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Optimal stopping and a non-zero-sum Dynkin game in discrete time with risk measures induced by BSDEs

Miryana Grigorova* Marie-Claire Quenez†

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Abstract

We first study an optimal stopping problem in which a player (an agent) uses a discrete stopping time in order to stop optimally a payoff process whose risk is evaluated by a (non-linear) g-expectation. We then consider a non-zero-sum game on discrete stopping times with two agents who aim at minimizing their respective risks. The payoffs of the agents are assessed by g-expectations (with possibly different drivers for the different players). By using the results of the first part, combined with some ideas of S. Hamadène and J. Zhang, we construct a Nash equilibrium point of this game by a recursive procedure. Our results are obtained in the case of a standard Lipschitz driver g without any additional assumption on the driver besides that ensuring the monotonicity of the corresponding g-expectation.

Keywords: optimal stopping, non-zero-sum Dynkin game, g-expectation, dynamic risk measure, game option, Nash equilibrium

1 Introduction

Initiated by Bismut Bismut (1976), Bismut (1973) (in the linear case), the theory of backward stochastic differential equations (BSDEs for short) has been further developed by Pardoux and Peng Pardoux and Peng (1990) in their seminal paper. The theory of BSDEs

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has found a number of applications in finance, among which pricing and hedging of European options, recursive utilities, risk measurement. BSDEs induce a family of operators, the so-called g-conditional expectations, which have proved useful in the literature on (non-linear) dynamic risk measures (cf., e.g., Peng (2004), Rosazza-Gianin (2006)). We recall that the g-conditional expectation at time $t \in [0, T]$ (where $T > 0$ is a fixed time horizon, and g is a Lipschitz driver) is the operator which maps a given terminal condition ξ (where ξ is a square-integrable random variable, measurable with respect to the information at time T) to the position at time t of (the first component of) the solution of the BSDE with parameters (g, ξ). This operator is denoted by $\mathcal{E}^g_{t,T}(\cdot)$. The operator $\mathcal{E}^g_{0,T}(\cdot)$ is called g-expectation.
On the other hand, zero-sum Dynkin games have been introduced by Dynkin in Dynkin (1969) in the discrete-time framework. Since then, there have been lots of contributions to zero-sum Dynkin games both in discrete time and in continuous time (cf., e.g., Bismut (1977), Neveu (1972), Alario-Nazaret et al. (1982), Lepeltier and Maingueneau (1984)). A prominent financial example is given by the pricing problem of game options (also known as Israeli options), introduced by Kifer in Kifer (2000). Compared to the zero-sum case, there have been fewer works on the non-zero-sum case. We can quote Hamadène and Zhang (2010), Laraki and Solan (2013), Hamadène and Hassani (2014) in the continuous-time setting, and Morimoto (1986), Ohtsubo (1987), Shmaya and Solan (2004), Hamadène and Hassani (2014) in the discrete-time setting. For a recent survey on zero-sum and non-zero-sum Dynkin games the reader is referred to Kifer (2013).
In all the above references the players' payoffs are assessed by "classical" mathematical expectations. In the recent years, some authors (cf. Dumitrescu et al. (2013), and Bayraktar and Yao (2015)) have considered "generalized" Dynkin games in continuous time where the "classical" expectations are replaced by more general (non-linear) functionals. All these extensions are limited to the zero-sum case.

In the present paper, we address a game problem with two "stoppers" whose profits (or payoffs) are assessed by non-linear dynamic risk measures and who aim at minimizing their risk. More concretely, the following situation is of interest to us: we are given two adapted processes X and Y with $X \leq Y$ and $X_T = Y_T$ a.s. We consider a game option in discrete time, that is, a contract between two "stoppers" (a seller and a buyer) who can act only at given times $0 = t_0 < t_1 < \ldots < t_n = T$, where $n \in \mathbb{N}$. The two agents can thus choose their strategies only among the discrete stopping times with values in the grid $\{t_0, t_1, \ldots, t_n\}$. We denote by $\mathcal{T}^d_{0,T}$ this set of stopping times. Recall that the game option gives the buyer the right *to exercise* at any (discrete) stopping time $\tau_1 \in \mathcal{T}^d_{0,T}$ and the seller the right *to cancel* at any (discrete) stopping time $\tau_2 \in \mathcal{T}^d_{0,T}$. In financial terms, we could say that both the seller's cancellation strategy and the buyer's exercise strategy are of *Bermudan type*. If the buyer exercises at time τ_1 before the seller cancels, then the seller pays to the buyer the amount X_{τ_1}; otherwise, the buyer receives from the seller the amount Y_{τ_2} at the

cancellation time τ_2. The difference $Y_{\tau_2} - X_{\tau_2} \geq 0$ is interpreted as a penalty which the seller pays to the buyer in the case of an early cancellation of the contract. To summarize, if the seller chooses a cancellation time τ_2 and the buyer chooses an exercise time τ_1, the former pays to the latter the payoff

$$I(\tau_1, \tau_2) := X_{\tau_1} \mathbb{I}_{\{\tau_1 \leq \tau_2\}} + Y_{\tau_2} \mathbb{I}_{\{\tau_2 < \tau_1\}}$$

at time $\tau_1 \wedge \tau_2$. The seller's payoff at time $\tau_1 \wedge \tau_2$ is equal to $-I(\tau_1, \tau_2)$.

We emphasize that our aim here is not to determine a "fair price" (or a "fair premium") for the game option, but rather to determine an "equilibrium" for the game problem related to the risk minimization of both the seller and the buyer.

The seller and the buyer are assumed to evaluate the risk of their payoffs in a (possibly) different manner.

The dynamic risk measure ρ^{f_1} (resp. ρ^{f_2}) of the buyer (resp. the seller) is induced by a BSDE with driver f_1 (resp. f_2). Up to a minus sign, ρ^{f_1} (resp. ρ^{f_2}) corresponds to the family of f_1-conditional expectations (resp. f_2-conditional expectations).

If, at time 0, the buyer chooses τ_1 as exercise time and the seller chooses τ_2 as cancellation time, the buyer's (resp. seller's) risk at time 0 is thus given by

$$\begin{aligned} \rho^{f_1}_{0,\tau_1\wedge\tau_2}(I(\tau_1,\tau_2)) &= -\mathcal{E}^{f_1}_{0,\tau_1\wedge\tau_2}(I(\tau_1,\tau_2)) \\ \text{resp.} \quad \rho^{f_2}_{0,\tau_1\wedge\tau_2}(-I(\tau_1,\tau_2)) &= -\mathcal{E}^{f_2}_{0,\tau_1\wedge\tau_2}(-I(\tau_1,\tau_2)). \end{aligned}$$

The goal of each of the agents is to minimize his/her risk. We are interested in finding an "equilibrium" pair of discrete stopping times (τ_1^*, τ_2^*) for this problem, that is, a pair $(\tau_1^*, \tau_2^*) \in \mathcal{T}^d_{0,T} \times \mathcal{T}^d_{0,T}$ such that the first agent's risk attains its minimum at τ_1^* when the strategy of the second one is fixed at τ_2^*, and the second agent's risk attains its minimum at τ_2^* when the strategy of the first one is fixed at τ_1^*. In other words, we are looking for a pair $(\tau_1^*, \tau_2^*) \in \mathcal{T}^d_{0,T} \times \mathcal{T}^d_{0,T}$ satisfying

$$\begin{aligned} \max_{\tau_1 \in \mathcal{T}^d_{0,T}} \mathcal{E}^{f_1}_{0,\tau_1\wedge\tau_2^*}(I(\tau_1,\tau_2^*)) &= \mathcal{E}^{f_1}_{0,\tau_1^*\wedge\tau_2^*}(I(\tau_1^*,\tau_2^*)) \\ \max_{\tau_2 \in \mathcal{T}^d_{0,T}} \mathcal{E}^{f_2}_{0,\tau_1^*\wedge\tau_2}(-I(\tau_1^*,\tau_2)) &= \mathcal{E}^{f_2}_{0,\tau_1^*\wedge\tau_2^*}(-I(\tau_1^*,\tau_2^*)). \end{aligned}$$

In the terminology of game theory, the above game problem is of a *non-zero-sum* type, and a pair (τ_1^*, τ_2^*) satisfying the above properties corresponds to a *Nash equilibrium point* of this non-zero-sum game. This game problem can be seen as a "generalized" non-zero-sum Dynkin game problem (the term "generalized" refers to the fact that our problem involves non-linear expectations instead of classical expectations). Note that in the trivial case where $f_1 = f_2 = 0$, our game reduces to a classical zero-sum Dynkin game (with classical expectations). Let us also mention that we can easily incorporate in our framework the situation where the seller and/or the buyer apply their respective risk measures to their

net gains (that is the payoff minus the initial price of the game option) instead of to their *payoffs*. If the initial price of the option is given by x (where $x > 0$), the buyer's (resp. seller's) net gain at time $\tau_1 \wedge \tau_2$ is given by $I(\tau_1, \tau_2) - x$ (resp. $x - I(\tau_1, \tau_2)$).

We show that there exists a Nash equilibrium point for the non-zero-sum Dynkin game problem described above by using a constructive approach similar to that of Hamadène and Zhang (2010), Hamadène and Hassani (2014), and Hamadène and Hassani (2014).

This approach requires some results on optimal stopping with one agent. We are thus led to considering first the following family of problems:

$$V(t_k) := ess \inf_{\tau \in \mathcal{T}^d_{t_k,T}} \rho^g_{t_k,\tau}(\xi_\tau) = -ess \sup_{\tau \in \mathcal{T}^d_{t_k,T}} \mathcal{E}^g_{t_k,\tau}(\xi_\tau), \text{ for all } k \in 0, 1, \ldots, n, \tag{1}$$

where $\mathcal{T}^d_{t_k,T}$ denotes the set of discrete stopping times valued in $\{t_k, \ldots, t_n\}$ and where ξ is a given square-integrable adapted process. We characterize the sequence of random variables $(V(t_k))_{k\in\mathbb{N}}$ via a backward recursive construction. We also show that the stopping time $\tau^* := \tau^*(t_k) := \inf\{t \in \{t_k, \ldots, t_n\}, V(t) = \xi_t\}$, which belongs to $\mathcal{T}^d_{t_k,T}$, is optimal for (1) at time t_k. To prove our results, we use a generalization of the martingale approach to the case of g-conditional expectations in discrete time. Our results are established without any additional assumption on the driver g besides that ensuring the monotonicity of the corresponding g-conditional expectations. In particular, we do not make an assumption of "groundedness" on g (that is, the assumption $g(t, 0, 0, 0) = 0$), nor do we make an assumption of concavity/convexity on g, nor an assumption of "independence from y" (that is, $g(t, y, z, k) = g(t, y', z, k)$, for all $y, y' \in \mathbb{R}$), which are sometimes made in the literature.
Optimal stopping problems with one agent whose payoff is assessed by a non-linear expectation have been largely studied. A continuous-time version of Problem (1) has been considered in El Karoui and Quenez (1997), Quenez and Sulem (2014) and Grigorova et al. (2015). Related works include, but are not limited to, Bayraktar et al. (2010), Bayraktar and Yao (2011). The discrete-time version of Problem (1) has been introduced by Krätschmer and Schoenmakers in Krätschmer and Schoenmakers (2010), Example 2.7, who address the problem under stronger assumptions on the driver g than those made in the present paper. In particular, the authors of Krätschmer and Schoenmakers (2010) need the zero-one law for g-expectation, and the property $\mathcal{E}^g_{t,T}(\xi) = \xi$, for all t, for all ξ square-integrable $\mathcal{F}_t$-measurable. These two properties do not hold for a general Lipschitz driver g. For this reason, the results from Krätschmer and Schoenmakers (2010) are not applicable in our framework; thus, we have been led to studying Problem (1) by using different techniques.

