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### 4 LABELED EMBEDDING OF (N, N – 2)-GRAPHS IN THEIR 5 COMPLEMENTS

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#### Abstract

Graph packing generally deals with unlabeled graphs. In [4], the authors have introduced a new variant of the graph packing problem, called the *labeled packing of a graph*. This problem has recently been studied on trees [6] and cycles [4]. In this note, we present a lower bound on the labeled packing number of any (n, n - 2)-graph into  $K_n$ . This result improves the bound given by Woźniak in [7].

28 **Keywords:** Packing of graphs, Labeled packing, Permutation.

29 **2010** Mathematics Subject Classification: 05C70.

#### 1. Context and definitions

#### 31 Graph theoretical definitions

All graphs considered in this paper are finite, undirected, without loops or multiple edges. If T is a rooted tree of order n, we define an *end vertex* as a vertex which does not have any son, and a *leaf-parent* as a vertex whose all of its sons are end vertices.

Given a positive integer n, the graphs  $K_n$ ,  $P_n$  and  $C_n$  will denote respectively the complete graph, the path and the cycle on n vertices. For a graph G, we will use V(G) and E(G) to denote its vertex and edge sets respectively. Given  $V' \subset V$ , the subgraph G[V'] denotes the subgraph of G induced by V', i.e., E(G[V'])contains all the edges of E which have both extremities in V'. If a graph G has order n and size m, we say that G is an (n, m)-graph.

An independent set of G is a subset of vertices  $X \subseteq V$ , such that no two vertices in X are adjacent. An independent set is said to be maximal if no independent set properly contains it. An independent set of maximum cardinality is called a maximum independent set. For undefined terms, we refer the reader to [2]. A permutation  $\sigma$  is a one-to-one mapping of  $\{1, \ldots, n\}$  into itself. We say that a permutation  $\sigma$  is fixed-point-free if  $\sigma(x) \neq x$  for all x of  $\{1, \ldots, n\}$ .

#### 48 The graph packing problem

The graph packing problem was introduced by Bollobás and Eldridge [1] and 49 Sauer and Spencer [5] in the late 1970s. Let  $G_1, \ldots, G_k$  be k graphs of order 50 n. We say that there is a packing of  $G_1, \ldots, G_k$  (into the complete graph  $K_n$ 51 ) if there exist bijections  $\sigma_i: V(G_i) \longrightarrow V(K_n)$ , where  $1 \leq i \leq k$ , such that 52  $\sigma_i^*(E(G_i)) \cap \sigma_j^*(E(G_j)) = \emptyset$  for  $i \neq j$ , and here the map  $\sigma_i^*: E(G_i) \longrightarrow E(K_n)$ 53 is the one induced by  $\sigma_i$ . A packing of k copies of a graph G will be called a 54 k-placement of G. A packing of two copies of G (*i.e.*, a 2-placement) is also called 55 an embedding of G (into its complement  $\overline{G}$ ). In other words, an embedding of 56 a graph G is a permutation  $\sigma$  on V(G) such that for each edge uv belonging to 57 E(G), its image  $\sigma(u)\sigma(v)$  does not belong to E(G). 58 59

- In the literature, the question of the existence of an embedding of a given graph received a great attention (see the survey papers [8, 9]). In [3], full characterizations of all the (n, n - 1) and (n, n) embeddable graphs are given. The case of (n, n - 2)-graphs was also solved independently in [1, 3, 5]. In particular, it is proved in [5] that any pair of (n, n - 2)-graphs can be packed into  $K_n$ .
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In [4], Duchêne *et al.* introduced and studied the graph packing problem for a vertex labeled graph. Roughly speaking, it consists of a graph packing which

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<sup>66</sup> preserves the labels of the vertices. We give below the formal definition of this
<sup>69</sup> problem.

**Definition** [4]. Given a positive integer p, let G be a graph of order n and f be a mapping from V(G) to the set  $\{1, \ldots, p\}$ . The mapping f is called a p-labeledpacking of k copies of G into  $K_n$  if there exist bijections  $\sigma_i : V(G) \longrightarrow V(K_n)$ for  $1 \le i \le k$ , such that:

74 1. 
$$\sigma_i^*(E(G)) \cap \sigma_i^*(E(G)) = \emptyset$$
 for all  $i \neq j$ .

