\left(X_s^\sigma\right) dh_s^{\sigma,\tilde{\sigma}} \\ &= \partial_\sigma X_T^\sigma.\tilde{\sigma}. \end{aligned}$$

Therefore, by Equality (10), Lemma 3.4 and [22], Proposition 1.2.3 (the chain rule of the Malliavin derivative) :

$$\begin{aligned} \partial_\sigma f_T(x,\sigma).\tilde{\sigma} &= \mathbb{E}[DF(X_T^\sigma).D^{\mathrm{FV}}_{(h^{\sigma,\tilde{\sigma}},0)} X_T^\sigma] \\ &= \mathbb{E}[DF(X_T^\sigma).\langle \mathbf{D}X_T^\sigma, I^{-1}(h^{\sigma,\tilde{\sigma}})\rangle_H] \\ &= \mathbb{E}[\langle \mathbf{D}(F\circ X_T^\sigma), I^{-1}(h^{\sigma,\tilde{\sigma}})\rangle_H]. \end{aligned}$$

□

Remark. In order to extend the formulas of Theorem 1.2 under the assumptions of Theorem 3.5, it is sufficient to show that $I^{-1}(h^{x,v})$ and $I^{-1}(h^{\sigma,\tilde{\sigma}})$ belong to $\mathbb{D}^{1,2}(H) \subset \mathrm{dom}(\delta)$. To do that, it is necessary to prove that

$$\|D_h^{\mathrm{FV}}\partial_x X^x.v\|_{\infty,T} \text{ and } \|D_h^{\mathrm{FV}}\partial_\sigma X^\sigma.\tilde{\sigma}\|_{\infty,T}$$

belong to $L^r(\Omega)$ for every $h \in H^1$ and $r > 0$, by using that $D_h^{\mathrm{FV}}\partial_x X^x.v$ and $D_h^{\mathrm{FV}}\partial_\sigma X^\sigma.\tilde{\sigma}$ are successive compositions of the Itô map. If $p \geqslant 2$, it is difficult for two technical reasons :

- At Definition 2.6, Y and h cannot be both of finite p-variation. Indeed, the direction h has to be smoother than the geometric rough path Y.
- The upper-bounds obtained at Section 2 for $\partial_x X^x.v$, $\partial_\sigma X^\sigma.\tilde{\sigma}$ and $D_h^{\mathrm{FV}}X$ with the uniform norm, tailor-made to apply the integrability results coming from [3], don't hold true with the p-variation norm (see [9], Lemma 10.63 and Remark 10.64).

If $p \in [1,2[$, μ and σ are three times differentiable, bounded and of bounded derivatives, $\sigma(a)$ is an invertible matrix for every $a \in \mathbb{R}^d$, and $\sigma^{-1} : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ is bounded, the statement of Theorem 3.5 holds true in the sense of Young, $I^{-1}(h^{x,v})$ and $I^{-1}(h^{\sigma,\tilde{\sigma}})$ belong to $\mathbb{D}^{1,2}(H)$, and

$$\begin{aligned} \partial_x f_T(x,\sigma).v &= \mathbb{E}\left[F(X_T^x)\delta\left[I^{-1}(h^{x,v})\right]\right] \text{ and} \\ \partial_\sigma f_T(x,\sigma).\tilde{\sigma} &= \mathbb{E}\left[F(X_T^\sigma)\delta\left[I^{-1}(h^{\sigma,\tilde{\sigma}})\right]\right]. \end{aligned}$$

See [20], Proposition 1.54 and Corollaire 2.23. Since the derivative of F don't involve anymore in these expressions of $\partial_x f_T(x,\sigma).v$ and $\partial_\sigma f_T(x,\sigma).\tilde{\sigma}$, by a usual regularization procedure, F has only to be measurable such that

$$\forall a \in \mathbb{R}^d,\ |F(a)| \leqslant C(1+\|a\|)^N$$

with $C>0$ and $N \in \mathbb{N}^*$.