The remainder of the paper is organized as follows: In Section 2, we set the framework and the notation. Section 3 is dedicated to the optimal stopping problem with one player. In Subsection 3.1, we define the notion of g-(super)martingales in discrete time and we give some of their properties; in Subsection 3.2, we characterize the value function of our optimal stopping problem and we show the existence of optimal stopping times. In Section 4, we formulate our non-zero-sum Dynkin game with two players and we show the existence

of a Nash equilibrium. In Section 5, we briefly comment upon possible extensions of our results. The Appendix contains a useful property of g-expectations (Prop. A.1), along with some related remarks, as well as the proof of an easy result from Section 3.

2 The framework

Let T be a fixed positive real number. Let $(E, \mathcal{K})$ be a measurable space equipped with a σ-finite positive measure ν. Let $(\Omega, \mathcal{F}, P)$ be a (complete) probability space equipped with a one-dimensional Brownian motion W and with an independent Poisson random measure $N(dt, de)$ with compensator $dt \otimes \nu(de)$. We denote by $\tilde{N}(dt, de)$ the compensated process, i.e. $\tilde{N}(dt, de) := \tilde{N}(dt, de) - dt \otimes \nu(de)$. Let $\mathbb{F} = \{\mathcal{F}_t \colon t \in [0, T]\}$ be the (complete) natural filtration associated with W and N.
We use the following notation:

- $L^2(\mathcal{F}_T)$ is the set of random variables which are $\mathcal{F}_T$-measurable and square-integrable.
- L^2_ν is the set of $\mathcal{K}$-measurable functions $\ell : E \to \mathbb{R}$ such that $\|\ell\|^2_\nu := \int_E |\ell(e)|^2 \nu(de) < \infty$. For $\ell \in L^2_\nu$, $k \in L^2_\nu$, we define $\langle \ell, k \rangle_\nu := \int_E \ell(e) k(e) \nu(de)$.
- $H^{2,T}$ is the set of real-valued predictable processes ϕ such that $\|\phi\|^2_{H^{2,T}} := E\left[(\int_0^T |\phi_t|^2 dt)\right] < \infty$.
- $H^{2,T}_\nu$ is the set of real-valued processes $l : (\omega, t, e) \in \Omega \times [0, T] \times E \mapsto l_t(\omega, e) \in \mathbb{R}$ which are *predictable*, that is $(\mathcal{P} \otimes \mathcal{K})$-measurable, and such that
$$\|l\|^2_{H^{2,T}_\nu} := E\left[\int_0^T \|l_t\|^2_\nu \, dt\right] < \infty.$$
- $\mathcal{S}^{2,T}$ is the set of real-valued RCLL adapted processes φ such that $\varphi^2_{\mathcal{S}^{2,T}} := E(\sup_{t \in [0,T]} |\varphi_t|^2) < \infty$.

We recall the following terminology from BSDE theory.

Definition 2.1 *(Lipschitz driver, standard data) A function g is said to be a* driver *if the following two conditions hold:*

- *(measurability) $g : \Omega \times [0, T] \times \mathbb{R}^2 \times L^2_\nu \to \mathbb{R}$ $(\omega, t, y, z; \ell) \mapsto g(\omega, t, y, z, \ell)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(L^2_\nu)-$ measurable, where $\mathcal{P}$ is the predictable σ-algebra on $\Omega \times [0, T]$, $\mathcal{B}(\mathbb{R}^2)$ is the Borel σ-algebra on $\mathbb{R}^2$, and $\mathcal{B}(L^2_\nu)$ is the Borel σ-algebra on L^2_ν.*
- *(integrability) $E\left[(\int_0^T |g(t, 0, 0, 0)|^2 dt)\right] < \infty$.*

A driver g is called a Lipschitz driver *(or a* standard Lipschitz driver*) if moreover there exists a constant $K \geq 0$ such that $dP \otimes dt$-a.e. , for each $(y_1, z_1, \ell_1) \in \mathbb{R}^2 \times L^2_\nu$, $(y_2, z_2, \ell_2) \in \mathbb{R}^2 \times L^2_\nu$,*

$$|g(\omega, t, y_1, z_1, \ell_1) - g(\omega, t, y_2, z_2, \ell_2)| \leq K(|y_1 - y_2| + |z_1 - z_2| + \|\ell_1 - \ell_2\|_\nu).$$

A pair (g, ξ) such that g is a Lipschitz driver and $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ is called a pair of standard data*, or a* pair of standard parameters*.*

Let (ξ, g) be a pair of standard data. The BSDE associated with Lipschitz driver g, terminal time T, and terminal condition ξ, is formulated as follows:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s, k_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E k_s(e)\tilde{N}(ds, de), \text{ for } t \in [0, T]. \quad (2)$$

We recall that the above BSDE admits a unique solution triplet (Y, Z, k) in the space $\mathcal{S}^{2,T} \times H^{2,T} \times H^{2,T}_\nu$. We denote by $\mathcal{E}^g_{\cdot,T}(\xi)$ the first component of the solution of that BSDE (i.e. $(\mathcal{E}^g_{t,T}(\xi))_{t\in[0,T]}$ is the family of g-conditional evaluations of ξ in the vocabulary of S. Peng).
Recall also (cf., e.g., El Karoui et al. El Karoui et al. (1997)) that if the terminal time is given by a stopping time $\tau \in \mathcal{T}_{0,T}$ and if ξ is $\mathcal{F}_\tau$-measurable, the solution of the BSDE associated with terminal time τ, terminal condition ξ and Lipschitz driver g is defined as the solution of the BSDE with (fixed) terminal time T, terminal condition ξ and Lipschitz driver g^τ given by

$$g^\tau(t, y, z, \ell) := g(t, y, z, \ell)\mathbb{I}_{\{t\leq\tau\}}.$$

The first component of this solution is thus equal to $(\mathcal{E}^{g^\tau}_{t,T}(\xi))_{t\in[0,T]}$. In the sequel it is also denoted by $(\mathcal{E}^g_{t,\tau}(\xi))_{t\in[0,T]}$. We have $\mathcal{E}^g_{t,\tau}(\xi) = \xi$ a.s. on the set $\{t \geq \tau\}$.

Recall that T is interpreted as the final time horizon. For each $T' \in [0, T]$ and $\eta \in L^2(\mathcal{F}_{T'})$, we set

$$\rho^g_{t,T'}(\eta) := -\mathcal{E}^g_{t,T'}(\eta), \quad 0 \leq t \leq T', \quad (3)$$

If T' represents a given maturity and η a financial position at time T', then $\rho^g_{t,T'}(\eta)$ is interpreted as the risk of η at time t. The functional $\rho^g : (\eta, T') \mapsto \rho^g_{\cdot,T'}(\eta)$ thus represents a *dynamic risk measure* induced by the BSDE with driver g. In order to ensure the *monotonicity* property of ρ^g, that is, the monotonicity property with respect to the financial position, which is naturally required for risk measures, we assume from now on that the driver g satisfies the following assumption (cf. (Quenez and Sulem, 2013, Thm. 4.2, combined with Prop. 3.2) and the references therein).

Assumption 2.1 *Assume that $dP \otimes dt$-a.e. for each $(y, z, \ell_1, \ell_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,*

$$g(t, y, z, \ell_1) - g(t, y, z, \ell_2) \geq \langle \theta^{y,z,\ell_1,\ell_2}_t \, , \, \ell_1 - \ell_2 \rangle_\nu,$$

where the mapping

$$\theta : [0,T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \to L^2_\nu \,;\, (\omega, t, y, z, \ell_1, \ell_2) \mapsto \theta_t^{y,z,\ell_1,\ell_2}(\omega, \cdot)$$

is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$*-measurable, and satisfies* $dP \otimes dt \otimes d\nu(e)$*-a.e. , for each* $(y, z, \ell_1, \ell_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$*,*

$$\theta_t^{y,z,\ell_1,\ell_2}(e) \geq -1. \tag{4}$$

Assume, moreover, that θ *is uniformly bounded, in the sense that,* $dP \otimes dt$*-a.e. , for each* (y, z, ℓ_1, ℓ_2)*,* $\|\theta_t^{y,z,\ell_1,\ell_2}\|_\nu \leq K$*, where* K *is a positive constant.*

3 Optimal stopping with g-expectations in discrete time

Let $(\xi_t)_{t\in[0,T]}$ be a given $\mathbb{F}$-adapted square-integrable process modelling an agent's dynamic financial position. The agent is allowed to "stop" only at given times $0 = t_0 < t_1 < \ldots < t_n = T$, where $n \in \mathbb{N}$. The agent's risk is assessed through a dynamic risk measure ρ^g induced by a BSDE with a given Lipschitz driver g; the dynamic risk measure ρ^g corresponds (up to a minus sign) to the family of g-conditional expectations . The agent's aim at time 0 is to choose his/her strategy in such a way that the risk of his/her position from "time 0-perspective" be minimal. The minimal risk at time 0 is defined by

$$V(0) := \inf_{\tau\in\mathcal{T}^d_{0,T}} \rho^g_{0,\tau}(\xi_\tau) = -\sup_{\tau\in\mathcal{T}^d_{0,T}} \mathcal{E}^g_{0,\tau}(\xi_\tau). \tag{5}$$

We have $V(0) = -V_0$, where

$$V_0 := \sup_{\tau\in\mathcal{T}^d_{0,T}} \mathcal{E}^g_{0,\tau}(\xi_\tau). \tag{6}$$

We are thus facing an optimal stopping problem in discrete time with g-expectation. Our purpose in this section is to compute or characterize the minimal risk measure $V(0)$ (or equivalently, V_0) and to study the question of the existence of an optimal stopping time.
In order to simplify the notation, we suppose from now on that the terminal time T is in $\mathbb{N}$ and that $t_k = k$, for all $k = 1, \ldots, n$. In this case, the set $\mathcal{T}^d_{t_k,T}$ corresponds to the set $\mathcal{T}^d_{k,T}$ of stopping times whose values are almost surely in the set $\{k, k+1, \ldots, T\}$. We will use the notation $\mathbb{F}^d$ for the filtration $(\mathcal{F}_k)_{k\in\{0,1,\ldots,T\}}$.

3.1 Discrete-time g-(super)martingales

We introduce the notion of discrete-time g-(super)martingales, which is to be compared with the definition of a $\mathcal{E}^g$-supermartingale (in continuous time), respectively $\mathcal{E}^g$-martingale (in continuous time).