2. For every vertex v of G, we have  $f(v)=f(\sigma_1(v))=f(\sigma_2(v))=\cdots=f(\sigma_k(v))$ .

The maximum positive integer p for which G admits a p-labeled-packing of kcopies of G is called the *labeled packing number* of k copies of G and is denoted by  $\lambda^k(G)$ . Throughout this paper, a labeled packing of two copies of G will be called a labeled embedding of G. It will be denoted by a pair  $(f, \sigma)$ .

Remark that the existence of a packing of k copies of a graph G is a necessary condition for the existence of p-labeled-packing of k copies of G. Indeed, it suffices to choose p = 1. Therefore, the result of Sauer and Spencer [5] ensures the existence of a p-labeled packing for (n, n - 2)-graphs. An estimation of the labeled packing number of such graphs is the main issue of the current paper.

The following result was proved in [4]. It gives an upper bound for the labeled packing number of two copies of a general graph.

**Lemma 1** (Duchêne et al., 11). Let G be a graph of order n and let I be a maximum independent set of G. If there exists an embedding of G into  $K_n$ , then

$$\lambda^2(G) \le |I| + \lfloor \frac{n - |I|}{2} \rfloor$$

In [4], exact values of  $\lambda^2(G)$  are given when G is a cycle or a path. In almost all cases, the upper bound of the above lemma is reached. More precisely, it is shown that for all  $n \ge 6$ ,

$$\lambda^2(P_n) \in \{\lfloor \frac{3n}{4} \rfloor, \lfloor \frac{3n}{4} \rfloor + 1\}$$
  
 $\lambda^2(C_n) = \lfloor \frac{3n}{4} \rfloor$ 

<sup>88</sup> The case of trees is also considered [6], but only a lower bound is proposed.

#### 2. LABELED EMBEDDING OF GRAPHS AND PERMUTATIONS

<sup>90</sup> In this section, we give a strong relationship between a labeled embedding and <sup>91</sup> its permutation structure.

A permutation  $\sigma$  of a finite set can be written as the disjoint union of cycles 92 (two cycles being disjoint if they do not have any common element). Here, a 93 cycle  $(a_1, \ldots, a_n)$  is a permutation sending  $a_i$  to  $a_{i+1}$  for  $1 \le i \le n-1$  and  $a_n$  to 94  $a_1$ . This representation is called the *cyclic decomposition of*  $\sigma$  and is denoted by 95  $C(\sigma)$ . According to this definition, the cycles of length one correspond to fixed 96 points of  $\sigma$ . For example, the cyclic decomposition of the permutation induced 97 by the labeled embedding of T (in Figure 1) is:  $\{(v_1), (v_2), (v_3), (v_4), (v_5), (v_6), (v_6),$ 98  $(v_7), (v_8, v_{10}), (v_{11}, v_{13}), (v_9, v_{12})\}.$ 99

We now recall a fundamental property of labeled embeddings (see [4]). For any labeled embedding  $(f, \sigma)$  of a graph G, one can remark that the vertices of every cycle of  $C(\sigma)$  share the same label. In other words, the labeled embedding number of G exactly corresponds to the maximum number of cycles induced by an embedding of G. It means that if G admits an embedding with k cycles, then  $\lambda^2(G) \geq k$ .

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<sup>107</sup> Although this correlation between labeled embeddings and the permutation's <sup>108</sup> number of cycles was rencently stated, several studies can be found about the <sup>109</sup> permutation structure of an embedding. In particular, the permutation structure <sup>110</sup> of embeddings of (n, n - 2)-graphs was investigated by Woźniak in [7]:

**Theorem 2** (Woźniak, 94). Let G be a graph of order n, different from  $K_3 \cup 2K_1$ and  $K_4 \cup 4K_1$ . If  $|E(G)| \le n-2$ , then there exists a permutation  $\sigma$  on V(G)such that  $\sigma_1$ ,  $\sigma_2$ ,  $\sigma_3$  define a 3-placement of G. Moreover,  $\sigma$  has all its cycles of length 3, except for one of length one if  $n \equiv 1 \mod 3$  or two of length one if  $n \equiv 2 \mod 3$ .

According to our previous remarks, the above theorem induces the following result in the context of labeled embeddings.

**Corollary 3.** Let G be a graph of order n, different from  $K_3 \cup 2K_1$  and  $K_4 \cup 4K_1$ . If  $|E(G)| \leq n-2$ , then

$$\lambda^2(G) \ge \lfloor \frac{n}{3} \rfloor + n \mod 3$$

In the next section, we will show that the lower bound of Corollary 3 can be improved (including for the excluded graphs).

#### 3. Main result

<sup>121</sup> We first define the notion of *good permutation* for a graph.

