Spreading in kinetic reaction-transport equations in higher velocity dimensions

Emeric Bouin, Nils Caillerie

▶ **To cite this version:**

Emeric Bouin, Nils Caillerie. Spreading in kinetic reaction-transport equations in higher velocity dimensions. European Journal of Applied Mathematics, 2019, 30 (2), pp.219-247. 10.1017/S0956792518000037 . hal-01518398v2

HAL Id: hal-01518398
https://hal.science/hal-01518398v2

Submitted on 10 Jul 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Spreading in kinetic reaction-transport equations in higher velocity dimensions

Emeric Bouin * Nils Caillerie †

July 10, 2017

Abstract

In this paper, we extend and complement previous works about propagation in kinetic reaction-transport equations. The model we study describes particles moving according to a velocity-jump process, and proliferating according to a reaction term of monostable type. We focus on the case of bounded velocities, having dimension higher than one. We extend previous results obtained by the first author with Calvez and Nadin in dimension one. We study the large time/large scale hyperbolic limit via an Hamilton-Jacobi framework together with the half-relaxed limits method. We deduce spreading results and the existence of travelling wave solutions. A crucial difference with the mono-dimensional case is the resolution of the spectral problem at the edge of the front, that yields potential singular velocity distributions. As a consequence, the minimal speed of propagation may not be determined by a first order condition.

Key-words: Kinetic equations, travelling waves, dispersion relation
AMS Class. No: 35Q92, 45K05, 35C07

1 Introduction

The model.

In this paper, we are interested in propagation phenomena occuring in the following reaction-transport equation

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = M(v)\rho(t,x) - f(t,x,v) + r\rho(t,x)\left(M(v) - f(t,x,v)\right), \\ \qquad\qquad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V, \\ f(0,x,v) = f_0(x,v), \qquad\qquad (x,v) \in \mathbb{R}^n \times V, \end{cases} \tag{1.1}$$

where $r > 0$. The mesoscopic density f depends on time $t \in \mathbb{R}^+$, position $x \in \mathbb{R}^n$ and velocity $v \in V$ and describes a population of individuals. The macroscopic density is $\rho(t,x) = \int_V f(t,x,v)\,dv$. The subset $V \subset \mathbb{R}^n$ is the set of all possible velocities. From now on, we assume

*CEREMADE - Université Paris-Dauphine, UMR CNRS 7534, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France. E-mail: bouin@ceremade.dauphine.fr

†Institut Camille Jordan (ICJ), Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France. E-mail: caillerie@math.univ-lyon1.fr

(H0) The velocity set $V \subset \mathbb{R}^n$ is compact.

For any given direction $e \in \mathbb{S}^{n-1}$, we define

$$\overline{v}(e) = \max\{v \cdot e, v \in V\}, \qquad \mu(p) = |p|\overline{v}\left(\frac{p}{|p|}\right), \quad \operatorname{Arg}\mu(p) = \{v \in V \mid v \cdot p = \mu(p)\}.$$

We set

$$v_{max} := \sup_{v \in V} |v|, \qquad |V| := \int_V dv.$$

Individuals move following a so-called velocity-jump process. That is, they alternate successively a run phase, with velocity $v \in V$, and a change of velocity at rate 1, which we call the tumbling. The new velocity is chosen according to the probability distribution M. Throughout the paper, we assume

(H1) $M \in L^1(V)$, and

$$\langle v \rangle_M := \int_V vM(v)dv = 0. \tag{1.2}$$

Note that it is challenging to replace the linear BGK operator $M\rho - f$ by a more general collision operator of the form $P(f) - \Sigma f$ where P is a positive operator. However, to remain consistent with [11], we will stick to their framework and leave this question for future work.

Remark 1.1. *In fact, our analysis can easily be extended to the case* $\langle v \rangle_M \in \mathbb{R}^n \setminus 0$. *Setting* $\mathbb{V} := V - \langle v \rangle_M$, $\mathbb{M}(w) := M(w + \langle v \rangle_M)$ *and* $\mathbb{F}(t, x, w) := f(t, x + \langle v \rangle_M t, w + \langle v \rangle_M)$, *for all* $(t, x, w) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{V}$, *we recover our assumptions in the new framework.*

The reproduction of individuals is taken into account through a reaction term of monostable type. The constant $r > 0$ is the growth rate in absence of any saturation. New individuals start with a velocity chosen at random with the same probability distribution M. The quadratic saturation term accounts for local competition between individuals, regardless of their speed.

We assume that initially $0 \leq f_0 \leq M$, so that this remains true for all times, see [11, 19].

Earlier works and related topics

It is relatively natural to address the question of spreading for (1.1) since there is a strong link between (1.1) and the classical Fisher-KPP equation [23, 30]. Indeed, a suitable parabolic rescaling

$$\varepsilon^2 \partial_t g_\varepsilon + \varepsilon v \cdot \nabla_x g_\varepsilon = (M(v)\rho_{g_\varepsilon} - g_\varepsilon) + \varepsilon^2 r \rho_{g_\varepsilon} (M(v) - g_\varepsilon), \tag{1.3}$$

leads to the Fisher-KPP equation (see [19] for example) in the limit $\varepsilon \to 0$,

$$\begin{aligned} &\partial_t \rho^0 - \left\langle v^2 \right\rangle_M \partial_{xx} \rho^0 = r\rho^0 \left(1 - \rho^0\right), \\ &g^0 := \lim_{\varepsilon \to 0} g_\varepsilon = M\rho^0, \end{aligned} \tag{1.4}$$

assuming that the two following conditions on M hold:

$$\int_V vM(v)dv = 0, \quad \left\langle v^2 \right\rangle_M := \int_V v^2 M(v)dv > 0.$$

We recall that for nonincreasing initial data decaying sufficiently fast at $x = +\infty$, the solution of (1.4) behaves asymptotically as a travelling front moving at the minimal speed $c^* = 2\sqrt{r\,\langle v^2\rangle_M}$ [1, 30]. However, even though the philosophy of the results will be the same in spirit, we emphasize that nothing related to this parabolic limit will be used in the present paper. Our argumentation does not rely on any perturbative analysis. Hence, we obtain results without any smallness assumption on the parameters. This will yield significant differences, regarding both the results and the methods of proof.

A short review of earlier results is now in order. Hadeler has worked on propagation for reaction-telegraph equations [26, 27], that can be seen as two-speeds kinetic models. Morever, a similar type of result was obtained by Cuesta, Hittmeir and Schmeiser [19] in the diffusive regime (*i.e.* for sufficiently small ε in (1.1)). Using a micro-macro decomposition, they constructed possibly oscillatory travelling waves of speed $c \geq 2\sqrt{rD}$ for ε small enough (depending on c). In addition, when the set of admissible speeds V is bounded, $c > 2\sqrt{rD}$, and ε is small enough, they prove that the travelling wave constructed in this way is indeed nonnegative.