4. Application to mathematical finance and simulations

In a first subsection, Theorem 3.5 is applied to the calculation of sensitivities in a financial market model with stochastic volatility, such that each equation is driven by a fractional Brownian motion of Hurst parameter belonging to $]1/4,1[$. In a second subsection, still with a fractional Brownian signal, some simulations of the sensitivities with respect to the initial condition and to the collection of vector fields are provided when the Hurst parameter of the fBm belongs to $]1/2,1[$.

4.1. **Calculation of sensitivities in a fractional stochastic volatility model**

In this subsection, the prices process of the risky assets is the solution of a fractional stochastic volatility model (taken in the sense of rough paths), and the sensitivity of the price of an option to some perturbations of the volatility is calculated by using Theorem 3.5.

Consider a stochastic process W, and $\mu : \mathbb{R}^d \to \mathbb{R}^d$, $\kappa : \mathbb{R}^d \to \mathbb{R}^d_+$, $\sigma, \vartheta : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ and $F : \mathbb{R}^d \to \mathbb{R}_+$ five functions satisfying the following assumption.

Assumption 4.1. *There exists two independent d-dimensional fBm B^{H_1} and B^{H_2}, of Hurst parameters $H_1 \in]1/4,1[$ and $H_2 \in]1/4,1[$ respectively, such that $W := (B^{H_1}, B^{H_2})$.*

The functions μ, σ and ϑ satisfy Assumption 3.1 for $p := 1/(H_1-\varepsilon) \vee 1/(H_2-\varepsilon) < 4$ and $\varepsilon > 0$ as close as possible to 0. The functions κ and F are such that $F \circ \kappa$ satisfies Assumption 3.2.

Consider the financial market model consisting of d risky assets, of prices S_t at the time $t \in [0,T]$ such that

$$\left\{ \begin{array}{rcl} S_t & := & \kappa\left(Y_t\right) \\ dY_t & = & \mu\left(Y_t\right)dt + \sigma\left(Z_t\right)dB_t^{H_1}\ ;\ Y_0 \in \mathbb{R}^d \\ dZ_t & = & \vartheta\left(Z_t\right)dB_t^{H_2}\ ;\ Z_0 \in \mathbb{R}^d \end{array} \right. ,$$

and an option of payoff $F(S_T) := (F \circ \kappa)(Y_T)$ over these assets.

Consider $X := (Y,Z)$, $\mathbb{W}$ the enhanced Gaussian process over W, and $V := (V_1, \ldots, V_{2d+1})$ the collection of γ-Lipschitz vector fields on $\mathbb{R}^d_1 \oplus \mathbb{R}^d_2$ ($\gamma > p$) defined by :

$$\forall a,b \in \mathbb{R}^d_1 \oplus \mathbb{R}^d_2,\ \forall c \in \mathbb{R},\ V(a)(b,c) := V_1(a)c + V_2(a)b$$

where,

$$V_1 := \begin{pmatrix} \mu \circ \pi_{\mathbb{R}^d_1} \\ 0 \end{pmatrix} \text{ and } V_2 := \begin{pmatrix} \sigma \circ \pi_{\mathbb{R}^d_2} & 0 \\ 0 & \vartheta \circ \pi_{\mathbb{R}^d_2} \end{pmatrix},$$

and $\pi_{\mathbb{R}^d_i}$ is the canonical projection from $\mathbb{R}^d_1 \oplus \mathbb{R}^d_2$ into $\mathbb{R}^d_i$ for $i=1,2$.

Precisely, $X = \pi_V(0, X_0; \mathbb{W}^g)$ with $\mathbb{W}^g := S_{[p]}(\mathbb{W} \oplus g)$ and $g := \mathrm{Id}_{[0,T]}$.

Corollary 4.2. *With the notations of Theorem 3.5, under Assumption 4.1, f_T is differentiable from $\mathbb{S}_p$ into $\mathbb{R}_+$, and for every $\vartheta, \tilde{\vartheta} \in \mathbb{S}_p$ satisfying Assumption 3.1 :*

$$\partial_\vartheta f_T(\vartheta).\tilde{\vartheta} = \mathbb{E}\left[\langle \mathbf{D}(F \circ S_T^\vartheta); ((I_{H_1}^{-1} \circ \pi_{H_1^1})(h^{\vartheta,\tilde{\vartheta}}), (I_{H_2}^{-1} \circ \pi_{H_2^1})(h^{\vartheta,\tilde{\vartheta}}))\rangle_H\right]$$