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- **Definition.** Given a graph G, a permutation  $\sigma$  on V(G) is said to be good if
- $\sigma$  is an embedding of G,
- $\sigma$  has at least  $\lfloor \frac{2n}{3} \rfloor$  cycles,
- every cycle of  $\sigma$  is of order at most 2, *i.e.*, for every pair of distinct vertices  $u, v \text{ of } G, \text{ if } \sigma(u) = v, \text{ then } \sigma(v) = u.$
- <sup>127</sup> The following lemma will be useful in a special case of our main result.
- Lemma 4. For k > 0, the graph  $kC_3 \cup 2K_1$  admits a good permutation.

<sup>129</sup> **Proof.** According to the diagram below (Figure 1), first remark that  $3C_3$  admits

- a good permutation. Indeed, the numbers inside the vertices correspond to a
  labeled embedding with 6 labels, with at most two vertices sharing the same
- 131 labeled embedding with 6 labels, with at most two vertices sharing the same label.



Figure 1. Good permutation for  $3C_3$ 

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Now let k be a positive integer and G be the graph  $kC_3 \cup 2K_1$ . Let u and t be the two isolated vertices of G. For  $1 \le i \le k$ , let  $\{v_{i1}, v_{i2}, v_{i3}\}$  be the vertices of the  $i^{th}$  triangle  $C_3$ . For k = 1, consider the permutation  $\sigma$  where  $v_{11}$  is a fixed point,  $v_{12}$  and u are mutual images, as well as  $v_{13}$  and t. One can easily check that  $\sigma$  is good for G. For k = 2, Figure 2 shows a good permutation (more precisely, the corresponding labeled embedding).



Figure 2. Good permutation for  $2C_3 \cup 2K_1$ 

For k = 3, consider the permutation  $\sigma$  corresponding to the labeled embedding of Figure 1, and extend it to G by setting  $\sigma(u) = u$  and  $\sigma(t) = t$ . Then  $\sigma$ remains good for G. For k > 3, we can now conclude to the existence of a good permutation for G by pairing good permutations of  $3C_3$  with a good permutation of  $rC_3 \cup 2K_1$  where r is in  $\{1, 2, 3\}$ . <sup>144</sup> We now present a lower bound for the labeled embedding number of any <sup>145</sup> (n, n-2)-graph.

**Theorem 5.** Let  $n \ge 2$  and G be an (n, m)-graph with  $m \le n-2$ . The following inequality holds:

$$\lambda^2(G) \geq \lfloor \frac{2n}{3} \rfloor$$

**Proof.** Let  $n \ge 2$  and G be an (n,m)-graph with  $m \le n-2$ . Without loss of generality, we can assume |E(G)| = n-2. We will show that G admits a good permutation by induction on n. If n = 2, 3, 4, then  $G \in \{2K_1, 3K_1, K_1 \cup K_2, 2K_2, K_{1,2} \cup K_1\}$ . In each case, one can quickly check that there exist good permutations with at least two cycles. The property still holds for n = 5, where  $G \in \{K_3 \cup 2K_1, K_1 \cup K_{1,3}, K_2 \cup K_{1,2}, P_4 \cup K_1\}$ . Good permutations with at least three cycles can be found.

Now let  $n \ge 6$  and assume there exists a good permutation for every (n', n' - 2)-graph of order n' < n with  $n' \ge 3$ . Since G is an (n, n - 2)-graph, at least two of its connected components are trees. Denote by T and H two trees of G of highest orders such that  $|V(T)| \ge |V(H)|$ . In what follows, we choose to consider T and H as rooted trees. We consider the following four cases:

159 **Case 1:**  $|V(T)| \ge 3$  and  $|V(H)| \ge 3$ . Two subcases are considered as follows:

**Subcase 1.1:** T or H admits a leaf-parent of degree 2. By symmetry, we may assume that T admits a leaf-parent say  $x_1$ , of degree 2. Let  $x_0$  and  $x_2$  be the two vertices of T such that  $(x_0, x_1, x_2)$  is an induced path of T and  $x_2$  is an end vertex. Let  $y_1$  be an end vertex of H and  $y_0$  its parent. Now consider the graph  $G' = G - \{x_1, x_2, y_1\}$ . Clearly, G' is an (n-3, n-5)-graph with  $n-3 \ge 3$ . Hence the induction hypothesis guarantees the existence of a good permutation  $\sigma'$  for G'. This permutation can be extended to a good permutation  $\sigma$  for G as follows:

$$\sigma(x_1) = \begin{cases} y_1 & \text{if } \sigma'(x_0) = x_0, \\ x_1 & otherwise. \end{cases} \quad \sigma(x_2) = \begin{cases} x_2 & \text{if } \sigma'(x_0) = x_0, \\ y_1 & otherwise. \end{cases}$$

