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A new family of distributions based on the hypoexponential distribution with fitting reliability data

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Abstract

In this paper, a new general family of distributions using the hypoexponential distribution is introduced and studied. A special case of this family is explored in detail, corresponding to a new finite generalized mixture of generalized exponential distributions. Some of their mathematical properties are provided. We investigate maximum likelihood estimation of the model parameters. Three real data sets are used to prove the potential of this distribution among some recent extensions of the exponential distribution.

Keywords: Hypoexponential distribution, Generalized mixture, Moments, Maximum likelihood estimation.

2000 MSC: 60E05, 62E15.

1. Introduction

Over the past decades, considerable efforts have been made to develop flexible distributions for modeling lifetime data. In many situations, the weighted exponential distributions reach this goal with a great success. They are characterized by probability density functions of the form: $f(x) = w(x)g(x)$, where $g(x)$ denotes a pdf of the exponential distribution with parameter $\lambda > 0$ and $w(x)$ denotes a weight function. A suitable choice for $w(x)$, based on practical or theoretical consideration, can lead a proper statistical model for analysis specific lifetime data. The literature covering weighted exponential distributions is vast and is growing fast. We refer to the review of Saghir *et al.* (2017), and the references therein.

In this paper, we introduce a new general family of distributions constructed from the so-called hypoexponential distribution and a transformation proposed by AL-Hussaini (2012). Let us recall that the hypoexponential distribution is characterized by the distribution of a sum of several independent exponential random variables with different parameters (see Amari and Misra (1997) and Akkouchi (2008)). It is used in many domains of application, including tele-traffic engineering and queueing systems. Several tuning parameters are thus necessary for a perfect model. Our new family of distributions profits of this flexibility to open the door of new applications.

By considering a specific configuration of this family, we exhibit a new weighted exponential distribution, called the $\text{EH}(\mathbf{a}, \lambda)$ distribution, having some interesting properties. The corresponding cumulative distribution function can also be viewed a special linear combination of several cumulative distribution functions associated to the well known exponentiated exponential distribution introduced by Gupta and Kundu (1999, 2001). Since some coefficients of the combination can be negative, it also belongs to the family of finite generalized mixture distributions (see Bartholomew (1969), Franco *et al.* (2014) and Bakouch and Popović (2016)). Mathematical properties of our distribution are studied, including moments, conditional moments, reversed conditional moments, moment generating function, Lorenz curve, Bonferroni curve, mean deviations and order statistics. Estimation of the parameters

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is determined using the method of maximum likelihood. Considering three real data sets of different natures, we show the superiority of our distribution in terms of some goodness-of-fit statistics in comparison to the well-known distributions: the Lindley distribution (see Ghitany *et al.* (2008)), the exponential distribution and the exponentiated exponential distribution (see Gupta and Kundu (1999, 2001)).

The rest of the paper is organized as follows: Section 2 presents our new distribution. Some of its mathematical properties are described in Section 3. Distributions of order statistics following our distribution are investigated in Section 4. Estimation of the parameters via the maximum likelihood is studied in Section 5. Applications to real data sets are performed in Section 6. Some concluding remarks are outlined in Section 7.

2. The EH(\mathbf{a}, λ) distribution

2.1. On the hypoexponential distribution

We now briefly present the so-called hypoexponential distribution. Let $m \geq 2$ be a positive integer and $\alpha_1, \alpha_2, \dots, \alpha_m$ be m different real numbers. Let X_1, X_2, \dots, X_m be m independent random variables, with X_i following the exponential distribution of parameter $\alpha_i > 0$; the probability distribution function (pdf) of X_i is given by

$$f_{X_i}(x) = \alpha_i e^{-\alpha_i x}, \quad x > 0.$$

Then the random variable $S = \sum_{i=1}^m X_i$ follows the hypoexponential distribution with a parameter vector $\mathbf{a} = (\alpha_1, \dots, \alpha_m)$; a pdf of S is given by

$$f_S(x) = \sum_{i=1}^m \frac{\alpha_i}{P_i} e^{-\alpha_i x}, \quad x > 0,$$

where

$$P_i = \prod_{\substack{j=1 \\ j \neq i}}^m \left(1 - \frac{\alpha_i}{\alpha_j}\right).$$

Note that P_i is completely determined by \mathbf{a} and can be positive or negative according to the sign of $\prod_{\substack{j=1 \\ j \neq i}}^m (\alpha_j - \alpha_i)$, offering a great richness in the construction of $f_S(x)$. The corresponding cumulative distribution function (cdf) is given by

$$F_S(x) = 1 - \sum_{i=1}^m \frac{1}{P_i} e^{-\alpha_i x}, \quad x > 0. \quad (1)$$

Further details on the hypoexponential distribution can be found in Amari and Misra (1997), Akkouchi (2008) and Smaili *et al.* (2013). Before introducing the EH(\mathbf{a}, λ) distribution, we present a more general family of distributions based on a general transformation of cdf and the hypoexponential distribution.

2.2. A family of distributions based on the hypoexponential distribution

Lemma 1 below presents an idea of construction of a new cdf using two existing cdfs.

Lemma 1 (AL-Hussaini (2012)). *Let $G(x)$ and $H(x)$ be two cdfs of continuous distributions with $G(x)$ of support $(0, +\infty)$. Let $K(x)$ be the function defined by*

$$K(x) = 1 - G(-\ln(H(x))), \quad x \in \mathbb{R}.$$

Then $K(x)$ is a cdf of a continuous distribution.

Remark 1. In the context of Lemma 1, one can observe that $K(x)$ is the cdf of the random variable $H^{-1}(e^{-U})$ where U denotes a random variable having the cdf $G(x)$. This idea was also used with specific distributions on U in Zografos and Balakrishnan (2009) and Ristić and Balakrishnan (2012).