where, for $i = 1, 2$, H_i^1 is the Cameron-Martin space of B^{H_i}, $\pi_{H_i^1}$ is the canonical projection from $H_1^1 \oplus H_2^1$ into H_i^1, and

$$h^{\vartheta,\tilde{\vartheta}} := \frac{1}{T}\int_0^\cdot V_2^{-1}\left(X_s^\vartheta\right) J_{s\leftarrow T}^{\mathbb{W}^g} \partial_\vartheta X_T^\vartheta.\tilde{\vartheta} ds.$$

Remark. By the final remark of Section 3, if $H_1, H_2 \in]1/2, 1[$, under some additional assumptions on μ, σ and ϑ, the statement of Corollary 4.2 holds true in the sense of Young, $I^{-1}(h^{\vartheta,\tilde{\vartheta}})$ belongs to $\mathbb{D}^{1,2}(H)$, and

$$\partial_\vartheta f_T(\vartheta).\tilde{\vartheta} = \mathbb{E}\left[F(S_T^\vartheta)\left[\delta_{H_1}\left[(I_{H_1}^{-1} \circ \pi_{H_1^1})(h^{\vartheta,\tilde{\vartheta}})\right] + \delta_{H_2}\left[(I_{H_2}^{-1} \circ \pi_{H_2^1})(h^{\vartheta,\tilde{\vartheta}})\right]\right]\right]$$

with $F \circ \kappa$ only measurable such that

$$\forall a \in \mathbb{R}^d,\ |(F \circ \kappa)(a)| \leqslant C(1 + \|a\|)^N$$

with $C > 0$ and $N \in \mathbb{N}^*$. Moreover, by Corollary A.4 and its remark :

$$\delta_{H_i}\left[(I_{H_i}^{-1} \circ \pi_{H_i^1})(h^{\vartheta,\tilde{\vartheta}})\right] = \delta_{1/2}\left[\left[(\varphi_{H_i} D^{H_i - 1/2}) \circ (\varphi_{H_i}^{-1} D^1) \circ \pi_{H_i^1}\right](h^{\vartheta,\tilde{\vartheta}})\right]$$

for $i = 1, 2$.

4.2. **Simulations**

In order to simulate the sensitivities studied in this paper, the results of [14] on the convergence of the explicit Euler scheme of differential equations driven by a α-Hölder continuous function from $[0, T]$ into $\mathbb{R}^d$ ($\alpha \in]1/2, 1[$) have to be reminded first.

Proposition 4.3. *Consider $x_0 \in \mathbb{R}^d$, $w : [0, T] \to \mathbb{R}^d$ a α-Hölder continuous function with $\alpha \in]1/2, 1[$, and $V := (V_1, \ldots, V_d)$ a differentiable collection of vector fields on $\mathbb{R}^d$ such that its derivative is γ-Hölder continuous from $\mathbb{R}^d$ into itself ($\gamma \in]0, 1[$ and $\gamma + 1 > 1/\alpha$). There exists a constant $C > 0$ such that for every $n \in \mathbb{N}^*$,*

$$\|x^n - \pi_V(0, x_0; w)\|_{\infty;T} \leqslant C n^{1-2/p}$$

where, x^n is the step-n explicit Euler scheme of $\pi_V(0, x_0; w)$ with respect to the dissection $D^n := \{r_k^n\} \in D_T$:

$$x_t^n := \sum_{k=0}^{n-1}\left[x_k^n + \frac{x_{k+1}^n - x_k^n}{r_{k+1}^n - r_k^n}(t - r_k^n)\right]\mathbf{1}_{[r_k^n, r_{k+1}^n[}(t)\ ;\ t \in [0, T]$$

with

$$\left\{\begin{array}{rcl} x_0^n & := & x_0 \\ x_{k+1}^n & = & x_k^n + V(x_k^n)(w_{r_{k+1}^n} - w_{r_k^n}) \end{array}\right.$$

for $k = 0, \ldots, n-1$.

See [14], Proposition 5 for a proof.

