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ABSTRACT. We describe an algorithm computing the monodromy and the pole order filtration on the Milnor fiber cohomology of any reduced projective plane curve C. The relation to the zero set of Bernstein-Sato polynomial of the defining homogeneous polynomial for C is also discussed. When C has some non weighted homogeneous singularities, then we have to assume that a conjecture holds in order to get some of our results. In all the examples computed so far this conjecture holds.

1. INTRODUCTION

Let $C : f = 0$ be a reduced plane curve of degree $d \geq 3$ in the complex projective plane $\mathbb{P}^2$, defined by a homogeneous polynomial $f \in S = \mathbb{C}[x, y, z]$. Consider the corresponding complement $U = \mathbb{P}^2 \setminus C$, and the global Milnor fiber F defined by $f(x, y, z) = 1$ in $\mathbb{C}^3$ with monodromy action $h : F \to F$,

$$h(x, y, z) = \exp(2\pi i/d) \cdot (x, y, z).$$

To determine the eigenvalues of the monodromy operators

$$(1.1) \qquad\qquad h^m : H^m(F, \mathbb{C}) \to H^m(F, \mathbb{C})$$

for $m = 1, 2$ starting from C or f is a rather difficult problem, going back to O. Zariski and attracting an extensive literature, see for instance [1, 3, 8, 9, 16, 22, 25, 26, 28, 11, 37, 13]. When the curve $C : f = 0$ is either free or nearly free, we have presented in [20] an efficient algorithm for listing the eigenvalues of the monodromy operator h^1, which determines completely the corresponding Alexander polynomial $\Delta_C(t)$, see Remark 4.4 below for its definition.

In this paper we explain an approach working in the general case. This time the results of our computation give not only the dimensions of the eigenspaces $H^m(F, \mathbb{C})_\lambda$ of the monodromy, but also the dimensions of the graded pieces $Gr_P^p H^m(F, \mathbb{C})_\lambda$, where P denotes the pole order filtration on $H^m(F, \mathbb{C})$, see section 2 below for the definition. More precisely, the algorithm described here gives the following.

(1) the dimensions of the eigenspaces $H^m(F, \mathbb{C})_\lambda$ for $m = 1, 2$ for any reduced curve $C : f = 0$, see Remark 4.4.
(2) the dimensions of the graded pieces $Gr_P^p H^1(F, \mathbb{C})_\lambda$, for any reduced curve $C : f = 0$. Note that the P^p filtration coincides to the Hodge filtration F^p on $H^1(F, \mathbb{C})$, see [21, Proposition 2.2].

(3) the dimensions of the graded pieces $Gr_P^p H^2(F, \mathbb{C})_\lambda$, for a reduced curve $C :$ $f = 0$ having only weighted homogeneous singularities. To achieve this efficiently one has to use the recent result by M. Saito stated below in Theorem 1.1.

(4) the dimensions of the graded pieces $Gr_P^p H^2(F, \mathbb{C})_\lambda$, for any reduced curve $C : f = 0$ under the assumption that a basic fact, stated as Conjecture 2.6, holds. This conjecture holds in all the examples we have computed so far, see Remark 6.2.

The new information on the pole order filtration P can be applied to describe the set of roots of $b_f(-s)$, where $b_f(s)$ is the Bernstein-Sato polynomial of f, see for details [31], [34]. In fact, using [31, Theorem 2], this comes down to checking whether $Gr_P^p H^2(F, \mathbb{C})_\lambda \neq 0$, see Theorem 7.2 for a precise statement and our applications described in Corollaries 7.3, 7.4, 7.5.

Here is in short how we proceed. Let Ω^j denote the graded S-module of (polynomial) differential j-forms on $\mathbb{C}^3$, for $0 \leq j \leq 3$. The complex $K_f^* = (\Omega^*, \mathrm{d}\, f\wedge)$ is just the Koszul complex in S of the partial derivatives f_x, f_y and f_z of the polynomial f. The general theory says that there is a spectral sequence $E_*(f)$, whose first term $E_1(f)$ is computable from the cohomology of the Koszul complex K_f^* and whose limit $E_\infty(f)$ gives us the action of monodromy operator on the graded pieces of the reduced cohomology $\tilde{H}^*(F, \mathbb{C})$ of F with respect to the pole order filtration P, see [12], [13, Chapter 6], [18].

Our approach takes a simpler form when C is assumed to have only weighted homogeneous singularities, e.g. when C is a line arrangement $\mathcal{A}$. This comes from the following result due to M. Saito [35], see for a more precise statement Theorem 2.1 below.

Theorem 1.1. *If the reduced plane curve $C : f = 0$ has only weighted homogeneous singularities, then the spectral sequence $E_*(f)$ degenerates at the E_2-term.*

This result has been conjectured already in [12] and has been checked in many cases using a computer program in [20]. The converse implication is known to hold, see [18, Theorem 5.2]. The algorithm described in [20] for free and nearly free curves actually computes the E_2-term of this spectral sequence.

In this paper we modify the algorithm in [20] such that it applies to any reduced curve. First we compute (a large part) of the E_2-term in Section 4, which is (more than) enough when C has only weighted homogeneous singularities, see Remark 4.3, or when we are interested only in the Alexander polynomial $\Delta_C(t)$. Then in Section 5 we compute the relevant part of the E_3-term of the above spectral sequence. Conjecture 2.6 tells us that essentially $E_3 = E_\infty$ and that we have missed no information on the P-filtration on $H^2(F, \mathbb{C})$ by looking only at some of the terms in the E_2 and E_3 pages.

The computations in this note were made using the computer algebra system Singular [10]. The corresponding codes are available at
`http://math.unice.fr/~dimca/singular.html`

We thank Morihiko Saito for his remarks that greatly improved the clarity of the presentation. In particular, Conjecture 2.6 and Proposition 6.1 were suggested by him, and they provide an efficient method to certify all the results given by our algorithm. For an alternative approach, see Morihiko Saito's paper [36].

We would also like to thank the referees for their very careful reading of our manuscript and their useful suggestions.

2. Gauss-Manin complexes, Koszul complexes, and Milnor fiber cohomology

Let S be the polynomial ring $\mathbb{C}[x, y, z]$ with the usual grading and consider a reduced homogeneous polynomial $f \in S$ of degree d. The graded Gauss-Manin complex C_f^* associated to f is defined by taking $C_f^j = \Omega^j[\partial_t]$, i.e. formal polynomials in ∂_t with coefficients in the space of differential forms Ω^j, where $\deg \partial_t = -d$ and the differential $\mathrm{d} : C_f^j \to C_f^{j+1}$ is $\mathbb{C}$-linear and given by

$$(2.1) \qquad \mathrm{d}(\omega \partial_t^q) = (\mathrm{d}\, \omega)\partial_t^q - (\mathrm{d}\, f \wedge \omega)\partial_t^{q+1},$$

see for more details [18, Section 4]. The complex C_f^* has a natural increasing filtration P'_* defined by

$$(2.2) \qquad P'_q C_f^j = \oplus_{i \leq q+j} \Omega^j \partial_t^i.$$

If we set $P'^q = P'_{-q}$ in order to get a decreasing filtration, then one has

$$(2.3) \qquad Gr_{P'}^q C_f^* = \sigma_{\geq q}(K_f^*((3-q)d)),$$

the truncation of a shifted version of the Koszul complex K_f^*, where σ denotes the stupid filtration, see for instance [14, Remark 1.1.15]. Moreover, this yields a decreasing filtration P' on the cohomology groups $H^j(C_f^*)$ and a spectral sequence

$$(2.4) \qquad E_1^{q,j-q}(f) \Rightarrow H^j(C_f^*).$$

On the other hand, the reduced cohomology $\tilde{H}^j(F, \mathbb{C})$ of the Milnor fiber F : $f(x, y, z) = 1$ associated to f has a pole order decreasing filtration P, see [18, Section 3], such that there is a natural identification for any integers q, j and $k \in [1, d]$

$$(2.5) \qquad P'^{q+1} H^{j+1}(C_f^*)_k = P^q \tilde{H}^j(F, \mathbb{C})_\lambda,$$

where $\lambda = \exp(-2\pi i k/d)$. Since the Milnor fiber F is a smooth affine variety, its cohomology groups $H^m(F, \mathbb{C})$ have a decreasing Hodge filtration F coming from the mixed Hodge structure constructed by Deligne, see [29]. The two filtrations P and F are related by the inclusion

$$(2.6) \qquad F^s H^m(F, \mathbb{C}) \subset P^s H^m(F, \mathbb{C}),$$

for any integers s, m, see formula (4.4.8) in [18]. Note also that

$$(2.7) \qquad F^0 H^m(F, \mathbb{C}) = H^m(F, \mathbb{C}) \text{ and } P^{m+1} H^m(F, \mathbb{C}) = 0,$$

for any integer m, where the first equality comes from the general properties of the Hodge filtration, see [29], while the second equality follows from the definition of the filtration P.