Let us now introduce a particular family of distributions which will be at the heart of this study. Let $H(x)$ be a cdf of a continuous distribution and $G(x)$ be the cdf associated to the hypoexponential distribution with a parameter vector $\mathbf{a} = (\alpha_1, \dots, \alpha_m)$ (1). It follows from Lemma 1 that the following function $K(x)$ is a cdf:

$$K(x) = 1 - G(-\ln(H(x))) = 1 - \left(1 - \sum_{i=1}^m \frac{1}{P_i} e^{-\alpha_i(-\ln(H(x)))}\right) = \sum_{i=1}^m \frac{1}{P_i} [H(x)]^{\alpha_i}.$$

From this expression, new distributions arise by taking specific cdf $H(x)$ (uniform distribution, normal distribution, Cauchy distribution, Pareto distribution ...). We will call the associated distribution the $\text{GH}(H; \mathbf{a})$ distribution (for general hypoexponential distribution with a parameter vector $\mathbf{a} = (\alpha_1, \dots, \alpha_m)$ using a cdf $H(x)$) for frequently used purpose in the present study or elsewhere.

Let us now consider a random variable Y following the $\text{GH}(H; \mathbf{a})$ distribution. Then the cdf of Y is given by

$$F_Y(x) = \sum_{i=1}^m \frac{1}{P_i} [H(x)]^{\alpha_i}, \quad x \in \mathbb{R}.$$

Using the equality $\sum_{i=1}^m \frac{1}{P_i} = 1$, the survival function (sf) of Y is given by

$$S_Y(x) = 1 - F_Y(x) = 1 - \sum_{i=1}^m \frac{1}{P_i} [H(x)]^{\alpha_i} = \sum_{i=1}^m \frac{1}{P_i} [1 - [H(x)]^{\alpha_i}], \quad x \in \mathbb{R}.$$

Denoting by $h(x)$ a pdf associated to $H(x)$, a pdf of Y is given by

$$f_Y(x) = h(x) \sum_{i=1}^m \frac{\alpha_i}{P_i} [H(x)]^{\alpha_i-1}, \quad x \in \mathbb{R}.$$

The hazard rate function (hrf) of Y is given by

$$h_Y(x) = \frac{f_Y(x)}{S_Y(x)} = h(x) \frac{\sum_{i=1}^m \frac{\alpha_i}{P_i} [H(x)]^{\alpha_i-1}}{\sum_{i=1}^m \frac{1}{P_i} [1 - [H(x)]^{\alpha_i}]}, \quad x \in \mathbb{R}.$$

Remark 2. The $\text{GH}(H; \mathbf{a})$ distribution belongs to the family of weighted distributions; we can express $f_Y(x)$ as: $f_Y(x) = w(x)g(x)$, where $g(x)$ denotes a pdf of the exponential distribution with parameter $\lambda > 0$ and $w(x) = \sum_{i=1}^m \frac{\alpha_i}{P_i} [H(x)]^{\alpha_i-1}$.

Remark 3. Since the coefficients $\frac{1}{P_1}, \frac{1}{P_2}, \dots, \frac{1}{P_m}$ have possible negative and positive values with $\sum_{i=1}^m \frac{1}{P_i} = 1$, the $\text{GH}(H; \mathbf{a})$ distribution also belongs to the family of finite generalized mixture of exponentiated baseline cdf $[H(x)]^\alpha$ (also known as Lehmann type-I distribution) (see Bartholomew (1969), Franco et al. (2014) and Bakouch and Popović (2016)).

Remark 4. In the particular case $m = 2$, $\mathbf{a} = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 > 0$, $\alpha_1 \neq \alpha_2$, we have $P_1 = \frac{\alpha_2 - \alpha_1}{\alpha_2}$ and $P_2 = \frac{\alpha_1 - \alpha_2}{\alpha_1}$,

$$F_Y(x) = \frac{\alpha_2}{\alpha_2 - \alpha_1} [H(x)]^{\alpha_1} + \frac{\alpha_1}{\alpha_1 - \alpha_2} [H(x)]^{\alpha_2}, \quad x \in \mathbb{R},$$

and

$$f_Y(x) = \frac{\alpha_1 \alpha_2}{\alpha_2 - \alpha_1} h(x) ([H(x)]^{\alpha_1 - 1} - [H(x)]^{\alpha_2 - 1}), \quad x \in \mathbb{R}.$$

Assuming that $\alpha_2 > \alpha_1$, note that $P_1 > 0$ and $P_2 < 0$.

The distribution of a sum of two independent $\text{GH}(H; \mathbf{a})$ distributions can be characterized. Let Y and Z be two independent random variables, Y follows the $\text{GH}(H_1; \mathbf{a})$ distribution with $\mathbf{a} = (\alpha_1, \alpha_2, \dots, \alpha_m)$ and Z follows the $\text{GH}(H_2; \mathbf{b})$ distribution with $\mathbf{b} = (\beta_1, \beta_2, \dots, \beta_m)$. Let $h_1(x)$ be a pdf associated to the cdf $H_1(x)$ and $h_2(x)$ be a pdf associated to the cdf $H_2(x)$. Set $Q_\ell = \prod_{\substack{j=1 \\ j \neq \ell}}^m \left(1 - \frac{\beta_\ell}{\beta_j}\right)$.

Then a pdf for $Y + Z$ is given by

$$f_{Y+Z}(x) = \sum_{i=1}^m \sum_{\ell=1}^m \frac{\alpha_i \beta_\ell}{P_i Q_\ell} \int_{-\infty}^{+\infty} h_1(t) h_2(x-t) [H_1(t)]^{\alpha_i - 1} [H_2(x-t)]^{\beta_\ell - 1} dt, \quad x \in \mathbb{R}.$$

This result is an immediate consequence of the continuous convolution formula.