Corollary 4.4. *Consider $x_0 \in \mathbb{R}$, a 1-dimensional fractional Brownian motion B^H of Hurst parameter $H \in]1/2, 1[$, μ and σ two functions from $\mathbb{R}$ into $\mathbb{R}$ satisfying Assumption 1.3 with $p := 1/(H-\varepsilon) < 2$ and $\varepsilon > 0$ as close as possible to 0, V the vector field on $\mathbb{R}$ such that $V(a)(b,c) := \mu(a)c + \sigma(a)b$ for every $a, b, c \in \mathbb{R}$, $X := \pi_V(0, x_0; B^H)$, $Y := \partial_x X^x.1$ and $Z := \partial_\sigma X^\sigma.\tilde{\sigma}$ for $\tilde{\sigma} \in \mathbb{S}_p$ arbitrarily chosen. For every $r \geqslant 1$,*

$$\lim_{n\to\infty} \mathbb{E}\left(\|X^n - X\|^r_{\infty;T}\right) = \lim_{n\to\infty} \mathbb{E}\left(\|Y^n - Y\|^r_{\infty;T}\right) = \lim_{n\to\infty} \mathbb{E}\left(\|Z^n - Z\|^r_{\infty;T}\right) = 0$$

where, for every $n \in \mathbb{N}^*$, X^n, Y^n and Z^n are respectively the explicit Euler schemes of X, Y and Z with respect to the dissection $D^n := \{r^n_k\} \in D_T$. Moreover, the rate of convergence of each sequence is $n^{r(1-2/p)}$.

Proof. The processes Y and Z satisfy respectively :

$$Y = \pi_{A_1}(0, 1; W^{\mu,\sigma}) \text{ and } Z = \pi_{A_2}\left[0, 0; (W^{\mu,\sigma}, W^{\tilde{\sigma}})\right]$$

where,

$$W^{\mu,\sigma} := \int_0^\cdot \dot{\mu}(X_s)\, ds + \int_0^\cdot \dot{\sigma}(X_s)\, dB^H_s \text{ and } W^{\tilde{\sigma}} := \int_0^\cdot \tilde{\sigma}(X_t)\, dB^H_t$$

and, A_1 and A_2 are the two collections of affine vector fields on $\mathbb{R}$ defined by :

$$\forall a, b, c \in \mathbb{R},\ A_1(a)b := ab \text{ and } A_2(a)(b,c) := ab + c.$$

Since the paths of B^H are almost surely $1/p$-Hölder continuous by the Kolmogorov continuity criterion, by [10], Theorem 6.8, the paths of $W^{\mu,\sigma}$ and $W^{\tilde{\sigma}}$ are also almost surely $1/p$-Hölder continuous. Then, X, Y and Z satisfy the conditions of Proposition 4.3, and there exists a random variable $C > 0$ such that for every $n \in \mathbb{N}^*$, $Cn^{1-2/p}$ is an upper-bound of

$$\|X^n - X\|_{\infty;T},\ \|Y^n - Y\|_{\infty;T} \text{ and } \|Z^n - Z\|_{\infty;T}.$$

By reading carefully the proof of [14], Proposition 5, C belongs to $L^r(\Omega)$ for every $r \geqslant 1$ by the Fernique theorem. Therefore, for every $r \geqslant 1$,

$$\begin{aligned}
\mathbb{E}(\|X^n - X\|^r_{\infty;T}) &\leqslant \mathbb{E}(C^r) n^{r(1-2/p)} \xrightarrow[n\to\infty]{} 0, \\
\mathbb{E}(\|Y^n - Y\|^r_{\infty;T}) &\leqslant \mathbb{E}(C^r) n^{r(1-2/p)} \xrightarrow[n\to\infty]{} 0 \text{ and} \\
\mathbb{E}(\|Z^n - Z\|^r_{\infty;T}) &\leqslant \mathbb{E}(C^r) n^{r(1-2/p)} \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

because $p < 2$. □

Remark. About the approximation of the solution of SDEs driven by a fBm, see also A. Neuenkirch and I. Nourdin [21].