Propagation for the full kinetic model (1.1) has then been investigated by the first author with Calvez and Nadin in [11]. In one dimension of velocities, and when the velocities are bounded, they proved the existence and stability of travelling waves solutions to (1.1). The minimal speed of propagation of the waves is determined by the resolution of a spectral problem in the velocity variable. In particular, it is not related with the KPP speed, except that the speeds coincide in the diffusive regime. It is worth mentioning that the case of unbounded velocities is significantly different as the front spreads with arbitrarily large speed [11]. This case shall not be discussed further in this paper. This phenomenon was newly appearing for this type of equations and unexpected from the macroscopic limit. One aim of this paper is to extend the construction of travelling waves solutions to any velocity dimension, which was left open after [11].

There is a strong link between this KPP type propagation phenomena and large deviations for the underlying velocity-jump process. Indeed, it is well known that fronts in Fisher-KPP equations are so-called *pulled fronts*, that is, are triggered by very small populations at the edge that are able to reproduce almost exponentially. Thus, studying large deviations for these type of processes at the kinetic level is an interesting problem in itself. In [7,8], the authors have combined Hamilton-Jacobi equations and kinetic equations to study large deviations (and propagation) from a PDE point of view. These works show that he asymptotics of large deviations in the kinetic equation do not coincide with the asymptotic of large deviations obtained after a diffusive approximation.

As a side note, the Hamilton-Jacobi technique (that will be described in the next subsection) has also much been used recently to study long time dynamics in all sorts of stuctured models. An interested reader could describe the evolution of dominant phenotypical traits in a given population reading [4, 12, 31] and the references therein), study different adaptative dynamics issues [20], describe propagation in reaction-diffusion models of kinetic types [10] but also in age renewal equations [16]. This approach has also recently been used to study large deviations of velocity jump-processes [8, 9, 15] or slow-fast systems [13, 14, 22, 29, 32].

The Hamilton-Jacobi limit

After the seminal paper by Evans and Souganidis [21, 24], an important technique to derive the propagating behavior in reaction-diffusion equations is to revisit the WKB expansion to study hyperbolic limits. We will directly present the technique on our problem for conciseness but one can find the original framework for the Fisher-KPP equation in [21] and complements in [2,3,18,34].

We perform the hyperbolic scaling $(t,x,v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$ in (1.1). Importantly, the velocity variable is not rescaled (it cannot be rescaled since it lies in a bounded set). The *kinetic Hopf-Cole transformation* (already used in [8, 15]) is written

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad f^\varepsilon(t,x,v) = M(v) e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}. \tag{1.5}$$

Thanks to the maximum principle [19], φ^ε is well defined and remains nonnegative for all times. Plugging (1.5) in (1.1), one obtains the following equation for φ^ε:

$$\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon + r = (1+r) \int_V M(v') \left(1 - e^{\frac{\varphi^\varepsilon(v) - \varphi^\varepsilon(v')}{\varepsilon}}\right) dv' + r\rho^\varepsilon. \tag{1.6}$$

Our aim is to pass to the limit in (1.6). To make the convergence result appear naturally, we shall start by providing formal arguments. Assuming Lipschitz bounds on φ^ε, and since ρ^ε is uniformly bounded, the boundedness of $\int_V M(v')(1 - \exp((\varphi^\varepsilon(v) - \varphi^\varepsilon(v'))/\varepsilon) dv'$ implies that we expect the limit φ^0 to be independent of v. To identify the limit φ^0, we shall thus perform the following expansion

$$\varphi^\varepsilon(t,x,v) = \varphi^0(t,x) + \varepsilon \eta(t,x,v). \tag{1.7}$$

Plugging the latter into (1.7) yields

$$\partial_t \varphi^0 + v \cdot \nabla_x \varphi^0 + r = (1+r) \int_V M(v') \left(1 - e^{\eta(v) - \eta(v')}\right) dv' + r e^{-\frac{\varphi^0}{\varepsilon}} \int_V e^{-\eta(v')} dv'.$$

As a consequence, for any $(t,x) \in \left\{\varphi^0 > 0\right\}$, we have

$$\partial_t \varphi^0 + v \cdot \nabla_x \varphi^0 = 1 - e^{\eta(v)} (1+r) \int_V M(v') e^{-\eta(v')} dv'. \tag{1.8}$$

One should read this equation as an eigenvalue problem in the velocity variable. Indeed, setting

$$p(t,x) = \nabla_x \varphi^0(t,x), \qquad \eta(t,x,v) := -\ln\left(\frac{Q_{p(t,x)}}{M(v)}\right), \qquad H(p(t,x)) := -\partial_t \varphi^0(t,x),$$

we see that (H,Q) are the principal eigenelements of the following spectral problem

$$(1+r) M(v) \int_V Q_p(v')\, dv' - (1 - v \cdot p)\, Q_p(v) = H(p) Q_p(v).$$

The dependency with respect to r can be identified by setting $p' := \frac{p}{1+r}$, $\mathcal{H}(\cdot) := \frac{H((1+r)\cdot) - r}{1+r}$ and $\widetilde{Q}_{p'} = Q_p$. Indeed, we have then that $\partial_t \varphi^0 + (r+1)\mathcal{H}(\frac{p}{r+1}) + r = 0$ and the Hamiltonian $\mathcal{H}$ is given by

$$\left(1 + \mathcal{H}\left(p'\right) - v \cdot p'\right) \widetilde{Q}_{p'}(v) = M(v) \int_V \widetilde{Q}_{p'}(v')\, dv'. \tag{1.9}$$

After these heuristics, we are now ready to define properly the Hamiltonian $\mathcal{H}$ involved.

Definition 1.2. *We define, for* $e \in \mathbb{S}^{n-1}$,

$$l(e) = \int_V \frac{M(v)}{\overline{v}(e) - v \cdot e} dv.$$

The so-called singular set is defined by

$$\mathrm{Sing}\,(M) := \left\{ p \in \mathbb{R}^n, \int_V \frac{M(v)}{\mu(p) - v\cdot p} dv \le 1 \right\} = \left\{ p \in \mathbb{R}^n,\, l\left(\frac{p}{|p|}\right) \le |p| \right\}. \tag{1.10}$$

Then, the Hamiltonian $\mathcal{H}$ involved in this paper is given as follows:

- *If $p \notin \mathrm{Sing}\,(M)$, then $\mathcal{H}$ is uniquely defined by the following implicit relation :*

$$\int_V \frac{M(v)}{1 + \mathcal{H}(p) - v\cdot p} dv = 1, \tag{1.11}$$

- *else, $\mathcal{H}(p) = \mu\,(p) - 1$.*

The relevancy of such a definition, *i.e.* the resolution of (1.9), will be discussed in Section 2 below. With this definition in hand, the convergence result for the sequence of functions φ^ε is as follows.