2.3. The $\text{EH}(\mathbf{a}, \lambda)$ distribution

Let us now consider the $\text{GH}(H; \mathbf{a})$ distribution defined with the cdf $H(x)$ associated to the exponential distribution of parameter $\lambda > 0$: $H(x) = 1 - e^{-\lambda x}$, $x > 0$. We will call the associated distribution the $\text{EH}(\mathbf{a}, \lambda)$ distribution (for exponential hypoexponential distribution with a parameter vector $\mathbf{a} = (\alpha_1, \dots, \alpha_m)$ and λ) for frequently used purpose in the present study or elsewhere. Let us now consider a random variable Y following the $\text{EH}(\mathbf{a}, \lambda)$ distribution. Then the cdf of Y given by

$$F_Y(x) = \sum_{i=1}^m \frac{1}{P_i} (1 - e^{-\lambda x})^{\alpha_i}, \quad x > 0.$$

This cdf can be viewed as a finite generalized mixture of generalized exponential distribution baseline cdf introduced by Gupta and Kundu (1999, 2001), with different parameters for the power (the standard cdf of the generalized exponential distribution is obtained by taking $m = 1$). Thanks to its several tuning parameters, the $\text{EH}(\mathbf{a}, \lambda)$ distribution has a great flexibility which can be useful for determine a nice statistical model.

The sf of Y is given by

$$S_Y(x) = \sum_{i=1}^m \frac{1}{P_i} [1 - (1 - e^{-\lambda x})^{\alpha_i}], \quad x > 0.$$

A pdf of Y is given by

$$f_Y(x) = \lambda e^{-\lambda x} \sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda x})^{\alpha_i - 1}, \quad x > 0.$$

The hrf of Y is given by

$$h_Y(x) = \lambda e^{-\lambda x} \frac{\sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda x})^{\alpha_i - 1}}{\sum_{i=1}^m \frac{1}{P_i} [1 - (1 - e^{-\lambda x})^{\alpha_i}]}, \quad x > 0.$$

Some asymptotic properties of the previous functions are given below: when $x \rightarrow 0$, we have

$$F_Y(x) \sim \sum_{i=1}^m \frac{1}{P_i} (\lambda x)^{\alpha_i}, \quad f_Y(x) \sim \lambda \sum_{i=1}^m \frac{\alpha_i}{P_i} (\lambda x)^{\alpha_i - 1}, \quad h_Y(x) \sim \lambda \sum_{i=1}^m \frac{\alpha_i}{P_i} (\lambda x)^{\alpha_i - 1}.$$

Note that, if $\inf(\alpha_1, \alpha_2, \dots, \alpha_m) > 1$, we have $\lim_{x \rightarrow 0} f_Y(x) = 0$. If $\inf(\alpha_1, \alpha_2, \dots, \alpha_m) < 1$, the limit depends on the signs of P_1, P_2, \dots, P_m ; we can have $\lim_{x \rightarrow 0} f(x) = +\infty$ in some configurations. We have $\lim_{x \rightarrow +\infty} f(x) = 0$. The limit for $h(x)$ when $x \rightarrow +\infty$ depends only on λ and P_1, P_2, \dots, P_m .

Remark 5. As already observed in Remark 3, the $EH(\mathbf{a}, \lambda)$ distribution belongs to the family of weighted exponential distributions; we can express $f_Y(x)$ as: $f_Y(x) = w(x)g(x)$, where $g(x)$ denotes a pdf of the exponential distribution with parameter $\lambda > 0$ and $w(x) = \sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda x})^{\alpha_i - 1}$.

Remark 6. We now adapt Remark 1 to the $EH(\mathbf{a}, \lambda)$ distribution. Let us observe that, for a random variable U following the hypoexponential distribution with parameter \mathbf{a} , the random variable $W = -\frac{1}{\lambda} \ln(1 - e^{-U})$ follows the $EH(\mathbf{a}, \lambda)$ distribution.

Remark 7. In the particular case $m = 2$, $\mathbf{a} = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 > 0$, $\alpha_1 \neq \alpha_2$, we have $P_1 = \frac{\alpha_2 - \alpha_1}{\alpha_2}$ and $P_2 = \frac{\alpha_1 - \alpha_2}{\alpha_1}$,

$$F_Y(x) = \frac{\alpha_2}{\alpha_2 - \alpha_1} [1 - e^{-\lambda x}]^{\alpha_1} + \frac{\alpha_1}{\alpha_1 - \alpha_2} [1 - e^{-\lambda x}]^{\alpha_2}, \quad x > 0,$$

and

$$f_Y(x) = \frac{\lambda \alpha_1 \alpha_2}{\alpha_2 - \alpha_1} e^{-\lambda x} ([1 - e^{-\lambda x}]^{\alpha_1 - 1} - [1 - e^{-\lambda x}]^{\alpha_2 - 1}), \quad x > 0.$$

Figures 1, 2 and 3 show the graphical features of pdfs, cdfs and hrfs of the $EH(\mathbf{a}, \lambda)$ distribution with $\mathbf{a} = (\alpha_1, \alpha_2)$, $\alpha_1 = \alpha$ and $\alpha_2 = \alpha + 0.1$, for several choices of parameters (α, λ) . In this particular case, let us precise that

$$F_Y(x) = (1 + 10\alpha)(1 - e^{-\lambda x})^\alpha - 10\alpha(1 - e^{-\lambda x})^{\alpha+0.1}, \quad (2)$$

a pdf is given by

$$f_Y(x) = \lambda \alpha (1 + 10\alpha) e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1} [1 - (1 - e^{-\lambda x})^{0.1}] \quad (3)$$

and the hrf is given by

$$h_Y(x) = \frac{\lambda \alpha (1 + 10\alpha) e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1} [1 - (1 - e^{-\lambda x})^{0.1}]}{1 - (1 + 10\alpha)(1 - e^{-\lambda x})^\alpha + 10\alpha(1 - e^{-\lambda x})^{\alpha+0.1}}. \quad (4)$$

We observe that the plots of the pdfs and hrfs of the $EH(\mathbf{a}, \lambda)$ distribution are very flexible; different shapes and curves can be of interest for modeling lifetime data.