Let $n \in \mathbb{N}^*$ be arbitrarily fixed. With the assumptions and notations of Corollary 4.4, at each iteration of the step-n explicit Euler schemes, the value of $B^H_{r^n_{\cdot+1}} - B^H_{r^n_\cdot}$ is computed via the Wood-Chang algorithm (see T. Dieker [6] about some simulation methods of the fBm).

Let $F : \mathbb{R} \to \mathbb{R}$ be a function satisfying Assumption 3.2. With the notations of Section 3, in order to approximate $\partial_x f_T(x, \sigma).1$ (resp. $\partial_\sigma f_T(x, \sigma).\tilde{\sigma}$),

$$\mathbb{E}\left[\dot{F}(X^n_T)\, Y^n_T\right] \text{ (resp. } \mathbb{E}\left[\dot{F}(X^n_T)\, Z^n_T\right] \text{)}$$

is estimated by the empirical mean $\Theta_m^n(Y)$ (resp. $\Theta_m^n(Z)$) of the m-sample from the distribution of $F^Y := \dot{F}(X_T^n)Y_T^n$ (resp. $F^Z := \dot{F}(X_T^n)Z_T^n$). By Corollary 4.4, F^Y and F^Z belong to $L^2(\Omega)$. Then,

(1) By the strong law of large numbers :

$$\begin{aligned}\Theta_m^n(Y) &\xrightarrow[m\to\infty]{\text{a.s.}} \theta^n(Y) := \mathbb{E}\left[\dot{F}\left(X_T^n\right)Y_T^n\right] \text{ and}\\ \Theta_m^n(Z) &\xrightarrow[m\to\infty]{\text{a.s.}} \theta^n(Z) := \mathbb{E}\left[\dot{F}\left(X_T^n\right)Z_T^n\right].\end{aligned}$$

(2) By the central limit theorem and the Slutsky lemma :

$$\begin{aligned}\sqrt{n}\frac{\Theta_m^n(Y)-\theta^n(Y)}{s_m^n(Y)} &\xrightarrow[m\to\infty]{\mathcal{D}} \mathcal{N}(0,1) \text{ and}\\ \sqrt{n}\frac{\Theta_m^n(Z)-\theta^n(Z)}{s_m^n(Z)} &\xrightarrow[m\to\infty]{\mathcal{D}} \mathcal{N}(0,1)\end{aligned}$$

where, $s_m^n(Y)$ (resp. $s_m^n(Z)$) is the empirical standard deviation of the m-sample from the distribution of F^Y (resp. F^Z).

At the level $\alpha \in]0,1[$, the second point provides the following confidence intervals :

$$\mathbb{P}\left[\Theta_m^n(Y) - \frac{t_\alpha}{\sqrt{m}}s_m^n(Y) \leqslant \theta^n(Y) \leqslant \Theta_m^n(Y) + \frac{t_\alpha}{\sqrt{m}}s_m^n(Y)\right] \simeq 1-\alpha$$

and

$$\mathbb{P}\left[\Theta_m^n(Z) - \frac{t_\alpha}{\sqrt{m}}s_m^n(Z) \leqslant \theta^n(Z) \leqslant \Theta_m^n(Z) + \frac{t_\alpha}{\sqrt{m}}s_m^n(Z)\right] \simeq 1-\alpha$$

where, $\Phi(t_\alpha) = 1-\alpha/2$ and Φ is the distribution function of $\mathcal{N}(0,1)$.

Example. Assume that $T := 1$, $H := 0.6$, $n := 2^{15}$, $m := 500$, $\mu \equiv 0$, $\sigma : a \mapsto 1 + e^{-a^2}$, $\tilde{\sigma} : a \mapsto 1 + \pi/2 + \arctan(a)$, $F : a \mapsto a^2$ and $x := 1$:

