Invariance for Rough Differential Equations

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INVARIANCE FOR ROUGH DIFFERENTIAL EQUATIONS

LAURE COUTIN* AND NICOLAS MARIE**

Abstract. In 1990, in Itô's stochastic calculus framework, Aubin and Da Prato established a necessary and sufficient condition of invariance of a nonempty compact or convex subset C of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) for stochastic differential equations (SDE) driven by a Brownian motion. In Lyons rough paths framework, this paper deals with an extension of Aubin and Da Prato's results to rough differential equations. A comparison theorem is provided, and the special case of differential equations driven by a fractional Brownian motion is detailed.

Contents

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1. Introduction

The invariance of a nonempty closed convex subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) for a (ordinary) differential equation was solved by Nagumo in [26], see also [2] for a simple proof. It was obtained by Aubin and Da Prato in [5] for stochastic differential equations. More explicit results in the special case of polyhedrons have been established in Milian [25]. In [10], Cresson, Puig and Sonner have introduced a stochastic generalization of the well-known Hodgkin-Huxley neuron model satisfying the assumptions of the stochastic viability theorem of Milian [25]. On the viability and the invariance of sets for stochastic differential equations, see also Milian [24], Gautier and Thibault [15], and Michta [23].

In [6], the results of [5] were extended by Aubin and Da Prato to the stochastic differential inclusions. The case of stochastic controlled differential equations was studied by Da Prato and Frankowska in [12] or more recently by Buckdahn,

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Quicampoix, Rainer and Teichmann in [8]. An unified approach which provides a viability theorem for stochastic differential equations, backward stochastic differential equations and partial differential equations is developed in Buckdahn et al. [7].

The invariance of a subset of $\mathbb{R}^d$ for a stochastic differential equation driven by a α-Hölder continuous process with $\alpha \in (1/2, 1)$ has been already studied by several authors in the fractional calculus framework developed by Nualart and Rascanu in [29]. In Ciotir and Rascanu [9] and Nie and Rascanu [27], the authors have proved a sufficient and necessary condition for the invariance of a closed subset of $\mathbb{R}^d$ for a stochastic differential equation driven by a fractional Brownian motion of Hurst parameter $H \in (1/2, 1)$. In [22], Melnikov, Mishura and Shevchenko have proved a sufficient condition for the invariance of a smooth and nonempty subset of $\mathbb{R}^d$ for a stochastic differential equation driven by a mixed process containing both a Brownian motion and a α-Hölder continuous process with $\alpha \in (1/2, 1)$.

The rough paths theory introduced by T. Lyons in 1998 in the seminal paper [20] provides a natural and powerful framework to study differential equations driven by α-Hölder signals with $\alpha \in (0, 1]$. The theory and its applications are widely studied by many authors. For instance, see the book of Friz and Victoir [14], the nice introduction of Friz and Hairer [13], or the approach of Gubinelli [16].

The main purpose of this article is to extend the viability theorem of Aubin and Da Prato [5] and to provide a comparison theorem for the rough differential equations. The paper deals also with an application of the viability theorem to stochastic differential equations driven by a fractional Brownian motion of Hurst parameter greater than $1/4$.

Let $T > 0$ be arbitrarily chosen, and consider the differential equation

$$y_t = y_0 + \int_0^t b(y_s)ds + \int_0^t \sigma(y_s)dw_s \ ; \ t \in [0, T] \tag{1}$$

where, $y_0 \in \mathbb{R}^d$, b (resp. σ) is a continuous map from $\mathbb{R}^d$ into itself (resp. $\mathcal{M}_{d,e}(\mathbb{R})$), and $w : [0, T] \to \mathbb{R}^e$ is a α-Hölder continuous signal with $e \in \mathbb{N}^*$ and $\alpha \in (0, 1]$.

At Section 2, some definitions and results on rough differential equations are stated in order to take Equation (1) in that sense. Section 3 deals with a viability theorem for Equation (1) taken in the sense of rough paths (see Friz and Victoir [14]) and a convex or compact set. At Section 4, a comparison theorem for the rough differential equations is proved by using the viability results of Section 3. At Section 5, the viability theorem is applied to stochastic differential equations driven by a fractional Brownian motion of Hurst parameter greater than $1/4$. Finally, Appendix A is a brief survey on convex analysis.

For the sake of readability, all results are proved on $[0, T]$, but they can be extended on $\mathbb{R}_+$ via some usual localization arguments.

The results established in this paper could be applied in stochastic analysis itself, and in other sciences as neurology. On the one hand, in stochastic analysis, one could study the viability of rough differential inclusions as in Aubin and Da Prato [6] in Itô's calculus framework, or could also compare the viability condition for rough differential equations to the reflecting boundary conditions for Ito's stochastic differential equations (see Lions and Sznitman [19]). On the other hand,

together with J.M. Guglielmi who is neurologist at the American Hospital of Paris, we are studying a fractional Hodgkin-Huxley neuron model, that extends the model of Cresson et al. [10], in order to model injured nerves membrane potential in some neuropathies.

The following notations are used throughout the paper.

Notations (general) :

- The Euclidean scalar product on $\mathbb{R}^d$ is denoted by $\langle .,. \rangle$, and the Euclidean norm on $\mathbb{R}^d$ is denoted by $\|.\|$. The canonical basis of $\mathbb{R}^d$ is denoted by $(e_k)_{k\in[\![1,d]\!]}$. For every $x \in \mathbb{R}^d$, its j-th coordinate with respect to $(e_k)_{k\in[\![1,d]\!]}$ is denoted by $x^{(j)}$ for every $j \in [\![1,d]\!]$.
- For every $x_0 \in \mathbb{R}^d$ and $r \in \mathbb{R}_+$, $B_d(x_0,r) := \{x \in \mathbb{R}^d : \|x - x_0\| \leqslant r\}$.
- The interior, the closure and the frontier of a set $S \subset \mathbb{R}^d$ are respectively denoted by $\mathrm{int}(S)$, $\overline{S}$ and ∂S.
- For every $k \in [\![1,d]\!]$, $D_k := \{x \in \mathbb{R}^d : x^{(k)} \geqslant 0\}$.
- For a nonempty closed set $S \subset \mathbb{R}^d$, and every $x \in \mathbb{R}^d$, $\Pi_K(x)$ denotes the set of best approximations of x by the elements of K :

$$\Pi_S(x) := \left\{ x^* \in S : \|x - x^*\| = \inf_{y\in S} \|x - z\| \right\}. \tag{2}$$

- The distance between $x \in \mathbb{R}^m$ and a nonempty closed set $S \subset \mathbb{R}^d$ is

$$d_S(x) := \inf_{y\in S} \|x - y\|.$$

- The space of the matrices of size $d \times e$ is denoted by $\mathcal{M}_{d,e}(\mathbb{R})$. The (euclidean) matrix norm on $\mathcal{M}_{d,e}(\mathbb{R})$ is denoted by $\|.\|_{\mathcal{M}_{d,e}(\mathbb{R})}$. If $d = e$, then $\mathcal{M}_d(\mathbb{R}) := \mathcal{M}_{d,e}(\mathbb{R})$. The canonical basis of $\mathcal{M}_{d,e}(\mathbb{R})$ is denoted by $(e_{k,l})_{(k,l)\in[\![1,d]\!]\times[\![1,e]\!]}$.
- Let E and F be two vector spaces. The space of the linear maps from E into F is denoted by $\mathcal{L}(E,F)$. If $E = F$, then $\mathcal{L}(E) := \mathcal{L}(E,F)$.
- The space of the continuous functions from $[0,T]$ into $\mathbb{R}^d$ is denoted by $C^0([0,T],\mathbb{R}^d)$ and equipped with the uniform norm $\|.\|_{\infty,T}$.
- The space of the continuous functions l from $(0,t_0)$ into $]0,\infty[$ with $t_0 > 0$, and such that

$$\lim_{t\to 0^+} \frac{t^\beta}{l(t)} = 0 \ ; \ \forall \beta > 0,$$

is denoted by $\mathcal{S}_{t_0}$.

Notations (rough paths). See Friz and Victoir [14], Chapters 5, 7, 8 and 9 :

- $\Delta_T := \{(s,t) \in \mathbb{R}_+^2 : 0 \leqslant s < t \leqslant T\}$.
- The space of the α-Hölder continuous functions from $[0,T]$ into $\mathbb{R}^d$ is denoted by $C^{\alpha\text{-Höl}}([0,T],\mathbb{R}^d)$ and equipped with the α-Hölder semi-norm $\|.\|_{\alpha\text{-Höl},T}$:

$$\|x\|_{\alpha\text{-Höl},T} := \sup_{(s,t)\in\Delta_T} \frac{\|x_t - x_s\|}{|t-s|^\alpha} \ ; \forall x \in C^{\alpha\text{-Höl}}([0,T],\mathbb{R}^d).$$

- The step-N signature of $x \in C^{1\text{-Höl}}([0,T],\mathbb{R}^d)$ with $N \in \mathbb{N}^*$ is denoted by $S_N(x)$:

$$S_N(x)_t := \left(1, \int_{0<u<t} dx_u, \ldots, \int_{0<u_1<\cdots<u_N<t} dx_{u_1} \otimes \cdots \otimes dx_{u_N}\right) \ ; \forall t \in [0,T].$$

- The step-N free nilpotent group over $\mathbb{R}^d$ is denoted by $G^N(\mathbb{R}^d)$:

$$G^N(\mathbb{R}^d) := \{S_N(\gamma)_1 \ ; \ \gamma \in C^{1\text{-Höl}}([0,1],\mathbb{R}^d)\}.$$

- The space of the geometric α-rough paths from $[0,T]$ into $G^{[1/\alpha]}(\mathbb{R}^d)$ is denoted by $G\Omega_{\alpha,T}(\mathbb{R}^d)$:

$$G\Omega_{\alpha,T}(\mathbb{R}^d) := \overline{\{S_{[1/\alpha]}(x) \ ; \ x \in C^{1\text{-Höl}}([0,T],\mathbb{R}^d)\}}^{d_{\alpha\text{-Höl},T}}$$

where, $d_{\alpha\text{-Höl},T}$ is the α-Hölder distance for the Carnot-Carathéodory metric.