Remark 8. The considered pdf $f_Y(x)$ (3) is a very special case of the $EH(\mathbf{a}, \lambda)$ distribution and the value 0.1 is subjective. It can also be viewed as a particular two parameters pdf of the McDonald modified weibull distribution introduced in Merovci and Elbatal (2013) or the Beta exponentiated Weibull distribution developed by Cordeiro et al. (2013). Note that the main differences between $f_Y(x)$ (3) and similar looking pdfs as those associated to the weighted exponentiated exponential distribution introduced by Mahdavi (2015) or the transmuted generalized exponential distribution proposed by Khan et al. (2017), is the last term $[1 - (1 - e^{-\lambda x})^{0.1}]$ which only depends on λ , not α .

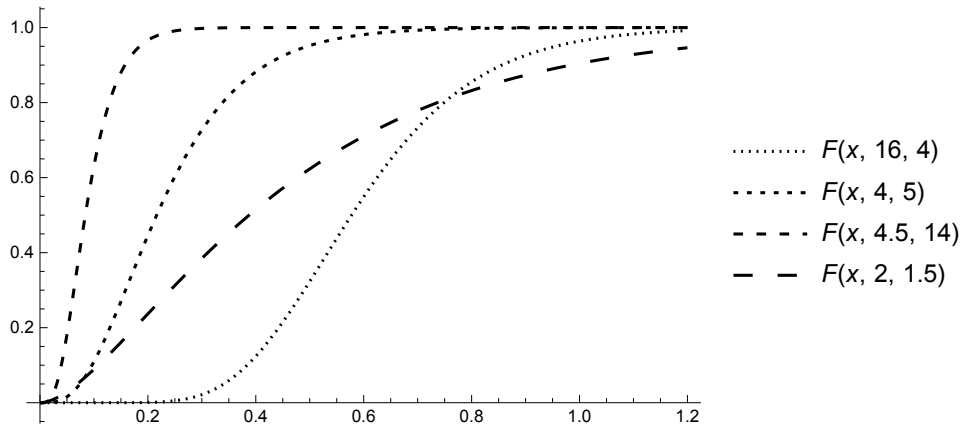


Figure 1: Some cdfs $F(x) = F(x; (\alpha, \lambda))$ (2) with various values for α and λ .

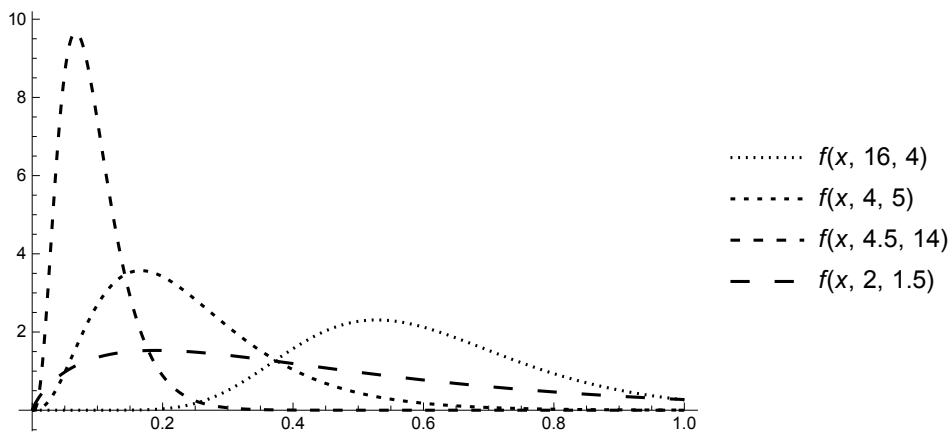


Figure 2: Some pdfs $f(x) = f(x; (\alpha, \lambda))$ (3) with various values for α and λ .

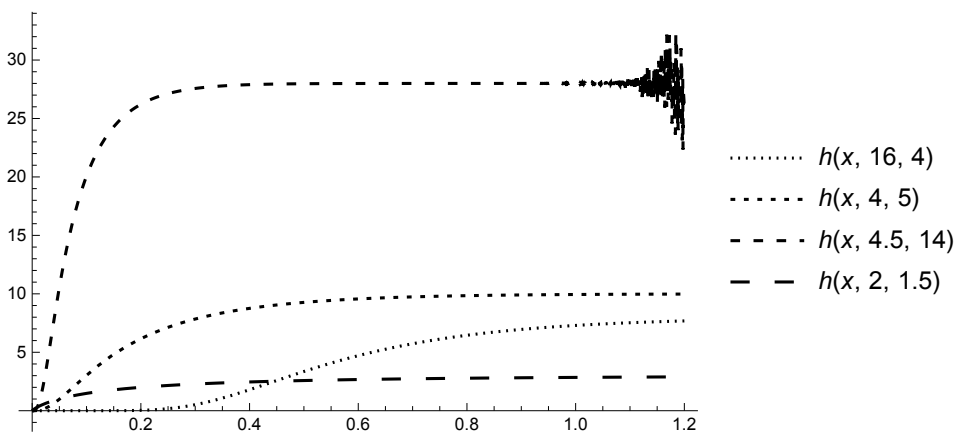


Figure 3: Some hrfs $h(x) = h(x; (\alpha, \lambda))$ (4) with various values for α and λ .

3. Mathematical properties related to the EH(a, λ) distribution

We now present some mathematical properties of our distribution.