Theorem 1.3. *Suppose that (H0) and (H1) hold, and that the initial data satisfies*

$$\forall (x,v) \in \mathbb{R}^n \times V, \qquad \varphi^\varepsilon(0,x,v) = \varphi_0(x,v).$$

Then, $(\varphi^\varepsilon)_\varepsilon$ converges uniformly on all compacts of $\mathbb{R}_+^* \times \mathbb{R}^n \times V$ towards φ^0, where φ^0 does not depend on v. Moreover φ^0 is the unique viscosity solution of the following Hamilton-Jacobi equation:

$$\begin{cases} \min\left\{ \partial_t \varphi^0 + (1+r)\mathcal{H}\left(\frac{\nabla_x \varphi^0}{1+r}\right) + r, \varphi^0 \right\} = 0, & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ \varphi^0(0,x) = \min\limits_{v\in V} \varphi_0(x,v), & x \in \mathbb{R}^n. \end{cases} \tag{1.12}$$

Let us now emphasize the differences between the result presented here and the very related works [7, 8, 15]. First, the results from [8] and [7] only hold for $n = 1$ and for $M \ge \delta > 0$. In [7], the first author successfully proved a convergence result in the case $r > 0$. It is worth mentioning that a much wider class of collision operators was considered in [7], but under the condition of existence of a L^1 eigenvector. We believe that the ideas of the present work could be used there, but with technicalities inherent from the spectral problem that would require a special study.

As explained before, the multidimensional case ($n > 1$) is more delicate since the relation (1.11) may not have a solution. We refer to our Example 2.3 for a situation where this happens. In [15], the second author generalized the convergence result of [8] in the multidimensional case, with no reaction term. However, the proof we design in this paper is simpler and more adaptable. For this we manage to use the half-relaxed limits of Barles and Perthame [5] in the spirit of [12]. We point out that an asymptotic preserving scheme has been developed by Hivert in [28] to numerically solve (1.6) using the Hamilton-Jacobi framework developed in [7]. We present the proof of Theorem 1.3 in Section 2 below.

Travelling wave solutions and spreading of planar like initial data

We then investigate the existence of travelling wave solutions of (1.1). As in the mono-dimensional case treated in [11], we will prove that there exists a minimal speed c^* for which travelling wave solutions exist. We will use the following definition throughout the paper.

Definition 1.1. *A function f is a travelling wave solution of speed $c \in \mathbb{R}_+$ and direction $e \in \mathbb{S}^{n-1}$ of equation* (1.1) *if it can be written $f(t,x,v) = \tilde{f}\,(x \cdot e - ct, v)$, where the profile $\tilde{f} \in \mathcal{C}^2\left(\mathbb{R}, L^1(V)\right)$ solves*

$$(v \cdot e - c)\,\partial_\xi \tilde{f} = M(v)\tilde{\rho} - \tilde{f} + r\tilde{\rho}\left(M(v) - \tilde{f}\right) \tag{1.13}$$

and satisfies

$$\forall (z,v) \in \mathbb{R} \times V\,, \quad 0 \leq \tilde{f}(z,v) \leq M(v)\,, \quad \lim_{z \to -\infty} \tilde{f}(z,v) = M(v)\,, \quad \lim_{z \to +\infty} \tilde{f}(z,v) = 0\,. \tag{1.14}$$

It is well known for this kind of Fisher-KPP type problems that propagation fronts are so-called pulled fronts, that is the speed of propagation is given by seeking exponentially decaying solutions of the linearized problem in a moving frame. As a consequence, for any $\lambda > 0$, one can define $c(\lambda, e)$ using the spectral problem solved in Definition 1.2. Indeed, we set

$$c(\lambda, e) = (1+r)\mathcal{H}\left(\frac{\lambda e}{1+r}\right) + r. \tag{1.15}$$

Then we have the formula for the minimal speed in the direction $e \in \mathbb{S}^{n-1}$.

$$c^*(e) = \inf_{\lambda > 0} c(\lambda, e)\,.$$

We obtain the following existence result.

Theorem 1.4. *Let $e \in \mathbb{S}^{n-1}$. For all $c \in [c^*(e), \overline{v}(e))$, there exists a travelling wave solution of* (1.1) *with speed c and direction e. Moreover, there exists no positive travelling wave solution of speed $c \in [0, c^*(e))$.*

Following very closely the proof used in the mono-dimensional case, we shall prove this Theorem using sub and super-solution and a comparison principle satisfied by (1.1). We shall construct these sub- and super-solution using travelling wave solutions of the linearized problem. The main difference concerning the travelling wave result is the way we prove the minimality of the speed $c^*(e)$. Indeed, it might happen that $c(\lambda, e)$ is singular at its minimum λ^* so that one can not reproduce the same argument as for the mono-dimensional case used in [11], that was based on the Rouché Theorem. Using the Hamilton-Jacobi framework above, in a similar fashion as in [**?**], we prove the following result.

Proposition 1.5. *Let f_0 be a non-zero initial data, compactly supported in some direction e_0, such that there exists $\gamma < 1$ such that*

$$\gamma M(v)\mathbf{1}_{[-x_m, x_m]\cdot e_0 + e_0^\perp}(x) \leq f_0(x,v) \leq M(v)\mathbf{1}_{[-x_M, x_M]\cdot e_0 + e_0^\perp}(x),$$

for all $(x,v) \in \mathbb{R}^n \times V$. Let f be the solution of the Cauchy problem (1.1) *associated to this initial data. Then we have*

$$\lim_{t \to +\infty} \sup_{x \cdot e_0 > ct} \rho(t,x) = 0, \ \text{if } c > c^*(e_0), \tag{1.16}$$

$$\lim_{t \to +\infty} f(t, e_0^\perp + cte_0, v) = M(v)\,, \ \text{if } c < c^*(e_0), \tag{1.17}$$

uniformly in $v \in V$.

Spreading of compactly supported initial data

Finally, we also deduce from the Hamilton-Jacobi framework a spreading result for initial conditions that are compactly supported. To this aim, let us first define the speed $w^*(e_0)$ associated to any direction $e_0 \in \mathbb{S}^{n-1}$ via the following Freidlin-Gärtner formula (see [25] for its first derivation).

$$w^*(e_0) = \min_{\substack{e\in\mathbb{S}^{n-1}\\ e_0\cdot e>0}} \left(\frac{c^*(e)}{e_0\cdot e}\right).$$

We obtain the following result.

Proposition 1.6. *Let f^0 be a non-zero compactly supported initial data such that $0 \leq f_0(x,v) \leq M(v)$ for all $(x,v) \in \mathbb{R}^n \times V$. Let f be the solution of the Cauchy problem (1.1) associated to this initial data. Then for any $e_0 \in \mathbb{S}^{n-1}$ and all $x \in \mathbb{R}^n$, we have*

$$\lim_{t\to\infty} f(t, x + cte_0, v) = 0, \qquad \text{if } c > w^*(e_0), \tag{1.18}$$

pointwise and

$$\lim_{t\to\infty} f(t, cte_0, v) = M(v), \qquad \text{if } 0 \leq c < w^*(e_0), \tag{1.19}$$

for all $v \in V$.