2. Preliminaries

The purpose of this section is to provide the appropriate formulation of Equation (1) in the rough paths framework. At the end of the section, a convergence result for the Euler scheme associated to Equation (1) is stated, and a definition of invariant sets for rough differential equations is provided.

The definitions and propositions stated in the major part of this section come from Lyons and Qian [21], Friz and Victoir [14], or Friz and Hairer [13].

First, the signal w is α-Hölder continuous with $\alpha \in (0,1]$. In addition, w has to satisfy the following assumption.

Assumption 2.1. *There exists* $\mathbf{w} \in G\Omega_{\alpha,T}(\mathbb{R}^e)$ *such that* $\mathbf{w}^{(1)} = w$.

Let $W : [0,T] \to \mathbb{R}^{e+1}$ be the signal defined by :

$$W_t := te_1 + \sum_{k=2}^{e+1} w_t^{(k-1)} e_k \ ; \forall t \in [0,T].$$

By Friz and Victoir [14], Theorem 9.26, there exists at least one $\mathbb{W} \in G\Omega_{\alpha,T}(\mathbb{R}^{e+1})$ such that $\mathbb{W}^{(1)} = W$.

Let us state the conditions the collection of vector fields of a rough differential equation has to satisfy in order to get at least the existence of solutions.

Notation. For every $\gamma > 0$, $\lfloor \gamma \rfloor$ is the largest integer strictly smaller than γ.

Definition 2.2. *Consider* $\gamma > 0$, $l, m \in \mathbb{N}^*$ *and a nonempty closed set* $V \subset \mathbb{R}^l$. *A map* $f : \mathbb{R}^l \to \mathcal{M}_{l,m}(\mathbb{R})$ *is* γ*-Lipschitz continuous (in the sense of Stein) from* V *into* $\mathcal{M}_{l,m}(\mathbb{R})$ *if and only if :*

(1) $f|_V \in C^{\lfloor \gamma \rfloor}(V, \mathcal{M}_{l,m}(\mathbb{R}))$.
(2) $f, Df, \ldots, D^{\lfloor \gamma \rfloor} f$ *are bounded on* V.
(3) $D^{\lfloor \gamma \rfloor} f$ *is* $(\gamma - \lfloor \gamma \rfloor)$*-Hölder continuous from* $\mathbb{R}^l$ *into* $\mathcal{L}(V^{\otimes \lfloor \gamma \rfloor}, \mathcal{M}_{l,m}(\mathbb{R}))$ *(i.e. there exists* $C > 0$ *such that for every* $x, y \in V$,

$$\|D^{\lfloor \gamma \rfloor} f(y) - D^{\lfloor \gamma \rfloor} f(x)\|_{\mathcal{L}(V^{\otimes \lfloor \gamma \rfloor}, \mathcal{M}_{l,m}(\mathbb{R}))} \leqslant C \|y - x\|^{\gamma - \lfloor \gamma \rfloor}).$$

The set of all such maps is denoted by $\mathrm{Lip}^{\gamma}(V, \mathcal{M}_{l,m}(\mathbb{R}))$.

The map f *is locally* γ*-Lipschitz continuous from* $\mathbb{R}^l$ *into* $\mathcal{M}_{l,m}(\mathbb{R})$, *if for every nonempty compact set* $K \subset \mathbb{R}^l$, f *is* γ*-Lipschitz continuous from* K *into* $\mathcal{M}_{l,m}(\mathbb{R})$. *The set of all such maps is denoted by* $\mathrm{Lip}^{\gamma}_{\mathrm{loc}}(\mathbb{R}^l, \mathcal{M}_{l,m}(\mathbb{R}))$.

In the sequel, b and σ satisfy the following assumption.

Assumption 2.3. *There exists* $\gamma \in (1/\alpha, [1/\alpha] + 1)$ *such that :*

(1) $b \in \mathrm{Lip}^{\gamma-1}_{\mathrm{loc}}(\mathbb{R}^d)$ *and* $\sigma \in \mathrm{Lip}^{\gamma-1}_{\mathrm{loc}}(\mathbb{R}^d, \mathcal{M}_{d,e}(\mathbb{R}))$.
(2) b *(resp.* σ*) is Lipschitz continuous from* $\mathbb{R}^d$ *into itself (resp.* $\mathcal{M}_{d,e}(\mathbb{R})$*).*

(3) $D^{[1/\alpha]}b$ *(resp.* $D^{[1/\alpha]}\sigma$*) is* $(\gamma - [1/\alpha])$*-Hölder continuous from* $\mathbb{R}^d$ *into* $\mathcal{L}((\mathbb{R}^d)^{\otimes[1/\alpha]}, \mathbb{R}^d)$ *(resp.* $\mathcal{L}((\mathbb{R}^d)^{\otimes[1/\alpha]}, \mathcal{M}_{d,e}(\mathbb{R}))$*).*

Let $f_{b,\sigma} : \mathbb{R}^d \to \mathcal{M}_{d,e+1}(\mathbb{R})$ be the map defined by :

$$f_{b,\sigma}(x) := \sum_{k=1}^{d} b^{(k)}(x) e_{k,1} + \sum_{l=2}^{e+1} \sum_{k=1}^{d} \sigma_{k,l}(x) e_{k,l} \ ; \forall x \in \mathbb{R}^d.$$

In the rough paths framework, $dy = f_{b,\sigma}(y)d\mathbb{W}$ with $y_0 \in \mathbb{R}^d$ as initial condition is the appropriate formulation of Equation (1).

Under the Assumptions 2.1 and 2.3, by Friz and Victoir Theorem 10.26, Exercice 10.55 and Exercice 10.56, the rough differential equation $dy = f_{b,\sigma}(y)d\mathbb{W}$ with $y_0 \in \mathbb{R}^d$ as initial condition has at least one solution y on $[0,T]$. Precisely, there exists a sequence $(W^n)_{n\in\mathbb{N}}$ of elements of $C^{1\text{-Höl}}([0,T], \mathbb{R}^{e+1})$ such that

$$\lim_{n\to\infty} d_{\alpha\text{-Höl},T}(S_{[1/\alpha]}(W^n), \mathbb{W}) = 0 \tag{3}$$

and

$$\lim_{n\to\infty} \|y - y^n\|_{\infty,T} = 0 \tag{4}$$

where, for every $n \in \mathbb{N}$, y^n is the solution on $[0,T]$ of the ordinary differential equation $dy^n = f_{b,\sigma}(y^n)dW^n$ with y_0 as initial condition.

Moreover, if b and σ satisfy the following assumption, the solution of Equation (1) is unique and denoted by $\pi_{f_{b,\sigma}}(0, y_0, \mathbb{W})$.

Assumption 2.4. $b \in \mathrm{Lip}^{\gamma}_{\mathrm{loc}}(\mathbb{R}^d)$ *and* $\sigma \in \mathrm{Lip}^{\gamma}_{\mathrm{loc}}(\mathbb{R}^d, \mathcal{M}_{d,e}(\mathbb{R}))$.

Let us now define the Euler scheme for Equation (1) and state a convergence result.

Let $D := (t_0, \dots, t_n)$ be a dissection of $[0,T]$ with $n \in \mathbb{N}^*$. The Euler scheme $\widehat{y}^n := (\widehat{y}^n_{t_0}, \dots, \widehat{y}^n_{t_n})$ for Equation (1) along the dissection D is defined by

$$\widehat{y}^n_{t_k} := \mathfrak{E}^{\mathbb{W}_{t_{k-1},t_k}} \circ \cdots \circ \mathfrak{E}^{\mathbb{W}_{t_0,t_1}} y_0 \ ; \forall k \in [\![1, n]\!]$$

with

$$\mathfrak{E}^g x := x + \mathcal{E}_{f_{b,\sigma}}(x, g)$$

and

$$\mathcal{E}_{f_{b,\sigma}}(x,g) := \sum_{k=1}^{[1/\alpha]} \sum_{i_1,\dots,i_k=1}^{e+1} f_{b,\sigma,i_1} \dots f_{b,\sigma,i_k} I(x) g^{(k),i_1,\dots,i_k}$$

for every $g \in G^{[1/\alpha]}(\mathbb{R}^{e+1})$ and $x \in \mathbb{R}^d$, and where I denotes the identity map from $\mathbb{R}^d$ into itself.

Theorem 2.5. *Let* $D := (t_0, \dots, t_n)$ *be a dissection of* $[0,t]$ *with* $t \in [0,T]$ *and* $n \in \mathbb{N}^*$*. Under the Assumptions 2.3 and 2.4, there exists a constant* $C > 0$ *depending only on* α*,* γ*,* $f_{b,\sigma}$ *and* $\|\mathbb{W}\|_{\alpha\text{-Höl},T}$ *such that*

$$\|\pi_{f_{b,\sigma}}(0, y_0; \mathbb{W})_t - \widehat{y}^n_t\| \leqslant Ct|D|^{\theta-1}$$

where, $\theta := (\lfloor\gamma\rfloor + 1)\alpha > 1$ *and* $|D|$ *is the mesh of* D*.*

See Friz and Victoir [14], Theorem 10.30.

Finally, let us state a definition of invariant sets for Equation (1).

Let S be a subset of $\mathbb{R}^d$.

Definition 2.6. *A function $\varphi : [0,T] \to \mathbb{R}^d$ is viable in S if and only if,*

$$\varphi(t) \in S \ ; \ \forall t \in [0,T].$$

The following definition provides a natural extension of the notion of invariant set in the rough paths theory setting.

Definition 2.7.