The E_1-term of the spectral sequence (2.4) is completely determined by the morphism of graded $\mathbb{C}$-vector spaces

$$(2.8) \qquad \qquad \mathrm{d}' : H^2(K_f^*) \to H^3(K_f^*),$$

induced by the exterior differentiation of forms, i.e. $\mathrm{d}' : [\omega] \mapsto [\mathrm{d}(\omega)]$. Note that this morphism d' coincides with the morphism $\mathrm{d}^{(1)} : N \to M$ considered in [18], up-to some shifts in gradings. More precisely, if we set for $k \in [1, d]$,

$$(2.9) \quad E_1^{s,t}(f)_k = H^{s+t+1}(K_f^*)_{td+k} \text{ and } \mathrm{d}_1 : E_1^{s,t}(f)_k \to E_1^{s+1,t}(f)_k, \mathrm{d}_1 : [\omega] \mapsto [\mathrm{d}(\omega)],$$

we get a new form of (a homogeneous component of) the spectral sequence (2.4), where $E_1^{s,t}(f)_k = 0$ for $s + t \notin \{1, 2\}$. Hence, with the above notation, one has
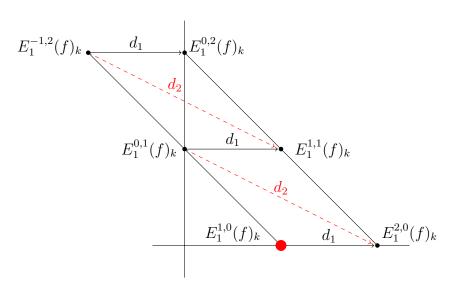
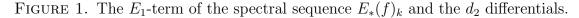$$(2.10) \qquad \qquad E_\infty^{s,t}(f)_k = Gr_P^s \tilde{H}^{s+t}(F, \mathbb{C})_\lambda.$$

The case $k = d$ is also discussed in [19], see also [36]. The second differential

$$(2.11) \qquad \qquad \mathrm{d}_2 : E_2^{s,t}(f)_k \to E_2^{s+2,t-1}(f)_k,$$

for $s + t = 1$ can be described as follows. If $[\omega] \in E_2^{s,t}(f)_k$, then $\mathrm{d}_1([\omega]) = 0$, which means that $\mathrm{d} f \wedge \omega = 0$ and there is a 2-form $\alpha \in \Omega^2$ such that $\mathrm{d}(\omega) = \mathrm{d} f \wedge \alpha$. Then one has

$$\mathrm{d}_2([\omega]) = [\mathrm{d}\,\alpha].$$

FIGURE 1. The E_1-term of the spectral sequence $E_*(f)_k$ and the d_2 differentials.

One has the following key result due to M. Saito [35].

Theorem 2.1. *The spectral sequences (2.9) degenerate at the E_2-term when the reduced plane curve $C : f = 0$ has only weighted homogeneous singularities.*

We need the following result, see [18, Theorem 5.3]. Let $\alpha_{p_i,j}$ with $j = 1, ..., \mu(C, p_i)$ be the spectral numbers of the plane curve singularity (C, p_i), where each spectral number is repeated as many times as its multiplicity in the spectrum of (C, p_i).

Theorem 2.2. *Assume that the plane curve $C : f = 0$ in $\mathbb{P}^2$ has only weighted homogeneous singularities and consider*

$$E_2^{1-t,t}(f)_k = \ker \left\{ \mathrm{d}' : H^2(K_f^*)_{td+k} \to H^3(K_f^*)_{td+k} \right\}.$$

Then $\dim E_2^{1-t,t}(f)_k \leq N(C,t,k)$, where $N(C,t,k)$ is the number of spectral numbers $\alpha_{p_i,j}$ equal to $\frac{td+k}{d}$, when p_i ranges over all the singularities of the curve C and $j = 1, ..., \mu(C,p_i)$.

Theorem 2.2 also implies the following weaker form of Theorem 2.1, see also [18, Corollary 5.5].

Corollary 2.3. *If the degree d reduced plane curve C has only weighted homogeneous singularities, then $E_2^{1,0}(f)_k = 0$ for $k < \alpha(C)d$, where $\alpha(C)$ is the minimum of the spectral numbers of the singularities of C. Moreover, the spectral sequences (2.9) degenerate at the E_3-term and the only possibly non-zero differentials in the E_2-terms are the differentials*

$$d_2 : E_2^{0,1}(f)_k \to E_2^{2,0}(f)_k$$

for $k = 1, ..., (1 - \alpha(C))d$, i.e. one has $E_2^{1-t,t}(f)_k = 0$ for $q = td + k > (2 - \alpha(C))d$.

Corollary 2.4. *Let $C : f = 0$ be a reduced plane curve of degree d having only weighted homogeneous singularities. Then $\dim Gr_P^1 H^1(F, \mathbb{C})_\lambda = 0$ for $k < \alpha(C)d$ and $\dim Gr_P^0 H^1(F, \mathbb{C})_\lambda = 0$ for $k > (1 - \alpha(C))d$.*

Corollary 2.3 implies that the following three related conjectures hold when all the singularities of C are weighted homogeneous.

Conjecture 2.5. The spectral sequences (2.9) degenerates at the E_3-term for any reduced plane curve $C : f = 0$.

We have no idea how to prove this conjecture, not even how to check that it holds on a specific example. For our practical purposes, we need only the following weaker version of this conjecture. This version can also be checked on a given example using the algorithm described below, see Proposition 6.1.

Conjecture 2.6. In the spectral sequences (2.9) of any reduced plane curve $C : f = 0$, the following equalities hold

$$\dim E_3^{2-t,t}(f)_k = \dim E_\infty^{2-t,t}(f)_k,$$

for any integer $k \in [1, d]$ and any $t = 0, 1, 2$.

To speed up the computations, the following more precise form of the above claims would be very useful.

Conjecture 2.7. For any reduced plane curve $C : f = 0$, there is a positive integer $q_0 \leq 3d + 1$, such that

$$\dim E_2^{1-t,t}(f)_k = \mu(C) - \tau(C) \text{ and } E_3^{1-t,t}(f)_k = 0$$

for any $q = td + k \geq q_0$, where $\mu(C)$ is the total Milnor number of C, that is the sum of all the Milnor numbers of the singular points of C. Similarly, $\tau(C)$ is the

total Tjurina number of C, that is the sum of all the Tjurina numbers of the singular points of C. The minimal positive integer q_0 satisfying the above property, if it exists, is denoted by $q_0(f)$.

It is clear that Conjecture 2.7 implies Conjecture 2.6, but again we have no idea how to prove Conjecture 2.7, not even how to check that it holds on a specific example.

Remark 2.8. Corollary 2.3 implies that when all the singularities of C are weighted homogeneous, one has $q_0(f) \leq (2 - \alpha(C))d + 1$. Note that by the semicontinuity of the spectrum, one has $\alpha(C) \geq 2/d$ and hence the vanishing $E_2^{1-t,t}(f)_k = 0$ holds in particular for $q = td + k \geq 2d - 1$.

However, there are curves for which $q_0(f) > 2d - 1$, see Example 8.3.

Remark 2.9. The two filtrations F and P coincide on $H^1(F, \mathbb{C})_\lambda$ always, see [21, Proposition 2.2]. On the other hand, the two filtrations F and P do not coincide on $H^2(F, \mathbb{C})_\lambda$ even in very simple cases, e.g. $C : f = (x^2 - y^2)(x^2 - z^2)(y^2 - z^2) = 0$ and $\lambda = -1$. A computation of the Hodge filtration on $H^2(F, \mathbb{C})$ in this case can be found in [5, 16]. Note also that the mixed Hodge structure on $H^2(F, \mathbb{C})_{\neq 1}$ is not pure in general. For a line arrangement, one can use the formulas for the spectrum given in [7] to study the interplay between monodromy and Hodge filtration on $H^2(F, \mathbb{C})_{\neq 1}$, see [21, Remark 2.5].