The distribution of a sum of two independent EH(a, λ) distributions can be characterized. Let Y and Z be two independent random variables, Y follows the EH(a, λ₁) distribution with $\mathbf{a} = (\alpha_1, \alpha_2, \dots, \alpha_m)$ and Z follows the EH(b, λ₂) distribution with $\mathbf{b} = (\beta_1, \beta_2, \dots, \beta_m)$. Set $Q_\ell = \prod_{\substack{j=1 \\ j \neq \ell}}^m \left(1 - \frac{\beta_j}{\beta_\ell}\right)$. Then a pdf for $Y + Z$ is given by

$$f_{Y+Z}(x) = \lambda_1 \lambda_2 \sum_{i=1}^m \sum_{\ell=1}^m \frac{\alpha_i \beta_\ell}{P_i Q_\ell} \int_0^x e^{-\lambda_1 t} e^{-\lambda_1(x-t)} [1 - e^{-\lambda_1 t}]^{\alpha_i - 1} [1 - e^{-\lambda_2(x-t)}]^{\beta_\ell - 1} dt, \quad x > 0.$$

Also, note that

$$\mathbb{P}(Y \leq Z) = \sum_{i=1}^m \sum_{\ell=1}^m \frac{\beta_\ell}{P_i Q_\ell (\alpha_i + \beta_\ell)}.$$

This probability can be of interest in the context of reliability the stress-strength model (see Kundu and Gupta (2005)).

Let us work with only a random variable Y following the EH(a, λ) distribution.

Owing to the binomial series, we have the following expansion for $F_Y(x)$:

$$F_Y(x) = \sum_{i=1}^m \frac{1}{P_i} (1 - e^{-\lambda x})^{\alpha_i} = \sum_{i=1}^m \sum_{k=0}^{+\infty} \binom{\alpha_i}{k} (-1)^k \frac{1}{P_i} e^{-k\lambda x}.$$

Using again the binomial series, we have the following expansion for $f_Y(x)$:

$$f_Y(x) = \lambda e^{-\lambda x} \sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda x})^{\alpha_i - 1} = \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} e^{-(k+1)\lambda x},$$

where

$$\eta_{i,k} = \lambda \binom{\alpha_i - 1}{k} (-1)^k \frac{\alpha_i}{P_i}.$$

We can obtain the quantile function $Q_Y(x)$ via the nonlinear equation:

$$F_Y(Q_Y(x)) = x \quad \Leftrightarrow \quad \sum_{i=1}^m \frac{1}{P_i} (1 - e^{-\lambda Q_Y(x)})^{\alpha_i} = x.$$

The EH(a, λ) distribution can be simulated by using $Y = Q_Y(U)$ where U is a random variable having the uniform distribution on $[0, 1]$.

Let us consider the Gamma function: $\Gamma(\nu) = \int_0^{+\infty} x^{\nu-1} e^{-x} dx$, $\nu > 0$. The r -th moment of Y is given by

$$E(Y^r) = \int_{-\infty}^{+\infty} x^r f_Y(x) dx = \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \int_0^{+\infty} x^r e^{-(k+1)\lambda x} dx = \frac{\Gamma(r+1)}{\lambda^{r+1}} \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \frac{1}{(k+1)^{r+1}}.$$

The moment generating function of Y is given by, for $t < \lambda$,

$$M_Y(t) = E(e^{tY}) = \int_{-\infty}^{+\infty} e^{tx} f_Y(x) dx = \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \int_0^{+\infty} e^{(t-(k+1)\lambda)x} dx = \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \frac{1}{(k+1)\lambda - t}.$$

Proceeding as in (Gupta and Kundu, 2001, Equation (2.3)), with the change of variable $y = e^{-\lambda x}$, we also have, for $t < \lambda$,

$$M_Y(t) = \sum_{i=1}^m \frac{1}{P_i} \frac{\Gamma(\alpha_i + 1) \Gamma\left(1 - \frac{t}{\lambda}\right)}{\Gamma\left(\alpha_i - \frac{t}{\lambda} + 1\right)}.$$

Let us consider the lower incomplete Gamma function: $\Gamma(t, \nu) = \int_0^t x^{\nu-1} e^{-x} dx$. Then we have

$$\int_0^t x^r f_Y(x) dx = \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \int_0^t x^r e^{-(k+1)\lambda x} dx = \frac{1}{\lambda^{r+1}} \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \frac{\Gamma((k+1)\lambda t, r+1)}{(k+1)^{r+1}}.$$

This equality will be useful in the next. For $t > 0$, the conditional r -th moment of Y is given by

$$\begin{aligned} E(Y^r | Y > t) &= \frac{1}{1 - F_Y(t)} \int_t^{+\infty} x^r f_Y(x) dx = \frac{1}{1 - F_Y(t)} \left(E(Y^r) - \int_0^t x^r f_Y(x) dx \right) \\ &= \frac{1}{\lambda^{r+1}(1 - F_Y(t))} \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \frac{\Gamma(r+1) - \Gamma((k+1)\lambda t, r+1)}{(k+1)^{r+1}}. \end{aligned}$$

For $t > 0$, the reversed conditional r -th moment of Y is given by

$$E(Y^r | Y \leq t) = \frac{1}{F_Y(t)} \int_0^t x^r f_Y(x) dx = \frac{1}{F_Y(t)\lambda^{r+1}} \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \frac{\Gamma((k+1)\lambda t, r+1)}{(k+1)^{r+1}}.$$

Let $\mu = E(Y)$. The Lorenz curve $L(F_Y(t))$ is given by

$$L(F_Y(t)) = \frac{1}{\mu} \int_0^t x f_Y(x) dx = \frac{1}{\mu\lambda^{r+1}} \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \frac{\Gamma((k+1)\lambda t, 2)}{(k+1)^{r+1}}.$$

The Bonferroni curve $B(F_Y(t))$ is given by

$$B(F_Y(t)) = \frac{1}{\mu F_Y(t)} \int_0^t x f_Y(x) dx = \frac{1}{\mu F_Y(t)\lambda^{r+1}} \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \frac{\Gamma((k+1)\lambda t, 2)}{(k+1)^{r+1}}.$$