(1) *The subset S is invariant for $(\sigma, \mathbf{w})$ if and only if, for any initial condition $y_0 \in S$, every solution on $[0,T]$ of the rough differential equation $dy = \sigma(y)d\mathbf{w}$ is viable in S.*
(2) *The subset S is invariant for $(b, \sigma, \mathbb{W})$ if and only if, for any initial condition $y_0 \in S$, every solution on $[0,T]$ of the rough differential equation $dy = f_{b,\sigma}(y)d\mathbb{W}$ is viable in S.*

3. An invariance theorem for rough differential equations

Consider a nonempty closed set $K \subset \mathbb{R}^d$. For every map $\varphi : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$ with $m \in \mathbb{N}^*$, consider

$$K_\varphi := \bigcap_{k=1}^{m} \{x \in \mathbb{R}^d : \forall x^* \in \Pi_K(x),\ \varphi_{\cdot,k}(x) \in T_K(x^*)^{\circ\circ}\}, \tag{5}$$

and then

$$K_{b,\pm\sigma} := K_b \cap K_\sigma \cap K_{-\sigma}. \tag{6}$$

The invariance of K for $(b, \sigma, \mathbb{W})$ is studied in this section under the two following assumptions on the maps b and σ, and the signal w.

Assumption 3.1. $K \subset K_{b,\pm\sigma}$.

Assumption 3.2. *There exists $\lambda, \mu \in]0,\infty[$, $\beta \in (0, 2\alpha \wedge 1)$, $t_0 \in (0,T]$, $l \in \mathcal{S}_{t_0}$ and a countable set $\mathcal{B}_e \subset \partial B_e(0,1)$ such that $\{\pm e_k ; k \in [\![1,e]\!]\} \subset \mathcal{B}_e$, $\overline{\mathcal{B}_e} = \partial B_e(0,1)$ and*

$$-\mu = \inf_{\delta \in \mathcal{B}_e} \liminf_{t \to 0^+} \frac{\langle \delta, w_t \rangle}{t^\beta l(t)} \leqslant \sup_{\delta \in \mathcal{B}_e} \liminf_{t \to 0^+} \frac{\langle \delta, w_t \rangle}{t^\beta l(t)} = -\lambda.$$

Consider also the following stronger assumption on the maps b and σ :

Assumption 3.3. $K_{b,\pm\sigma} = \mathbb{R}^d$.

Now, let us state the main result of the paper ; the invariance theorem.

Theorem 3.4. *Under the Assumptions 2.1 and 2.3 on b and σ :*

(1) *Under Assumption 3.3, K is invariant for $(b, \sigma, \mathbb{W})$.*
(2) *When K is convex :*
   (a) *Under Assumption 3.1, K is invariant for $(b, \sigma, \mathbb{W})$.*
   (b) *Under the Assumptions 2.4 and 3.2, if K is invariant for $(b, \sigma, \mathbb{W})$, then Assumption 3.1 is fulfilled.*
(3) *When K is compact and $b \equiv 0$, under Assumption 3.2, if K is invariant for $(\sigma, \mathbf{w})$, then Assumption 3.1 is fulfilled.*

Remark 3.5.

(1) *By Remark A.3, for any map $\varphi : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$ with $m \in \mathbb{N}^*$, $\mathrm{int}(K) \subset K_\varphi$. So, in particular, $\mathrm{int}(K) \subset K_{b,\pm\sigma}$. Therefore, Assumption 3.1 is satisfied if and only if $\partial K \subset K_{b,\pm\sigma}$.*

(2) *If K is convex, then*

$$K_\varphi = \bigcap_{k=1}^{m} \{x \in \mathbb{R}^d : \varphi_{.,k}(x) \in T_K(p_K(x))\}$$

for every $\varphi : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$ with $m \in \mathbb{N}^*$, and where $p_K(x)$ is the unique projection of $x \in \mathbb{R}^d$ on K.

(3) *As in Aubin and Da Prato* [5]*, when K is not convex, the sufficient condition involves all $x \in \mathbb{R}^d$, and not only all $x \in K$ (see the statement of Theorem 1.5 and its remark page 601).*

(4) *Assumption 3.1 for some usual convex subsets of $\mathbb{R}^d$:*

- *When K is a vector subspace of $\mathbb{R}^d$, Assumption 3.1 means that*

$$b(K) \subset K$$

and

$$\sigma_{.,k}(K) \subset K \ ; \ \forall k \in [\![1, e]\!].$$

- *When K is the unit ball of $\mathbb{R}^d$, Assumption 3.1 means that for every $x \in \mathbb{R}^d$ such that $\|x\| = 1$,*

$$\langle x, b(x) \rangle \leqslant 0$$

and

$$\langle \sigma_{.,k}(x), x \rangle = 0 \ ; \ \forall k \in [\![1, e]\!].$$

- *Consider the polyhedron*

$$K = \bigcap_{i \in I} \{x \in \mathbb{R}^d : \langle s_i, x - a_i \rangle \leqslant 0\}$$

where, $I \subset \mathbb{N}$ is a nonempty finite set, and $(a_i)_{i \in I}$ and $(s_i)_{i \in I}$ are two families of elements of $\mathbb{R}^d$ such that $s_i \neq 0$ for every $i \in I$. Here, Assumption 3.1 means that for every $x \in K$ and $i \in I$ such that $\langle s_i, x - a_i \rangle = 0$,

$$\langle s_i, b(x) \rangle \leqslant 0$$

and

$$\langle s_i, \sigma_{.,k}(x) \rangle = 0 \ ; \ \forall k \in [\![1, e]\!].$$

These conditions on b and σ are quite natural, and the same as in Milian [25] *or Cresson et al.* [10]*, where the driving signal of the main equation is a Brownian motion.*

(5) *Assumption 3.2 is close to the notion of "signal rough at time 0" stated at* [13]*, Chapter 6.*

(6) *Almost all the paths of the e-dimensional fractional Brownian motion satisfy Assumption 3.2 (see Proposition 5.2).*

At Subsection 3.1, the invariance of K for $(b, \sigma, \mathbb{W})$ is proved under Assumption 3.3, and under Assumption 3.1 when K is convex. At Subsection 3.2, when K is compact and $b \equiv 0$, under Assumption 3.2, the necessity of Assumption 3.1 to get the invariance of K for $(\sigma, \mathbf{w})$ is proved. At Subsection 3.3, when K is convex, under the Assumptions 2.4 and 3.2, the necessity of Assumption 3.1 to get the invariance of K for $(b, \sigma, \mathbb{W})$ is proved.

3.1. **Sufficient condition of invariance.** The main purpose of this subsection is to prove the invariance of K for $(b, \sigma, \mathbb{W})$ under Assumption 3.3, and under Assumption 3.1 when K is convex (Theorem 3.4.(1,2.a)). As in Aubin and Da Prato [5], the proof deeply relies on the fact that $t \in [0,T] \mapsto d_K^2(y_t)$ has a nonpositive epiderivative (see J.P. Aubin et al. [4], Section 18.6.2), where y is the solution of $dy = f_{b,\sigma}(y)dW$ with $\alpha = 1$ and $y_0 \in K$ as initial condition (see Lemma 3.6). Finally, when K is convex and compact, Corollary 3.7 allows to relax the regularity assumptions on b and σ.

Lemma 3.6. *Let K be nonempty closed subset of $\mathbb{R}^d$. Under the Assumptions 2.3 and 3.3 with $\alpha = 1$, the solution y on $[0,T]$ of the ordinary differential equation $dy = f_{b,\sigma}(y)dW$ with $y_0 \in K$ as initial condition is viable in K.*

Proof. In order to show that y is viable in K in a second step, as in Aubin and Da Prato [5], the following inequality is proved in a first step :

$$\liminf_{h \to 0^+} \frac{d_K^2(y_{t+h}) - d_K^2(y_t)}{h} \leqslant 0. \tag{7}$$

Step 1. For $t \in [0,T]$ and $h > 0$,

$$y_{t+h} - y_t = b(y_t)h + \sigma(y_t)(w_{t+h} - w_t) + R_{t,h} \tag{8}$$

with

$$R_{t,h} := \int_t^{t+h} [b(y_s) - b(y_t)]ds + \int_t^{t+h} [\sigma(y_s) - \sigma(y_t)]dw_s.$$

Since y (resp. b) is Lipschitz continuous from $[0,T]$ (resp. $\mathbb{R}^d$) into $\mathbb{R}^d$, there exists $C_1 > 0$ such that

$$\left\| \int_t^{t+h} [b(y_s) - b(y_t)]ds \right\| \leqslant C_1 h^2.$$

The function w is Lipschitz continuous from $[0,T]$ into $\mathbb{R}^e$, so there exists $\dot{w} \in L^\infty([0,T], \mathbb{R}^e)$ such that :

$$w_s = w_0 + \int_0^s \dot{w}_u du \ ; \forall s \in [0,T].$$

Then,

$$\begin{aligned} \left\| \int_t^{t+h} [\sigma(y_s) - \sigma(y_t)]dw_s \right\| &= \left\| \int_t^{t+h} [\sigma(y_s) - \sigma(y_t)]\dot{w}_s ds \right\| \\ &\leqslant \|\dot{w}\|_{\infty,T} \int_t^{t+h} \|\sigma(y_s) - \sigma(y_t)\|_{\mathcal{M}_{d,e}(\mathbb{R})} ds. \end{aligned}$$

Since y (resp. σ) is Lipschitz continuous from $[0,T]$ (resp. $\mathbb{R}^d$) into $\mathbb{R}^d$ (resp. $\mathcal{M}_{d,e}(\mathbb{R})$), there exists $C_2 > 0$ such that :

$$\int_t^{t+h} \|\sigma(y_s) - \sigma(y_t)\|_{\mathcal{M}_{d,e}(\mathbb{R})} ds \leqslant C_2 h^2.$$

Therefore,

$$\|R_{t,h}\| \leqslant C_3 h^2 \tag{9}$$

with $C_3 := C_1 + C_2 \|\dot{w}\|_{\infty,T}$.