3. Jacobian syzygies of plane curves

Consider the graded S−submodule $AR(f) \subset S^3$ of *all relations* involving the derivatives of f, namely

$$\rho = (a, b, c) \in AR(f)_q$$

if and only if $af_x + bf_y + cf_z = 0$ and a, b, c are in S_q, the space of homogeneous polynomials of degree q. Let $d_1 = mdr(f)$ be the minimal degree of a relation in $AR(f)$. We assume in the sequel that $d_1 > 0$, which is equivalent to saying that C is not the union of d lines passing through one point, a case easy to handle directly.

To each syzygy $\rho = (a, b, c) \in AR(f)_q$ we associate a differential 2-form

$$(3.1) \qquad \omega(\rho) = a \, \mathrm{d} y \wedge \mathrm{d} z - b \, \mathrm{d} x \wedge \mathrm{d} z + c \, \mathrm{d} x \wedge \mathrm{d} y \in \Omega_{q+2}^2$$

such that the relation $af_x + bf_y + cf_z = 0$ becomes $\mathrm{d} f \wedge \omega(\rho) = 0$. A relation $\rho = (a, b, c)$ is called a Koszul relation if the associated differential form $\omega(\rho)$ is of the form $\mathrm{d} f \wedge \alpha$, for some $\alpha \in \Omega^1$. The set $KR(f)$ of these Koszul relations forms a graded S-submodule in $AR(f)$.

Hence, up to a shift in degrees, for any polynomial f there is an identification

$$AR(f)(-2) = Syz(f) := \ker\{\mathrm{d} f \wedge : \Omega^2 \to \Omega^3\},$$

such that the Koszul relations $KR(f)$ inside $AR(f)$ correspond to the submodule $\mathrm{d} f \wedge \Omega^1$ in $Syz(f)$. Since $C : f = 0$ has only isolated singularities, it follows that $H^1(K_f^*) = 0$, i.e. the following sequence, where the morphisms are the wedge product by $\mathrm{d} f$, is exact for any j

$$0 \to \Omega_{j-2d}^0 \to \Omega_{j-d}^1 \to (\mathrm{d} f \wedge \Omega^1)_j \to 0.$$

In particular, one has

$$(3.2) \qquad \dim(\mathrm{d}\, f \wedge \Omega^1)_j = 0 \text{ for } j \leq d,$$

$$(3.3) \qquad \dim(\mathrm{d}\, f \wedge \Omega^1)_j = 3 \binom{j - d + 1}{2} \text{ for } d < j < 2d.$$

and

$$(3.4) \qquad \dim(\mathrm{d}\, f \wedge \Omega^1)_j = 3 \binom{j - d + 1}{2} - \binom{j - 2d + 2}{2} \text{ for } j \geq 2d.$$

Remark 3.1. Let J_f be the Jacobian ideal spanned by f_x, f_y, f_z in S, and denote by $M(f) = S/J_f$ the corresponding Jacobian (or Milnor) algebra of f. Let $m(f)_j = \dim M(f)_j$ for $j \geq 0$ and recall the formulas

$$(3.5) \qquad \dim H^2(K_f^*)_j = m(f)_{j+d-3} - m(f_s)_{j+d-3} \text{ for } 2 \leq j \leq 2d - 3,$$

and $\dim H^2(K_f^*)_j = \tau(C)$ for $j \geq 2d - 2$, where $C_s : f_s = 0$ denotes a smooth curve of degree d, see [15]. Since $H^2(K_f^*)_j = Syz(f)_j/(\mathrm{d}\, f \wedge \Omega^1)_j$, the combination of the formulas (3.2), (3.3), (3.4) and (3.5) above gives us formulas for the dimensions $syz(f)_j = \dim Syz(f)_j$ for any $j \geq 3$. Note that $Syz(f)_j = AR(f)_{j-2} = 0$ for any $j < 3$ by our assumption $d_1 = mdr(f) > 0$.

Definition 3.2. For a reduced plane curve $C : f = 0$ of degree d, we recall the following invariants.
(i) the *coincidence threshold*

$$ct(f) = \max\{q : m(f)_k = m(f_s)_k \text{ for all } k \leq q\},$$

with f_s a homogeneous polynomial in S of the same degree d as f and such that $C_s : f_s = 0$ is a smooth curve in $\mathbb{P}^2$.
(ii) the *stability threshold* $st(f) = \min\{q : m(f)_k = \tau(C) \text{ for all } k \geq q\}$.

4. The first cycle in the algorithm: computing $E_2^{s,t}(f)_k$

4.1. The computation of $E_2^{s,t}(f)_k$ for $s + t = 1$, $0 \leq t \leq 3$. For $3 \leq q \leq 4d$, we set $q_1 = q - d$ and consider the linear mapping

$$(4.1) \qquad \phi_q' : \Omega_q^2 \times \Omega_{q_1}^2 \to \Omega_{q+d}^3 \times \Omega_q^3,$$

given by

$$(\omega, \alpha) \mapsto (\mathrm{d}\, f \wedge \omega, \mathrm{d}\, \omega - \mathrm{d}\, f \wedge \alpha).$$

This map puts together the differential in the Koszul complex K_f^* and d_1 from the spectral sequence (2.9). Indeed, if $\omega \in \Omega^2$ (resp. $\alpha \in \Omega^2$) is given by the formula (3.1) starting with the triple $(a, b, c) \in S_{q-2}^3$ (resp. the triple $(u, v, w) \in S_{q_1-2}^3$), one sees that essentially we get a map

$$(4.2) \qquad \phi_q' : S_{q-2}^3 \times S_{q_1-2}^3 \to S_{q-3+d} \times S_{q-3},$$

given by

$$((a, b, c), (u, v, w)) \mapsto (af_x + bf_y + cf_z, a_x + b_y + c_z - uf_x - vf_y - wf_z).$$

Since $S_j = 0$ for $j < 0$, this map has a simpler form for $q < d + 2$. The following claims are straightforward.

    i) One has $((a, b, c), (u, v, w)) \in K'_q := \ker \phi'_q$ if and only if $\mathrm{d} f \wedge \omega = 0$ and $\mathrm{d} \omega = \mathrm{d} f \wedge \alpha$.

    ii) Note that if $\omega = \mathrm{d} f \wedge \eta$, one can take $\alpha = -\mathrm{d} \eta$ and the corresponding pair (ω, α) gives rise to an element in K'_q.

    iii) Consider the projection $B_q \subset S^3_{q-2}$ of K'_q on the first component and note that $B_q/(\mathrm{d} f \wedge \Omega^1)_q$ can be identified to $E_2^{1,0}(f)_q$, for $3 \leq q \leq d$, to $E_2^{0,1}(f)_{q-d}$, for $d + 1 \leq q \leq 2d$, to $E_2^{-1,2}(f)_{q-2d}$, for $2d + 1 \leq q \leq 3d$, and respectively to $E_2^{-2,3}(f)_{q-3d}$, for $3d + 1 \leq q \leq 4d$.

    iv) The kernel of the projection $K'_q \to B_q$ can be identified to the set of forms $\alpha' \in \Omega^2$ such that $\mathrm{d} f \wedge \alpha' = 0$.

It follows that if we set $k'_q = \dim K'_q$ and

$$\epsilon'_q = k'_q - syz(f)_{q_1} - \dim(\mathrm{d} f \wedge \Omega^1)_q,$$

we have $\epsilon'_q = \dim E_2^{1,0}(f)_q$ for $3 \leq q \leq d$, $\epsilon'_q = \dim E_2^{0,1}(f)_{q-d}$ for $d + 1 \leq q \leq 2d$, $\epsilon'_q = \dim E_2^{-1,2}(f)_{q-2d}$ for $2d + 1 \leq q \leq 3d$, and respectively $\epsilon'_q = \dim E_2^{-2,3}(f)_{q-3d}$ for $3d + 1 \leq q \leq 4d$

By convention, we set $K'_j = 0$ and $k'_j = \epsilon'_j = 0$ for $j = 0, 1, 2$.