The mean deviation of Y about the mean μ can be expressed as

$$\begin{aligned} \delta_1(Y) &= E(|Y - \mu|) = 2\mu F_Y(\mu) - 2 \int_0^\mu x f_Y(x) dx \\ &= 2\mu F_Y(\mu) - \frac{2}{\lambda^{r+1}} \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \frac{\Gamma((k+1)\lambda \mu, 2)}{(k+1)^{r+1}}. \end{aligned}$$

The mean deviation of Y about the median M can be expressed as

$$\delta_2(Y) = E(|Y - M|) = \mu - 2 \int_0^M x f_Y(x) dx = \mu - \frac{2}{\lambda^{r+1}} \sum_{i=1}^m \sum_{k=0}^{+\infty} \eta_{i,k} \frac{\Gamma((k+1)\lambda M, 2)}{(k+1)^{r+1}}.$$

4. Order statistics

Order statistics are essential in many areas of statistics. They naturally appear in the probabilistic analysis of reliability of a system. We explicit here the distributions of these order statistics in the context of our new distribution. Let Y_1, Y_2, \dots, Y_n be n independent and identically distributed random variables following the EH(\mathbf{a}, λ) distribution. Let us consider its order statistics is $Y_{1:n}, Y_{2:n}, \dots, Y_{n:n}$. A pdf of the i -th order statistic $Y_{i:n}$ is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} [F_Y(x)]^{i-1} [1 - F_Y(x)]^{n-i} f_Y(x) \\ &= \frac{n!}{(i-1)!(n-i)!} \left[\sum_{i=1}^m \frac{1}{P_i} (1 - e^{-\lambda x})^{\alpha_i} \right]^{i-1} \left[1 - \sum_{i=1}^m \frac{1}{P_i} (1 - e^{-\lambda x})^{\alpha_i} \right]^{n-i} \\ &\times \lambda e^{-\lambda x} \sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda x})^{\alpha_i - 1}, \quad x > 0. \end{aligned}$$

In particular, a pdf of $Y_{1:n} = \inf(Y_1, Y_2, \dots, Y_n)$ is given by

$$f_{1:n}(x) = \lambda n e^{-\lambda x} \left[1 - \sum_{i=1}^m \frac{1}{P_i} (1 - e^{-\lambda x})^{\alpha_i} \right]^{n-1} \sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda x})^{\alpha_i - 1}$$

and a pdf of $Y_{n:n} = \sup(Y_1, Y_2, \dots, Y_n)$ is given by

$$f_{n:n}(x) = \lambda n e^{-\lambda x} \left[\sum_{i=1}^m \frac{1}{P_i} (1 - e^{-\lambda x})^{\alpha_i} \right]^{n-1} \sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda x})^{\alpha_i - 1}.$$

Consequently, the cdf of the i -th order statistic $Y_{i:n}$ is given by

$$\begin{aligned} F_{i:n}(x) &= \int_{-\infty}^x f_{i:n}(t) dt = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{i+k} [F_Y(x)]^{i+k} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{i+k} \left[\sum_{i=1}^m \frac{1}{P_i} (1 - e^{-\lambda x})^{\alpha_i} \right]^{i+k}, \quad x > 0. \end{aligned}$$

Further, for $j < k$, a joint pdf of $(Y_{j:n}, Y_{k:n})$ is given by

$$\begin{aligned} &f_{(j:n, k:n)}(x_j, x_k) \\ &= \frac{n!}{(j-1)!(n-k)!(k-j-1)!} [F_Y(x_j)]^{j-1} [F_Y(x_k) - F_Y(x_j)]^{k-j-1} [1 - F_Y(x_k)]^{n-k} f_Y(x_j) f_Y(x_k) \\ &= \frac{n!}{(j-1)!(n-k)!(k-j-1)!} \left[\sum_{i=1}^m \frac{1}{P_i} (1 - e^{-\lambda x_j})^{\alpha_i} \right]^{j-1} \left[\sum_{i=1}^m \frac{1}{P_i} [(1 - e^{-\lambda x_k})^{\alpha_i} - (1 - e^{-\lambda x_j})^{\alpha_i}] \right]^{k-j-1} \\ &\times \left[1 - \sum_{i=1}^m \frac{1}{P_i} (1 - e^{-\lambda x_k})^{\alpha_i} \right]^{n-k} \lambda^2 e^{-\lambda(x_j + x_k)} \sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda x_j})^{\alpha_i - 1} \sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda x_k})^{\alpha_i - 1}, \\ &0 < x_j < x_k. \end{aligned}$$

5. Maximum likelihood estimation

Let Y_1, Y_2, \dots, Y_n be a random sample from the EH(\mathbf{a}, λ) distribution with unknown parameters $\alpha_1, \dots, \alpha_m, \lambda$. We consider the maximum likelihood estimation providing the maximum likelihood estimators (MLEs) $\hat{\alpha}_1, \dots, \hat{\alpha}_m, \hat{\lambda}$ for $\alpha_1, \dots, \alpha_m, \lambda$. Let us recall that the MLEs have some statistical desirable properties (under regularity conditions) as the sufficiency, invariance, consistency, efficiency and asymptotic normality. Using the observed information matrix, asymptotic confidence interval for $\alpha_1, \dots, \alpha_m, \lambda$ can be constructed. Details can be found in Larsen and Marx (2000).