For $y_t^* \in \Pi_K(y_t)$ and $y_{t+h}^* \in \Pi_K(y_{t+h})$ arbitrarily chosen :

$$\begin{aligned} d_K^2(y_{t+h}) &= \|y_{t+h} - y_{t+h}^*\|^2 \\ &\leqslant \|y_{t+h} - y_t^*\|^2 \\ &= \|y_{t+h} - y_t\|^2 + 2\langle y_{t+h} - y_t, y_t - y_t^* \rangle + \|y_t - y_t^*\|^2. \end{aligned}$$

So,

$$d_K^2(y_{t+h}) - d_K^2(y_t) \leqslant \|y_{t+h} - y_t\|^2 + 2\langle y_{t+h} - y_t, y_t - y_t^*\rangle. \tag{10}$$

The function y is Lipschitz continuous from $[0,T]$ into $\mathbb{R}^d$, so there exists $C_4 > 0$ such that :

$$\|y_{t+h} - y_t\|^2 \leqslant C_4 h^2.$$

By Equality (8) and Inequality (9) :

$$\langle y_{t+h} - y_t, y_t - y_t^*\rangle \leqslant C_3 h^2 + \langle b(y_t), y_t - y_t^*\rangle h + \sum_{k=1}^{e} \langle \sigma_{.,k}(y_t), y_t - y_t^*\rangle (w_{t+h}^{(k)} - w_t^{(k)}).$$

Moreover,

$$\langle b(y_t), y_t - y_t^*\rangle h + \sum_{k=1}^{e} \langle \sigma_{.,k}(y_t), y_t - y_t^*\rangle (w_{t+h}^{(k)} - w_t^{(k)}) \leqslant 0$$

because $y_t - y_t^* \in T_K(y_t^*)^\circ$ by Proposition A.4, and $b(y_t), \pm\sigma_{.,k}(y_t) \in T_K(y_t^*)^{\circ\circ}$ for every $k \in [\![1, e]\!]$ by Assumption 3.3. So,

$$\langle y_{t+h} - y_t, y_t - y_t^*\rangle \leqslant C_3 h^2.$$

Therefore, by Inequality (10) :

$$d_K^2(y_{t+h}) - d_K^2(y_t) \leqslant C_5 h^2$$

with $C_5 := 2C_3 + C_4$. This achieves the first step.

Step 2. Consider the function $\varphi : [0,T] \to \mathbb{R}_+$ defined by :

$$\varphi(t) := d_K^2(y_t) \ ; \ \forall t \in [0,T].$$

Assume that there exists $\tau \in (0,T]$ such that $\varphi(\tau) > 0$. Since φ is continuous on $[0,T]$, the set

$$\{t \in [0,\tau) : \forall s \in (t,\tau],\ \varphi(s) > 0\}$$

is not empty, and its infimum is denoted by t_*. Moreover, if $\varphi(t_*) > 0$, then there exists $t_{**} \in [0, t_*)$ such that $\varphi(t) > 0$ for every $t \in (t_{**}, t_*]$. So, necessarily, $\varphi(t_*) = 0$.

By Inequality (7), for every $t \in [t_*, \tau]$,

$$D_\uparrow \varphi(t)(1) := \liminf_{h \to 0^+, u \to 1} \frac{\varphi(t + hu) - \varphi(t)}{h} \leqslant 0.$$

So, by Aubin [2] :

$$0 < \varphi(\tau) = \varphi(\tau) - \varphi(t_*) \leqslant 0.$$

There is a contradiction, then φ is nonpositive on $[0,T]$. Since $\varphi([0,T]) \subset \mathbb{R}_+$, necessarily :

$$\varphi(t) = d_K^2(y_t) = 0 \ ; \ \forall t \in [0,T].$$

In other words, y is viable in K. □

Via Lemma 3.6, let us prove Theorem 3.4.(1,2).

Proof. Theorem 3.4.(1,2a). Theorem 3.4.(1) is proved at the first step, and Theorem 3.4.(2a) is proved at the second step.

Step 1. Assume that $\alpha \in (0,1]$ and $K_{b,\pm\sigma} = \mathbb{R}^d$. Since K is a closed subset of $\mathbb{R}^d$, every solution on $[0,T]$ of the rough differential equation $dy = f_{b,\sigma}(y)d\mathbb{W}$ with $y_0 \in K$ as initial condition is viable in K by Lemma 3.6 together with Equality (4).

Step 2. Assume that $\alpha = 1$, K is convex and $K \subset K_{b,\pm\sigma}$ (see (6) for a definition). Let y be the solution on $[0,T]$ of the ordinary differential equation $dy = f_{b,\sigma}(y)dW$ with $y_0 \in K$ as initial condition. Consider the maps $B := b \circ p_K$, $S := \sigma \circ p_K$ and $F : \mathbb{R}^d \to \mathcal{M}_{d,e+1}(\mathbb{R})$ such that :

$$F(x) := \sum_{k=1}^{d} B^{(k)}(x) e_{k,1} + \sum_{l=2}^{e+1} \sum_{k=1}^{d} S_{k,l}(x) e_{k,l} \ ; \ \forall x \in \mathbb{R}^d.$$

Since $f_{b,\sigma}$ (resp. p_K) is Lipschitz continuous from $\mathbb{R}^d$ into $\mathcal{M}_{d,e+1}(\mathbb{R})$ (resp. K), F is Lipschitz continuous from $\mathbb{R}^d$ into $\mathcal{M}_{d,e+1}(\mathbb{R})$. So, the ordinary differential equation $dY = F(Y)dW$ with $y_0 \in K$ as initial condition has a unique solution Y on $[0,T]$.

Since $K_{B,\pm S} = \mathbb{R}^d$, Y is viable in K by Lemma 3.6.

Therefore, the solution y of the ordinary differential equation $dy = f_{b,\sigma}(y)dW$ with y_0 as initial condition coincides with Y on $[0,T]$ because $f_{b,\sigma}$ coincides with F on K. In particular, y is viable in K.

Assume now that $\alpha \in (0,1]$. Since K is a closed subset of $\mathbb{R}^d$, every solution on $[0,T]$ of the rough differential equation $dy = f_{b,\sigma}(y)d\mathbb{W}$ with $y_0 \in K$ as initial condition is viable in K by Equality (4). □

Corollary 3.7. *Under the Assumptions 2.1 and 3.1, if K is a nonempty convex and compact subset of $\mathbb{R}^d$, $b \in \mathrm{Lip}_{\mathrm{loc}}^{\gamma-1}(\mathbb{R}^d)$ and $\sigma \in \mathrm{Lip}_{\mathrm{loc}}^{\gamma-1}(\mathbb{R}^d, \mathcal{M}_{d,e}(\mathbb{R}))$ with $\gamma > 1/\alpha$, then all the solutions of the rough differential equation $dy = f_{b,\sigma}(y)d\mathbb{W}$ with $y_0 \in K$ as initial condition are defined on $[0,T]$ and viable in K.*

Proof. Since $f_{b,\sigma}$ is locally $(\gamma-1)$-Lipschitz continuous from $\mathbb{R}^d$ into $\mathcal{M}_{d,e+1}(\mathbb{R})$, there exists $\tau \in (0,T]$ such that the rough differential equation $dy = f_{b,\sigma}(y)d\mathbb{W}$ with $y_0 \in K$ as initial condition has at least one solution y on $[0,\tau)$.

Since b and σ satisfy Assumption 3.1, by Theorem 3.4.(2) applied to $(b,\sigma,\mathbb{W})$ on $[0,\tau[$:

$$y_t \in K \ ; \ \forall t \in [0,\tau[.$$

So, y is bounded on $[0,\tau[$ by at least one continuous function from $[0,T]$ into $\mathbb{R}^d$ because K is a bounded subset of $\mathbb{R}^d$.

Therefore, by Friz and Victoir [14], Theorem 10.21, y is defined on $[0,T]$, and by Theorem 3.4.(2), it is viable in K. □

3.2. **Necessary condition of invariance : compact case.** When K is compact and $b \equiv 0$, the purpose of this subsection is to prove that under Assumption 3.2, if K is invariant for $(\sigma, \mathbf{w})$, then Assumption 3.1 is fulfilled (Theorem 3.4.(3)).

Lemma 3.8. *Under Assumption 3.2, for $T_0 := t_0 \wedge T$,*

$$M := \max_{k \in [\![1,e]\!]} \sup_{t \in [0,T_0]} \left| \frac{w_t^{(k)}}{t^\beta l(t)} \right| < \infty.$$

Proof. By Assumption 3.2 :

$$\begin{aligned}\min_{k\in[\![1,e]\!]} \liminf_{t\to 0^+} \frac{w_t^{(k)}}{t^\beta l(t)} &= \min_{k\in[\![1,e]\!]} \liminf_{t\to 0^+} \frac{\langle e_k, w_t\rangle}{t^\beta l(t)} \\ &\geqslant \inf_{\delta\in\mathcal{B}_e} \liminf_{t\to 0^+} \frac{\langle \delta, w_t\rangle}{t^\beta l(t)} = -\mu.\end{aligned}$$

Moreover, the function

$$t \in (0,T_0] \longmapsto \frac{w_t^{(k)}}{t^\beta l(t)}$$

is continuous. So, there exists $r>0$ such that :

$$\min_{k\in[\![1,e]\!]} \inf_{t\in[0,T_0]} \frac{w_t^{(k)}}{t^\beta l(t)} \geqslant -r\mu.$$

Similarly, there exists $R>0$ such that :

$$\max_{k\in[\![1,e]\!]} \sup_{t\in[0,T_0]} \frac{w_t^{(k)}}{t^\beta l(t)} \leqslant R\mu.$$

This achieves the proof. □

Proposition 3.9. *Under Assumption 3.2 :*

$$-\mu = \inf_{\delta\in\partial B_e(0,1)} \liminf_{t\to 0^+} \frac{\langle \delta, w_t\rangle}{t^\beta l(t)} \leqslant \sup_{\delta\in\partial B_e(0,1)} \liminf_{t\to 0^+} \frac{\langle \delta, w_t\rangle}{t^\beta l(t)} = -\lambda.$$

Proof. Let $\delta\in\partial B_e(0,1)$ be arbitrarily chosen. Since $\overline{\mathcal{B}_e} = \partial B_e(0,1)$, there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{B}_e$ such that :

$$\lim_{n\to\infty} \|\delta - \delta_n\| = 0.$$

By Lemma 3.8 :

$$M := \max_{k\in[\![1,e]\!]} \sup_{t\in[0,T_0]} \left| \frac{w_t^{(k)}}{t^\beta l(t)} \right| < \infty.$$