This result is interesting since contrary to the case of the usual Fisher-KPP equation in heterogeneous domains, where the Freidlin-Gärtner formula holds, see [33], here there is no heterogeneity in space. The heterogeneity coming potentially from the velocity set, this would not be present in the macroscopic limit (the Fisher-KPP equation, see above). Of course, if V is rotationally symmetric, the speed w^* is independent of the direction, and the propagation is radial.

One could wonder how the shape of V (*e.g.* its topological properties or its convexity) influences the shape of the front and the speed of propagation. We will investigate this question in a future work.

The rest of this paper is organized as follows. In Section 2, we prove the Hamilton-Jacobi limit. We discuss the construction of travelling waves and the spreading results in Section 3.

Acknowledgements

This work has started after a discussion with Olivier Benichou and Raphaël Voituriez. The authors thank warmly Vincent Calvez for early discussions about this problem, and for a careful reading of the manuscript. This work has benefited of an invaluable insight from Guy Barles, which, while originally meant for another manuscript, found a direct application in the present one. The authors are more than deeply grateful to Guy Barles for this. EB acknowledges the green hospitality of the University of Cambridge during the second semester of the academic year 2015-2016. EB and NC acknowledge the support of the ERC Grant MATKIT (ERC-2011-StG). NC thanks the University of Cambridge for its hospitality during his three week stay over Spring 2016. In addition, NC has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No 639638). EB dedicates this work to Paul-Antoine Girard, as an encouragement to never give up and always slide towards his highest ambitions.

2 The Hamilton-Jacobi limit

In this Section, we present the proof of the convergence result Theorem 1.3. We then prove a convergence result for ρ^ε in the region $\{\varphi^0 = 0\}$. This result will help us to show that the speed of propagation is still the minimal speed of existence of travelling waves, despite the singularity of $\mathcal{H}$.

2.1 The spectral problem

In this Section, we discuss the resolution of the spectral problem given by (1.9). We also provide examples for which the singular set of M is not empty.

2.1.1 The resolution

For any $p' > 0$, we look for an eigenvalue $\mathcal{H}(p')$ associated to a positive eigenvector $\widetilde{Q}_p$ such that

$$\left(1 + \mathcal{H}(p') - v \cdot p'\right) \widetilde{Q}_{p'}(v) = M(v) \int_V \widetilde{Q}_{p'}(v')\, dv', \qquad v \in V.$$

Note that it may happen that $\widetilde{Q}_{p'}$ has a singular part. Since the problem is linear, one can always assume that $\widetilde{Q}_{p'}$ is a probability measure. We are thus led to find an eigenvalue $\mathcal{H}(p')$ such that there exists a probability measure $\widetilde{Q}_{p'}$[1] such that

$$\left(1 + \mathcal{H}(p') - v \cdot p'\right) \widetilde{Q}_{p'}(v) = M(v), \qquad v \in V.$$

To make the singular set $\mathrm{Sing}\,(M)$ appear naturally, let us first investigate the case when $\widetilde{Q}_{p'} \in L^1(V)$. If a solution exists, then the profile $\widetilde{Q}_{p'}$ necessarily satisfies the following equation:

$$\widetilde{Q}_{p'}(v) = \frac{M(v)}{1 + \mathcal{H}(p') - v \cdot p'}, \qquad v \in V. \tag{2.20}$$

This is only possible if such an expression defines a probability measure. As a consequence, one shall look for conditions under which there exists $\mathcal{H}(p')$ such that

$$I\left(\mathcal{H}(p'), p'\right) := \int_V \frac{M(v')}{1 + \mathcal{H}(p') - v' \cdot p'} dv' = 1,$$

with $1 + \mathcal{H}(p') - v' \cdot p > 0$ for all $v' \in V$, that is $\mathcal{H}(p') > \mu(p') - 1$.

For any $p' \notin \mathrm{Sing}(M)$, since the function $\xi \mapsto I(\xi, p')$ is decreasing over $(\mu(p') - 1, +\infty)$, $\mathcal{H}(p')$ exists and is unique in this interval since $I(\mu(p') - 1, p') > 1$, by the definition of p' not being in the singular set.

However, for any $p' \in \mathrm{Sing}(M)$, it is not possible to solve $I\left(\mathcal{H}(p'), p'\right) = 1$ since $I(\mu(p') - 1, p') \leq 1$. After Theorem 1.2 in [17], there exists a solution to (1.9), given by the couple $(\mathcal{H}(p'), \widetilde{Q}_{p'})$ where $\mathcal{H}(p') = \mu(p') - 1$ and $\widetilde{Q}_{p'}$ is a positive measure given by:

$$\widetilde{Q}_{p'} := \frac{M(v)}{\mu(p') - v \cdot p'} dv + \left(1 - \int_V \frac{M(v)}{\mu(p') - v \cdot p'} dv\right) \delta_w,$$

where δ_w is the dirac mass located at $w \in \mathrm{Arg}\,\mu(p')$.

From [15], we know that the set $\mathrm{Sing}(M)^c$ is convex and contains 0. To identify the different cases where such a singularity set may occur, we detail three examples hereafter.

[1] To avoid too many notation, we identify $\widetilde{Q}_{p'}$ to its density when relevant.

2.1.2 Examples

Example 2.1. *In the one-dimensional case ($n = 1$), we have* $\mathrm{Sing}(M) = \emptyset$ *when* $\inf_{v\in V} M(v) > 0$ *since*

$$\int_{-\overline{v}}^{\overline{v}} \frac{M(v')}{\mu(p') - vp'} dv = \int_{-\overline{v}}^{\overline{v}} \frac{M(v)}{|p'|v_{max} - vp'} dv \geq \frac{\inf M(v)}{|p'|\overline{v}} \cdot \int_{-1}^{1} \frac{dv}{1-v} = +\infty. \tag{2.21}$$

By monotone convergence we have

$$\lim_{H\to\mu(p')-1} \int_{-\overline{v}}^{\overline{v}} \frac{M(v)}{1 + H - vp'} dv = +\infty,$$

hence, for all $p' \in \mathbb{R}$, there exists a unique $\mathcal{H}(p')$ that solves the spectral problem in $L^1(V)$.