4.2. The computation of $E_2^{s,t}(f)_k$ for $s + t = 2$, $0 \leq t \leq 3$.

Recall that, setting $q = td + k$, we have $1 \leq q \leq 4d$ and

$$\dim E_1^{2-t,t}(f)_k = \dim H^3(K_f^*)_q = \dim M(f)_{q-3} = m(f)_{q-3}.$$

It follows that

$$\theta_q := \dim E_2^{2-t,t}(f)_k = \dim E_1^{2-t,t}(f)_k - (\dim E_1^{1-t,t}(f)_k - \dim E_2^{1-t,t}(f)_k) =$$

$$= m(f)_{q-3} - syz(f)_q + \dim(\mathrm{d} f \wedge \Omega^1)_q + \epsilon'_q.$$

This can be rewritten as

$$\theta_q = m(f)_{q-3} - m(f)_{q+d-3} + m(f_s)_{q+d-3} + \epsilon'_q$$

for $2 \leq q \leq 2d - 3$ and

$$\theta_q = m(f)_{q-3} - \tau(C) + \epsilon'_q$$

for $2d - 2 \leq q \leq 3d - 4$, resp. $\theta_q = \epsilon'_q$ for $3d - 3 \leq q \leq 4d$ in view of Remark 3.1 above and since $st(f) \leq 3d - 6$.

Remark 4.3. When the degree d reduced plane curve C has only weighted homogeneous singularities, we have

$$\epsilon'_q = 0$$

for any $q > (2 - \alpha(C))d$ as implied by Corollary 2.3. Moreover, Theorem 1.1 implies that in this case

$$E_2^{s,t}(f)_k = E_\infty^{s,t}(f)_k$$

and hence to determine the monodromy action and the pole order filtration on Milnor fiber cohomology in this case it is enough to compute the integers ϵ'_q only for $q \leq$

$2d-2$. Indeed, all the terms $E_2^{s,t}(f)_k = E_\infty^{s,t}(f)_k$ for $s+t = 1$ and $q > 2d-2$ are trivial, as it follows from (2.10) and the properties of the pole order filtration P, see (2.6) and (2.7). Therefore for a reduced plane curve C having only weighted homogeneous singularities the algorithm stops at this stage. Note that when C is a line arrangement one can compute only θ_q for $q \leq 2d$, since $P^1 H^2(F, \mathbb{C}) = H^2(F, \mathbb{C})$ as shown in Corollary 6.3, but in general one may need some values θ_q for $2d < q \leq st(f) + 2$ as shown by Corollary 6.5 and Example 8.1, where $P^1 H^2(F, \mathbb{C}) \neq H^2(F, \mathbb{C})$.

Remark 4.4. One can consider the characteristic polynomials of the monodromy, namely

$$(4.3) \qquad \Delta_C^j(t) = \det(t \cdot Id - h^j | H^j(F, \mathbb{C})),$$

for $j = 0, 1, 2$. It is clear that, when the curve C is reduced, one has $\Delta_C^0(t) = t - 1$, and moreover

$$(4.4) \qquad \Delta_C^0(t)\Delta_C^1(t)^{-1}\Delta_C^2(t) = (t^d - 1)^{\chi(U)},$$

where $\chi(U)$ denotes the Euler characteristic of the complement U, see for instance [25, 27] or [13, Proposition 4.1.21]. Since

$$\chi(U) = (d - 1)(d - 2) + 1 - \mu(C)$$

it follows that the polynomial $\Delta_C(t) = \Delta_C^1(t)$, also called the Alexander polynomial of C, see [30], determines the remaining polynomial $\Delta_C^2(t)$. Note that the computation of the dimension of the term $E_2^{1,0}(f)_k$, for $k = 1, ..., d$ described in this section is enough to determine the Alexander polynomial $\Delta_C(t)$ for all reduced plane curves. Indeed, the equality of the pole order filtration P^p with the Hodge filtration F^p on $H^1(F, \mathbb{C})$, see [21, Proposition 2.2] and the obvious equality

$$\dim H^{0,1}(F, C)_\lambda = \dim H^{1,0}(F, C)_{\bar\lambda},$$

with $\bar\lambda$ the complex conjugate of λ, imply that

$$\dim H^{1,0}(F, C)_\lambda = \dim E_2^{1,0}(f)_k \text{ and } \dim H^{0,1}(F, C)_\lambda = \dim E_2^{1,0}(f)_{d-k},$$

where $\lambda = \exp(-2\pi i k/d)$.

Note that the results by H. Esnault in [22], A. Libgober in [25], F. Loeser and M. Vaquié in [27], and E. Artal Bartolo in [1], can be used to compute the Alexander polynomials in many cases. However these results do not seem to be easily implementable as algorithms to performed computer aided computations.

We also note that our results, reducing the computation of the Alexander polynomial to linear algebra, imply in a clear way that this polynomial is not changed when we apply an automorphism of $\mathbb{C}$ over $\mathbb{Q}$ to our defining polynomial f. In particular, the Alexander polynomial cannot be used to distinguished Galois conjugate Zariski pairs as noticed already in [2].

Remark 4.5. As explained in Remark 4.4, if we are interested only in the Alexander polynomial $\Delta_C(t)$, then it is enough to compute the dimensions of the terms $E_2^{1,0}(f)_k$ for all $k \in [1, d]$. This justifies our approach above via the study of the *kernels* of the differentials d_1's. On the other hand, if the main interest is in the study of the polar filtration on the top cohomology group of the Milnor fiber, and this is

often the case when we consider higher dimensional projective hypersurfaces, as in [21, 35, 36], it turns out to be more efficient to look at the dimensions of the *cokernels* of the differentials d_j's, for $j = 1, 2$. This passage from kernels to cokernels was first suggested by Morihiko Saito.

Remark 4.6. In practice, the computation of $\epsilon'_q = \dim E_2^{-1,2}(f)_{q-2d}$ for $2d + 1 \leq q \leq 4d$ takes a lot of time. A way to estimate in general the numerical invariant $q_0(f)$ introduced in Conjecture 2.7 would be of a great help. All the examples computed so far suggest that Conjecture 2.7 holds. Some of these examples are given in the final section.

5. The second cycle in the algorithm: computing $E_3^{s,t}(f)_k$

In this section we explain how the algorithm described in the previous section has to be continued in the case of the presence of non weighted homogeneous singularities, and if one wants to control the P-filtration on $H^2(F, \mathbb{C})$. The general construction is described in the subsection 5.4, but for the clarity of exposition and the optimization of the computer time we discuss several cases.

5.1. **The case $3 \leq q \leq d + 1$.** We consider the linear mapping

$$(5.1) \qquad \phi_q : S_{q-2}^3 \to S_{q-3+d} \times S_{q-3},$$

given by

$$(a, b, c) \mapsto (af_x + bf_y + cf_z, a_x + b_y + c_z).$$

This map is a simpler version of the map in (4.2), and in fact $\phi_q = \phi'_q$ in this range.

It follows that (a, b, c) is in the kernel $K_q := \ker \phi_q$ if and only if the corresponding form ω is in $E_2^{1,0}(f)_q$ for $q = 3, ..., d$ or in $E_2^{0,1}(f)_1$ for $q = d + 1$. Note that one obviously has $E_2^{1,0}(f)_q = E_\infty^{1,0}(f)_q$ for $q = 3, ..., d$, and also $E_2^{0,1}(f)_1 = E_\infty^{0,1}(f)_1$ as explained in [20, Remark 2.4]. We set $k_q = \dim K_q$ and $\epsilon_q = k_q - \dim(\mathrm{d}\, f \wedge \Omega^1)_q$. Since the class $[\omega]$ of the form ω is determined up to an element of $(\mathrm{d}\, f \wedge \Omega^1)_q$, it follows that

$$\epsilon_q = \dim E_\infty^{1,0}(f)_q = \dim Gr_P^1 H^1(F, \mathbb{C})_\lambda$$

for $q = 3, ..., d$ and $\epsilon_{d+1} = \dim E_\infty^{0,1}(f)_q = \dim Gr_P^0 H^1(F, \mathbb{C})_\lambda$, with $\lambda = \exp(-2\pi i q/d)$. By convention, we set $K_j = 0$ and $k_j = \epsilon_j = 0$ for $j = 0, 1, 2$.