Let $\Theta = (\alpha_1, \dots, \alpha_m, \lambda)$ and y_1, y_2, \dots, y_n be the observed values. The likelihood function is given by

$$L(\Theta) = \lambda^n e^{-\lambda \sum_{u=1}^n y_u} \prod_{u=1}^n \left(\sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha_i - 1} \right).$$

The log-likelihood function is given by

$$\ell(\Theta) = \log(L(\Theta)) = n \log(\lambda) - \lambda \sum_{u=1}^n y_u + \sum_{u=1}^n \log \left(\sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha_i - 1} \right).$$

The nonlinear log-likelihood equations given by $\frac{\partial \ell(\Theta)}{\partial \Theta} = 0$ are listed below

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{u=1}^n y_u + \sum_{u=1}^n y_u e^{-\lambda y_u} \frac{\sum_{i=1}^m \frac{\alpha_i(\alpha_i - 1)}{P_i} (1 - e^{-\lambda y_u})^{\alpha_i - 2}}{\sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha_i - 1}} = 0 \quad (5)$$

and, for any $q \in \{1, \dots, m\}$,

$$\frac{\partial \ell(\Theta)}{\partial \alpha_q} = \sum_{u=1}^n \frac{\frac{\partial}{\partial \alpha_q} \left(\sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha_i - 1} \right)}{\sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha_i - 1}} = 0. \quad (6)$$

Let us now investigate the numerator. We have

$$\sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha_i - 1} = \frac{\alpha_q}{P_q} (1 - e^{-\lambda y_u})^{\alpha_q - 1} + \sum_{\substack{i=1 \\ i \neq q}}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha_i - 1}.$$

Hence

$$\begin{aligned} & \frac{\partial}{\partial \alpha_q} \left(\sum_{i=1}^m \frac{\alpha_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha_i - 1} \right) \\ &= \frac{\partial}{\partial \alpha_q} \left(\frac{\alpha_q}{P_q} \right) (1 - e^{-\lambda y_u})^{\alpha_q - 1} + \frac{\alpha_q}{P_q} (1 - e^{-\lambda y_u})^{\alpha_q - 1} \log(1 - e^{-\lambda y_u}) \\ &+ \sum_{\substack{i=1 \\ i \neq q}}^m \alpha_i (1 - e^{-\lambda y_u})^{\alpha_i - 1} \frac{\partial}{\partial \alpha_q} \left(\frac{1}{P_i} \right). \end{aligned}$$

Observe that

$$\frac{\partial}{\partial \alpha_q} \left(\frac{\alpha_q}{P_q} \right) = \frac{\partial}{\partial \alpha_q} \left(\alpha_q \prod_{\substack{j=1 \\ j \neq q}}^m \frac{\alpha_j}{\alpha_j - \alpha_q} \right) = \prod_{\substack{j=1 \\ j \neq q}}^m \frac{\alpha_j}{\alpha_j - \alpha_q} + \alpha_q \left(\prod_{\substack{j=1 \\ j \neq q}}^m \frac{\alpha_j}{\alpha_j - \alpha_q} \right) \left(\sum_{\substack{v=1 \\ v \neq q}}^m \frac{1}{\alpha_v - \alpha_q} \right).$$

On the other hand, for $i \neq q$, we have

$$\begin{aligned} \frac{\partial}{\partial \alpha_q} \left(\frac{1}{P_i} \right) &= \frac{\partial}{\partial \alpha_q} \left(\prod_{\substack{j=1 \\ j \neq i}}^m \frac{\alpha_j}{\alpha_j - \alpha_i} \right) = \left(\prod_{\substack{j=1 \\ j \neq \{i, q\}}}^m \frac{\alpha_j}{\alpha_j - \alpha_i} \right) \frac{\partial}{\partial \alpha_q} \left(\frac{\alpha_q}{\alpha_q - \alpha_i} \right) \\ &= - \left(\prod_{\substack{j=1 \\ j \neq \{i, q\}}}^m \frac{\alpha_j}{\alpha_j - \alpha_i} \right) \frac{\alpha_i}{(\alpha_q - \alpha_i)^2}. \end{aligned}$$

Putting these equalities together, we obtain an unified equation for (6). The MLEs are solutions of (5) and (6). These equations are not solvable analytically, but some numerical iterative methods, as Newton-Raphson method, can be used. The solutions can be approximate numerically by using software such as MATHEMATICA, MAPLE and R. Here we work with MATHEMATICA.

Study of a particular case. Let us now consider a simple two parameters EH(\mathbf{a}, λ) distribution where $\mathbf{a} = (\alpha_1, \dots, \alpha_m)$, for any $i \in \{1, \dots, m\}$, $\alpha_i = \alpha + \epsilon_i$, with $\epsilon_1, \dots, \epsilon_m$ denote different fixed positive real numbers. Then $\alpha > 0$ and $\lambda > 0$ can be estimated via the maximum likelihood method. It is enough to set $\Theta = (\alpha, \lambda)$, using (5) with $\alpha_i = \alpha + \epsilon_i$ and consider the new equation:

$$\frac{\partial \ell(\Theta)}{\partial \alpha} = \sum_{u=1}^n \frac{\frac{\partial}{\partial \alpha} \left(\sum_{i=1}^m \frac{\alpha + \epsilon_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha + \epsilon_i - 1} \right)}{\sum_{i=1}^m \frac{\alpha + \epsilon_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha + \epsilon_i - 1}} = 0. \quad (7)$$

We can explicit the numerator by observing that

$$\begin{aligned} \frac{\partial}{\partial \alpha} \left(\sum_{i=1}^m \frac{\alpha + \epsilon_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha + \epsilon_i - 1} \right) &= \sum_{i=1}^m \frac{\partial}{\partial \alpha} \left(\frac{\alpha + \epsilon_i}{P_i} \right) (1 - e^{-\lambda y_u})^{\alpha + \epsilon_i - 1} \\ &+ \sum_{i=1}^m \frac{\alpha + \epsilon_i}{P_i} (1 - e^{-\lambda y_u})^{\alpha + \epsilon_i - 1} \ln(1 - e^{-\lambda y_u}), \end{aligned}$$

with

$$\frac{\partial}{\partial \alpha} \left(\frac{\alpha + \epsilon_i}{P_i} \right) = \left(\prod_{\substack{j=1 \\ j \neq i}}^m \frac{1}{\epsilon_j - \epsilon_i} \right) \frac{\partial}{\partial \alpha} \left(\prod_{j=1}^m (\alpha + \epsilon_j) \right) = \left(\prod_{\substack{j=1 \\ j \neq i}}^m \frac{1}{\epsilon_j - \epsilon_i} \right) \left(\prod_{j=1}^m (\alpha + \epsilon_j) \right) \left(\sum_{j=1}^m \frac{1}{\alpha + \epsilon_j} \right).$$

Again, these equations can be solve numerically. Thanks to its simplicity, this particular EH(\mathbf{a}, λ) distribution will be considered in the applications presented in the next section.