This latter framework is the one used in [11]. In fact, we can only require that M does not cancel in a neighborhood of $v = \overline{v}$ in order to get $\mathrm{Sing}(M) = \emptyset$. Indeed, the integral in (2.21) will also diverge in that scenario. If $M(\overline{v}) = 0$, this argument may not work out. Consider for example:

Example 2.2. *Let $n = 1$, $V = [-1, 1]$ and $M(v) = \frac{3}{2}(1 - |v|)^2$. Then,*

$$\begin{aligned} l(1) = \int_{-1}^{1} \frac{M(v)}{1-v}\, dv &= \frac{3}{2}\int_{-1}^{1} \frac{(1-|v|)^2}{1-v}\, dv = 3\int_0^1 \frac{(1-v)^2}{(1-v)(1+v)}\, dv \\ &= 3\int_0^1 \frac{1-v}{1+v}\, dv = 3\int_0^1 \frac{2-(1+v)}{1+v}\, dv \\ &= 3\left(\int_0^1 \frac{2}{1+v}\, dv - 1\right) = 3(2\ln(2) - 1). \end{aligned}$$

Hence, $|p'| \geq 3(2\ln(2) - 1)$ if and only if

$$\int_{-1}^{1} \frac{M(v)}{\mu(p') - vp'} dv = \frac{3}{2|p'|}\int_{-1}^{1} \frac{(1-|v|)^2}{1-v} dv \leq 1,$$

therefore, $\mathrm{Sing}(M) = (-3(2\ln(2) - 1), 3(2\ln(2) - 1))^c$. *Let us also notice that*

$$\begin{aligned} \int_V \frac{M(v)}{(\overline{v}(1) - v)^2} dv &= \frac{3}{2}\int_{-1}^{1} \frac{(1-|v|)^2}{(1-v)^2} dv = 3\int_0^1 \frac{1+v^2}{(1+v)^2} dv \\ &= 3\int_0^1 \frac{(1+v)^2 - 2(1+v) + 2}{(1+v)^2} dv \\ &= 3\int_0^1 \left(1 - \frac{2}{1+v} + \frac{2}{(1+v)^2}\right) dv \\ &= 3(1 - 2\ln(2) + 1) = 6(1 - \ln(2)) < +\infty. \end{aligned}$$

We will make a use of this result later.

In the multi-dimensional case, we may encounter a singular set, even when $\inf_{v\in V} M(v) > 0$. These singularities can occur in the simplest cases.

Example 2.3. *Let $n \geq 1$, let $V = B(0,1)$ be the n-dimensional unit ball. Let $e = e_1$ and $M = \omega_n^{-1}.\mathbb{1}_{\overline{B(0,1)}}$, where ω_n is the Lebesgue measure of V. For $n = 1$, since $M > 0$ we have $\mathrm{Sing}(M) = \emptyset$ (recall example 2.1). Suppose now that $n > 1$. Then,*

$$\begin{aligned} l(e_1) &= \int_{B(0,1)} \frac{M(v)}{\overline{v}(e_1) - v \cdot e_1} dv \\ &= \frac{1}{\omega_n} \int_{B(0,1)} \frac{1}{1 - v_1} dv \\ &= \frac{1}{\omega_n} \int_{-1}^{1} \frac{1}{1 - v_1} \left(\int \mathbb{1}_{\left\{v_1^2 + v_2^2 + \ldots + v_n^2 \leq 1\right\}} (v_2, \ldots, v_n) dv_2 \ldots dv_n \right) dv_1. \end{aligned}$$

Now, for fixed v_1, the quantity $\int \mathbb{1}_{\left\{v_1^2 + v_2^2 + \ldots + v_n^2 \leq 1\right\}} (v_2, \ldots, v_n) dv_2 \ldots dv_n$ is the Lebesgue measure of the $(n-1)$-dimensional ball of radius $\sqrt{1 - v_1^2}$, hence

$$\int \mathbb{1}_{\left\{v_1^2 + v_2^2 + \ldots + v_n^2 \leq 1\right\}} (v_2, \ldots, v_n) dv_2 \ldots dv_n = \omega_{n-1} \times \left(\sqrt{1 - v_1^2} \right)^{n-1}.$$

Finally,

$$\begin{aligned} l(e_1) &= \frac{\omega_{n-1}}{\omega_n} \int_{-1}^{1} \frac{\left(1 - v_1^2\right)^{\frac{n-1}{2}}}{1 - v_1} dv_1 \\ &= \frac{2\omega_{n-1}}{\omega_n} \int_{0}^{1} \left(1 - v_1^2\right)^{\frac{n-3}{2}} dv_1 \\ &= \frac{2\omega_{n-1}}{\omega_n} \int_{0}^{\frac{\pi}{2}} (\cos(\theta))^{n-2} d\theta, \\ &= \frac{n}{n-1}, \end{aligned}$$

where we have used, for example, the relationship between the volume of the unit ball and the Wallis integrals. By rotational invariance, $\mathrm{Sing}(M) = B\left(0, \frac{n-1}{n}\right)^c$.

2.2 Proof of Theorem 1.3

In this Section, we now prove Theorem 1.3. We will use the half-relaxed limits method of Barles and Perthame [5]. In addition to that, and similarly to the papers [8, 15], we need to use the perturbed test-function method. We emphasize that the corrected test function is defined thanks to the spectral problem (1.9), keeping only the regular part of the eigenfunction (recall that it may have singularities).

Since the sequence φ^ε is uniformly bounded by the maximum principle (check Proposition 5 in [7]), we can define its upper- and lower- semi continuous envelopes by the following formulas

$$\varphi^*(t,x,v) = \limsup_{\substack{\varepsilon \to 0 \\ (s,y,w) \to (t,x,v)}} \varphi^\varepsilon(s,y,w), \qquad \varphi_*(t,x,v) = \liminf_{\substack{\varepsilon \to 0 \\ (s,y,w) \to (t,x,v)}} \varphi^\varepsilon(s,y,w)$$

Recall that φ^* is upper semi-continuous, φ_* is lower semi-continuous and that from their definition, one has $\varphi_* \leq \varphi^*$. We have the following:

Proposition 2.4. *Let φ^ε be a solution to* (1.6).

(i) The upper semi-limit φ^* is constant with respect to the velocity variable on $\mathbb{R}_+^* \times \mathbb{R}^n$.

(ii) The function $(t,x) \mapsto \varphi^*(t,x)$ is a viscosity sub-solution to (1.12) *on $\mathbb{R}_+^* \times \mathbb{R}^n$.*

(iii) The function $(t,x) \mapsto \min_{w\in V} \varphi_*(t,x,w)$ is a viscosity super-solution to (1.12) *on $\mathbb{R}_+^* \times \mathbb{R}^n$.*

We recall that for all (t,x), the minimum $\min_{w\in V} \varphi_*(t,x,w)$ is attained since V is bounded and φ_* is lower semi-continuous. We point out here that if $r = 0$, that is, the case of [8, 15], it is not necessary to prove that φ^* is constant in the velocity variable. One can replace this by proving that $\max_{w\in V} \varphi^*$ is a sub-solution to (1.12). The fact that φ^* is constant in the velocity variable is needed to control the limit of ρ^ε.