Definition 3.1 *Let* $(\phi_k)_{k\in\{0,1,\ldots,T\}}$ *be a sequence of square-integrable random variables, adapted to* $(\mathcal{F}_k)_{k\in\{0,1,\ldots,T\}}$*. We say that the sequence* (ϕ_k) *is a* g-supermartingale *(resp.* g-martingale*)* in discrete time *if* $\phi_k \geq \mathcal{E}^g_{k,k+1}(\phi_{k+1})$*, for all* $k \in \{0, 1, \ldots, T-1\}$*.*

Remark 3.1 *We note that if $(\phi_t)_{t\in[0,T]}$ is a $\mathcal{E}^g$-martingale (in continuous time) with respect to the filtration $\mathbb{F}$, then $(\phi_k)_{k\in\{0,1,\dots,T\}}$ is a g-martingale in discrete time in the sense of Definition 3.1.*

Remark 3.2 *In the case where $g \equiv 0$ (corresponding to the classical expectation), if $(\phi_k)_{k\in\{0,1,\dots,T\}}$ is a* discrete-time *martingale with respect to $(\mathcal{F}_k)_{k\in\{0,1,\dots,T\}}$, then $(\phi_k)_{k\in\{0,1,\dots,T\}}$ can be extended into a* continuous-time *martingale with respect to $\mathbb{F}$ (with time parameter t in $[0,T]$) by setting $\phi_t := \phi_k$, for all $t \in (k, k+1)$, for all $k \in \{0,\dots,T-1\}$. This statement does not necessarily hold true in the case of a general driver g.*

Remark 3.3 *Let (ϕ_t) be a square-integrable adapted process. We recall that by definition $\mathcal{E}^g_{t,s}(\phi_s) = \phi_s$, for all $T \geq t \geq s \geq 0$.*
If ϕ_t is $\mathcal{F}_t$-measurable, then $\mathcal{E}^{g^t}_{t,T}(\phi_t) = \mathcal{E}^g_{t,t}(\phi_t) = \phi_t$.

Theorem 3.1 *Let $(\phi_k)_{k\in\{0,1,\dots,T\}}$ be a g-supermartingale (resp. a g-martingale) in discrete time. Let $\tau \in \mathcal{T}^d_{0,T}$. Then, the stopped process $(\phi_{k\wedge\tau})_{k\in\{0,1,\dots,T\}}$ is a g^τ-supermartingale (resp. a g^τ-martingale) in discrete time.*

Proof: Let $k \in \{0,1,\dots,T-1\}$. Since $\mathcal{E}^{g^\tau}_{k,k+1}(\phi_{(k+1)\wedge\tau}) = \mathcal{E}^g_{k,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau})$, it is sufficient to prove the following:

$$\mathcal{E}^g_{k,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}) \leq \phi_{k\wedge\tau}. \tag{7}$$

We write

$$\mathcal{E}^g_{k,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}) = \mathbb{I}_{\{\tau\leq k\}}\mathcal{E}^g_{k,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}) + \mathbb{I}_{\{\tau\geq k+1\}}\mathcal{E}^g_{k,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}). \tag{8}$$

Due to the definition of the solution of a standard BSDE with a stopping time as a terminal time, we have

$$\mathbb{I}_{\{\tau\leq k\}}\mathcal{E}^g_{k,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}) = \mathbb{I}_{\{\tau\leq k\}}\phi_{(k+1)\wedge\tau} = \mathbb{I}_{\{\tau\leq k\}}\phi_\tau. \tag{9}$$

For the second term on the right-hand side of equation (8) we have

$$\mathbb{I}_{\{\tau\geq k+1\}}\mathcal{E}^g_{k,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}) \leq \mathbb{I}_{\{\tau\geq k+1\}}\phi_{k\wedge\tau}. \tag{10}$$

Indeed, after noticing that $\mathbb{I}_{\{\tau\geq k+1\}}$ is $\mathcal{F}_k$-measurable, we apply Proposition A.1 to obtain

$$\begin{aligned}\mathbb{I}_{\{\tau\geq k+1\}}\mathcal{E}^g_{k,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}) &= \mathbb{I}_{\{\tau\geq k+1\}}\mathcal{E}^{g^{(k+1)\wedge\tau}}_{k,T}(\phi_{(k+1)\wedge\tau}) \\ &= \mathcal{E}^{g^{(k+1)\wedge\tau}\mathbb{I}_{\{\tau\geq k+1\}}}_{k,T}(\mathbb{I}_{\{\tau\geq k+1\}}\phi_{k+1}).\end{aligned} \tag{11}$$

By using the fact that $g^{(k+1)\wedge\tau}\mathbb{I}_{\{\tau\geq k+1\}} = g^{k+1}\mathbb{I}_{\{\tau\geq k+1\}}$ and by applying Proposition A.1 again, we get

$$\begin{aligned}\mathcal{E}_{k,T}^{g^{(k+1)\wedge\tau}\mathbb{I}_{\{\tau\geq k+1\}}}(\mathbb{I}_{\{\tau\geq k+1\}}\phi_{k+1}) &= \mathcal{E}_{k,T}^{g^{k+1}\mathbb{I}_{\{\tau\geq k+1\}}}(\mathbb{I}_{\{\tau\geq k+1\}}\phi_{k+1}) \\ &= \mathbb{I}_{\{\tau\geq k+1\}}\mathcal{E}_{k,T}^{g^{k+1}}(\phi_{k+1}) = I_{\{\tau\geq k+1\}}\mathcal{E}_{k,k+1}^{g}(\phi_{k+1}).\end{aligned} \tag{12}$$

As ϕ is a g-supermartingale in discrete time, we have $\mathbb{I}_{\{\tau\geq k+1\}}\mathcal{E}_{k,k+1}^{g}(\phi_{k+1}) \leq \mathbb{I}_{\{\tau\geq k+1\}}\phi_k = \mathbb{I}_{\{\tau\geq k+1\}}\phi_{k\wedge\tau}$, which proves the inequality (10). From (9) and (10) we get the desired inequality (7). The theorem is thus proved. □

Remark 3.4 *We know that a "classical" (super)martingale in discrete time, stopped at a stopping time τ, is again a (super)martingale. A g-(super)martingale in discrete time, stopped at a stopping time $\tau \in \mathcal{T}_{0,T}^d$, is generally not a g-(super)martingale, but a g^τ-(super)martingale (in virtue of the previous Theorem 3.1). This is illustrated by the following example. Let g be a driver which does not depend on y, z, and ℓ (i.e. $g(\omega,t,y,\ell) \equiv g(\omega,t)$). Recall that in this case the solution Y of the BSDE with driver g and terminal condition ξ is given explicitly by*

$$Y_t = \mathbb{E}(\int_t^T g(s)ds + \xi|\mathcal{F}_t), \text{ for all } t \in [0,T]. \tag{13}$$

Assume that g is positive. Let ϕ be a g-martingale in discrete time and take $\tau \equiv k$, where $k \in \{0,\ldots,T-1\}$. By applying (13) with $\xi := \phi_k$, we obtain

$$\mathcal{E}_{k,T}^{g}(\phi_{(k+1)\wedge k}) = \mathcal{E}_{k,T}^{g}(\phi_k) = \mathbb{E}(\int_k^T g(s)ds + \phi_k|\mathcal{F}_k) = \mathbb{E}(\int_k^T g(s)ds|\mathcal{F}_k) + \phi_k > \phi_k,$$

the inequality being due to the positivity of g. Hence, ϕ stopped at k is not a g-martingale in discrete time.

We now establish an "optional sampling" result for g-supermartingales (resp. for g-martingales). The result can be obtained as a corollary of the previous theorem.

Corollary 3.1 *Let $(\phi_k)_{k\in\{0,1,\ldots,T\}}$ be a g-supermartingale (resp. a g-martingale) in discrete time. Then, for σ,τ in $\mathcal{T}_{0,T}^d$ such that $\sigma \leq \tau$ a.s., we have*

$$\mathcal{E}_{\sigma,\tau}^{g}(\phi_\tau) \leq \phi_\sigma \text{ (resp. } = \phi_\sigma) \text{ a.s.}$$

Proof: We prove the result for the case of a g-supermartingale; the case of a g-martingale can be treated similarly. Let σ,τ in $\mathcal{T}_{0,T}^d$ be such that $\sigma \leq \tau$ a.s. We notice that it suffices to prove the following property:

$$\mathcal{E}_{k\wedge\tau,\tau}^{g}(\phi_\tau) \leq \phi_{k\wedge\tau}, \text{ for all } k \in \{0,1,\ldots,T\}. \tag{14}$$

Indeed, this property proven, we will have

$$\mathcal{E}^g_{\sigma,\tau}(\phi_\tau) = \mathcal{E}^g_{\sigma\wedge\tau,\tau}(\phi_\tau) = \sum_{k=0}^{T} \mathbb{I}_{\{\sigma=k\}}\mathcal{E}^g_{k\wedge\tau,\tau}(\phi_\tau) \leq \sum_{k=0}^{T} \mathbb{I}_{\{\sigma=k\}}\phi_{k\wedge\tau} = \phi_{\sigma\wedge\tau} = \phi_\sigma,$$

which will conclude the proof. Let us now prove property (14). We proceed by backward induction. At the final time T we have

$$\mathcal{E}^g_{T\wedge\tau,\tau}(\phi_\tau) = \mathcal{E}^g_{\tau,\tau}(\phi_\tau) = \phi_\tau = \phi_{T\wedge\tau}.$$

We suppose that the property (14) holds true for $k+1$. Then, by using this induction hypothesis, the time-consistency and the monotonicity of the g-conditional expectation (the latter property holds under Assumption 2.1), we get

$$\mathcal{E}^g_{k\wedge\tau,\tau}(\phi_\tau) = \mathcal{E}^g_{k\wedge\tau,(k+1)\wedge\tau}(\mathcal{E}^g_{(k+1)\wedge\tau,\tau}(\phi_\tau)) \leq \mathcal{E}^g_{k\wedge\tau,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}).$$

In order to conclude, it remains to prove

$$\mathcal{E}^g_{k\wedge\tau,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}) \leq \phi_{k\wedge\tau}. \tag{15}$$

We have

$$\mathbb{I}_{\{\tau\geq k\}}\mathcal{E}^g_{k\wedge\tau,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}) = \mathbb{I}_{\{\tau\geq k\}}\mathcal{E}^g_{k,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}) \leq \mathbb{I}_{\{\tau\geq k\}}\phi_{k\wedge\tau}, \tag{16}$$

where we have used Theorem 3.1 to obtain the inequality.

By Proposition A.1, we have

$$\mathbb{I}_{\{\tau<k\}}\mathcal{E}^g_{k\wedge\tau,(k+1)\wedge\tau}(\phi_{(k+1)\wedge\tau}) = \mathcal{E}^{g^{(k+1)\wedge\tau}\mathbb{I}_{\{\tau<k\}}}_{\tau,T}(\phi_\tau\mathbb{I}_{\{\tau<k\}}). \tag{17}$$

According to the convention given in Remark A.2, the "driver" $g^{(k+1)\wedge\tau}(s,y,z,\ell)\mathbb{I}_{\{\tau<k\}}$ is here equal to $g^{(k+1)\wedge\tau}(s,y,z,\ell)\mathbb{I}_{\{\tau<k\}}\mathbb{I}_{]\tau,T]}(s)$, which is equal to zero. Hence, we have

$$\mathcal{E}^{g^{(k+1)\wedge\tau}\mathbb{I}_{\{\tau<k\}}}_{\tau,T}(\phi_\tau\mathbb{I}_{\{\tau<k\}}) = \mathcal{E}^0_{\tau,T}(\phi_\tau\mathbb{I}_{\{\tau<k\}}) = \phi_\tau\mathbb{I}_{\{\tau<k\}}.$$

where the last equality is due to the $\mathcal{F}_\tau$-measurability of $\phi_\tau\mathbb{I}_{\{\tau<k\}}$.
From equations (16) and (17) we deduce (15). The proposition is thus proved. □

3.2 Discrete-time g-Snell envelope and optimal stopping times

We now turn to the optimal stopping problem of the beginning of the section. As is usual in optimal control, we embed the above optimization problem (6) in a larger class of problems by considering

$$V_k := \text{ess sup}_{\tau\in\mathcal{T}^d_{k,T}}\mathcal{E}^g_{k,\tau}(\xi_\tau), \text{ for } k\in\{0,1,\ldots,T\}. \tag{18}$$

The following definition is analogous to the definition of the Snell envelope of a given process in discrete time, where we have replaced the mathematical expectation of the classical setting by a g-expectation. We define the process $(U_k)_{k\in\{0,1,\dots,T\}}$ by backward induction as follows:

$$\begin{cases} U_T = \xi_T, \\ U_k = \max\left(\xi_k; \mathcal{E}^g_{k,k+1}(U_{k+1})\right), \text{ for } k \in \{0,1,\dots,T-1\}. \end{cases} \tag{19}$$

From (19) we see by backward induction that (U_k) is a well-defined, $(\mathcal{F}_k)$-adapted sequence of square integrable random variables. The sequence (U_k) will be called the *g-Snell envelope in discrete time* of (ξ_k).