For every $n\in\mathbb{N}$ and $t\in(0,T_0]$,

$$\begin{aligned}\frac{\langle \delta, w_t\rangle}{t^\beta l(t)} &= \frac{\langle \delta-\delta_n, w_t\rangle}{t^\beta l(t)} + \frac{\langle \delta_n, w_t\rangle}{t^\beta l(t)} \\ &\leqslant \|\delta-\delta_n\| \cdot \frac{\|w_t\|}{t^\beta l(t)} + \frac{\langle \delta_n, w_t\rangle}{t^\beta l(t)} \\ &\leqslant \|\delta-\delta_n\| M + \frac{\langle \delta_n, w_t\rangle}{t^\beta l(t)}.\end{aligned}$$

So, for every $n\in\mathbb{N}$, by Assumption 3.2 :

$$\begin{aligned}\liminf_{t\to 0^+} \frac{\langle \delta, w_t\rangle}{t^\beta l(t)} &\leqslant \|\delta-\delta_n\| M + \liminf_{t\to 0^+} \frac{\langle \delta_n, w_t\rangle}{t^\beta l(t)} \\ &\leqslant \|\delta-\delta_n\| M - \lambda \\ &\xrightarrow[n\to\infty]{} -\lambda.\end{aligned}$$

Since the right hand side of the previous inequality is not depending on δ :

$$\sup_{\delta\in\partial B_e(0,1)} \liminf_{t\to 0^+} \frac{\langle \delta, w_t\rangle}{t^\beta l(t)} \leqslant -\lambda.$$

Moreover, by Assumption 3.2 and since $\mathcal{B}_e \subset \partial B(0,1)$:

$$-\lambda = \sup_{\delta\in\mathcal{B}_e} \liminf_{t\to 0^+} \frac{\langle \delta, w_t\rangle}{t^\beta l(t)} \leqslant \sup_{\delta\in\partial B_e(0,1)} \liminf_{t\to 0^+} \frac{\langle \delta, w_t\rangle}{t^\beta l(t)}.$$

Therefore,

$$\sup_{\delta\in\partial B_e(0,1)} \liminf_{t\to 0^+} \frac{\langle \delta, w_t\rangle}{t^\beta l(t)} = -\lambda.$$

Similarly,

$$\inf_{\delta\in\partial B_e(0,1)} \liminf_{t\to 0^+} \frac{\langle \delta, w_t\rangle}{t^\beta l(t)} = -\mu.$$

□

Proof. Theorem 3.4.(3). Let y be a solution of the rough differential equation $dy = \sigma(y)d\mathbf{w}$ with $y_0 \in K$ as initial condition, and assume that y is viable in K.

Consider $s \in T_K(y_0)^\circ$ and $\varepsilon > 0$. Since K is a nonempty compact subset of $\mathbb{R}^d$, by Proposition A.5, there exists $\delta_\varepsilon > 0$ such that for every $x \in K \cap B_d(y_0, \delta_\varepsilon)$,

$$\langle x - y_0, s\rangle \leqslant \varepsilon\|x - y_0\|.$$

Since y is continuous, there exists $t_\varepsilon \in [0, T]$ such that $y([0, t_\varepsilon]) \subset B_d(y_0, \delta_\varepsilon)$. So, for every $t \in [0, t_\varepsilon]$,

$$\langle y_t - y_0, s\rangle \leqslant \varepsilon\|y_t - y_0\|.$$

Then,

$$\limsup_{t\to 0^+} \frac{\langle y_t - y_0, s\rangle}{t^\beta l(t)} \leqslant \varepsilon \limsup_{t\to 0^+} \frac{\|y_t - y_0\|}{t^\beta l(t)}.$$

For every $t \in [0, T]$, by Theorem 2.5 applied with the dissection $(0, t)$ of $[0, t]$, there exists a constant $C > 0$, depending on T but not on t, such that :

$$\begin{aligned}\|y_t - y_0 - \sigma(y_0)w_t\| &\leqslant Ct|(0,t)|^{(\lfloor\gamma\rfloor+1)\alpha-1}\\ &\leqslant CT^{(\lfloor\gamma\rfloor-1)\alpha}t^{2\alpha}.\end{aligned}$$

Since $\beta < 2\alpha$,

$$\limsup_{t\to 0^+} \frac{\langle \sigma(y_0)w_t, s\rangle}{t^\beta l(t)} \leqslant \varepsilon \limsup_{t\to 0^+} \frac{\|\sigma(y_0)w_t\|}{t^\beta l(t)}.$$

Therefore, by duality in $\mathbb{R}^d$:

$$\liminf_{t\to 0^+} \frac{\langle u(s), w_t\rangle}{t^\beta l(t)} \geqslant 0 \ ; \forall s \in T_K(y_0)^\circ \tag{11}$$

where, $u : T_K(y_0)^\circ \to \mathbb{R}^e$ is the map defined by

$$u(s) := -\sigma(y_0)^\tau s \ ; \forall s \in T_K(y_0)^\circ,$$

and $\sigma(y_0)^\tau$ is the transpose of the matrix $\sigma(y_0)$.

Assume that there exists $s \in T_K(y_0)^\circ$ such that $u(s) \neq 0$, and put

$$v(s) := \frac{u(s)}{\|u(s)\|} \in \partial B_e(0, 1).$$

By Inequality (11) :

$$\liminf_{t\to 0^+} \frac{\langle v(s), w_t\rangle}{t^\beta l(t)} \geqslant 0.$$

There is a contradiction with Assumption 3.2 by Proposition 3.9. So, necessarily :

$$u(s) = 0 \ ; \forall s \in T_K(y_0)^\circ.$$

Therefore, since $(e_k)_{k\in[\![1,e]\!]}$ is a basis of $\mathbb{R}^e$:

$$\langle \sigma_{\cdot,k}(y_0), s\rangle = 0 \ ; \forall k \in [\![1, e]\!].$$

In particular, $\pm\sigma_{\cdot,k}(y_0) \in T_K(y_0)^{\circ\circ}$ for every $k \in [\![1, e]\!]$. This achieves the proof because $y_0 \in K$ has been arbitrarily chosen. □

3.3. **Necessary condition of invariance : convex case.** When K is convex, the purpose of this subsection is to prove that under the Assumptions 2.4 and 3.2, if K is invariant for $(b, \sigma, \mathbb{W})$, then b and σ satisfy Assumption 3.1 (Theorem 3.4.(2b)). First, that result is proved for the half-hyperplanes.

Lemma 3.10. *Under the Assumptions 2.1, 2.3, 2.4 and 3.2, if there exists $\nu \in [\![1, d]\!]$ such that $D_\nu = \{x \in \mathbb{R}^d : x^{(\nu)} \geqslant 0\}$ is invariant for $(b, \sigma, \mathbb{W})$, then*

$$b^{(\nu)}(x - x^{(\nu)}e_\nu) \geqslant 0 \text{ and } \sigma_{\nu,.}(x - x^{(\nu)}e_\nu) = 0$$

for every $x \in \mathbb{R}^d$.

Proof. Let $\widehat{y} : [0, T] \to \mathbb{R}^d$ be the map defined by :

$$\widehat{y}_t := \mathfrak{E}^{\mathbb{W}_{0,t}} y_0 \ ; \forall t \in [0, T].$$

For every $t \in [0, T]$, $(y_0, \widehat{y}_t)$ coincides with the Euler scheme for the rough differential equation $dy = f_{b,\sigma}(y)d\mathbb{W}$ with $y_0 \in \mathbb{R}^d$ as initial condition along the dissection $D_t := (0, t)$ of $[0, t]$.

In a first step, it is proved that if there exists $y_0 \in D_\nu$ such that

$$\liminf_{t \to 0^+} \frac{\widehat{y}_t^{(\nu)}}{t^\beta l(t)} < 0, \tag{12}$$

then D_ν is not invariant for $(b, \sigma, \mathbb{W})$. In a second step, it is proved that if there exists $y_0 \in \partial D_\nu$ such that $\sigma_{\nu,.}(y_0) \neq 0$ or $b^{(\nu)}(y_0) < 0$, then $\widehat{y}$ satisfies Inequality (12).

Step 1. Assume that there exists $y_0 \in D_\nu$ such that :

$$\liminf_{t \to 0^+} \frac{\widehat{y}_t^{(\nu)}}{t^\beta l(t)} < 0.$$

For every $t \in [0, T]$, by Theorem 2.5 applied with the dissection $(0, t)$ of $[0, t]$, there exists a constant $C_1 > 0$, depending on T but not on t, such that

$$\begin{aligned} \|\pi_{f_{b,\sigma}}(0, y_0; \mathbb{W})_t - \widehat{y}_t\| &\leqslant C_1 t |D_t|^{\theta - 1} \\ &= C_1 t^\theta \end{aligned}$$

with $\theta := (\lfloor \gamma \rfloor + 1)\alpha > 1$. So,

$$\frac{\pi_{f_{b,\sigma}}^{(\nu)}(0, y_0; \mathbb{W})_t}{t^\beta l(t)} \leqslant C_1 \frac{t^{\theta - \beta}}{l(t)} + \frac{\widehat{y}_t^{(\nu)}}{t^\beta l(t)} \ ; \forall t \in (0, T_0].$$

Moreover, since $\theta > 1 > \beta$ and $l \in \mathcal{S}_{t_0}$:

$$\liminf_{t \to 0^+} \left[C_1 \frac{t^{\theta - \beta}}{l(t)} + \frac{\widehat{y}_t^{(\nu)}}{t^\beta l(t)} \right] = \liminf_{t \to 0^+} \frac{\widehat{y}_t^{(\nu)}}{t^\beta l(t)}.$$

Therefore, by Inequality (12) :

$$\liminf_{t \to 0^+} \frac{\pi_{f_{b,\sigma}}^{(\nu)}(0, y_0; \mathbb{W})_t}{t^\beta l(t)} < 0.$$

In conclusion, there exists $t_1 \in [0, T_0]$ such that :

$$\pi_{f_{b,\sigma}}^{(\nu)}(0, y_0; \mathbb{W})_{t_1} < 0.$$

The path $\pi_{f_{b,\sigma}}^{(\nu)}(0, y_0; \mathbb{W})$ is not viable in D_ν.