Proof of Proposition 2.4. We start with the proof of (i). Take $(t^0,x^0,v^0) \in \mathbb{R}_+^* \times \mathbb{R}^n \times V$. Let ψ be a test function such that $\varphi^* - \psi$ has a strict local maximum at (t^0,x^0,v^0). Then there exists a sequence $(t^\varepsilon,x^\varepsilon,v^\varepsilon)$ of maximum points of $\varphi^\varepsilon - \psi$ satisfying $(t^\varepsilon,x^\varepsilon,v^\varepsilon) \to (t^0,x^0,v^0)$. From this we deduce that $\lim_{\varepsilon\to 0} \varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^\varepsilon) = \varphi^*(t,x,v)$. Recalling (1.6), we have at $(t^\varepsilon,x^\varepsilon,v^\varepsilon)$:

$$\partial_t\psi + v^\varepsilon \cdot \nabla_x\psi + r = (1+r)\left(1 - \int_V M(v') e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon'}}{\varepsilon}} dv'\right) + r\rho^\varepsilon.$$

From this, we deduce that

$$\int_{V'} M(v') e^{\frac{\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^\varepsilon) - \varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v')}{\varepsilon}} dv'$$

is uniformly bounded for any $V' \subset V$. By the Jensen inequality,

$$\exp\left(\frac{1}{\varepsilon\,|V'|_M} \int_{V'} \left(\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^\varepsilon) - \varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v')\right) M(v')dv'\right) \le \frac{1}{|V'|_M} \int_{V'} M(v') e^{\frac{\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^\varepsilon) - \varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v')}{\varepsilon}} dv',$$

where $|V'|_M := \int_{V'} M(v)dv$. Thus,

$$\limsup_{\varepsilon\to 0} \left(\int_{V'} \left(\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^\varepsilon) - \varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v')\right) M(v')dv'\right) \le 0.$$

We write

$$\begin{aligned}
\int_{V'} \left(\varphi^\varepsilon(v^\varepsilon) - \varphi^\varepsilon(v')\right) M(v')dv' &= \int_{V'} \left[\left(\varphi^\varepsilon(v^\varepsilon) - \psi(v^\varepsilon)\right) - \left(\varphi^\varepsilon(v') - \psi(v')\right) + \left(\psi(v^\varepsilon) - \psi(v')\right)\right] M(v')dv' \\
&= \int_{V'} \left[\left(\varphi^\varepsilon(v^\varepsilon) - \psi(v^\varepsilon)\right) - \left(\varphi^\varepsilon(v') - \psi(v')\right)\right] M(v')dv' \\
&+ \int_{V'} \left(\psi(v^\varepsilon) - \psi(v')\right) M(v')dv'
\end{aligned}$$

We can thus use the Fatou Lemma, together with $-\limsup_{\varepsilon\to 0}\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v')\geq -\varphi^*(t^0,x^0,v')$ to get

$$\begin{aligned}\left(\int_{V'}M(v')dv'\right)\varphi^*(v^0)-\int_{V'}\varphi^*(v')M(v')dv' &= \int_{V'}\left(\varphi^*(v^0)-\varphi^*(v')\right)M(v')dv'\\ &\leq \int_{V'}\liminf_{\varepsilon\to 0}\left(\varphi^\varepsilon(v^\varepsilon)-\varphi^\varepsilon(v')\right)M(v')dv'\\ &\leq \liminf_{\varepsilon\to 0}\left(\int_{V'}\left(\varphi^\varepsilon(v^\varepsilon)-\varphi^\varepsilon(v')\right)M(v')dv'\right)\\ &\leq \limsup_{\varepsilon\to 0}\left(\int_{V'}\left(\varphi^\varepsilon(v^\varepsilon)-\varphi^\varepsilon(v')\right)M(v')dv'\right)\\ &\leq 0,\end{aligned}$$

We shall deduce, since the latter is true for any $|V'|$ that

$$\varphi^*(t^0,x^0,v^0)\leq \inf_V \varphi^*(t^0,x^0,\cdot)$$

and thus φ^* is constant in velocity.

We now continue with the proof of (ii). We have to prove that on $\{\varphi^*>0\}\cap(\mathbb{R}_+^*\times\mathbb{R}^n)$, the function φ^* is a viscosity subsolution of (1.12). To this aim, let $\psi\in C^2(\mathbb{R}_+^*\times\mathbb{R}^n)$ be a test function such that $\varphi^*-\psi$ has a local maximum in $(t^0,x^0)\in(\mathbb{R}_+^*\times\mathbb{R}^n)\cap\{\varphi^*>0\}$. We denote by $p^0(t^0,x^0)=\frac{\nabla_x\psi(t^0,x^0)}{1+r}$.

\# **First case :** $p^0(t^0,x^0)\notin \operatorname{Sing} M$**.**

We define a corrector η according to the following formula:

$$\eta(v)=\ln\left(1+\mathcal{H}\left(p^0(t^0,x^0)\right)-v\cdot p^0(t^0,x^0)\right)$$

Let us define the perturbed test function $\psi^\varepsilon:=\psi+\varepsilon\eta$. We recall the fact that in this case $\int_V M(v')\exp(-\eta(v'))dv'=1$. The function ψ^ε converges uniformly to ψ since η is bounded on V. As a consequence, there exists a sequence $(t^\varepsilon,x^\varepsilon,v^\varepsilon)$ of maximum points of $\varphi^\varepsilon-\psi^\varepsilon$ satisfying $(t^\varepsilon,x^\varepsilon)\to(t^0,x^0)$ and such that $\lim_{\varepsilon\to 0}\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^\varepsilon)=\varphi^*(t^0,x^0)$. Recalling (1.6), we have at $(t^\varepsilon,x^\varepsilon,v^\varepsilon)$:

$$\partial_t\psi^\varepsilon+v^\varepsilon\cdot\nabla_x\psi^\varepsilon+r=(1+r)\left(1-\int_V M(v')e^{\frac{\varphi^\varepsilon-\varphi^{\varepsilon'}}{\varepsilon}}dv'\right)+r\rho^\varepsilon.$$

Since $(t^\varepsilon,x^\varepsilon,v^\varepsilon)$ is a maximum point, we may rearrange the r.h.s. of the latter so that the previous equation may be rewritten as follows

$$\begin{aligned}\partial_t\psi^\varepsilon+v^\varepsilon\cdot\nabla_x\psi^\varepsilon+r &\leq (1+r)\left(1-\int_V M(v')\exp\left(\eta(v^\varepsilon)-\eta(v')\right)dv'\right)+r\rho^\varepsilon,\\ &= (1+r)\left(1-\left(\int_V M(v')\exp\left(-\eta(v')\right)dv'\right)\exp\left(\eta(v^\varepsilon)\right)\right)+r\rho^\varepsilon,\\ &= (1+r)\left(1-\exp\left(\eta(v^\varepsilon)\right)\right)+r\rho^\varepsilon,\\ &= -(1+r)\mathcal{H}\left(p^0(t^0,x^0)\right)+v^\varepsilon\cdot\nabla_x\psi^0(t^0,x^0)+r\rho^\varepsilon.\end{aligned}$$