We now give a characterization of the g-Snell envelope in discrete time of (ξ_k).

Proposition 3.1 *The sequence $(U_k)_{k\in\{0,1,\dots,T\}}$ defined in equation (19) is the smallest g-supermartingale in discrete time dominating the sequence $(\xi_k)_{k\in\{0,1,\dots,T\}}$.*

The proof of the above proposition is similar to the proof in the case of a classical expectation and is given in the Appendix for reader's convenience.

Let $k \in \{0,1,\dots,T\}$ be given. We define the following stopping time:

$$\nu_k := \inf\{l \in \{k,\dots,T\} : U_l = \xi_l\}. \tag{20}$$

The following propositions hold true.

Proposition 3.2 *Let $k \in \{0,1,\dots,T-1\}$. Let ν_k be the stopping time defined in (20). The sequence $(U_{l\wedge\nu_k})_{l\in\{k,\dots,T\}}$ is a g^{ν_k}-martingale in discrete time.*

Proof: Let $l \in \{k,\dots,T-1\}$. We show that $U_{l\wedge\nu_k} = \mathcal{E}^g_{l,(l+1)\wedge\nu_k}(U_{(l+1)\wedge\nu_k})$ We write

$$\mathcal{E}^g_{l,(l+1)\wedge\nu_k}(U_{(l+1)\wedge\nu_k}) = \mathbb{I}_{\{\nu_k\le l\}}\mathcal{E}^g_{l,(l+1)\wedge\nu_k}(U_{(l+1)\wedge\nu_k}) + \mathbb{I}_{\{\nu_k\ge l+1\}}\mathcal{E}^g_{l,(l+1)\wedge\nu_k}(U_{(l+1)\wedge\nu_k}). \tag{21}$$

As in the proof of Theorem 3.1, we have, by definition of the solution of the BSDE with a stopping time as a terminal time,

$$\mathbb{I}_{\{\nu_k\le l\}}\mathcal{E}^g_{l,(l+1)\wedge\nu_k}(U_{(l+1)\wedge\nu_k}) = \mathbb{I}_{\{\nu_k\le l\}}U_{(l+1)\wedge\nu_k} = \mathbb{I}_{\{\nu_k\le l\}}U_{l\wedge\nu_k}. \tag{22}$$

For the second term on the right-hand side of equation (21) we use again the same arguments as those of the proof of Theorem 3.1 (cf. equations (11) and (12)) to show

$$\mathbb{I}_{\{\nu_k\ge l+1\}}\mathcal{E}^g_{l,(l+1)\wedge\nu_k}(U_{(l+1)\wedge\nu_k}) = \mathbb{I}_{\{\nu_k\ge l+1\}}\mathcal{E}^g_{l,l+1}(U_{l+1}).$$

From the definition of ν_k we see that $U_l > \xi_l$ on the set $\{\nu_k \ge l+1\}$. Combining this observation with the definition of U gives $U_l = \mathcal{E}^g_{l,l+1}(U_{l+1})$ on the set $\{\nu_k \ge l+1\}$. Hence,

$$\mathbb{I}_{\{\nu_k\ge l+1\}}\mathcal{E}^g_{l,(l+1)\wedge\nu_k}(U_{(l+1)\wedge\nu_k}) = \mathbb{I}_{\{\nu_k\ge l+1\}}U_l = \mathbb{I}_{\{\nu_k\ge l+1\}}U_{l\wedge\nu_k}. \tag{23}$$

Plugging (22) and (23) in (21) gives the desired result.

□

In the following proposition we show that the stopping time ν_k defined in (20) is optimal for the optimization problem (18) at time k and that the value function V_k of the problem is equal to U_k (the g-Snell envelope in discrete time of (ξ_k)).

Theorem 3.2 *For $k \in \{0, 1, \ldots, T\}$,*

$$U_k = \mathcal{E}^g_{k,\nu_k}(\xi_{\nu_k}) = ess\ sup_{\nu \in \mathcal{T}_{k,T}} \mathcal{E}^g_{k,\nu}(\xi_\nu), \tag{24}$$

where $\nu_k := \inf\{l \in \{k, \ldots, T\} : U_l = \xi_l\}$.

Proof: In the case where $k = T$ the result is trivially true. Suppose $k \in \{0, 1, \ldots, T-1\}$. By using Proposition 3.2 and Corollary 3.1 (applied with $\sigma = k$ and $\tau = T$), and the fact that $U_{\nu_k} = \xi_{\nu_k}$, we obtain

$$U_k = U_{k \wedge \nu_k} = \mathcal{E}^{g^{\nu_k}}_{k,T}(U_{T \wedge \nu_k}) = \mathcal{E}^g_{k,\nu_k}(U_{\nu_k}) = \mathcal{E}^g_{k,\nu_k}(\xi_{\nu_k}). \tag{25}$$

Let $\nu \in \mathcal{T}_{k,T}$. By using Proposition 3.1 and Corollary 3.1 (applied with $\sigma = k$ and $\tau = \nu$), as well as the monotonicity of the functional $\mathcal{E}^g_{0,\nu}(\cdot)$, we get

$$U_k \geq \mathcal{E}^g_{k,\nu}(U_\nu) \geq \mathcal{E}^g_{k,\nu}(\xi_\nu). \tag{26}$$

Combining equations (25) and (26) gives the desired conclusion.

□

Remark 3.5 *Families of non-linear operators $\{\mathcal{E}_t(\cdot) : t \in [0, T]\}$ indexed by a single index (as opposed to doubly-indexed families) are considered in several papers in the literature on optimal stopping with non-linear functionals (cf., e.g., Krätschmer and Schoenmakers (2010), and Bayraktar and Yao (2011)). We note that we work here with the doubly-indexed family of operators $\{\mathcal{E}^g_{t,t'}(\cdot) : t, t' \in [0, T]\}$. This family of operators reduces to the family $\{\mathcal{E}^g_{t,T}(\cdot) : t \in [0, T]\}$, indexed by a single index, under the additional assumption $\mathcal{E}^g_{t,t'}(\eta) = \mathcal{E}^g_{t,T}(\eta)$, $0 \leq t \leq t'$, for all $t' \in [0, T]$, for all $\eta \in L^2(\mathcal{F}_{t'})$. By using the consistency property of g-conditional expectations, it can be shown that this assumption is equivalent to the assumption (mentioned in the introduction) $\mathcal{E}^g_{t,T}(\eta) = \eta$, for all $t \in [0, T]$, for all $\eta \in L^2(\mathcal{F}_t)$.*

4 A non-zero-sum Dynkin game in discrete time related to risk minimization

We now consider a game problem which is slightly more general than that of the introduction. We are given two agents A^1 and A^2 whose payoffs/financial positions are defined via four $\mathbb{F}$-adapted sequences X^1, X^2, Y^1, Y^2.

We make the following assumptions:

(A1) $X^1 \leq Y^1$ and $X^2 \leq Y^2$ (that is, $X^1_k \leq Y^1_k$ and $X^2_k \leq Y^2_k$, $\forall k \in \{0, \ldots, T\}$)

(A2) $X^1_T = Y^1_T$ and $X^2_T = Y^2_T$.

(A3) The processes Y^1, Y^2, X^1, X^2 satisfy $\mathbb{E}(\max_{k\in\{0,\ldots,T\}} |Y^1_k|^2) < \infty$, $\mathbb{E}(\max_{k\in\{0,\ldots,T\}} |Y^2_k|^2) < \infty$, $\mathbb{E}(\max_{k\in\{0,\ldots,T\}} |X^1_k|^2) < \infty$, $\mathbb{E}(\max_{k\in\{0,\ldots,T\}} |X^2_k|^2) < \infty$.

The set of strategies of each of the agents at time 0 is $\mathcal{T}^d_{0,T}$. We emphasize that both agents use discrete stopping times as strategies. If the first agent's strategy is $\tau_1 \in \mathcal{T}^d_{0,T}$ and the second agent's strategy is $\tau_2 \in \mathcal{T}^d_{0,T}$, the payoff of the first (resp. second) agent at time $\tau_1 \wedge \tau_2$ is given by:

$$X^1_{\tau_1}\mathbb{I}_{\{\tau_1\leq\tau_2\}} + Y^1_{\tau_2}\mathbb{I}_{\{\tau_2<\tau_1\}} \quad (\text{resp. } X^2_{\tau_2}\mathbb{I}_{\{\tau_2<\tau_1\}} + Y^2_{\tau_1}\mathbb{I}_{\{\tau_1\leq\tau_2\}}),$$

where we have adopted the following convention: when $\tau_1 = \tau_2$, it is the first player who is responsible for stopping the game.

The agents A_1 and A_2 evaluate the risk of their respective payoffs in a (possibly) different manner.

More precisely, we are now given two standard Lipschitz drivers f_1 and f_2.

The dynamic risk measure of the first agent is equal to $\rho^{f_1} = -\mathcal{E}^{f_1}$ and the dynamic risk measure of the second agent is equal to $\rho^{f_2} = -\mathcal{E}^{f_2}$.

If the first agent's strategy is $\tau_1 \in \mathcal{T}^d_{0,T}$ and the second agent's strategy is $\tau_2 \in \mathcal{T}^d_{0,T}$, the first agent's (resp. second agent's) risk at time 0 is thus given by $-J_1(\tau_1, \tau_2)$ (resp. $-J_2(\tau_1, \tau_2)$) where

$$J_1(\tau_1, \tau_2) := \mathcal{E}^{f_1}_{0,\tau_1\wedge\tau_2}(X^1_{\tau_1}\mathbb{I}_{\{\tau_1\leq\tau_2\}} + Y^1_{\tau_2}\mathbb{I}_{\{\tau_2<\tau_1\}})$$

$$(\text{resp. } J_2(\tau_1, \tau_2) := \mathcal{E}^{f_2}_{0,\tau_1\wedge\tau_2}(X^2_{\tau_2}\mathbb{I}_{\{\tau_2<\tau_1\}} + Y^2_{\tau_1}\mathbb{I}_{\{\tau_1\leq\tau_2\}})).$$

The two agents aim at minimizing the risk of their payoffs.

The problem with the game option presented in the introduction can be seen as a particular case of the game described above, with the first agent A_1 corresponding to the buyer of the game option, the second agent A_2 corresponding to the seller, and with $X^1 = -Y^2 = X$ and $Y^1 = -X^2 = Y$. Let us emphasize that even in this particular case where the payoffs of the two agents are equal up to a minus sign, the game is of a non-zero-sum type due to the non-linearity of the dynamic risk measures. The situation, also mentioned in the introduction, where the seller and/or the buyer of the option apply their risk measures to their *net gains*, also enters in the above general framework. For instance, if the seller of the option takes into account his/her *net gain*, while the buyer considers *the payoff of the option* only, we set: $X^1 = X$, $Y^1 = Y$, $Y^2 = -X + x$, and $X^2 = -Y + x$, where $x > 0$ is the initial price of the option.