5.2. **The case $d + 2 \leq q \leq d + d_1 + 1$ where $d_1 = mdr(f)$.** We set $q_1 = q - d$ and we consider the linear mapping

$$(5.2) \qquad \phi_q : S_{q-2}^3 \times S_{q_1-2}^3 \to S_{q-3+d} \times S_{q-3} \times S_{q_1-3},$$

given by

$$((a, b, c), (u, v, w)) \mapsto (af_x + bf_y + cf_z, a_x + b_y + c_z - uf_x - vf_y - wf_z, u_x + v_y + w_z).$$

This map can be regarded as an extension of the map (4.2) obtained by adding the last component. Then it is clear that

$$((a, b, c), (u, v, w)) \in K_q = \ker \phi_q$$

if and only if $\mathrm{d}\,f \wedge \omega = 0$, $\mathrm{d}\,\omega = \mathrm{d}\,f \wedge \alpha$ and $\mathrm{d}\,\alpha = 0$. Since $q_1 \leq d_1 + 1$, it follows that $E_1^{1,0}(f)_{q_1} = 0$, and hence $E_2^{2,0}(f)_{q_1} = E_1^{2,0}(f)_{q_1} = M(f)_{q_1-3}$. Consider the projection $A_q \subset S_{q-2}^3$ of K_q on the first component and note that $A_q/(\mathrm{d}\,f \wedge \Omega^1)_q$ can be identified to $E_3^{0,1}(f)_{q_1}$. Moreover $E_3^{0,1}(f)_{q_1} = E_\infty^{0,1}(f)_{q_1}$ since $q_1 \leq d$. The kernel of the projection $K_q \to A_q$ can be identified to the set of forms $\alpha' \in \Omega^2$ such that $\mathrm{d}\,f \wedge \alpha' = 0$ and $\mathrm{d}\,\alpha' = 0$. Since $q_1 \leq d_1 + 1$, the fist condition implies $\alpha' = 0$, and hence $k_q = \dim K_q = \dim A_q$ in this case. It follows that if we set

$$\epsilon_q = k_q - \dim(\mathrm{d}\,f \wedge \Omega^1)_q,$$

we have again

$$\epsilon_q = \dim E_\infty^{0,1}(f)_{q_1} = \dim Gr_P^0 H^1(F, \mathbb{C})_\lambda.$$

5.3. **The case** $d + d_1 + 2 \leq q \leq 2d$ **where** $d_1 = mdr(f)$. We set again $q_1 = q - d$ and we consider the linear mapping

(5.3) $$\phi_q : S_{q-2}^3 \times S_{q_1-2}^3 \times S_{q_1-2}^3 \to S_{q-3+d} \times S_{q-3}^2 \times S_{q_1-3},$$

given by

$$((a, b, c), (u, v, w), (u', v', w')) \mapsto (\Phi_1, \Phi_2, \Phi_3, \Phi_4),$$

where $\Phi_1 = af_x + bf_y + cf_z$, $\Phi_2 = a_x + b_y + c_z - uf_x - vf_y - wf_z$, $\Phi_3 = u'f_x + v'f_y + w'f_z$ and $\Phi_4 = u_x + v_y + w_z - u'_x - v'_y - w'_z$. With the notation from the previous case, let $\beta \in \Omega^2$ be the form associated to the triple (u', v', w'). Then it is clear that

$$((a, b, c), (u, v, w), (u', v', w')) \in K_q = \ker \phi_q$$

if and only if $\mathrm{d}\,f \wedge \omega = 0$, $\mathrm{d}\,\omega = \mathrm{d}\,f \wedge \alpha$, $\mathrm{d}\,f \wedge \beta = 0$ and $\mathrm{d}\,\alpha = \mathrm{d}\,\beta$. Since $q_1 \leq d$, it follows that $\Phi_4 = 0$ has the same meaning in both S_{q_1-3} and in $M(f)_{q_1-3} = S_{q_1-3}$.

Consider the projection $A_q \subset S_{q-2}^3$ of K_q on the first component and note that $A_q/(\mathrm{d}\,f \wedge \Omega^1)_q$ can be identified to $E_3^{0,1}(f)_{q_1}$, which again, is clearly the same as $E_\infty^{0,1}(f)_{q_1}$. The kernel of the projection $K_q \to A_q$ can be identified to the set of forms $\alpha', \beta' \in \Omega^2$ such that $\mathrm{d}\,f \wedge \alpha' = \mathrm{d}\,f \wedge \beta' = 0$ and $\mathrm{d}\,\alpha' = \mathrm{d}\,\beta'$. By setting $\gamma = \alpha' - \beta'$, we see that this is the same as the set of forms $\alpha', \gamma \in \Omega^2$ such that $\mathrm{d}\,f \wedge \alpha' = \mathrm{d}\,f \wedge \gamma = 0$ and $\mathrm{d}\,\gamma = 0$. This says exactly that $\alpha' \in Syz(f)_{q_1}$ and $\gamma \in K_{q_1} = K'_{q_1}$, where K'_{q_1} was introduced in the first cycle above. It follows that if we set

$$\epsilon_q = k_q - syz(f)_{q_1} - k'_{q_1} - \dim(\mathrm{d}\,f \wedge \Omega^1)_q,$$

we have again

$$\epsilon_q = \dim E_\infty^{0,1}(f)_{q_1} = \dim Gr_P^0 H^1(F, \mathbb{C})_\lambda.$$

Note that the value for $syz(f)_{q_1}$ is determined in Remark 3.1, the value for k_{q_1} is computed in the first step of our algorithm, and the value of $\dim(\mathrm{d}\,f \wedge \Omega^1)_q$ is given in the equations (3.2), (3.3), and (3.4).

5.4. **The case** $2d + 1 \leq q \leq 4d$. We set $q_1 = q - d$ and $q_2 = q - 2d$, and we consider the linear mapping

$$(5.4) \qquad \phi_q : S_{q-2}^3 \times S_{q_1-2}^3 \times S_{q_1-2}^3 \times S_{q_2-2}^3 \to S_{q-3+d} \times S_{q-3}^2 \times S_{q_1-3},$$

given by

$$((a,b,c),(u,v,w),(u',v',w'),(u'',v'',w'')) \mapsto (\Phi_1, \Phi_2, \Phi_3, \Phi_4),$$

where $\Phi_1 = af_x + bf_y + cf_z$, $\Phi_2 = a_x + b_y + c_z - uf_x - vf_y - wf_z$, $\Phi_3 = u'f_x + v'f_y + w'f_z$ and $\Phi_4 = u_x + v_y + w_z - u'_x - v'_y - w'_z - u''f_x - v''f_y - w''f_z$. With the notation from the previous case, let $\eta \in \Omega^2$ be the form associated to the triple (u'', v'', w''). Then it is clear that

$$((a,b,c),(u,v,w),(u',v',w'),(u'',v'',w'')) \in K_q = \ker \phi_q$$

if and only if $d\, f \wedge \omega = 0$, $d\, \omega = d\, f \wedge \alpha$, $d\, f \wedge \beta = 0$ and $d\, \alpha - d\, \beta = d\, f \wedge \eta$. As above, one considers the projection $K_q \to A_q$ and note that the kernel of this projection can be identified to the space K'_{q_1}, whose dimension k'_{q_1} was determined in the first cycle above. Hence

$$\epsilon_q = k_q - syz(f)_{q_1} - k'_{q_1} - \dim(d\, f \wedge \Omega^1)_q,$$

will give again $\dim E_3^{-1,2}(f)_{q_2}$ for $2d + 1 \leq q \leq 3d$, resp. $\dim E_3^{-2,3}(f)_{q-3d}$ for $3d + 1 \leq q \leq 4d$

6. Computation of $E_\infty^{s,t}(f)_k$ for $s + t = 2$ under some assumptions

We start with the following very useful result.

Proposition 6.1. *Conjecture 2.6 holds if and only if*

$$\dim E_3^{2,0}(f)_k + \dim E_3^{1,1}(f)_k + \dim E_3^{0,2}(f)_k - \dim E_3^{1,0}(f)_k - \dim E_3^{0,1}(f)_k + \delta_d^k = \chi(U),$$

for any integer $k \in [1, d]$, *where* $\delta_d^k = 0$ *if* $k \neq d$ *and* $\delta_d^d = 1$.