Since $(t^0,x^0)\in\{\varphi^*>0\}$ and $\lim_{\varepsilon\to 0}\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^\varepsilon)=\varphi^*(t^0,x^0)$, we have that, eventually, $\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^\varepsilon)>\varphi^*(t^0,x^0)/2>0$ for ε sufficiently small. Since

$$r\rho^\varepsilon\left(e^{\frac{\varphi^\varepsilon}{\varepsilon}}-1\right)=\left(1-\int_V M(v')e^{\frac{\varphi^\varepsilon-\varphi^{\varepsilon'}}{\varepsilon}}dv'\right)-\left(\partial_t\psi^\varepsilon+v^\varepsilon\cdot\nabla_x\psi^\varepsilon\right),$$

and the latter r.h.s. is uniformly bounded from above in ε, we deduce that $\lim_{\varepsilon\to 0}\rho^\varepsilon(t^\varepsilon,x^\varepsilon)=0$. Taking the limit $\varepsilon\to 0$, we get

$$\partial_t\psi\left(t^0,x^0\right)+(1+r)\mathcal{H}\left(\frac{\nabla_x\psi(t^0,x^0)}{1+r}\right)+r\leq 0.$$

\# **Second case :** $p^0(t^0,x^0)\in \operatorname{Sing} M$**.**

Let $v^*\in \operatorname{Arg}\mu(p^0(t^0,x^0))$. The function $(t,x)\mapsto\varphi^\varepsilon(t,x,v^*)-\psi(t,x)$ has a local maximum at a point $(t^\varepsilon,x^\varepsilon)$ satisfying $(t^\varepsilon,x^\varepsilon)\to(t^0,x^0)$ as $\varepsilon\to 0$. We then have:

$$\begin{aligned}\partial_t\psi(t^\varepsilon,x^\varepsilon)+v^*\cdot\nabla_x\psi(t^\varepsilon,x^\varepsilon)+r &= \partial_t\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^*)+v^*\cdot\nabla_x\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^*)+r\\ &= (1+r)\int_V M(v')\left(1-e^{\frac{\varphi^\varepsilon(v^*)-\varphi^\varepsilon(v')}{\varepsilon}}\right)dv'+r\rho^\varepsilon\\ &\leq (1+r)+r\rho^\varepsilon.\end{aligned}$$

Since $(t^0,x^0)\in\{\varphi^*>0\}$, we have $\rho^\varepsilon(t^\varepsilon,x^\varepsilon)\to 0$. As a consequence, taking the limit $\varepsilon\to 0$, we get

$$\partial_t\psi(t^0,x^0)+\mu(\nabla_x\psi(t^0,x^0))\leq 1.$$

We finally turn to the proof of (iii). That is, the fact that on $\mathbb{R}^*_+\times\mathbb{R}^n$, the function $\min_{w\in V}\varphi_*(\cdot,w)$ is a viscosity supersolution of (1.12).

Let $\psi\in C^1(\mathbb{R}^*_+\times\mathbb{R}^n)$ be a test function such that $\min_{w\in V}\varphi_*-\psi$ has a local minimum in $(t^0,x^0)\in\mathbb{R}^*_+$. We denote by $p^0(t^0,x^0)=\frac{\nabla_x\psi(t^0,x^0)}{1+r}$. We define the truncated corrector η_δ,

$$\begin{aligned}\eta(v) &= \ln\left(1+\mathcal{H}\left(p^0(t,x)\right)-v\cdot p^0(t,x)\right),\\ \eta_\delta(v) &= \max\left(\eta(v),-1/\delta\right).\end{aligned}$$

Let us define the perturbed test function $\psi^\varepsilon:=\psi+\varepsilon\eta_\delta$. For any $\delta>0$, the function ψ^ε converges uniformly to ψ as $\varepsilon\to 0$ since η_δ is bounded on V. Since $\varphi_*(t^0,x^0,\cdot)$ attains its minimum at, say, v^0, we have, for all $v\in V$ and locally in the (t,x) variables,

$$\varphi_*(t^0,x^0,v^0)-\psi(t^0,x^0)=\min_{w\in V}\varphi_*(t^0,x^0)-\psi(t^0,x^0)\leq\min_{w\in V}\varphi_*(t,x)-\psi(t,x)\leq\varphi_*(t,x,v)-\psi(t,x),$$

and thus (t^0,x^0,v^0) is a local minimum of $\varphi_*-\psi$, strict in the (t,x) variables. By the definition of the lower semi-limit, there exists a sequence $(t^\varepsilon_\delta,x^\varepsilon_\delta,v^\varepsilon_\delta)$ of minimum points of $\varphi^\varepsilon-\psi^\varepsilon$ satisfying $(t^\varepsilon_\delta,x^\varepsilon_\delta)\to(t^0,x^0)$. We obtain, after (1.6), at the point $(t^\varepsilon_\delta,x^\varepsilon_\delta,v^\varepsilon_\delta)$,

$$\begin{aligned}\partial_t\psi^\varepsilon+v^\varepsilon_\delta\cdot\nabla_x\psi^\varepsilon+r &\geq (1+r)\left(1-\int_V M(v')\exp\left(\eta_\delta(v^\varepsilon)-\eta_\delta(v')\right)dv'\right),\\ &= (1+r)\left(1-\left(\int_V M(v')\exp\left(-\eta_\delta(v')\right)dv'\right)\exp\left(\eta_\delta(v^\varepsilon_\delta)\right)\right).\end{aligned}$$

Since the sequence v^ε_δ lies in a compact set, taking the limit $\varepsilon\to 0$ (up to extraction), we obtain v^0_δ such that

$$\partial_t\psi+v^0_\delta\cdot\nabla_x\psi+r \geq (1+r)\left(1-\left(\int_V M(v')e^{-\eta_\delta(v')}dv'\right)e^{\eta_\delta(v^0_\delta)}\right).$$

By construction, $\eta_\delta \geq \eta$. As a consequence, $\int_V M(v')e^{-\eta_\delta(v')}dv' \leq \int_V M(v')e^{-\eta(v')}dv' \leq 1$. Thus,

$$\partial_t\psi + v_\delta^0 \cdot \nabla_x\psi + r \quad \geq \quad (1+r)\left(1 - e^{\eta_\delta(v_\delta^0)}\right).$$

We now pass to the limit $\delta \to 0$. By compactness of V, one can extract a converging subsequence from $(v_\delta^0)_\delta$, we denote by v^* the limit.