In this section, we investigate the question of the existence of a Nash equilibrium point for the general game described above.

Definition 4.1 *(Nash equilibrium point) A pair of stopping times* $(\tau_1^*, \tau_2^*) \in \mathcal{T}_{0,T}^d \times \mathcal{T}_{0,T}^d$ *is called a* Nash equilibrium point *for the above non-zero-sum Dynkin game if* $J_1(\tau_1^*, \tau_2^*) \geq J_1(\tau_1, \tau_2^*)$ *and* $J_2(\tau_1^*, \tau_2^*) \geq J_2(\tau_1^*, \tau_2)$*, for any pair* (τ_1, τ_2) *of stopping times in* $\mathcal{T}_{0,T}^d \times \mathcal{T}_{0,T}^d$*.*

In other words, a pair of strategies is a Nash equilibrium of the game if any unilateral deviation from that strategy on the part of one of the agents (the other agent's strategy remaining fixed) does not reduce his/her risk.

4.1 A preliminary result

We begin by a preliminary proposition in which we show that interchanging the strict and large inequalities in the expression of the payoff process does not change the corresponding value functions. This result will be used in the construction of a Nash equilibrium point in the following sub-section.

Proposition 4.1 *Let* (X_k) *and* (Y_k) *be two* $\mathbb{F}^d$*-adapted sequences of square-integrable random variables such that* $X_k \leq Y_k$*, for all* $k \in \{0, \ldots, T\}$*,* $X_T = Y_T$ *and* $\mathbb{E}(\max_{k\in\{0,\ldots,T\}} |X_k|^2) < \infty$*. Let* g *be a standard driver and* $\mu \in \mathcal{T}_{0,T}^d$*. For each* $k \in \{0, 1, \ldots, T\}$*, let*

$$\bar{\xi}_k := X_k \mathbb{I}_{\{k\leq\mu\}} + Y_\mu \mathbb{I}_{\{\mu<k\}} \quad \text{and} \quad \xi_k := X_k \mathbb{I}_{\{k<\mu\}} + Y_\mu \mathbb{I}_{\{\mu\leq k\}}.$$

For each $k \in \{0, 1, \ldots, T\}$*, let*

$$\bar{U}_k := \textit{ess sup}_{\tau\in\mathcal{T}_{k,T}^d} \mathcal{E}_{k,\tau\wedge\mu}^g(\bar{\xi}_\tau) \quad \text{and} \quad U_k := \textit{ess sup}_{\tau\in\mathcal{T}_{k,T}^d} \mathcal{E}_{k,\tau\wedge\mu}^g(\xi_\tau),$$

which correspond to the g^μ*-Snell envelope in discrete time of* $(\bar{\xi}_k)$ *and* (ξ_k) *respectively.*

Then, the following properties hold true:

(i) $\bar{U}_k = Y_\mu = U_k$ *a.s. on* $\{\mu \leq k-1\}$*.*

(ii) $\bar{U}_k = U_k$ *a.s. for all* $k \in \{0, \ldots, T\}$*.*

Remark 4.1 *The result can be seen as an analogue in our framework of a result by Lepeltier and Maingueneau (1984)(lemma 5) shown in a continuous-time framework with right-continuous payoffs and classical expectation.*

Proof: Let us prove (i). We proceed by backward induction. A direct computation gives

$$U_T = \xi_T = Y_\mu \quad \text{and} \quad \bar{U}_T = \bar{\xi}_T = Y_\mu \qquad \text{on} \quad \{\mu \leq T-1\}.$$

We suppose now that $\bar{U}_{k+1} = Y_\mu = U_{k+1}$ a.s. on $\{\mu \leq k\}$.
We show that $\bar{U}_k = Y_\mu = U_k$ a.s. on $\{\mu \leq k-1\}$.
By using the definition of U_k, we have

$$U_k \mathbb{I}_{\{\mu\leq k-1\}} = \max\left(\xi_k \mathbb{I}_{\{\mu\leq k-1\}}; \mathbb{I}_{\{\mu\leq k-1\}} \mathcal{E}_{k,k+1}^{g^\mu}(U_{k+1})\right). \tag{27}$$

Now, by the definition of ξ_k, we get

$$\xi_k \mathbb{I}_{\{\mu \leq k-1\}} = Y_\mu \mathbb{I}_{\{\mu \leq k-1\}} \tag{28}$$

Moreover, by Proposition A.1 and the induction hypothesis, we get

$$\mathbb{I}_{\{\mu \leq k-1\}} \mathcal{E}^{g^\mu}_{k,k+1}(U_{k+1}) = \mathcal{E}^{g^\mu \mathbb{I}_{\{\mu \leq k-1\}}}_{k,k+1}(U_{k+1}\mathbb{I}_{\{\mu \leq k-1\}}) = \mathcal{E}^{g^\mu \mathbb{I}_{\{\mu \leq k-1\}}}_{k,k+1}(Y_\mu \mathbb{I}_{\{\mu \leq k-1\}})$$

Now, the "driver" $g^\mu(\omega, s, y, z)\mathbb{I}_{\{\mu(\omega) \leq k-1\}}$ is equal to $g(\omega, s, y, z)\mathbb{I}_{\{s \leq \mu(\omega) \leq k-1\}}\mathbb{I}_{]k,T]}(s)$ (according to the convention used in Prop. A.1), which is equal to zero. Moreover, $Y_\mu \mathbb{I}_{\{\mu \leq k-1\}}$ is $\mathcal{F}_k$-measurable. Therefore, $\mathcal{E}^{g^\mu \mathbb{I}_{\{\mu \leq k-1\}}}_{k,k+1}(Y_\mu \mathbb{I}_{\{\mu \leq k-1\}}) = Y_\mu \mathbb{I}_{\{\mu \leq k-1\}}$ a.s. By combining this observation with equations (27) and (28), we get $U_k = Y_\mu$ a.s. on $\{\mu \leq k-1\}$. By similar arguments we show that $\bar{U}_k = Y_\mu$ a.s. on $\{\mu \leq k-1\}$ (to obtain this claim, it is sufficient to replace U by $\bar{U}$, and ξ by $\bar{\xi}$ in equation (27)). Property (i) is thus proven.
Let us prove property (ii). We proceed again by backward induction. At the final time T we have $U_T = \xi_T = \bar{\xi}_T = \bar{U}_T$ (due to the assumption $X_T = Y_T$). Suppose that $\bar{U}_{k+1} = U_{k+1}$. Let us prove $\bar{U}_k = U_k$. We note that $\xi_k = \bar{\xi}_k$ on the set $\{\mu = k\}^c$. This observation, the induction hypothesis, and the definitions of U_k and $\bar{U}_k$ lead to the equality $U_k = \bar{U}_k$ on the set $\{\mu = k\}^c$. It remains to show that the equality also holds true on the set $\{\mu = k\}$. By using the definition of U_k, the definition of ξ_k and Proposition A.1, we get

$$\begin{aligned} U_k \mathbb{I}_{\{\mu=k\}} &= \max\left(\xi_k \mathbb{I}_{\{\mu=k\}}; \mathbb{I}_{\{\mu=k\}} \mathcal{E}^{g^\mu}_{k,k+1}(U_{k+1})\right) \\ &= \max\left(Y_k \mathbb{I}_{\{\mu=k\}}; \mathcal{E}^{g^\mu \mathbb{I}_{\{\mu=k\}}}_{k,k+1}(U_{k+1}\mathbb{I}_{\{\mu=k\}})\right). \end{aligned} \tag{29}$$

Now, by property (i) which we have just proved, we have

$$\mathcal{E}^{g^\mu \mathbb{I}_{\{\mu=k\}}}_{k,k+1}(U_{k+1}\mathbb{I}_{\{\mu=k\}}) = \mathcal{E}^{g^\mu \mathbb{I}_{\{\mu=k\}}}_{k,k+1}(Y_k \mathbb{I}_{\{\mu=k\}}).$$

As the "driver" $g^\mu \mathbb{I}_{\{\mu=k\}}$ is equal to 0 and $Y_k \mathbb{I}_{\{\mu=k\}}$ is $\mathcal{F}_k$-measurable,

$$\mathcal{E}^{g^\mu \mathbb{I}_{\{\mu=k\}}}_{k,k+1}(Y_k \mathbb{I}_{\{\mu=k\}}) = \mathcal{E}^{0}_{k,k+1}(Y_k \mathbb{I}_{\{\mu=k\}}) = Y_k \mathbb{I}_{\{\mu=k\}}.$$

From the previous three expressions, we obtain $U_k \mathbb{I}_{\{\mu=k\}} = Y_k \mathbb{I}_{\{\mu=k\}}$. Similarly, by using the definition of $\bar{U}_k$, the definition of $\bar{\xi}_k$, Proposition A.1, and property (i), we get

$$\begin{aligned} \bar{U}_k \mathbb{I}_{\{\mu=k\}} &= \max\left(\bar{\xi}_k \mathbb{I}_{\{\mu=k\}}; \mathbb{I}_{\{\mu=k\}} \mathcal{E}^{g^\mu}_{k,k+1}(\bar{U}_{k+1})\right) \\ &= \max\left(X_k \mathbb{I}_{\{\mu=k\}}; \mathcal{E}^{g^\mu \mathbb{I}_{\{\mu=k\}}}_{k,k+1}(\bar{U}_{k+1}\mathbb{I}_{\{\mu=k\}})\right) \\ &= \max\left(X_k \mathbb{I}_{\{\mu=k\}}; \mathcal{E}^{g^\mu \mathbb{I}_{\{\mu=k\}}}_{k,k+1}(Y_k \mathbb{I}_{\{\mu=k\}})\right), \end{aligned}$$

Moreover, $\mathcal{E}^{g^\mu \mathbb{I}_{\{\mu=k\}}}_{k,k+1}(Y_k \mathbb{I}_{\{\mu=k\}}) = \mathcal{E}^{0}_{k,k+1}(Y_k \mathbb{I}_{\{\mu=k\}}) = Y_k \mathbb{I}_{\{\mu=k\}}$. Hence, $\bar{U}_k \mathbb{I}_{\{\mu=k\}} = \max(X_k; Y_k)\mathbb{I}_{\{\mu=k\}}$. As $X_k \leq Y_k$ by assumption, we obtain $\bar{U}_k \mathbb{I}_{\{\mu=k\}} = Y_k \mathbb{I}_{\{\mu=k\}}$.
Thus, the equality $\bar{U}_k \mathbb{I}_{\{\mu=k\}} = U_k \mathbb{I}_{\{\mu=k\}}$ holds, which concludes the proof. □

4.2 Construction of a Nash equilibrium point

Following ideas of Hamadène and Zhang (2010), and Hamadène and Hassani (2014), we construct a Nash equilibrium point of the game described above by a recursive procedure. We also rely on the preliminary result of the previous subsection (Prop. 4.1) and on the results of Section 3.