Proof. Note that, using the properties of the filtrations F and P given in (2.6) and (2.7), it follows that $E_\infty^{s,t}(f)_k$, with $s + t = 2$, can be nonzero only for $t = 0, 1, 2$. Moreover, as explained above, one has

$$\dim E_3^{1,0}(f)_k + \dim E_3^{0,1}(f)_k = \dim E_\infty^{1,0}(f)_k + \dim E_\infty^{0,1}(f)_k = \dim H^1(F, \mathbb{C})_\lambda,$$

where $\lambda = \exp(-2\pi i k/d)$. On the other hand

$$\dim E_3^{2,0}(f)_k + \dim E_3^{1,1}(f)_k + \dim E_3^{0,2}(f)_k \geq$$

$$\dim E_\infty^{2,0}(f)_k + \dim E_\infty^{1,1}(f)_k + \dim E_\infty^{0,2}(f)_k = \dim H^2(F, \mathbb{C})_\lambda,$$

and the equality holds if and only if Conjectures 2.6 holds. This equality is easy to check in practice as soon as we have the dimensions of the 3-terms $E_3^{2,0}(f)_k, E_3^{1,1}(f)_k$ and $E_3^{0,2}(f)_k$, since

$$\dim H^0(F, \mathbb{C})_\lambda - \dim H^1(F, \mathbb{C})_\lambda + \dim H^2(F, \mathbb{C})_\lambda = \chi(U)$$

for any λ, as follows from the formula (4.4). Note that $\dim H^0(F, \mathbb{C})_\lambda = \delta_d^k$, which completes our proof.

$$\square$$

Remark 6.2. Let $C : f = 0$ be a reduced plane curve of degree d. To compute $\chi(U)$, it is enough to determine the Milnor number $\mu(C)$. This can be achieved in practice as follows. We choose the coordinates x, y, z such that the line $z = 0$ contains no singular point of C. It follows that $\mu(C) = \mu(C_a)$, where C_a is the affine curve in $\mathbb{C}^2$ defined by $g(x, y) = f(x, y, 1) = 0$. Then one has

$$\mu(C_a) = \dim \frac{\mathbb{C}[x, y]}{(g_x, g_y, g^2)}.$$

Here we have added g^2 to the Jacobian ideal (g_x, g_y) since we want to take the sum of the Milnor numbers of the singularities of the polynomial g situated on the curve $g = 0$. For such a singularity p, the Briançon–Skoda Theorem, see [6], shows that the local Milnor number at p is not affected by the addition of the generator g_p^2, since $g_p^2 \in (g_{p,x}, g_{p,y})$. Here we denote by g_p the analytic germ induced by the polynomial g at the point p.

We assume mainly in this section that the curve $C : f = 0$ has some non weighted homogeneous singularity, i.e. $\tau(f) = \tau(C) < \mu(C) = \mu(f)$. However, the formulas (6.1) and (6.2) hold for any reduced plane curves satisfying the stronger Conjecture 2.7, which explains why Corollaries 6.3 and 6.5 treat curves with weighted homogeneous singularities. We also assume that Conjecture 2.6 holds. When the stronger Conjecture 2.7 holds, the algorithm is much faster, since instead of computing the invariants till $q = 4d$ we can stop practically at $q = q_0(f) \le 3d + 1$.

Note that for $t = 0$ and $k = 1, ..., d$ we get the following

$$\dim Gr_P^2 H^2(F, \mathbb{C})_\lambda = \dim E_\infty^{2,0}(f)_k = \dim E_3^{2,0}(f)_k =$$
$$= \dim E_2^{2,0}(f)_k - \dim E_2^{0,1}(f)_k + \epsilon_{d+k} = \theta_k - \epsilon'_{d+k} + \epsilon_{d+k}.$$

When the stronger Conjecture 2.7 holds, and $t \ge 1$, one has also

$$\dim Gr_P^{2-t} H^2(F, \mathbb{C})_\lambda = \dim E_\infty^{2-t,t}(f)_k = \dim E_3^{2-t,t}(f)_k =$$
$$= \dim E_2^{2-t,t}(f)_k - \mu(C) + \dim E_2^{-t,t+1}(f)_k = \theta_q - \epsilon'_{q+d},$$

for $t = 1, 2$ and $k = 1, 2, ..., d$, where we set $\epsilon'_q = \mu(C) - \tau(C)$ for $q > 3d$. In particular, for $q \ge q_1(f) = \max(2d - 2, q_0(f))$, we have

$$\dim Gr_P^{2-t} H^2(F, \mathbb{C})_\lambda = m(f)_{q-3} - \tau(C).$$

Note that, when $t = 2$, this implies

(6.1) $\qquad\qquad \dim Gr_P^0 H^2(F, \mathbb{C})_\lambda = m(f)_{2d+k-3} - \tau(C) = 0$

if $q \ge q_2(f) = \max(st(f) + 3, q_1(f))$ and

$$\dim Gr_P^0 H^2(F, \mathbb{C})_\lambda = m(f)_{2d+k-3} - \tau(C) \ne 0$$

if $q = st(f) + 2 \ge q_1(f)$. Similarly, for $t = 1$ and $k = d - 2$ or $k = d - 1$ we get

(6.2) $\qquad\qquad \dim Gr_P^1 H^2(F, \mathbb{C})_\lambda = m(f)_{d+k-3} - \tau(C) = 0,$

if $q = d + k \ge q_2(f)$.

Corollary 6.3. *Let $\mathcal{A} : f = 0$ be an arrangement of d lines in $\mathbb{P}^2$ with Milnor fiber F. Then $P^1 H^2(F, \mathbb{C}) = H^2(F, \mathbb{C})$ and $Gr_P^1 H^2(F, \mathbb{C})_\lambda = 0$ for $\lambda = \exp(-2\pi i(d-1)/d)$.*

Proof. The first part of the claim is clearly equivalent to $\dim Gr_P^0 H^2(F, \mathbb{C})_\lambda = 0$ for any λ. Since $st(f) \le 2d - 4$ for any line arrangement, see [17], the claim follows from (6.1). The second part of the claim follows from (6.2). $\square$

Corollary 6.4. *Let $C : f = 0$ be a reduced plane curve which is either free or nearly free, and for which Conjecture 2.7 hold, with $q_0(f) \le 2d + 1$. Then*

$$P^1 H^2(F, \mathbb{C}) = H^2(F, \mathbb{C}).$$

Proof. When $C : f = 0$ is a free (resp. nearly free) curve with exponents $d_1 \le d_2$, it is known that $st(f) = d_2 + d - 3 \le 2d - 4$ (resp. $st(f) = d_2 + d - 2 \le 2d - 3$. We conclude as above. $\square$

Corollary 6.5. *Let $C : f = 0$ be a reduced plane curve of degree $d \ge 5$ having only one singularity, which is a node. Then $\Delta_C^1(t) = 1$ and*

$$\dim Gr_P^0 H^2(F, \mathbb{C})_\lambda = m(f)_{2d+k-3} - \tau(C) = m(f)_{d-k-3} - 1 = \binom{d-k-1}{2} - 1 \ne 0$$

for $k = 1, 2, ..., d - 4$.

Proof. For the first claim, see for instance [13, Theorem 6.4.17]. To prove the second claim, note that for such a curve $ct(f) = st(f) = 3d - 6$, see [19]. This implies in particular

$$m(f)_{2d+k-3} = m(f_s)_{2d+k-3} = m(f_s)_{d-k-3} = m(f)_{d-k-3},$$

where the equality in the middle follows from the Grothendieck duality of the Artinian Milnor algebra $M(f_s)$. $\square$

Note that for this uninodal curve, the vanishing bound given in (6.1) is sharp. For more on such uninodal curves, see Corollary 7.5 below.