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$$\sigma(y_1) = \begin{cases} x_1 & \text{if } \sigma'(x_0) = x_0, \\ x_2 & otherwise. \end{cases} \qquad \sigma(v) = \sigma'(v) \text{ if } v \in V(G')$$

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Since the number of cycles of  $\sigma_{|G-G'}$  equals two, and they all are of length at most 2, it ensures that  $\sigma$  is a good permutation for G.

**Subcase 1.2:** T and H do not have any leaf-parent of degree 2 and T is a star. Let  $\ell$  be a leaf of H and v be its neighbour (of degree at least 3). Let ube the unique vertex of degree at least 3 in T. Now consider the graph G' = $G - T - \{\ell, v\}$ . Clearly, G' is an (k, j)-graph with  $k - 2 \ge j$  and  $k \ge 2$ . Hence the induction hypothesis guarantees the existence of a good permutation  $\sigma'$  for <sup>177</sup> G'. This permutation can be extended to a good permutation  $\sigma$  for G by setting <sup>178</sup>  $\sigma(u) = v, \sigma(v) = u, \sigma(\ell) = \ell$  and  $\sigma(l_i) = l_i$  for all leaf  $l_i$  of T.

**Subcase 1.3:** T and H do not have any leaf-parent of degree 2 and T is not a star. Hence T admits two leaf parents, say u and t. Let  $u_1, \ldots, u_k$  (resp.  $t_1, \ldots, t_{k'}$ ) be the leaves adjacent to u (resp. t), with  $k, k' \ge 2$ . Let  $\ell$  be a leaf of H and v be its neighbour (of degree at least 3). Let x be the non-leaf neighbour of u (it exists since T is not a star and x = t if T is a bistar). See Figure 3 for a better view of these notations. Without loss of generality, we will assume  $k \ge k'$ .



Figure 3. Subcase (1.3)

If  $k \geq 3$ , consider the graph  $G' = G - \{u, u_1, \ldots, u_k, t_1, \ell\}$ . Hence G' is a (n - k - 3, n - k - 5)-graph satisfying  $|V(G')| \geq 3$  (since  $|V(H)| \geq 4$ ), and the induction hypothesis can be applied. Let  $\sigma'$  be a good permutation for G'. We now define a permutation  $\sigma$  for G as follows:

$$\sigma(u) = \begin{cases} t_1 & \text{if } \sigma'(x) = v, \\ \ell & otherwise. \end{cases} \quad \sigma(t_1) = \begin{cases} u & \text{if } \sigma'(x) = v, \\ u_1 & otherwise. \end{cases}$$

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$$\sigma(u_1) = \begin{cases} \ell & \text{if } \sigma'(x) = v, \\ t_1 & otherwise. \end{cases} \quad \sigma(\ell) = \begin{cases} u_1 & \text{if } \sigma'(x) = v, \\ u & otherwise. \end{cases}$$

 $\sigma(u_i)_{i>1} = u_i$ 

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This permutation remains good as (k+1) cycles of lengths 1 or 2 and (k+3)vertices have been added to  $\sigma'$  et G' respectively.

 $\sigma(v) = \sigma'(v)$  if  $v \in V(G')$ 

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If k = k' = 2, we first consider the case where x = t (i.e., T is a bistar). Consider the graph  $G' = G - \{u_1, t_1, \ell\}$  and proceed by induction as above. Since ut is an edge of T, we have  $\sigma'(u) \neq \sigma'(t)$  and u and t cannot be fixed points simultaneously. Therefore, at least  $u_1$  or  $t_1$  can be set as a fixed point in  $\sigma$ and the two remaining vertices in  $\{u_1, t_1, \ell\}$  can be images of each other (thus implying that  $\sigma$  is good).