Theorem 4.1 *Our non-zero-sum game with g-expectations in discrete time admits a Nash equilibrium point.*

To prove this theorem, we construct a pair $(\tau_{2n+1}, \tau_{2n+2})_{n\in\mathbb{N}}$ of non-increasing sequences of stopping times and we show that the limit (as $n \to \infty$) is a NEP of the game defined in Section 2.
We set $\tau_1 := T$ and $\tau_2 := T$. We suppose that the stopping times τ_{2n-1} and τ_{2n} have been defined.
We define $f_1^{2n+1}(\omega, t, \cdot, \cdot, \cdot) := f_1^{\tau_{2n}}(\omega, t, \cdot, \cdot, \cdot) := f_1(\omega, t, \cdot, \cdot, \cdot)\mathbb{I}_{\{t\leq\tau_{2n}(\omega)\}}$. We note that f_1^{2n+1} is a standard Lipschitz driver, as f_1 is a standard Lipschitz driver.
We set, for all $k \in \{0, \dots, T\}$,

$$\begin{aligned}
\xi_k^{2n+1} &:= X_k^1 \mathbb{I}_{\{k<\tau_{2n}\}} + Y_{\tau_{2n}}^1 \mathbb{I}_{\{\tau_{2n}\leq k\}} \\
W_k^{2n+1} &:= \text{ess sup}_{\tau\in\mathcal{T}_{k,T}^d} \mathcal{E}_{k,\tau}^{f_1^{2n+1}}(\xi_\tau^{2n+1}) \\
\tilde{\tau}_{2n+1} &:= \inf\{k \in \{0, \dots, T\} : W_k^{2n+1} = \xi_k^{2n+1}\} \\
\tau_{2n+1} &:= (\tilde{\tau}_{2n+1} \wedge \tau_{2n-1})\mathbb{I}_{\{\tilde{\tau}_{2n+1}\wedge\tau_{2n-1}<\tau_{2n}\}} + \tau_{2n-1}\mathbb{I}_{\{\tilde{\tau}_{2n+1}\wedge\tau_{2n-1}\geq\tau_{2n}\}}.
\end{aligned} \tag{30}$$

Due to Theorem 3.2, the stopping time $\tilde{\tau}_{2n+1}$ is optimal for the above optimization problem (30) at time 0: more precisely, we have $W_0^{2n+1} = \sup_{\tau\in\mathcal{T}_{0,T}^d} \mathcal{E}_{0,\tau}^{f_1^{2n+1}}(\xi_\tau^{2n+1}) = \mathcal{E}_{0,\tilde{\tau}_{2n+1}}^{f_1^{2n+1}}(\xi_{\tilde{\tau}_{2n+1}}^{2n+1})$.
Moreover, thanks to Proposition 4.1 applied with $\mu := \tau_{2n}$ and $g := f_1$, we have

$$W_0^{2n+1} = \sup_{\tau\in\mathcal{T}_{0,T}^d} J_1(\tau, \tau_{2n}). \tag{31}$$

We gather some more observations on the objects defined above in the following Remark 4.2 and Proposition 4.2.

Remark 4.2 *(i) For all $n \in \mathbb{N}$, τ_{2n+1} is a stopping time (this observation follows directly from the definition of τ_{2n+1}).*

(ii) For all $n \in \mathbb{N}$, for all $\tau \in \mathcal{T}_{0,T}^d$, $\xi_\tau^{2n+1} = \xi_{\tau\wedge\tau_{2n}}^{2n+1}$.

Moreover, we have the following proposition.

Proposition 4.2 *(i) $W_k^{2n+1}\mathbb{I}_{\{\tau_{2n}\leq k\}} = Y_{\tau_{2n}}^1 \mathbb{I}_{\{\tau_{2n}\leq k\}}$.*

(ii) For all $n \in \mathbb{N}$, $\tilde{\tau}_{2n+1} = \inf\{k \in \{0,\dots,T\} : W_k^{2n+1} = X_k^1\} \wedge \tau_{2n}$. *In particular,* $\tilde{\tau}_{2n+1} \leq \tau_{2n}$, *for all* $n \in \mathbb{N}$.

Proof: Let us prove (i). From the definition of ξ^{2n+1} we have

$$\xi_\tau^{2n+1}\mathbb{I}_{\{\tau_{2n}\leq k\}} = Y_{\tau_{2n}}^1\mathbb{I}_{\{\tau_{2n}\leq k\}}, \text{ for all } \tau \in \mathcal{T}_{k,T}^d. \tag{32}$$

Due to the definition of the solution of a standard BSDE with a stopping time as a terminal time, we have

$$\mathbb{I}_{\{\tau_{2n}\leq k\}}\mathcal{E}_{k,\tau\wedge\tau_{2n}}^{f_1^{2n+1}}(\xi_\tau^{2n+1}) = \mathbb{I}_{\{\tau_{2n}\leq k\}}\xi_\tau^{2n+1}, \text{ for all } \tau \in \mathcal{T}_{k,T}^d. \tag{33}$$

Thus,

$$\begin{aligned}\mathbb{I}_{\{\tau_{2n}\leq k\}}W_k^{2n+1} &= \mathbb{I}_{\{\tau_{2n}\leq k\}}\text{ess sup}_{\tau\in\mathcal{T}_{k,T}^d}\mathcal{E}_{k,\tau\wedge\tau_{2n}}^{f_1^{2n+1}}(\xi_\tau^{2n+1})\\ &= \text{ess sup}_{\tau\in\mathcal{T}_{k,T}^d}\mathbb{I}_{\{\tau_{2n}\leq k\}}\mathcal{E}_{k,\tau\wedge\tau_{2n}}^{f_1^{2n+1}}(\xi_\tau^{2n+1}) = \mathbb{I}_{\{\tau_{2n}\leq k\}}\xi_\tau^{2n+1} = \mathbb{I}_{\{\tau_{2n}\leq k\}}Y_{\tau_{2n}}^1,\end{aligned}$$

where we have used equation (33) to obtain the last but one equality, and equation (32) to obtain the last.
Let us prove (ii). By using the definition of $\tilde{\tau}_{2n+1}$, and that of ξ^{2n+1}, we have

$$\tilde{\tau}_{2n+1} = \inf\{k \in \{0,\dots,T\} : W_k^{2n+1}\mathbb{I}_{\{k<\tau_{2n}\}} + W_k^{2n+1}\mathbb{I}_{\{\tau_{2n}\leq k\}} = X_k^1\mathbb{I}_{\{k<\tau_{2n}\}} + Y_{\tau_{2n}}^1\mathbb{I}_{\{\tau_{2n}\leq k\}}\}.$$

Thanks to this observation and to the previous property (i), we obtain

$$\begin{aligned}\tilde{\tau}_{2n+1} &= \inf\{k \in \{0,\dots,T\} : W_k^{2n+1}\mathbb{I}_{\{k<\tau_{2n}\}} = X_k^1\mathbb{I}_{\{k<\tau_{2n}\}}\}\\ &= \inf\{k \in \{0,\dots,T\} : W_k^{2n+1} = X_k^1\} \wedge \tau_{2n}.\end{aligned}$$

□

We define $f_2^{2n+2}(\omega,t,\cdot,\cdot,\cdot) := f_2^{\tau_{2n+1}}(\omega,t,\cdot,\cdot,\cdot) := f_2(\omega,t,\cdot,\cdot,\cdot)\mathbb{I}_{\{t\leq\tau_{2n+1}(\omega)\}}$. Similarly to the definitions of (30), we set

$$\begin{aligned}\xi_k^{2n+2} &:= X_k^2\mathbb{I}_{\{k<\tau_{2n+1}\}} + Y_{\tau_{2n+1}}^2\mathbb{I}_{\{\tau_{2n+1}\leq k\}}\\ W_k^{2n+2} &:= \text{ess sup}_{\tau\in\mathcal{T}_{k,T}^d}\mathcal{E}_{k,\tau}^{f_2^{2n+2}}(\xi_\tau^{2n+2})\\ \tilde{\tau}_{2n+2} &:= \inf\{k \in \{0,\dots,T\} : W_k^{2n+2} = \xi_k^{2n+2}\}\\ \tau_{2n+2} &:= (\tilde{\tau}_{2n+2}\wedge\tau_{2n})\mathbb{I}_{\{\tilde{\tau}_{2n+2}\wedge\tau_{2n}<\tau_{2n+1}\}} + \tau_{2n}\mathbb{I}_{\{\tilde{\tau}_{2n+2}\wedge\tau_{2n}\geq\tau_{2n+1}\}}.\end{aligned} \tag{34}$$

We note that the objects defined in the previous equation (34) satisfy properties analogous to those of Remark 4.2 and Proposition 4.2.

Proposition 4.3 *For all* $m \geq 1$, $\tilde{\tau}_{m+2} \leq \tau_m$ *P-a.s..*

Proof: We suppose, by way of contradiction, that there exists $m \geq 1$ such that $P(\tau_m < \tilde{\tau}_{m+2}) > 0$, and we set $n := \min\{m \geq 1 : P(\tau_m < \tilde{\tau}_{m+2}) > 0\}$. We have $n \geq 3$.
The definition of n implies $\tilde{\tau}_{n+1} \leq \tau_{n-1}$. This observation, combined with the definition of τ_{n+1} and with the inequality of part (ii) of proposition 4.2, gives

$$\tau_{n+1} = \tilde{\tau}_{n+1}\mathbb{I}_{\{\tilde{\tau}_{n+1}<\tau_n\}} + \tau_{n-1}\mathbb{I}_{\{\tilde{\tau}_{n+1}=\tau_n\}}. \tag{35}$$

For similar reasons we have

$$\tau_n = \tilde{\tau}_n\mathbb{I}_{\{\tilde{\tau}_n<\tau_{n-1}\}} + \tau_{n-2}\mathbb{I}_{\{\tilde{\tau}_n=\tau_{n-1}\}}. \tag{36}$$

For the easing of the presentation, we set $\Gamma := \{\tau_n < \tilde{\tau}_{n+2}\}$. On the set Γ, we have:

1. $\tau_n < \tilde{\tau}_{n+2} \leq \tau_{n+1}$ (the last inequality being again due to property (ii) of Prop. 4.2).
2. $\tau_{n+1} = \tau_{n-1}$. This observation is due to 1, combined with the equality (35).
3. $\xi^{n+2} = \xi^n$. This is a direct consequence of 2 and the definitions of ξ^{n+2} and ξ^n.
4. $\tau_n = \tilde{\tau}_n$. We prove that $\{\tilde{\tau}_n = \tau_{n-1}\}\cap\Gamma = \varnothing$ which, together with the expression (36), gives the desired statement. Due to (36) we have $\{\tilde{\tau}_n = \tau_{n-1}\} = \{\tilde{\tau}_n = \tau_{n-1}\} \cap \{\tau_n = \tau_{n-2}\}$. Thus, $\{\tilde{\tau}_n = \tau_{n-1}\} \cap \Gamma = \{\tilde{\tau}_n = \tau_{n-1}, \tau_n = \tau_{n-2} < \tilde{\tau}_{n+2}\}$. Now, we have $\tilde{\tau}_n \leq \tau_{n-2}$ (due to the definition of n). Thus, $\{\tilde{\tau}_n = \tau_{n-1}\} \cap \Gamma = \{\tilde{\tau}_n = \tau_{n-1} \leq \tau_n = \tau_{n-2} < \tilde{\tau}_{n+2}\} = \varnothing$, the last equality being due to $\tilde{\tau}_{n+2} \leq \tau_{n-1}$.