Step 2. Let us show that if there exists $y_0 \in \partial D_\nu$ such that $\sigma_{\nu,.}(y_0) \neq 0$ or

$b^{(\nu)}(y_0) < 0$, then $\widehat{y}$ satisfies Inequality (12).

Case 1. Assume that there exists $y_0 \in \partial D_\nu$ such that $\sigma_{\nu,.}(y_0) \neq 0$. By Lemma 3.8 :

$$M := \max_{k\in[\![1,e]\!]} \sup_{t\in[0,T_0]} \left| \frac{w_t^{(k)}}{t^\beta l(t)} \right| < \infty.$$

On the one hand, since $\|\sigma_{\nu,.}(y_0)\|^{-1}\sigma_{\nu,.}(y_0) \in \partial B_e(0,1)$ and $\overline{\mathcal{B}_e} = \partial B_e(0,1)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{B}_e$ such that :

$$\lim_{n\to\infty} \left\| u_n - \frac{\sigma_{\nu,.}(y_0)}{\|\sigma_{\nu,.}(y_0)\|} \right\| = 0.$$

So, there exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N} \cap [n_0, \infty[$,

$$\|\sigma_{\nu,.}(y_0) - v_n\| \leqslant \frac{\lambda}{4eM} \|\sigma_{\nu,.}(y_0)\|$$

and $v_n := \|\sigma_{\nu,.}(y_0)\| u_n$ where, λ is defined in Assumption 3.2. Then, for every $n \in \mathbb{N} \cap [n_0, \infty[$,

$$\begin{aligned} \sup_{t\in[0,T_0]} \frac{1}{t^\beta l(t)} |\langle \sigma_{\nu,.}(y_0) - v_n, w_t\rangle| &\leqslant eM \|\sigma_{\nu,.}(y_0) - v_n\| \\ &\leqslant \frac{\lambda}{4} \|\sigma_{\nu,.}(y_0)\|. \end{aligned} \tag{13}$$

On the other hand, by the definition of the Euler scheme for Equation (1), there exists $C_2 > 0$ such that :

$$|\mathcal{E}^{(\nu)}_{f_{b,\sigma}}(y_0, \mathbb{W}_{0,t}) - \langle \sigma_{\nu,.}(y_0), w_t\rangle| \leqslant C_2(t^{2\alpha} \vee t) \ ; \ \forall t \in [0,T].$$

Since $l \in \mathcal{S}_{t_0}$ and $\beta \in (0, 2\alpha \wedge 1)$:

$$\begin{aligned} \lim_{t\to 0^+} \frac{1}{t^\beta l(t)} |\mathcal{E}^{(\nu)}_{f_{b,\sigma}}(y_0, \mathbb{W}_{0,t}) - \langle \sigma_{\nu,.}(y_0), w_t\rangle| &\leq C_2 \lim_{t\to 0^+} \frac{t^{2\alpha-\beta}}{l(t)} \\ &= 0. \end{aligned} \tag{14}$$

For every $n \in \mathbb{N} \cap [n_0, \infty[$, by Assumption 3.2, Inequality (13) and Equality (14) together :

$$\begin{aligned} \liminf_{t\to 0^+} \frac{\widehat{y}_t^{(\nu)}}{t^\beta l(t)} &= \liminf_{t\to 0^+} \frac{\langle \sigma_{\nu,.}(y_0), w_t\rangle}{t^\beta l(t)} \\ &\leqslant \sup_{t\in[0,T_0]} \frac{1}{t^\beta l(t)} |\langle \sigma_{\nu,.}(y_0) - v_n, w_t\rangle| + \liminf_{t\to 0^+} \frac{\langle v_n, w_t\rangle}{t^\beta l(t)} \\ &\leqslant \frac{\lambda}{4} \|\sigma_{\nu,.}(y_0)\| - \lambda \|\sigma_{\nu,.}(y_0)\| = -\frac{3\lambda}{4} \|\sigma_{\nu,.}(y_0)\|. \end{aligned}$$

So,

$$\liminf_{t\to 0^+} \frac{\widehat{y}_t^{(\nu)}}{t^\beta l(t)} < 0.$$

By the first step of the proof, it means that D_ν is not invariant for $(b, \sigma, \mathbb{W})$. In other words, if K is invariant for $(b, \sigma, \mathbb{W})$, then

$$\sigma_{\nu,.}(x) = 0 \ ; \ \forall x \in \partial D_\nu. \tag{15}$$

Case 2. Assume that D_ν is invariant for $(b, \sigma, \mathbb{W})$ and there exists $y_0 \in \partial D_\nu$ such that $b^{(\nu)}(y_0) < 0$. By the first case, since D_ν is invariant for $(b, \sigma, \mathbb{W})$, Equation (15) is true.

Let $t \in [0, T]$ be arbitrarily chosen.

- If $\alpha \in (1/2, 1)$, then

$$\begin{aligned} \widehat{y}_t^{(\nu)} &= b^{(\nu)}(y_0)t + \langle \sigma_{\nu,.}(y_0), w_t \rangle \\ &= b^{(\nu)}(y_0)t. \end{aligned}$$

- If $\alpha \in (0, 1/2]$, then

$$\widehat{y}_t^{(\nu)} = b^{(\nu)}(y_0)t + \langle \sigma_{\nu,.}(y_0), w_t \rangle + \sum_{k=2}^{[1/\alpha]} \sum_{i_1,\dots,i_k=1}^{e} [\sigma_{i_1} \dots \sigma_{i_k} I(y_0)]^{(\nu)} \mathbf{w}^{(k),i_1,\dots,i_k}.$$

Consider $k \in [\![2, [1/\alpha]]\!]$ and $i_1, \dots, i_k \in [\![1, e]\!]$. There exists a real family $(\rho_{i_2,\dots,i_{k-1},l_1,\dots,l_{k-1}}(y_0))_{(l_1,\dots,l_{k-1}) \in [\![1,d]\!]^{k-1}}$ such that :

$$[\sigma_{i_1} \dots \sigma_{i_k} I(y_0)]^{(\nu)} = \sum_{l_1,\dots,l_{k-1}=1}^{d} \sigma_{l_1,i_1}(y_0) \partial^{k-1}_{l_1,\dots,l_{k-1}} \sigma_{\nu,i_k}(y_0) \rho_{i_2,\dots,i_{k-1},l_1,\dots,l_{k-1}}(y_0).$$

Consider the set

$$I_{k-1} := \{(l_1, \dots, l_{k-1}) \in [\![1, d]\!]^{k-1} : \exists \kappa \in [\![1, k-1]\!],\, l_\kappa \neq \nu\}.$$

By (15) together with Schwarz's lemma :

$$\partial^{k-1}_{l_1,\dots,l_{k-1}} \sigma_{\nu,.}(x) = 0 \ ; \ \forall x \in \partial D_\nu,\, \forall (l_1, \dots, l_{k-1}) \in I_{k-1}.$$

Then,

$$\begin{aligned} [\sigma_{i_1} \dots \sigma_{i_k} I(y_0)]^{(\nu)} &= \sigma_{\nu,i_1}(y_0) \partial^{k-1}_{\nu,\dots,\nu} \sigma_{\nu,i_k}(y_0) \rho_{i_2,\dots,i_{k-1},\nu,\dots,\nu}(y_0) + \\ &\qquad \sum_{l_1,\dots,l_{k-1} \in I_{k-1}} \sigma_{l_1,i_1}(y_0) \partial^{k-1}_{l_1,\dots,l_{k-1}} \sigma_{\nu,i_k}(y_0) \rho_{i_2,\dots,i_{k-1},l_1,\dots,l_{k-1}}(y_0) \\ &= 0. \end{aligned}$$

So,

$$\widehat{y}_t^{(\nu)} = b^{(\nu)}(y_0)t.$$

Since $l \in \mathcal{S}_{t_0}$ and $1 > \beta$, there exists $t_2 \in (0, T_0]$ such that :

$$0 < \frac{t^{1-\beta}}{l(t)} < 1 \ ; \ \forall t \in (0, t_2].$$

Therefore,

$$\begin{aligned} \liminf_{t \to 0^+} \frac{\widehat{y}_t^{(\nu)}}{t^\beta l(t)} &\leqslant \liminf_{t \to 0^+} \frac{\widehat{y}_t^{(\nu)}}{t} \\ &\leqslant b^{(\nu)}(y_0) < 0. \end{aligned}$$

This achieves the proof because there is a contradiction by the first step of the proof. □

Via Lemma 3.10, let us prove that Assumption 3.1 is necessary to get the invariance of K for $(b, \sigma, \mathbb{W})$.

Proof. Theorem 3.4.(2b). In a first step, it is proved that if the half-space

$$H_{a,s} := \{x \in \mathbb{R}^d : \langle s, x - a \rangle \leqslant 0\}$$

with $a \in \mathbb{R}^d$ and $s \in \mathbb{R}^d \backslash \{0\}$ is invariant for $(b, \sigma, \mathbb{W})$, then

$$\langle b(x), s \rangle \leqslant 0$$

and

$$\langle \sigma_{.,k}(x), s \rangle = 0 \ ; \ \forall k \in [\![1, e]\!]$$

for every $x \in \partial H_{a,s}$. In a second step, this result is used to show that if K is invariant for $(b, \sigma, \mathbb{W})$, then $K \subset K_{b,\pm\sigma}$.