 $\sigma(v) = \sigma'(v)$  if  $v \in V(G')$ 

In the case where  $x \neq t$ , consider the graph  $G' = G - \{u, u_1, u_2, t_1, t_2, \ell\}$ , and proceed as previously with the following permutation:

$$\sigma(u) = \begin{cases} t_1 & \text{if } \sigma'(t) = t, \\ t_1 & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ \ell & otherwise. \end{cases} \quad \sigma(t_1) = \begin{cases} u & \text{if } \sigma'(t) = t, \\ u & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ t_1 & otherwise. \end{cases}$$

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$$\sigma(t_2) = \begin{cases} \ell & \text{if } \sigma'(t) = t, \\ t_2 & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ t_2 & otherwise. \end{cases} \quad \sigma(\ell) = \begin{cases} t_2 & \text{if } \sigma'(t) = t, \\ \ell & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ u & otherwise. \end{cases}$$

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One can easily check that  $\sigma$  remains good since 6 vertices and at least 4 cycles have been added to  $\sigma'$ .

206 **Case 2:**  $|V(T)| \ge 3$  and  $H = K_1$ .

 $\sigma(u_i)_{i=1,2} = u_i$ 

**Subcase 2.1:** There exists a leaf-parent, say x, of degree at least 3. Let  $\ell$  be one of its leaves, and y be the unique vertex of H. Now consider the graph  $G' = G - \{x, \ell, y\}$ , which satisfies  $|E(G')| \leq |V(G')| - 2$ . Hence it admits a good permutation  $\sigma'$  by induction hypothesis. A good permutation  $\sigma$  of G can thus be extended from G' by setting  $\sigma(x) = y$ ,  $\sigma(y) = x$  and  $\sigma(\ell) = \ell$ .

**Subcase 2.2:** All the vertices of T which are adjacent to leaves are of degree 2. Let  $x_0$  be such a vertex (it exists since  $|V(T)| \ge 3$ ), let  $\ell_1$  be its adjacent leaf, and  $x_1$  its second neighbor. Now let  $\ell_2$  be a distinct leaf from  $\ell_1$  in T, and  $x_2$  be its neighbor. If  $x_1 \ne \ell_2$  and  $x_1 \ne x_2$ , then consider  $G' = G - \{x_0, \ell_1, \ell_2\}$ , which admits a good permutation  $\sigma'$  by induction hypothesis. Then set  $\sigma_{|G'} = \sigma'$  and

• If 
$$\sigma'(x_1) = x_1$$

$$\sigma(x_0) = \ell_2, \ \sigma(\ell_2) = x_0, \ \text{and} \ \sigma(\ell_1) = \ell_1.$$

• If 
$$\sigma'(x_1) \neq x_1$$
:

$$\sigma(x_0) = x_0, \, \sigma(\ell_2) = \ell_1, \text{ and } \sigma(\ell_1) = \ell_2$$

One can now easily check that  $\sigma$  is good for G. If  $x_1 = \ell_2$  or  $x_1 = x_2$ , then Tis either a  $P_3$  or a  $P_4$ . Since  $n \ge 6$ , it implies that G admits at least another connected component which is an (n, n - 1) or an (n, n) connected graph with  $n \ge 2$  (since  $|E(G)| \ge 4$ , there exists at least one edge in G - T). In other words, this component is either a tree T', or a tree with an edge  $T' \cup \{e\}$ . Let  $\ell_3$  be a leaf in T'. Note that we do not care whether  $\ell_3$  is adjacent to e or not. By considering  $G' = G - \{x_0, \ell_1, \ell_3\}$  together with the above permutation where  $\ell_2$  <sup>229</sup> is replaced by  $\ell_3$ , we find a good permutation for G.

**Case 3:** |V(T)| = 2. Let  $T = (x_0, x_1)$  and let y be a vertex of degree 2 of G. Such a vertex exists since  $n \ge 6$ . Consider the graph  $G' = G - \{x_0, x_1, y\}$ . By induction hypothesis, there exists a good permutation for G', say  $\sigma'$ . We set  $\sigma(x_0) = y, \sigma(y) = x_0, \sigma(x_1) = x_1$  and for every vertex  $v \in V(G'), \sigma(v) = \sigma'(v)$ , which defines a good permutation for G.

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**Case 4:** |V(T)| = 1. In this case, G contains isolated vertices (at least two) and non-tree connected components. Two subcases are considered as follows:

**Subcase 4.1:** *G* has a vertex, say *x*, of degree at least 3. Let *y* and *z* be two isolated vertices of *G*. Consider the graph  $G' = G - \{x, y, z\}$ . The induction hypothesis guarantees the existence of a good permutation  $\sigma'$  for G'. By putting  $\sigma(x) = y, \sigma(y) = x, \sigma(z) = z$  and for every vertex  $v \in V(G'), \sigma(v) = \sigma'(v)$ , we get a good permutation for *G*.