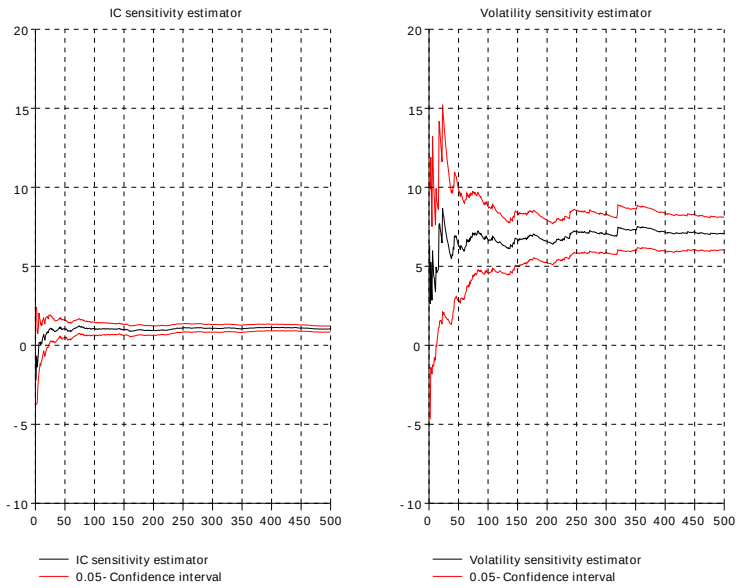

FIGURE 1. Estimators convergence

Statistics	Values
$\Theta_m^n(Y)$	1.042
0.05-confidence intervals	$[0.851; 1.232]$
Length of the confidence interval	0.381
$\Theta_m^n(Z)$	7.112
0.05-confidence intervals	$[6.071; 8.154]$
Length of the confidence interval	2.083

APPENDIX A. FRACTIONAL BROWNIAN MOTION

Essentially inspired by [22] and [5], this appendix provides the basics on the fractional Brownian motion, and the explicit expression of the associated isometry I^{-1} defined at Lemma 3.3.

Definition A.1. *A fractional Brownian motion of Hurst parameter $H \in]0,1]$ is a centered Gaussian process B^H of covariance function R_H defined by :*

$$R_H(s,t) := \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) \ ; \ s,t \in [0,T].$$

Let B^H be a fractional Brownian motion of Hurst parameter $H \in]0,1[$. Its reproducing kernel Hilbert space is denoted by $\mathcal{H}$, the Wiener integral with respect to B^H is denoted by $\mathbf{B}^H$, and the isometry provided at Lemma 3.3 is denoted by I_H.

Definition A.2. *Consider $\varphi : \mathbb{R}_+ \to \mathbb{R}$ and $\alpha \in]0,1]$:*

(1) *If*

$$l^\alpha(\varphi)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds$$

exists for every $t \in \mathbb{R}_+$, $l^\alpha(\varphi)$ is the α-fractional integral of φ.

(2) *If*

$$D^{\alpha}(\varphi)(t) := \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \times \dfrac{d}{dt}\displaystyle\int_0^t (t-s)^{-\alpha}\varphi(s)ds & \text{if} \quad \alpha \in]0,1[\\ \dot{\varphi}(t) & \text{if} \quad \alpha = 1 \end{cases}$$

exists for every $t \in \mathbb{R}_+$, $D^{\alpha}(\varphi)$ *is the* α*-fractional derivative of* φ.

(3) *If they are both defined :*

$$(l^{\alpha} \circ D^{\alpha})(\varphi) = (D^{\alpha} \circ l^{\alpha})(\varphi) = \varphi.$$

On the fractional operators, see S. Samko et al. [25].

Notation. $\mathcal{E}$ is the set of functions from $[0,T]$ into $\mathbb{R}$ of the form

$$\sum_{k=1}^{n} a_k \mathbf{1}_{[0,s_k]} \; ; \; n \in \mathbb{N}^*, \; (s_1,\ldots,s_n) \in [0,T]^n, \; (a_1,\ldots,a_n) \in \mathbb{R}^n.$$

Theorem A.3. *Let* K_H^* *be the operator defined on* $\mathcal{E}$ *by :*

$$\forall (s,t) \in \Delta_T, \; K_H^*(\mathbf{1}_{[0,t]})(s) := K_H(t,s)\mathbf{1}_{[0,t]}(s)$$

where,

$$K_H(t,s) := \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)} \mathbf{F}\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, 1-\frac{t}{s}\right)\mathbf{1}_{[0,t[}(s)$$