Step 1. Let $y_0 \in H_{a,s}$ be arbitrarily chosen. Since $s \in \mathbb{R}^d \backslash \{0\}$, there exists $\nu \in [\![1, d]\!]$ such that $s^{(\nu)} \neq 0$. Consider the map $U : \mathbb{R}^d \to \mathbb{R}^d$ defined by :

$$U(x) := -(x-a) + (x^{(\nu)} - a^{(\nu)} - \langle x-a, s\rangle)e_\nu \ ; \forall x \in \mathbb{R}^d.$$

The map U is one to one from $\mathbb{R}^d$ into itself, and

$$U^{-1}(x) = -x + a - \frac{1}{s^{(\nu)}}(x^{(\nu)} - \langle x, s\rangle)e_\nu \ ; \forall x \in \mathbb{R}^d.$$

Moreover, $U|_{H_{a,s}}$ (resp. $U|_{\partial H_{a,s}}$) is one to one from $H_{a,s}$ (resp. $\partial H_{a,s}$) into D_ν (resp. ∂D_ν) where, D_ν is defined in Lemma 3.10. For every $x, h \in \mathbb{R}^d$,

$$\begin{aligned} DU(x).h &= -h + (h^{(\nu)} - \langle h, s\rangle)e_\nu \\ &= M_U h \end{aligned}$$

with

$$M_U := -I + e_{\nu,\nu} - \sum_{k=1}^{d} s^{(k)} e_{\nu,k}.$$

Consider the maps $B : \mathbb{R}^d \to \mathbb{R}^d$ and $S : \mathbb{R}^d \to \mathcal{M}_{d,e}(\mathbb{R})$ defined by

$$\begin{aligned} B(x) &:= M_U b(U^{-1}(x)) \\ &= (e_{\nu,\nu} - I)b(U^{-1}(x)) - \langle b(U^{-1}(x)), s\rangle e_\nu \end{aligned}$$

and

$$\begin{aligned} S(x) &:= M_U \sigma(U^{-1}(x)) \\ &= (e_{\nu,\nu} - I)\sigma(U^{-1}(x)) - \sum_{k=1}^{e} \langle \sigma_{.,k}(U^{-1}(x)), s\rangle e_{\nu,k} \end{aligned}$$

for every $x \in \mathbb{R}^d$. Let $F : \mathbb{R}^d \to \mathcal{M}_{d,e+1}(\mathbb{R})$ be the map defined by :

$$F(x) := \sum_{k=1}^{d} B^{(k)}(x) e_{k,1} + \sum_{l=2}^{e+1} \sum_{k=1}^{d} S_{k,l}(x) e_{k,l} \ ; \forall x \in \mathbb{R}^d.$$

Since $U^{-1} \in \mathcal{L}(\mathbb{R}^d)$ and b and σ satisfy assumptions 2.3 and 2.4, B and S also. So, by Friz and Victoir [14], Theorem 10.26, Exercice 10.55 and Exercice 10.56, the rough differential equation $dz = F(z)d\mathbb{W}$ with $U(y_0)$ as initial condition has a unique solution z on $[0, T]$. By the (rough) change of variable formula, for every $t \in [0, T]$,

$$\begin{aligned} U^{-1}(z_t) &= y_0 + \int_0^t M_U^{-1} dz_s \\ &= y_0 + \int_0^t f_{b,\sigma}(U^{-1}(z_s))dW_s. \end{aligned}$$

Therefore, since $d_y = f_{b,\sigma}(y)dW$ has a unique solution,

$$\pi_F(0, .; \mathbb{W}) = U(\pi_{f_{b,\sigma}}(0, U^{-1}(.); \mathbb{W})). \tag{16}$$

Assume that $H_{a,s}$ is invariant for $(b, \sigma, \mathbb{W})$. Since $U|_{H_{a,s}}$ is one to one from $H_{a,s}$ into D_ν, by Equality (16), D_ν is invariant for $(B, S, \mathbb{W})$. So, by Lemma 3.10 :

$$B^{(\nu)}(x - x^{(\nu)}e_\nu) \geqslant 0 \text{ and } S_{\nu,.}(x - x^{(\nu)}e_\nu) = 0$$

for every $x \in \mathbb{R}^d$. Let $k \in [\![1, e]\!]$ and $x \in \partial H_{a,s}$ be arbitrarily chosen. Since $U|_{\partial H_{a,s}}$ is one to one from $\partial H_{a,s}$ into ∂D_ν, $U(x) \in \partial D_\nu$. Therefore, by construction of B and S :

$$\langle b(x), s\rangle = -B^{(\nu)}(U(x)) \leqslant 0$$

and

$$\begin{aligned}\langle \sigma_{.,k}(x), s\rangle &= -\sum_{j=1}^{d} \sigma_{j,k}(x) s^{(j)} \\ &= -S_{\nu,k}(U(x)) = 0.\end{aligned}$$

Step 2. Assume that there exists $y_0 \in \partial K$ such that $y_0 \notin K_{b,\pm\sigma}$. Then, there exists $s \in N_K(y_0)$ such that :

$$\langle b(y_0), s\rangle > 0 \text{ or } (\exists k \in [\![1, e]\!] : \langle \sigma_{.,k}(y_0), s\rangle \neq 0). \tag{17}$$

Consider the half-space

$$H_{y_0,s} := \{x \in \mathbb{R}^d : \langle s, x - y_0\rangle \leqslant 0\}.$$

By the first step of the proof, (17) implies that there exists $t \in [0, T]$ such that $\pi_{f_{b,\sigma}}(0, y_0; \mathbb{W})_t \notin H_{y_0,s}$. Moreover, since $y_0 \in \partial K$ and $s \in N_K(y_0)$, $K \subset H_{y_0,s}$. Therefore, $\pi_{f_{b,\sigma}}(0, y_0; \mathbb{W})_t \notin K$. This achieves the proof by contraposition. □

4. A comparison theorem for rough differential equations

In this section, a comparison theorem for rough differential equations is proved by using the viability theorem of Section 3.

Consider a nonempty set $I \subset [\![1, d]\!]$, and

$$K := \{(x_1, x_2) \in (\mathbb{R}^d)^2 : \forall i \in I,\, x_1^{(i)} \leqslant x_2^{(i)}\}.$$

The following comparison theorem is a consequence of Theorem 3.4.

Proposition 4.1. *For $j \in \{1, 2\}$, consider $b_j : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma^j : \mathbb{R}^d \to \mathcal{M}_{d,e}(\mathbb{R})$ satisfying assumptions 2.3 and 2.4, and the map $f^j : \mathbb{R}^d \to \mathcal{M}_{d,e+1}(\mathbb{R})$ defined by :*

$$f^j(x) := \sum_{k=1}^{d} b_j^{(k)}(x) e_{k,1} + \sum_{l=2}^{e+1}\sum_{k=1}^{d} \sigma_{k,l}^j(x) e_{k,l} \ ;\ \forall x \in \mathbb{R}^d.$$

Under the Assumptions 2.1 and 3.2, the two following conditions are equivalent :

1. *For every $(y_0^1, y_0^2) \in K$, $i \in I$ and $t \in [0, T]$, $(y_t^1)^{(i)} \leqslant (y_t^2)^{(i)}$ where y^j is the solution of the rough differential equation $dy^j = f^j(y^j)d\mathbb{W}$ with y_0^j as initial condition for $j \in \{1, 2\}$.*
2. *For every $(x_1, x_2) \in K$ and $i \in I$, if $x_1^{(i)} = x_2^{(i)}$, then*

$$b_1^{(i)}(x_1) \leqslant b_2^{(i)}(x_2)$$

and

$$\sigma_{i,k}^1(x_1) = \sigma_{i,k}^2(x_2) \ ;\ \forall k \in [\![1, e]\!].$$

Proof. The set K is (isomorphe to) a nonempty closed convex polyhedron of $\mathbb{R}^{2d}$. Indeed,

$$K \cong \bigcap_{i \in I} \{x \in \mathbb{R}^{2d} : \langle s_i, x\rangle \leqslant 0\}$$

with $s_i := e_i - e_{d+i}$ for every $i \in I$. Let $F : \mathbb{R}^{2d} \to \mathcal{M}_{2d,e+1}(\mathbb{R})$ be the map defined by :

$$F(x_1, x_2) := \sum_{l=1}^{e+1}\sum_{k=1}^{d} [f_{k,l}^1(x_1) e_{k,l} + f_{k,l}^2(x_2) e_{d+k,l}] \ ;\ \forall (x_1, x_2) \in (\mathbb{R}^d)^2.$$

Since b_j and σ^j satisfy Assumption 2.3 for $j \in \{1, 2\}$, $B := F_{.,1}$ and $S := (F_{.,k})_{k \in [\![2,e+1]\!]}$ also.

The first condition is equivalent to the invariance of K for $(B, S, \mathbb{W})$, and the second condition means that $K \subset K_{B,\pm S}$ (see Milian [25], Theorem 2 (proof)). Therefore, these conditions are equivalent by Theorem 3.4. □

5. Invariance for differential equations driven by a fractional Brownian motion

At Subsection 4.1, it is shown that the fractional Brownian motion satisfies assumptions 2.1 and 3.2. Subsection 4.2 deals with the viability of the solutions of a multidimensional logistic equation driven by a fractional Brownian motion of Hurst parameter belonging to $(1/4, 1)$.

5.1. Fractional Brownian motion.

In this subsection, it is proved that the fractional Brownian motion satisfies assumptions 2.1 and 3.2. So, the viability theorem proved at Section 3 (Theorem 3.4) can be applied to differential equations driven by a fractional Brownian motion. In particular, it extends the results of Aubin and Da Prato [5].

First of all, let us remind the definition of the fractional Brownian motion.

Definition 5.1. *Let $(B_t)_{t\in[0,T]}$ be an e-dimensional centered Gaussian process. It is a fractional Brownian motion of Hurst parameter $H \in (0,1)$ if and only if,*

$$\mathrm{cov}(B_s^{(i)}, B_t^{(j)}) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})\delta_{i,j}$$

for every $(i,j) \in [\![1,e]\!]^2$ and $(s,t) \in [0,T]^2$.

About the fractional Brownian motion, the reader can refer to Nualart [28], Chapter 5.

Let $B := (B_t)_{t\in[0,T]}$ be an e-dimensional fractional Brownian motion of Hurst parameter $H \in (1/4, 1)$. The associated canonical probability space is denoted by $(\Omega, \mathcal{A}, \mathbb{P})$.