We note that combining properties 1 and 4 gives $\tilde{\tau}_n < \tilde{\tau}_{n+2}$ on Γ. We will obtain a contradiction with this property. To this end, let us show that

$$\mathbb{I}_\Gamma W^{n+2}_{\tilde{\tau}_n} = \mathbb{I}_\Gamma \xi^{n+2}_{\tilde{\tau}_n}. \tag{37}$$

By definition of $\tilde{\tau}_n$, we have $W^n_{\tilde{\tau}_n} = \xi^n_{\tilde{\tau}_n}$. This observation combined with property 3 gives $W^n_{\tilde{\tau}_n} = \xi^n_{\tilde{\tau}_n} = \xi^{n+2}_{\tilde{\tau}_n}$ on Γ. Thus, in order to show equality (37) it suffices to show

$$\mathbb{I}_\Gamma W^{n+2}_{\tilde{\tau}_n} = \mathbb{I}_\Gamma W^n_{\tilde{\tau}_n}. \tag{38}$$

In the following computations f_i^{n+2} is equal to f_1^{n+2} (resp. f_2^{n+2}) if $n+2$ is an odd (resp. even) number; similarly, f_i^n is equal to f_1^n (resp. f_2^n) if n is an odd (resp. even) number. By using property 4 and Proposition A.1 of the appendix, applied with $A := \Gamma$ which is $\mathcal{F}_{\tau_n}$-measurable, we obtain $\mathbb{I}_\Gamma W^{n+2}_{\tilde{\tau}_n} = \mathbb{I}_\Gamma W^{n+2}_{\tau_n} = \text{ess sup}_{\tau\in\mathcal{T}_{\tau_n,T}} \mathbb{I}_\Gamma \mathcal{E}^{f_i^{n+2}}_{\tau_n,\tau}(\xi^{n+2}_\tau) = \text{ess sup}_{\tau\in\mathcal{T}_{\tau_n,T}} \mathcal{E}^{f_i^{n+2}\mathbb{I}_\Gamma}_{\tau_n,\tau}(\mathbb{I}_\Gamma\xi^{n+2}_\tau)$. Now, by using the definitions of f_i^{n+2} and f_i^n, as well as property 2, we have:
$f_i^{n+2}(\omega,t,y,z,k)\mathbb{I}_\Gamma(\omega) = f_i(\omega,t,y,z,k)\mathbb{I}_{\{t\leq\tau_{n+1}(\omega)\}}\mathbb{I}_\Gamma(\omega) = f_i(\omega,t,y,z,k)\mathbb{I}_{\{t\leq\tau_{n-1}(\omega)\}}\mathbb{I}_\Gamma(\omega) = f_i^n(\omega,t,y,z,k)\mathbb{I}_\Gamma(\omega)$. By using this observation and property 3, we obtain $\mathcal{E}^{f_i^{n+2}\mathbb{I}_\Gamma}_{\tau_n,\tau}(\mathbb{I}_\Gamma\xi^{n+2}_\tau) =$

$\mathcal{E}^{f_i^n \mathbb{I}_\Gamma}_{\tau_n,\tau}(\mathbb{I}_\Gamma \xi^n_\tau)$. By using again Proposition A.1 from the appendix, we get $\mathcal{E}^{f_i^n \mathbb{I}_\Gamma}_{\tau_n,\tau}(\mathbb{I}_\Gamma \xi^n_\tau) = \mathbb{I}_\Gamma \mathcal{E}^{f_i^n}_{\tau_n,\tau}(\xi^n_\tau)$. Combining the previous equalities leads to $\mathbb{I}_\Gamma W^{n+2}_{\tilde{\tau}_n} = \text{ess sup}_{\tau \in \mathcal{T}_{\tau_n,T}} \mathbb{I}_\Gamma \mathcal{E}^{f_i^n}_{\tau_n,\tau}(\xi^n_\tau) = \mathbb{I}_\Gamma \text{ess sup}_{\tau \in \mathcal{T}_{\tau_n,T}} \mathcal{E}^{f_i^n}_{\tau_n,\tau}(\xi^n_\tau) = \mathbb{I}_\Gamma W^n_{\tau_n} = \mathbb{I}_\Gamma W^n_{\tilde{\tau}_n}$, where we have again used property 4 to obtain the last equality. The proof of equality (38), and hence also of equality (37), is thus completed. From equality (37) and from the definition of $\tilde{\tau}_{n+2}$ we deduce $\tilde{\tau}_{n+2} \leq \tilde{\tau}_n$ on Γ, which is in contradiction with $\tilde{\tau}_n < \tilde{\tau}_{n+2}$ on Γ. The proposition is thus proved. □

Corollary 4.1 *(i) For all $n \geq 2$, $\tau_{n+1} = \tilde{\tau}_{n+1} \mathbb{I}_{\{\tilde{\tau}_{n+1} < \tau_n\}} + \tau_{n-1} \mathbb{I}_{\{\tilde{\tau}_{n+1} = \tau_n\}}$.*

(ii) For all $n \geq 2$, $\tilde{\tau}_{n+1} = \tau_{n+1} \wedge \tau_n$.

Proof: The proof of assertion (i) is a direct consequence of the definition of τ_{n+1}, combined with the previous proposition 4.3 and with property (ii) of proposition 4.2.
Let us prove (ii). By using the previous assertion (i), we obtain

$$\begin{aligned}\tau_{n+1} \wedge \tau_n &= (\tilde{\tau}_{n+1} \wedge \tau_n) \mathbb{I}_{\{\tilde{\tau}_{n+1} < \tau_n\}} + (\tau_{n-1} \wedge \tau_n) \mathbb{I}_{\{\tilde{\tau}_{n+1} = \tau_n\}} \\ &= \tilde{\tau}_{n+1} \mathbb{I}_{\{\tilde{\tau}_{n+1} < \tau_n\}} + (\tau_{n-1} \wedge \tilde{\tau}_{n+1}) \mathbb{I}_{\{\tilde{\tau}_{n+1} = \tau_n\}}.\end{aligned}$$

By using the previous proposition 4.3 and property (ii) of proposition 4.2, we conclude that $\tau_{n+1} \wedge \tau_n = \tilde{\tau}_{n+1} \mathbb{I}_{\{\tilde{\tau}_{n+1} < \tau_n\}} + \tilde{\tau}_{n+1} \mathbb{I}_{\{\tilde{\tau}_{n+1} = \tau_n\}} = \tilde{\tau}_{n+1}$. □

Lemma 4.1 *On $\{\tau_n = \tau_{n-1}\}$ we have $\tau_m = T$, $\forall m \in \{1, \ldots, n\}$.*

Proof: The proof is similar to that of lemma 3.1 in Hamadène and Zhang (2010) and is given for reader's convenience. We proceed by induction. The result is trivially true for $n = 2$. Assume that the result holds for $n - 1$, where $n \geq 3$. From the expression of τ_n from the previous Corollary 4.1, part (i), we see that $\tau_n = \tau_{n-2}$ on $\{\tau_n = \tau_{n-1}\}$, and we conclude by using the induction hypothesis. □

Proposition 4.4 *The following inequalities hold true:*

$$J_1(\tau, \tau_{2n}) \leq J_1(\tau_{2n+1}, \tau_{2n}), \text{ for all } \tau \in \mathcal{T}^d_{0,T}.$$
$$J_2(\tau_{2n+1}, \tau) \leq J_2(\tau_{2n+1}, \tau_{2n+2}), \text{ for all } \tau \in \mathcal{T}^d_{0,T}.$$

In other words, the strategy τ_{2n+1} is optimal for the first agent at time 0 when the second agent's strategy is fixed at τ_{2n}. The strategy τ_{2n+2} is optimal for the second agent at time 0 when the first agent's strategy is fixed at τ_{2n+1}.

Proof: We will prove the first inequality. The second one can be proved by means of similar arguments. Due to equation (31), we have

$$J_1(\tau, \tau_{2n}) \leq W^{2n+1}_0. \tag{39}$$

On the other hand,

$$J_1(\tau_{2n+1},\tau_{2n}) = \mathcal{E}^{f_1^{2n+1}}_{0,\tau_{2n+1}\wedge\tau_{2n}}\big(X^1_{\tau_{2n+1}}\mathbb{I}_{\{\tau_{2n+1}\leq\tau_{2n}\}} + Y^1_{\tau_{2n}}\mathbb{I}_{\{\tau_{2n}<\tau_{2n+1}\}}\big) = \mathcal{E}^{f_1^{2n+1}}_{0,\tau_{2n+1}\wedge\tau_{2n}}(\xi^{2n+1}_{\tau_{2n+1}}),$$

where Lemma 4.1 and Assumption (A2) (that is, the assumption $X^1_T = Y^1_T$) have been used to obtain the last equality. We use Remark 4.2, Corollary 4.1 (part (ii)), and the optimality of $\tilde{\tau}_{2n+1}$ to obtain

$$\mathcal{E}^{f_1^{2n+1}}_{0,\tau_{2n+1}\wedge\tau_{2n}}(\xi^{2n+1}_{\tau_{2n+1}}) = \mathcal{E}^{f_1^{2n+1}}_{0,\tau_{2n+1}\wedge\tau_{2n}}(\xi^{2n+1}_{\tau_{2n+1}\wedge\tau_{2n}}) = \mathcal{E}^{f_1^{2n+1}}_{0,\tilde{\tau}_{2n+1}}(\xi^{2n+1}_{\tilde{\tau}_{2n+1}}) = W_0^{2n+1}.$$

Thus, $J_1(\tau_{2n+1},\tau_{2n}) = W_0^{2n+1}$, which, combined with (39), gives the desired result. □

Remark 4.3 *As a by-product of the above proof we obtain:* $W_0^{2n+1} = \mathcal{E}^{f_1^{2n+1}}_{0,\tau_{2n+1}\wedge\tau_{2n}}(\xi^{2n+1}_{\tau_{2n+1}})$. *We conlude that the stopping time* τ_{2n+1} *is also optimal (at time* $t=0$*) for the optimization problem* (30).

We define τ_1^* **and** τ_2^* **by** $\tau_1^* := \lim_{n\to\infty}\tau_{2n+1}$ **and** $\tau_2^* := \lim_{n\to\infty}\tau_{2n}$.
We note that τ_1^* and τ_2^* are stopping times in $\mathcal{T}^d_{0,T}$.
We now prove that we can pass to the limit in the inequalities of the previous proposition.

Proposition 4.5 *(i) For all* $\tau\in\mathcal{T}^d_{0,T}$, $\lim_{n\to\infty}J_1(\tau,\tau_{2n}) = J_1(\tau,\tau_2^*)$ *and* $\lim_{n\to\infty}J_2(\tau_{2n+1},\tau) = J_2(\tau_1^*,\tau)$.

(ii) $\lim_{n\to\infty}J_1(\tau_{2n+1},\tau_{2n}) = J_1(\tau_1^*,\tau_2^*)$ *and* $\lim_{n\to\infty}J_2(\tau_{2n+1},\tau_{2n+2}) = J_2(\tau_1^*,\tau_2^*)$.