and $\mathbf{F}$ *is the Gauss hyper-geometric function.*

(1) *Let* $J_H : L^2([0,T]) \to H^1$ *be the map defined by :*

$$\forall h \in L^2([0,T]), \; J_H(h) := \int_0^{\cdot} h(s)K_H(.,s)ds.$$

For every $h \in \mathcal{H}$,

$$J_H(h) = \begin{cases} l^{2H} \circ (\varphi_H^{-1} l^{1/2-H}) \circ (\varphi_H h) & \text{if} \quad H \leqslant 1/2 \\ l^1 \circ (\varphi_H l^{H-1/2}) \circ (\varphi_H^{-1} h) & \text{if} \quad H \geqslant 1/2 \end{cases}$$

where, φ_H *is the map defined by* $\varphi_H(a) := a^{H-1/2}\mathbf{1}_{a \geqslant 0}$ *for every* $a \in \mathbb{R}$.

(2) *The operator* K_H^* *can be extended as an isometry from* $\mathcal{H}$ *into the closed subspace* $\mathcal{G} := K_H^*(\mathcal{H})$ *of* $L^2([0,T])$.

(3) *The process* $B := \mathbf{B}^H[(K_H^*)^{-1}(\mathbf{1}_{[0,.]})]$ *is a standard Brownian motion, and*

$$\forall t \in [0,T], \; B_t^H = \int_0^t K_H(t,s)dB_s.$$

(4) *The divergence operator* δ_H *associated to* $\mathbf{B}^H$ *satisfies* $\delta_H = \delta_{1/2} \circ K_H^*$.

See [5], Theorem 2.1 and Corollary 3.1, and to [22], Proposition 5.2.2 for a proof.

Remark. At [5], Theorem 3.3, L. Decreusefond and S. Ustunel proved that :

$$H^1 = \left\{ J_H(\dot{h}); \dot{h} \in L^2([0,T]) \right\}.$$

Corollary A.4. *The isometry I_H satisfies $I_H = J_H \circ K_H^*$. In particular,*

$$I_H^{-1} = \begin{cases} (K_H^*)^{-1} \circ (\varphi_H^{-1} D^{1/2-H}) \circ (\varphi_H D^{2H}) & \text{if} \quad H \leqslant 1/2 \\ (K_H^*)^{-1} \circ (\varphi_H D^{H-1/2}) \circ (\varphi_H^{-1} D^1) & \text{if} \quad H \geqslant 1/2 \end{cases}.$$

Proof. On one hand, by the isometry property of the Itô stochastic integral together with the third point of Theorem A.3, for every $s, t \in [0,T]$,

$$\int_0^{s\wedge t} K_H(t,u) K_H(s,u) du = \mathbb{E}(B_t^H B_s^H).$$

So, by the definitions of $\mathbf{B}^H$ and I_H :

$$\int_0^{s\wedge t} K_H^*(\mathbf{1}_{[0,t]})(u) K_H(s,u) du = \mathbb{E}\left[\mathbf{B}^H(\mathbf{1}_{[0,t]}) B_s^H\right].$$

Then, the construction of I_H at Lemma 3.3 implies that :

$$I_H(\mathbf{1}_{[0,t]}) = (J_H \circ K_H^*)(\mathbf{1}_{[0,t]}).$$

That equality can be extended on $\mathcal{H}$ by a classical continuity argument.

On the other hand, since $K_H^* : \mathcal{H} \to \mathcal{G}$ and $I_H : \mathcal{H} \to H^1$ are two invertible maps, the restriction $(J_H)_{|\mathcal{G}} = I_H \circ (K_H^*)^{-1}$ is also invertible. Then, by the first point of Theorem A.3 :

$$I_H^{-1} = \begin{cases} (K_H^*)^{-1} \circ (\varphi_H^{-1} D^{1/2-H}) \circ (\varphi_H D^{2H}) & \text{if} \quad H \leqslant 1/2 \\ (K_H^*)^{-1} \circ (\varphi_H D^{H-1/2}) \circ (\varphi_H^{-1} D^1) & \text{if} \quad H \geqslant 1/2 \end{cases}.$$

□

Remark. By the fourth point of Theorem A.3 and Corollary A.4 :

$$\delta_H \circ I_H^{-1} \quad = \quad \delta_{1/2} \circ (J_H)_{|\mathcal{G}}^{-1}.$$

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