7. Application to the study of Bernstein-Sato polynomials

Let $(D, 0) : g = 0$ be a complex analytic hypersurface germ at the origin of $\mathbb{C}^n$ and denote by $b_{g,0}(s)$ the corresponding (local) Bernstein-Sato polynomial. If the analytic germ g is given by a homogeneous polynomial, then one can define also the global Bernstein-Sato polynomial $b_g(s)$ of g, and one has an equality $b_g(s) = b_{g,0}(s)$, see for more details [31], [34]. Let $R_{g,0}$ (resp. $\tilde{R}_{g,0}$) be the set of roots of the polynomial $b_{g,0}(-s)$ (resp. of the polynomial $b_{g,0}(-s)/(1-s)$). Recall that one has

$$\tilde{R}_{g,0} \subset [\tilde{\alpha}_{g,0}, n - \tilde{\alpha}_{g,0}],$$

where $\tilde{\alpha}_{g,0} = \min \tilde{R}_{g,0} > 0$. Moreover, $\alpha_{g,0} = \min R_{g,0} > 0$ coincides with the log canonical threshold $lct(g)$ of the germ g, a.k.a. the log canonical threshold $lct(\mathbb{C}^n, D)$ of the pair $(\mathbb{C}^n, D)$, see [24]. When g is a homogeneous polynomial, we use the simpler notation $R_g = R_{g,0}$, $\tilde{R}_g = \tilde{R}_{g,0}$ and so on.

Example 7.1 (Cones over smooth projective hypersurfaces). Let g be a homogeneous polynomial of degree d in n variables such that the corresponding projective hypersurface $g = 0$ is smooth. Using the relation between the zero set $\tilde{R}_g$ and the

spectrum in the case of an isolated weighted homogeneous hypersurface singularity, see for instance [32], one has

$$\tilde{R}_g = \left\{ \frac{j}{d} \ : \ n \le j \le n(d-1) \right\}.$$

In this section we consider mainly the case when $n = 3$ and $g = f$ is the defining equation of a reduced curve C in $\mathbb{P}^2$. Let D be the surface in $\mathbb{C}^3$ defined by $f = 0$ and note that at a point $a \in D$, $a \ne 0$, the germ (D, a) is analytically a product between a plane curve singularity and a smooth 1-dimensional germ. It follows that

$$\tilde{R}_{f,a} \subset [\tilde{\alpha}_{f,a}, 2 - \tilde{\alpha}_{f,a}].$$

Recall M. Saito's fundamental result [31, Theorem 2] quoted here in the case $n = 3$.

Theorem 7.2. *Let $C : f = 0$ be a reduced curve in $\mathbb{P}^2$, let $\alpha > 0$ be a rational number and set $\lambda = \exp(-2\pi i \alpha)$.*

(1) *If $Gr_P^p H^2(F, \mathbb{C})_\lambda \ne 0$, where $p = \lfloor 3 - \alpha \rfloor$, then $\alpha \in R_f$*

(2) *If the sets $\alpha + \mathbb{N}$ and $\cup_{a \in D, a \ne 0} R_{f,a}$ are disjoints, then the converse of the assertion (1) holds.*

The following result is due to M. Saito in arbitrary dimension, see [33, Theorem 1]. We give a new, simple proof below to point out the relation with Corollary 6.3.

Corollary 7.3. *Let $\mathcal{A} : f = 0$ be an arrangement of d lines in $\mathbb{P}^2$ with Milnor fiber F. Then*

$$\max R_f < 2 - \frac{1}{d}.$$

Proof. We prove the result by discussing the possible cases for α. If $\alpha = 2$, the result is clear by Theorem 7.2, since

$$H^2(F, \mathbb{C})_1 = H^2(U, \mathbb{C}) = F^2 H^2(U, \mathbb{C}) = F^2 H^2(F, \mathbb{C})_1 \subset P^2 H^2(F, \mathbb{C})_1$$

and hence $H^2(F, \mathbb{C})_1 = P^2 H^2(F, \mathbb{C})_1$, which implies $Gr_P^1 H^2(F, \mathbb{C})_1 = 0$. If $\alpha \ne 2$, then we know that $d\alpha \in \mathbb{N}$, and hence to prove the claim we have to consider the case $\alpha' = 1 + (d-1)/d$ and the case $\alpha'' = 2 + k/d$ for $k = 1, ..., d-1$. Both cases follow from Theorem 7.2 (2) and Corollary 6.3. To see this, the only point to explain is why

$$\alpha' \notin \bigcup_{a \in D, a \ne 0} R_{f,a}.$$

Indeed, note that a singular point $a \in D$ correspond to a point of multiplicity $m \ge 2$ in the line arrangement. For such a point one has $\alpha_{f,a} = 2/m$, and this implies our claim. $\square$

The following result can be proved along the same lines using Corollary 6.4.

Corollary 7.4. *Let $C : f = 0$ be a reduced plane curve which is either free or nearly free, and for which Conjectures 2.5 and 2.7 hold. Then $\max R_f \le 2$.*

The next result shows that a nodal curve behaves quite differently from a line arrangement with respect to the zero set R_f.

Corollary 7.5. *Let $C : f = 0$ be a reduced plane curve of degree $d \geq 5$ having only one singularity, which is a node. Then*

$$R_f = \left\{ \frac{j}{d} \; : \; 3 \leq j \leq 3d - 4 \right\}.$$

In particular

$$\max R_f = 2 + \frac{d - 4}{d}.$$

Proof. Recall that for a uninodal plane curve $C : f = 0$ of degree d one has the following: $ct(f) = st(f) = 3d - 6$ and $\dim H^2(K_f^*)_q = 0$ for $q < 2d - 2$ and $\dim H^2(K_f^*)_q = 1$ for $q \geq 2d - 2$, see [19, Example 4.3 and Equation (2.17)]. Using this, the fact that R_f contains the given set of rational numbers follows Theorem 2.1 and Theorem 7.2 (1). To show that $\alpha = (3d - 3)/d$ is not in R_f one uses Theorem 7.2 (2). □

Remark 7.6.   (1) Corollary 7.5 can be restated as

$$R_f = R_{f_s} \setminus \left\{ \frac{d - 3}{d} \right\},$$

where f_s is a generic polynomial of degree d in x, y, z as in Example 7.1.
   (2) In the case of a line arrangement $\mathcal{A} : f = 0$ in $\mathbb{P}^2$, the zero set R_f is not determined by the combinatorics, see Walther [38] and Saito [33]. In fact, there is a pair of line arrangements $\mathcal{A}_1 : f_1 = 0$ and $\mathcal{A}_2 : f_2 = 0$ of degree $d = 9$, going back to Ziegler [39], having the same combinatorics but different sets R_f and different Hilbert functions for their Milnor algebras $M(f_1)$ and $M(f_2)$.
   (3) The zero set R_f is not determined by the Hilbert function for its Milnor algebra $M(f)$ for a reduced plane curve $C : f = 0$, see [34] and Example 8.4 below.

8. EXAMPLES

Example 8.1 (A torus curve of type $(2, 4)$)**.** Consider the curve $C : f = (x^2 + y^2)^4 + (y^4 + z^4)^2 = 0$. This curve has 8 weighted homogeneous singularities of type A_3 with local equation $u^2 + v^4 = 0$, and hence $\mu(C) = \tau(C) = 8 \times 3 = 24$. A direct computation shows that $\epsilon_q' = 0$ for $q \geq q_0(f) = 11$ as in Conjecture 2.7. On the other hand we have $\theta_q = 0$ for $q \geq 18$, which is exactly the bound predicted by (6.1) which is in this case $q \geq q_2(f) = st(f) + 3 = 15 + 3 = 18$. To state the full result, we consider the pole order spectrum defined by

$$(8.1) \qquad\qquad Sp_P^j(f) = \sum_{\alpha > 0} n_{P,f,\alpha}^j t^\alpha$$

for $j = 0, 1$, where

$$n_{P,f,\alpha}^j = \dim Gr_P^p H^{2-j}(F, \mathbb{C})_\lambda$$

with $p = [3 - \alpha]$ and $\lambda = \exp(-2\pi i\alpha)$. With this notation, we have the following for our torus curve of type $(2,4)$.

$$(8.2) \qquad Sp_P^1(f) = t^{\frac{14}{8}} + t^{\frac{16}{8}} + t^{\frac{18}{8}}.$$

$$(8.3) \qquad Sp_P^0(f) = t^{\frac{3}{8}} + 3t^{\frac{4}{8}} + 6t^{\frac{5}{8}} + 10t^{\frac{6}{8}} + 12t^{\frac{7}{8}} + 16t^{\frac{8}{8}} + 18t^{\frac{9}{8}} + 20t^{\frac{10}{8}} +$$
$$+ 18t^{\frac{11}{8}} + 16t^{\frac{12}{8}} + 13t^{\frac{13}{8}} + 10t^{\frac{14}{8}} + 7t^{\frac{15}{8}} + 3t^{\frac{16}{8}} + t^{\frac{17}{8}}.$$

Note in particular that, unlike the case of line arrangements treated in Corollary 6.3, here $P^1 H^2(F, \mathbb{C}) \neq H^2(F, \mathbb{C})$.