By Garcia-Rodemich-Rumsey's lemma (see Nualart [28], Lemma A.3.1), almost all the paths of B are α-Hölder continuous with $\alpha \in (0, H)$. The following proposition ensures that almost all the paths of B satisfy also Assumption 3.2.

Proposition 5.2. *For any countable set $\mathcal{B}_e \subset \partial B_e(0,1)$, almost surely,*

$$\liminf_{t\to 0^+} \frac{\langle x, B_t\rangle}{t^\beta l(t)} = -1 \ ; \ \forall x \in \mathcal{B}_e$$

with $\beta = H$ and $l \in \mathcal{S}_{e^{-1}}$ defined by

$$l(t) := \sqrt{2\log\left(\log\left(\frac{1}{t}\right)\right)} \ ; \ \forall t \in (0, e^{-1}].$$

Proof. By the law of the iterated logarithm for the 1-dimensional fractional Brownian motion (see Arcones [1] and Viitasaari [30], Remark 2.3.3) :

$$\mathbb{P}\left[\liminf_{t\to 0^+} \frac{B_t^{(1)}}{t^\beta l(t)} = -1\right] = 1.$$

Consider $x \in \partial B_e(0,1)$. Since $(\langle x, B_t\rangle)_{t\in[0,T]} \overset{d}{=} B^{(1)}$:

$$\mathbb{P}\left[\liminf_{t\to 0^+} \frac{\langle x, B_t\rangle}{t^\beta l(t)} = -1\right] = 1.$$

Therefore, since $\mathcal{B}_e$ is a countable subset of $\partial B_e(0,1)$:

$$\mathbb{P}\left[\bigcap_{x\in\mathcal{B}_e}\left\{\liminf_{t\to 0^+}\frac{\langle x,B_t\rangle}{t^\beta l(t)}=-1\right\}\right]=1.$$

This achieves the proof. □

By Friz and Victoir [14], Proposition 15.5 and Theorem 15.33, there exists an enhanced Gaussian process $\mathbf{B}:(\Omega,\mathcal{A})\to G\Omega_{\alpha,T}(\mathbb{R}^e)$ such that $\mathbf{B}^{(1)}=B$. So, the signal B satisfies assumptions 2.1 and 3.2.

Let $W:=(W_t)_{t\in[0,T]}$ be the stochastic process defined by :

$$W_t:=te_1+\sum_{k=1}^{e}B_t^{(k)}e_{k+1}\ ;\ \forall t\in[0,T].$$

By Friz and Victoir [14], Theorem 9.26, there exists an enhanced stochastic process $\mathbb{W}:(\Omega,\mathcal{A})\to G\Omega_{\alpha,T}(\mathbb{R}^{e+1})$ such that $\mathbb{W}^{(1)}:=W$.

Consider $\alpha\in(0,H)$ and a nonempty closed set $K\subset\mathbb{R}^d$.

Proposition 5.3. *Under the Assumptions 2.3 and 2.4 on b and σ :*

1. *Under Assumption 3.3, K is invariant for $(b,\sigma,\mathbb{W})$.*
2. *When K is compact and $b\equiv 0$, if K is invariant for $(\sigma,\mathbf{B})$, then Assumption 3.1 is fulfilled.*
3. *When K is convex, it is invariant for $(b,\sigma,\mathbb{W})$ if and only if Assumption 3.1 is fulfilled.*

Proof. Straightforward application of Theorem 3.4. □

Proposition 5.4. *Under Assumption 3.1, if K is convex and compact, $b\in\mathrm{Lip}_{\mathrm{loc}}^{\gamma-1}(\mathbb{R}^d)$ and $\sigma\in\mathrm{Lip}_{\mathrm{loc}}^{\gamma-1}(\mathbb{R}^d,\mathcal{M}_{d,e}(\mathbb{R}))$ with $\gamma>1/\alpha$, then all the solutions of the rough differential equation $dY=f_{b,\sigma}(Y)d\mathbb{W}$ with $y_0\in K$ as initial condition are defined on $[0,T]$ and viable in K.*

Proof. Straightforward application of Corollary 3.7. □

Remark 5.5.

1. *The Brownian motion is a fractional Brownian motion of Hurst parameter $H=1/2$.*
2. *The rough differential equations driven by a Brownian motion are stochastic differential equations in the sense of Stratonovich. Let B be an e-dimensional Brownian motion. In order to consider the stochastic differential equation*

$$dy_t=b(y_t)dt+\sigma(y_t)dB_t$$

in the sense of Itô, one has to consider the rough differential equation

$$dy_t=\left[b(y_t)-\frac{1}{2}\sum_{i,j=1}^{e}\sigma_{.,i}\sigma_{.,j}(y_t)\right]dt+\sigma(y_t)dB_t$$

where,

$$\sigma_{.,i}\sigma_{.,j}:=\sum_{k=1}^{d}\sigma_{k,i}\partial_k\sigma_{.,j}\ ;\ \forall i,j\in[\![1,e]\!].$$

(see Friz and Victoir [14]*, p. 510, Equation (17.3)).*

5.2. **A logistic equation driven by a fractional Brownian motion.** The logistic equation is a typical example of differential equation with a non-Lipschitz vector field, but with solutions viable in a nonempty convex and compact subset of $\mathbb{R}^d$.

Consider $K := [0,1]^d$, $\gamma > 1/\alpha$, a locally $(\gamma-1)$-Lipschitz continuous map $\sigma : \mathbb{R}^d \to \mathcal{M}_{d,e}(\mathbb{R})$ such that $K \subset K_\sigma \cap K_{-\sigma}$, $m \in \mathbb{R}^d$ and $b_m : \mathbb{R}^d \to \mathbb{R}^d$ the map defined by :

$$b_m^{(i)}(x) := m^{(i)} x^{(i)}(1 - x^{(i)}) \ ; \ \forall i \in [\![1,d]\!], \ \forall x \in \mathbb{R}^d.$$

The set K is a nonempty compact convex polyhedron of $\mathbb{R}^d$. Indeed,

$$K = K_1 \cap K_2$$

with

$$K_1 := \bigcap_{i=1}^{d} \{x \in \mathbb{R}^d : \langle -e_i, x \rangle \leqslant 0\}$$

and

$$K_2 := \bigcap_{i=1}^{d} \{x \in \mathbb{R}^d : \langle e_i, x - e_i \rangle \leqslant 0\}.$$

Consider $i \in [\![1,d]\!]$ and $x \in \mathbb{R}^d$. If $\langle -e_i, x \rangle = 0$, then $x^{(i)} = 0$ and $\langle b_m(x), -e_i \rangle = 0$. If $\langle e_i, x - e_i \rangle = 0$, then $x^{(i)} = 1$ and $\langle b_m(x), e_i \rangle = 0$. Therefore, $K \subset K_{b_m}$. In other words, b_m and σ satisfy Assumption 3.1.

Consider $y_0 \in K$. Since K is convex and compact, by Proposition 5.4, the logistic equation

$$Y_t = y_0 + \int_0^t b_m(Y_s) ds + \int_0^t \sigma(Y_s) dB_s \ ; \ t \in [0,T]$$

has at least one solution defined on $[0,T]$ and viable in K. For instance, one can put

$$\sigma(x) := \sum_{i=1}^{d} x^{(i)}(1 - x^{(i)}) e_{i,i} \ ; \ \forall x \in \mathbb{R}^d.$$

Appendix A. Tangent and normal cones

This Appendix is a brief survey on convex analysis.

The definitions and propositions stated in this subsection come from Hiriart-Urrut and Lemaréchal [17], Chapter A, and Aubin et al. [4], Chapter 18.

First, let us define the polar and bipolar sets of a closed cone.

Definition A.1.

(1) *The polar set of a closed cone $K \subset \mathbb{R}^d$ is the closed cone*

$$K^\circ = \{s \in \mathbb{R}^d : \forall \delta \in K, \ \langle s, \delta \rangle \leqslant 0\}.$$

(2) *The bipolar set of K is the closed cone $K^{\circ\circ} := (K^\circ)^\circ$.*

Let us now define the tangent and normal cones to a nonempty closed set $S \subset \mathbb{R}^d$ at $x \in S$.

Definition A.2.

(1) *A vector $\delta \in \mathbb{R}^d$ is tangent to S at x if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of S, and a real sequence $(t_n)_{n\in\mathbb{N}}$ such that when $n \to \infty$,*

$$\|x_n - x\| \to 0,\ t_n \downarrow 0 \text{ and } \left\| \frac{x_n - x}{t_n} - \delta \right\| \to 0.$$

The set of the tangent vectors to S at x is a closed cone of $\mathbb{R}^d$, called the tangent cone to S at x, and denoted by $T_S(x)$.

(2) *A vector $s \in \mathbb{R}^d$ is normal to S at x if and only if,*

$$\langle s, \delta \rangle \leqslant 0 \ ; \ \forall \delta \in T_S(x).$$

The set of the normal vectors to S at x is the normal cone to S at x, denoted by $N_S(x)$.

Remark A.3. *If $x \in \text{int}(S)$, then $T_S(x) = T_S(x)^{\circ\circ} = \mathbb{R}^d$.*

The two following properties are crucial in the proof of Theorem 3.4.

Proposition A.4. *For every $y \in \mathbb{R}^d$ and $y^* \in \Pi_S(y)$ (see (2) for a definition), $y - y^* \in T_S(y^*)^\circ$.*

See [3], Proposition 3.2.3 p. 85.

Proposition A.5. *If S is compact, then*

$$T_S(x)^\circ = \{s \in \mathbb{R}^d : \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall y \in S \cap B_d(x, \delta),\ \langle y - x, s\rangle \leqslant \varepsilon \|y - x\|\}.$$

The two last propositions provide some properties of the tangent and normal cones when S is a nonempty closed convex set.

Proposition A.6. *The tangent cone $T_S(x)$ is a closed convex cone such that $S \subset \{x\} + T_S(x)$.*

Proposition A.7. *A vector $s \in \mathbb{R}^d$ is normal to S at x if and only if,*

$$\langle s, y - x\rangle \leqslant 0 \ ; \ \forall y \in S.$$

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