6. Illustrative real data examples

In this section, we analysis real data sets to show that the EH(\mathbf{a}, λ) distribution can be a better model than other existing distributions. We consider the following well-known weighted exponential distributions for comparison purpose:

Lindley distribution: A pdf associated to the Lindley distribution of parameter $\theta > 0$ is given by

$$f(x) = \frac{\theta^2}{\theta + 1} (1 + x) e^{-\theta x}, \quad x > 0.$$

Theory and applications related to this distribution can be found in Ghitany *et al.* (2008).

Exponential distribution: A pdf associated to the exponential distribution of parameter $\lambda > 0$ is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

Exponentiated exponential distribution (E Exponential): A pdf associated to the Exponentiated exponential distribution of parameters $\alpha, \lambda > 0$ is given by

$$f(x) = \lambda \alpha e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha - 1}, \quad x > 0.$$

Details about this distribution can be found in Gupta and Kundu (1999).

Exponential Hypoexponential distribution (Exp Hypo): To simplify the situation, we consider a particular simple two parameters EH(\mathbf{a}, λ) distribution with $m = 2$, $\mathbf{a} = (\alpha_1, \alpha_2)$, $\alpha_1 = \alpha$ and $\alpha_2 = \alpha + 0.1$, $\alpha > 0$ and $\lambda > 0$ are the parameters to be estimated; the value 0.1 is subjective. The corresponding pdf is given by

$$f(x) = \lambda \alpha (1 + 10\alpha) e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha - 1} [1 - (1 - e^{-\lambda x})^{0.1}], \quad x > 0.$$

Remark 9. Another two parameters EH(\mathbf{a}, λ) distribution can be used, as the one defined with $m = 3$, say $\mathbf{a} = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_1 = \alpha$, $\alpha_2 = \alpha + \epsilon$ and $\alpha_3 = \alpha + v$, for fixed ϵ and v with $\epsilon \neq v$, the parameters to be estimated are $\alpha > 0$ and $\lambda > 0$.

The three considered real data sets are described as follows:

Data set 1: This data set contains $n = 31$ measures of the strength data of glass of the aircraft window. It is extracted from Fuller *et al.* (1994):

18.83, 20.8, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.69, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381

	Model	Estimate of parameters	$-2\log(L)$	AIC	AICC	BIC
Data set 1	Lindley	$\hat{\theta} = 0.062988$	253.99	255.99	256.13	257.42
	Exponential	$\hat{\lambda} = 0.032455$	274.53	276.53	276.67	277.96
	E Exponential	$(\hat{\alpha}, \hat{\lambda}) = (93.7873, 0.166016)$	208.26	212.26	212.69	215.13
	Exp Hypo	$(\hat{\alpha}, \hat{\lambda}) = (41.288, 0.107838)$	208.09	212.09	212.52	214.96
Data set 2	Lindley	$\hat{\theta} = 0.996116$	162.56	164.56	164.62	166.70
	Exponential	$\hat{\lambda} = 0.663647$	177.66	179.66	179.73	181.80
	E Exponential	$(\hat{\alpha}, \hat{\lambda}) = (31.3489, 2.61157)$	62.76	66.76	66.96	71.05
	Exp Hypo	$(\hat{\alpha}, \hat{\lambda}) = (24.0816, 1.83894)$	55.67	59.67	59.87	63.96
Data set 3	Lindley	$\hat{\theta} = 0.196$	839.05	841.06	841.09	843.89
	Exponential	$\hat{\lambda} = 0.106773$	828.68	830.68	830.71	833.53
	E Exponential	$(\hat{\alpha}, \hat{\lambda}) = (1.21795, 0.121167)$	826.15	830.15	830.25	835.85
	Exp Hypo	$(\hat{\alpha}, \hat{\lambda}) = (1.44399, 0.057924)$	825.49	829.49	829.59	835.20

Table 1: $-2\log(L)$, AIC, AICC and BIC of the fitted distributions of Data sets 1, 2 and 3.

Data set 2: The data set contains $n = 63$ measures related to the strength of 1.5cm glass fibers. It is reported in Smith and Naylor (1987):

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89

Data set 3: The data set contains $n = 128$ measures on the remission times in months of bladder cancer patients. It is extracted from Lee and Wang (2003):

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69, 5.49

For each data set, we compare the fitted distributions using the four criteria: $-2\log(L)$, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected) and BIC (Bayesian Information Criterion). Let us precise that $\log(L)$ is the log-likelihood taking with the estimate values, $AIC = -2\log(L) + 2k$, $AICC = AIC + \frac{2k(k+1)}{n-k-1}$ and $BIC = -2\log(L) + k\log(n)$, where k denotes the number of estimated parameters and n denotes the sample size. The best fitted distribution corresponds to lower $-2\log(L)$, AIC, AICC and BIC.

We see in Table 1 that the $EH(\mathbf{a}, \lambda)$ distribution has the smallest $-2\log(L)$, AIC, AICC and BIC for the three data sets, indicating that it is a serious competitor to the other considered distributions.

7. Conclusion

A new weighted exponential distribution based on the hypoexponential distribution is introduced. Some of its structural properties are studied. The analysis of three real data sets shows that the fit of the model related to our new distribution can be superior to other models.

As future work, we plan to study the $GH(H; \mathbf{a})$ distribution with another cdf $H(x)$ and also to provide more applications for the $EH(\mathbf{a}, \lambda)$ distribution with $m \geq 3$. Inverted or transmuted transformations of $EH(\mathbf{a}, \lambda)$ can be of interest too for some applications.

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