Proof: Let us prove the first assertion of part (i). The other assertions can be proved by similar arguments; the details are left to the reader. For the easing of the presentation, we set $\bar{\xi}^{2n+1}_t := X^1_t\mathbb{I}_{\{t\leq\tau_{2n}\}} + Y^1_{\tau_{2n}}\mathbb{I}_{\{\tau_{2n}<t\}}$ (the process $\bar{\xi}^{2n+1}$ corresponds to the reward process of the first agent when the second agent's strategy is τ_{2n}). We also set $\bar{\xi}^{\tau_2^*}_t := X^1_t\mathbb{I}_{\{t\leq\tau_2^*\}} + Y^1_{\tau_2^*}\mathbb{I}_{\{\tau_2^*<t\}}$. With this notation, we have $J_1(\tau,\tau_{2n}) = \mathcal{E}^{f_1}_{0,\tau\wedge\tau_{2n}}(\bar{\xi}^{2n+1}_\tau)$ and $J_1(\tau,\tau_2^*) = \mathcal{E}^{f_1}_{0,\tau\wedge\tau_2^*}(\bar{\xi}^{\tau_2^*}_\tau)$. We note that the sequence $(\tau\wedge\tau_{2n})$ converges from above to $\tau\wedge\tau_2^*$. Suppose that we have shown

$$\bar{\xi}^{2n+1}_\tau \underset{n\to\infty}{\longrightarrow} \bar{\xi}^{\tau_2^*}_\tau \quad \text{a.s. and} \quad \mathbb{E}(\sup_n(\bar{\xi}^{2n+1}_\tau)^2) < +\infty. \tag{40}$$

Then, by the continuity property of the solutions of BSDEs with respect to both terminal time and terminal condition (see proposition A.6 in Quenez and Sulem (2013)), we get $\lim_{n\to\infty}J_1(\tau,\tau_{2n}) = J_1(\tau,\tau_2^*)$, which is the desired result.
It remains to check (40). Now, the sequence $(\tau\wedge\tau_{2n})$ converges a.s. and $\tau\wedge\tau_{2n}$ is valued in the finite set $\{0,\dots,T\}$. It follows that for almost every ω, the sequence of reals $(\tau(\omega)\wedge\tau_{2n}(\omega))$ is stationary, which implies that the sequence $(\bar{\xi}^{2n+1}_\tau(\omega))$ is also stationary

and converges to $\bar{\xi}_{\tau}^{\tau_2^*}(\omega)$. Finally, we check that $\mathbb{E}(\sup_n(\bar{\xi}_{\tau}^{2n+1})^2) < +\infty$, which is due to the inequality $|\bar{\xi}_{\tau}^{2n+1}|^2 \leq 2|X_{\tau}^1|^2 + 2|Y_{\tau_{2n}}^1|^2$, to the assumption (A3) , and to the square integrability of X^1. The proof is thus complete. □

Conclusion: We deduce from the previous two propositions (Prop. 4.4 and Prop. 4.5) that $J_1(\tau, \tau_2^*) \leq J_1(\tau_1^*, \tau_2^*)$, for all $\tau \in \mathcal{T}_{0,T}^d$, and $J_2(\tau_1^*, \tau) \leq J_2(\tau_1^*, \tau_2^*)$, for all $\tau \in \mathcal{T}_{0,T}^d$; in other words, (τ_1^*, τ_2^*) is a NEP of our Dynkin game.

Remark 4.4 *We note that the proof of Proposition 4.5 relies on the fact that the stopping times in the framework of our paper are valued in a finite set. Proposition 4.5 (more specifically, statement (ii)) seems difficult to establish in a continuous time framework. More precisely, due to the fact that a convergent sequence of reals in* $[0, T]$ *is not necessarily stationary, it is not so clear that it is possible to derive statement (ii) of Proposition 4.5 from Proposition 4.4, contrary to the discrete case.*

5 Further developments

The results given in the present paper can be generalized to the case of strategies valued in a finite set of stopping times. More specifically, let us consider the following setting: Let T be a positive real number. Let $K \in \mathbb{N}$. Let θ_0, θ_1, ..., θ_K be $K+1$ (distinct) $\mathbb{F}$-stopping times with values in $[0, T]$ such that $0 = \theta_0 \leq \theta_1 \leq \ldots \leq \theta_K = T$ a.s. We consider a stopper who, in each scenario $\omega \in \Omega$, can act only at times $\theta_0(\omega)$, $\theta_1(\omega)$, ...,$\theta_K(\omega)$. In other words, the stopper can choose his/her strategy among the stopping times τ of the form $\tau = \sum_{i=0}^{K} \theta_i \mathbf{1}_{A_i}$, where $(A_i)_{i \in \{0,\ldots,K\}}$ is a partition of Ω such that $A_i \in \mathcal{F}_{\theta_i}$, for all $i \in \{0, \ldots, K\}$. We denote by Θ this set of stopping times. We are also given an $\mathbb{F}$- adapted square-integrable payoff process $(\xi_t)_{t \in [0,T]}$. In this framework the optimal stopping problem of Section 3 becomes: $V_0 := \sup_{\tau \in \Theta} \mathcal{E}_{0,\tau}^g(\xi_\tau)$. A game problem analogous to that of Section 4, where the set of stopping times $\mathcal{T}_{0,T}^d$ is replaced by the set Θ, can also be formulated. In the particular case where the stopping times θ_0, θ_1, ..., θ_K are strictly ordered (that is, $0 = \theta_0 < \theta_1 < \ldots < \theta_K = T$ a.s.), the two problems can be addressed by using techniques similar to those used in the present paper, combined with a change of variables. For the general case (cf. our ongoing work Grigorova and Quenez (2016)), we need some additional arguments related to the work of Kobylanski and Quenez (2012),Kobylanski et al. (2011).

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A Appendix

Remark A.1 *Let (ζ, g) be standard parameters and let Y be the solution of the BSDE with parameters (ζ, g). Let $\tau \in \mathcal{T}_{0,T}$ be a stopping time. Let $\bar{Y}$ be the solution associated with driver $g\mathbf{1}_{]\tau,T]}$ and terminal condition ζ. We have $\bar{Y}_t = Y_t\mathbf{1}_{[\tau,T]}$ a.s. Thus, the process $\bar{Y}$ can be seen as the restriction of Y on $[\tau, T]$.*

Remark A.2 *Let $\tau \in \mathcal{T}_{0,T}$ be a stopping time . By $]\tau, T]$ we denote the set $\{(\omega, t) \in \Omega \times [0, T] : \tau(\omega) < t \leq T\}$. Let us recall the following: for $A \in \mathcal{F}_\tau$, the process $\mathbb{I}_A\mathbb{I}_{]\tau,T]}$ is adapted left-continuous and thus predictable. Thus, if g is a standard Lipschitz driver, then $g\mathbb{I}_A\mathbb{I}_{]\tau,T]}$ is also a standard Lipschitz driver. For notational simplicity, the driver $g\mathbb{I}_A\mathbb{I}_{]\tau,T]}$ will be denoted by $g\mathbb{I}_A$. This makes sense if we consider the BSDE restricted to $[\tau, T]$, which will be the case in the sequel.*

The following easy proposition is used in the proof of some of the results of the main part.

Proposition A.1 *Let (g, ζ) be standard parameters. Let $\tau \in \mathcal{T}_{0,T}$ be a stopping time and let $A \in \mathcal{F}_\tau$. We have $\mathbb{I}_A\mathcal{E}^g_{\tau,T}(\zeta) = \mathcal{E}^{g\mathbb{I}_A}_{\tau,T}(\mathbb{I}_A\zeta)$, where we have used the notational convention of Remark A.2.*

Remark A.3 *Proposition A.1 is to be compared with the "zero-one law" for g-expectations. We note that the assumption $g(s, 0, 0, 0) = 0$, required in the "zero-one law" for g-expectations, is not required in the above proposition.*

Proof: The proof, which is similar to that of the "zero-one law" for g-expectations (cf., for instance, (Peng, 2004, page 30)), is given for the convenience of the reader. Let (Y, Z, k) be the unique solution of the BSDE with standard parameters (g, ζ). Thus, (Y, Z, k) satisfies the equation

$$Y_{u\vee\tau} = \zeta + \int_{u\vee\tau}^T g(s, Y_s, Z_s, k_s)ds - \int_{u\vee\tau}^T Z_s dW_s - \int_{u\vee\tau}^T \int_E k_s(e)\tilde{N}(ds, de), \text{ for all } u \in [0, T]. \tag{41}$$

We note that $\mathbb{I}_A g(s, Y_s, Z_s, k_s) = \mathbb{I}_A g(s, \mathbb{I}_A Y_s, \mathbb{I}_A Z_s, \mathbb{I}_A k_s)$. By multiplying both sides of the equation (41) by $\mathbb{I}_A$ and by using the previous observation, we obtain: for all $u \in [0, T]$,

$$\begin{aligned}
\mathbb{I}_A Y_{u\vee\tau} &= \mathbb{I}_A \zeta + \int_{u\vee\tau}^T \mathbb{I}_A g(s, \mathbb{I}_A Y_s, \mathbb{I}_A Z_s, \mathbb{I}_A k_s) ds - \int_{u\vee\tau}^T \mathbb{I}_A Z_s dW_s \\
&\quad - \int_{u\vee\tau}^T \int_E \mathbb{I}_A k_s(e) \tilde{N}(ds, de) \\
&= \mathbb{I}_A \zeta + \int_{u\vee\tau}^T \mathbb{I}_A g(s, \mathbb{I}_A Y_s, \mathbb{I}_A Z_s, \mathbb{I}_A k_s) \mathbb{I}_{]\tau,T]}(s) ds - \int_{u\vee\tau}^T \mathbb{I}_A Z_s dW_s \\
&\quad - \int_{u\vee\tau}^T \int_E \mathbb{I}_A k_s(e) \tilde{N}(ds, de).
\end{aligned}$$

Hence, for a.e. $\omega \in \Omega$, for all u such that $\tau(\omega) \leq u \leq T$,

$$\begin{aligned}
\mathbb{I}_A Y_u &= \mathbb{I}_A \zeta + \int_u^T \mathbb{I}_A g(s, \mathbb{I}_A Y_s, \mathbb{I}_A Z_s, \mathbb{I}_A k_s) \mathbb{I}_{]\tau,T]}(s) ds - \int_u^T \mathbb{I}_A Z_s dW_s \\
&\quad - \int_u^T \int_E \mathbb{I}_A k_s(e) \tilde{N}(ds, de).
\end{aligned}$$

From this and the uniqueness of the solution of the BSDE with standard parameters, we get that the triple $(\mathbb{I}_A Y, \mathbb{I}_A Z, \mathbb{I}_A k)$ is the unique solution on $[\tau, T]$ of the BSDE with standard parameters $(\mathbb{I}_A \zeta, g\mathbb{I}_A \mathbb{I}_{]\tau,T]})$. In terms of g-expectations we can thus write the following: $\mathbb{I}_A \mathcal{E}^g_{\tau,T}(\zeta) = \mathcal{E}^{g\mathbb{I}_A}_{\tau,T}(\mathbb{I}_A \zeta)$, where we have used the notational convention of Remark A.2. □

Proof of Proposition 3.1: From the definition of (U_k), we get $U_k \geq \xi_k$, for all $k \in \{0, 1, \ldots, T\}$ and $U_k \geq \mathcal{E}^g_{k,k+1}(\xi_{k+1})$, for all $k \in \{0, 1, \ldots, T-1\}$. Hence, the sequence (U_k) is a g-supermartingale in discrete time dominating the sequence (ξ_k). Let $(\tilde{U}_k)_{k\in\{0,1,\ldots,T\}}$ be a g-supermartingale in discrete time dominating the sequence (ξ_k). We show that $\tilde{U}_k \geq U_k$, for all $k \in \{0, 1, \ldots, T\}$, by backward induction. At time T we have $\tilde{U}_T \geq \xi_T = U_T$. We suppose that $\tilde{U}_{k+1} \geq U_{k+1}$. By using the g-supermartingale property of $\tilde{U}$, the induction hypothesis and the monotonicity property of the operator $\mathcal{E}^g_{k,k+1}(\cdot)$, we get $\tilde{U}_k \geq \mathcal{E}^g_{k,k+1}(\tilde{U}_{k+1}) \geq \mathcal{E}^g_{k,k+1}(U_{k+1})$. On the other hand, $\tilde{U}_k \geq \xi_k$ by definition of $\tilde{U}$. Thus, we get $\tilde{U}_k \geq \max\left(\xi_k; \mathcal{E}^g_{k,k+1}(U_{k+1})\right)$. We conclude by recalling that the right-hand side of the previous inequality is equal to U_k.