**Subcase 4.2:** The complementary subcase to (4.1), i.e., G is the sum of two isolated vertices and an union of cycles. This case is solved as follows:

(a)  $G = kC_3 \cup 2K_1$  for some  $k \ge 1$ . Lemma 4 allows us to conclude.

(b) G has at least one cycle, say H, of order at least 4, and one cycle, say Q, of order at least 3: let  $(x_1, x_2, x_3)$  be an induced path of H, let  $x_4$  be a vertex of Q and z, t be the two isolated vertices of G. Denote by x (resp. y) the neighbor of  $x_1$  (resp.  $x_3$ ) different from  $x_2$ . Note that we may have x = y in the case  $H = C_4$ . See Figure 4 for a graphical depiction of these notations.



Figure 4. Case (4.2.b)

<sup>252</sup> Consider the graph  $G' = G - \{x_1, x_2, x_3, x_4, z, t\}$ . Since  $|V(G)| \ge 9$ , we <sup>253</sup> have  $|V(G')| \ge 3$  and the induction hypothesis guarantees the existence of a <sup>254</sup> good permutation  $\sigma'$  for G'. The permutation  $\sigma'$  can be extended to a good <sup>255</sup> permutation  $\sigma$  of G by setting  $\sigma(t) = t$ , and

Hence  $\sigma_{|G-G'|}$  has four cycles of size at most 2, and  $\sigma$  is thus good for G.

(c) G is the sum of  $C_m$  (for some  $m \ge 4$ ) and two isolated vertices. If m < 8, then Figure 5 shows labeled embeddings corresponding to good permutations.

$$\sigma(x_1) = \begin{cases} x_1 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_4 & \text{if } \sigma'(x) = x, \\ z & \text{otherwise.} \end{cases} \qquad \sigma(x_2) = \begin{cases} x_4 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_2 & \text{otherwise.} \end{cases}$$

$$\sigma(x_3) = \begin{cases} x_3 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ z & \text{if } \sigma'(x) = x, \\ x_4 & \text{otherwise.} \end{cases} \qquad \sigma(x_4) = \begin{cases} x_2 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_1 & \text{if } \sigma'(x) = x, \\ x_3 & \text{otherwise.} \end{cases}$$

$$\sigma(z) = \begin{cases} z & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_3 & \text{if } \sigma'(x) = x, \\ x_1 & \text{otherwise.} \end{cases} \qquad (4)$$

Figure 5. Case  $C_m \cup 2K_1$  for  $m = 4, \ldots, 7$ 

If  $m \geq 8$ , let  $(x_1, \ldots, x_8)$  be a path of  $C_m$ . Let z, t be the two isolated vertices of G. We consider the graph  $G' = G - \{x_2, x_3, x_6, x_7, z, t\}$  which admits a good permutation  $\sigma'$  by induction hypothesis. Since  $v_4$  and  $v_5$  are adjacent, at least one of them is not a fixed point under  $\sigma'$ . Without loss of generality, assume  $\sigma'(x_4) \neq x_4$ . The permutation  $\sigma'$  can be extended to a good permutation  $\sigma$  for G as follows: set  $x_3$  and t as fixed points. If  $\sigma'(x_5) \neq x_1$ , we set  $\sigma(x_2) = x_6$ ,  $\sigma(x_6) = x_2, \ \sigma(x_7) = z$ , and  $\sigma(z) = x_7$ . Otherwise, we set  $\sigma(x_2) = x_7, \ \sigma(x_7) = x_6$ . For the same reasons as in case (4.2.*b*), this permutation is good for *G*.

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#### CONCLUSION

Theorem 5 gives a first lower bound about the labeled embedding number of 270 (n, n-2)-graphs. Yet, the computation of the exact value remains an open 271 question, as this bound is not exact for many families of (n, n-2)-graphs. As an 272 example, consider a cycle  $C_n$  without two edges. Its labeled packing number is 273 at least the one of  $C_n$ , (i.e., |3n/4|). Yet, for any large value of n, we can find an 274 (n, n-2)-graph for which the bound is tight. Indeed, consider G as an union of k 275 disjoint triangles with  $K_2 \cup K_1$ . The size of a maximum independent set for this 276 graph equals k+2. According to Lemma 1, we have that  $\lambda_2(G) = 2k+2 = \lfloor 2n/3 \rfloor$ . 277 In addition, we mention that this result can be used to study the labeled 278 embedding of (n, n-1)-graphs. One can show for example that the same bound 279 is valid for the union of cycles with a single tree. 280

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