Example 8.2 (A free curve with non weighted homogeneous singularities). Consider the curve $C : f = (y^2 z^2 - x^4)^2 y^2 - x^{10}$, introduced in [4]. This curve is free with exponents $(4, 5)$, and hence $st(f) = 12$. Moreover, one has $\mu(C) = 70$ and $\tau(C) = 61$. A direct computation shows that $\epsilon_q' = \mu(C) - \tau(C) = 9$ for $18 = q_0(f) \leq q \leq 30$. Moreover, we have $Gr_P^1 H^2(F, \mathbb{C})_\lambda = 0$ for $q = d + k \geq 15$, which is stronger that the bound predicted by (6.2) which is in this case $q = d + k \geq d + (d - 2) = 18 \geq st(f) + 3 = 15$. Moreover Conjecture 2.6 holds for this curve since the condition in Proposition 6.1 is fulfilled. In this case we have

$$(8.4) \qquad Sp_P^1(f) = t^{\frac{16}{10}} + t^{\frac{17}{10}} + t^{\frac{18}{10}} + t^{\frac{19}{10}} + t^{\frac{20}{10}} + t^{\frac{21}{10}} + t^{\frac{22}{10}} + t^{\frac{23}{10}} + t^{\frac{24}{10}}$$

and

$$(8.5)$$
$$Sp_P^0(f) = t^{\frac{3}{10}} + 2t^{\frac{4}{10}} + 3t^{\frac{5}{10}} + 4t^{\frac{6}{10}} + 4t^{\frac{7}{10}} + 4t^{\frac{8}{10}} + 4t^{\frac{9}{10}} + 3t^{\frac{10}{10}} + 4t^{\frac{11}{10}} + 4t^{\frac{12}{10}} + 3t^{\frac{13}{10}} + 2t^{\frac{14}{10}}.$$

Moreover, $P^1 H^2(F, \mathbb{C}) = H^2(F, \mathbb{C})$ as predicted by Corollary 6.4.

The free curve C above has two irreducible components. To get a similar example with an irreducible free curve, one may consider

$$C' : f' = x^4 y^2 + y^6 - 3xy^4 z + 3x^2 y^2 z^2 - x^3 z^3 = 0.$$

This curve has a simple cusp A_2 at $p = (1 : 0 : 0)$ and a cusp with multiplicity sequence $[3_3]$, and hence Milnor number $\mu(C, q) = 12$ at $q = (0 : 0 : 1)$. Moreover, C' occurs as a type C_4 curve in Fenske's classification, see [23, Corollary 1.5]. Then C' is free with exponents $(2, 3)$ and one has $\mu(C') = 20$, $\tau(C') = 19$, $q_0(f) = 10$. Our algorithm gives

$$(8.6) \qquad Sp_P^1(f') = t^{\frac{11}{6}} + t^{\frac{13}{6}}$$

and, respectively,

$$(8.7) \qquad Sp_P^0(f') = t^{\frac{3}{6}} + t^{\frac{4}{6}} + 2t^{\frac{5}{6}} + 2t^{\frac{7}{6}} + t^{\frac{8}{6}}.$$

Example 8.3 (A curve with $\mu(C) \neq \tau(C)$ and high $q_0(f) = 2d + 2$). Consider the curve $C : f = x^4 y^4 z^4 + x^{12} + y^{12} = 0$. This curve is far from being free, since $ct(f) + st(f) - T = 11$, where $T = 3(d - 2)$, and it has $\mu(C) = 73$ and $\tau(C) = 64$. A direct computation shows that $\epsilon_q' = \mu(C) - \tau(C) = 9$ for $q \geq q_0(f) = 26 = 2d + 2 < 3d$, as required in Conjecture 2.7. In fact, Conjecture 2.6 holds for this curve since the condition in Proposition 6.1 is fulfilled. Note also that in this example the sequence

ϵ'_q contains the sequence $\epsilon'_{20} = 3$, $\epsilon'_{21} = \epsilon'_{22} = 2$, $\epsilon'_{23} = 3$, and hence it is not, as in most other examples, an increasing sequence. Note that $H^1(F, \mathbb{C}) = 0$ in this case, $b_2(F) = 455$ is large, and $P^1 H^2(F, \mathbb{C}) = H^2(F, \mathbb{C})$.

Example 8.4 (Some examples considered by M. Saito in [34]). Consider the curves $C_1 : f_1 = x^5 + y^4 z + x^4 y = 0$ and $C_2 : f_2 = x^5 + y^4 z + x^3 y^2 = 0$. Then the Milnor algebras $M(f_1)$ and $M(f_2)$ have the same Hilbert functions, and one has $\mu(f_1) = \mu(f_2) = 12$ and $\tau(f_1) = \tau(f_2) = 11$. Our algorithm shows that in both cases $H^1(F, \mathbb{C}) = 0$, $q_0(f_1) = 9$, $q_0(f_2) = 8$, while the spectrum on $H^2(F, \mathbb{C})$ is given by

$$(8.8) \qquad\qquad Sp^0(f_1) = t^{\frac{3}{5}} + t^{\frac{4}{5}} + t^{\frac{6}{5}} + t^{\frac{7}{5}},$$

and, respectively,

$$(8.9) \qquad\qquad Sp^0(f_2) = t^{\frac{4}{5}} + t^{\frac{6}{5}} + t^{\frac{7}{5}} + t^{\frac{8}{5}},$$

hence exactly the same formulas as in [34], equations (1.4.5). Using Theorem 7.2, it follows that

$$\frac{8}{5} \in R_{f_2} \setminus R_{f_1}$$

as proved in [34]. Notice also that $C_1 : f_1 = 0$ and $C_2 : f_2 = 0$ are nearly free with exponents $(2, 3)$. Both curves have a unique unibranch singulary located at $p = (0 : 0 : 1)$, which is semi-weighted-homogeneous with weights $(1/5, 1/4)$. In particular, the singularities (C_1, p) and (C_2, p) are topologically equivalent.

In [34, Remark 2.5], M. Saito also considers the curves $C_3 : f_3 = x^5 + xy^3 z + y^4 z + xy^4 = 0$ and $C_4 : f_4 = x^5 + xy^3 z + y^4 z = 0$. In this case one has $\mu(f_3) = \mu(f_4) = 11$, $\tau(f_3) = \tau(f_4) = 10$. Our algorithm shows that again $H^1(F, \mathbb{C}) = 0$, $q_0(f_3) = 10$, $q_0(f_4) = 9$, while the spectrum on $H^2(F, \mathbb{C})$ is given by

$$(8.10) \qquad Sp^0(f_3) = t^{\frac{3}{5}} + 2t^{\frac{4}{5}} + t + 2t^{\frac{6}{5}} + 2t^{\frac{7}{5}} + t^{\frac{8}{5}},$$

and, respectively,

$$(8.11) \qquad Sp^0(f_4) = t^{\frac{3}{5}} + t^{\frac{4}{5}} + t + 2t^{\frac{6}{5}} + 2t^{\frac{7}{5}} + t^{\frac{8}{5}} + t^{\frac{9}{5}}.$$

Using Theorem 7.2, it follows that

$$\frac{9}{5} \in R_{f_4} \setminus R_{f_3}$$

as proved in [34]. Notice also that $C_3 : f_3 = 0$ and $C_4 : f_4 = 0$ are neither free nor nearly free, they satisfy $ct(f_j) + st(f_j) - T = 3$ for $j = 3, 4$. In both cases, there are independent syzygies of degree 2 and 3. Note that Conjecture 2.6 holds for all these curves, since the condition in Proposition 6.1 is fulfilled. Both curves have a unique two-branch singulary located at $p = (0 : 0 : 1)$, which is semi-weighted-homogeneous with weights $(1/5, 4/15)$. Again, the singularities (C_3, p) and (C_4, p) are topologically equivalent.

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