\# **First case :** $p^0(t^0, x^0) \notin \operatorname{Sing} M$**.**

In this case, since η is bounded, $\eta_\delta = \eta$ for δ sufficiently small. Thus, passing to the limit $\delta \to 0$, one gets

$$\begin{aligned} \partial_t\psi + v^* \cdot \nabla_x\psi + r &\geq (1+r)\left(1 - \exp\left(\eta(v^*)\right)\right), \\ &= -(1+r)\mathcal{H}\left(p^0(t^0, x^0)\right) + v^* \cdot \nabla_x\psi(t^0, x^0), \end{aligned}$$

from which we deduce

$$\partial_t\psi\left(t^0, x^0\right) + (1+r)\mathcal{H}\left(\frac{\nabla_x\psi(t^0, x^0)}{1+r}\right) + r \geq 0.$$

\# **Second case :** $p^0(t^0, x^0) \in \operatorname{Sing} M$**.**

In this case, the corrector η_δ is

$$\eta_\delta(v) = \max\left(\ln\left(\mu\left(\nabla_x\psi(t^0, x^0)\right) - v \cdot \nabla_x\psi(t^0, x^0)\right), -1/\delta\right).$$

If $v^* \notin \operatorname{Arg}\mu(p^0(t^0, x^0))$, since η is bounded on all compacts of $V \setminus \operatorname{Arg}\mu(p^0(t^0, x^0))$, $\eta_\delta(v_\delta^0) = \eta(v_\delta^0)$ for δ sufficiently small and we recover the first case.

If $v^* \in \operatorname{Arg}\mu(p^0(t^0, x^0))$, then take $\delta' > 0$, one has when $\delta < \delta'$ is sufficiently small,

$$-\frac{1}{\delta'} = \eta_{\delta'}(v_\delta^0) \geq \eta_\delta(v_\delta^0),$$

and thus $\lim_{\delta\to 0}\eta_\delta(v_\delta^0) = -\infty$. From that we conclude

$$\partial_t\psi + \mu\left(\nabla_x\psi(t^0, x^0)\right) \geq 1.$$

□

We now conclude with the proof of the convergence result. For this, we need to input initial conditions. Obviously, one cannot get any uniqueness result for the Hamilton-Jacobi equation (1.12) without imposing any initial condition. We now check the initial condition of (1.12) in the viscosity sense.

Proposition 2.5. *If one assumes that $\varphi_0^\varepsilon = \varphi_0$, the sequence φ^ε converges uniformly on compact subsets of $\mathbb{R}_+^* \times \mathbb{R}^n$ to φ^0, the unique viscosity solution of*

$$\begin{cases} \min\left\{\partial_t\varphi^0 + (1+r)\mathcal{H}\left(\frac{\nabla_x\varphi^0}{1+r}\right) + r, \varphi^0\right\} = 0, & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ \varphi^0(0,x) = \min\limits_{v\in V}\varphi_0(x,v), & x \in \mathbb{R}^n. \end{cases}$$

Proof of Proposition 2.5. We extend the definition of φ^* to $\{t=0\}\times\mathbb{R}^n$ by the formula

$$\varphi^*(0,x)=\limsup_{\substack{t\searrow 0^+\\ x'\to x}}\varphi^*(t,x').$$

One has to prove the following

$$\min\left(\min\left\{\partial_t\varphi^*+(1+r)\mathcal{H}\left(\frac{\nabla_x\varphi^*}{1+r}\right)+r,\varphi^*\right\},\varphi^*-\min_{v\in V}\varphi_0(\cdot,v)\right)\leq 0, \tag{2.22}$$

on $\{t=0\}\times\mathbb{R}^n$ in the viscosity sense.

Let $\psi\in\mathcal{C}^1(\mathbb{R}^+\times\mathbb{R})$ be a test function such that $\varphi^*-\psi$ has a strict local maximum at $(t^0=0,x^0)$. We now prove that either

$$\varphi^*(0,x^0)\leq\min_{v\in V}\varphi_0(x,v),$$

or

$$\partial_t\psi+(1+r)\mathcal{H}\left(\frac{\nabla_x\psi}{1+r}\right)+r\leq 0\qquad\text{when}\qquad\varphi^*(0,x^0)>0.$$

Suppose then that

$$\varphi^*(0,x^0)>\min_{v\in V}\varphi_0(x,v).$$

We shall now prove that

$$\partial_t\psi+(1+r)\mathcal{H}\left(\frac{\nabla_x\psi}{1+r}\right)+r\leq 0,$$

since then $\varphi^*(0,x^0)>0$. We now go through the same steps as for the proof of Proposition 2.4, but with slight changes due to the present situation. We keep the same notations.

\# **First case :** $p^0(t^0,x^0)\notin\operatorname{Sing}M$**.**

The function ψ^ε converges uniformly to ψ since η is bounded on V. Adding this fact to the definition of $\varphi^*(0,x^0)$, we get the existence of a sequence $(t^\varepsilon,x^\varepsilon,v^\varepsilon)$ of maximum points of $\varphi^\varepsilon-\psi^\varepsilon$ satisfying $t^\varepsilon>0$, $(t^\varepsilon,x^\varepsilon)\to(0,x^0)$ and such that $\lim_{\varepsilon\to 0}\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^\varepsilon)=\varphi^*(0,x^0)$. The rest of the proof is similar.

\# **Second case :** $p^0(t^0,x^0)\in\operatorname{Sing}M$**.**

Let $v^*\in\operatorname{Arg}\mu(p^0(t^0,x^0))$. As for the previous case, due to the definition of φ^*, the function $(t,x)\mapsto\varphi^\varepsilon(t,x,v^*)-\psi(t,x)$ has a local maximum at a point $(t^\varepsilon,x^\varepsilon)$ satisfying $(t^\varepsilon>0,x^\varepsilon)\to(t^0,x^0)$ as $\varepsilon\to 0$. The conclusion is the same.

We shall now prove that the initial condition for $\min_w\varphi_*$ is given by

$$\max\left(\min\left\{\partial_t\left(\min_w\varphi_*\right)+(1+r)\mathcal{H}\left(\frac{\nabla_x\left(\min_w\varphi_*\right)}{1+r}\right)+r,\min_w\varphi_*\right\},\min_w\varphi_*-\min_{v\in V}\varphi_0\right)\geq 0, \tag{2.23}$$

on $\{t=0\}\times\mathbb{R}^n$ in the viscosity sense.

Let us prove (2.23). Let $\psi \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R})$ be a test function such that $\min_{w\in V} \varphi_* - \psi$ has a strict local minimum at $(t^0 = 0, x^0)$. We now prove that either

$$\min_{w\in V} \varphi_*(0, x^0, w) \geq \min_{v\in V} \varphi_0(x^0, v),$$

or

$$\partial_t \psi + (1+r)\mathcal{H}\left(\frac{\nabla_x \psi}{1+r}\right) + r \geq 0 \qquad \text{and} \qquad \min_{w\in V} \varphi_*(0, x^0, w) \geq 0.$$

Suppose that $\min_{w\in V} \varphi_*(0, x^0, w) < \min_{v\in V} \varphi_0(x^0, v)$. The argument now starts similarly as in the proof above. Let us define the perturbed test function $\psi^\varepsilon := \psi + \varepsilon\eta_\delta$. For any $\delta > 0$, the function ψ^ε converges uniformly to ψ since η_δ is bounded on V. Since $\varphi_*(0, x^0, \cdot)$ attains its minimum at, say, v^0, we have, for all $v \in V$ and locally in the (t, x) variables,

$$\varphi_*(0, x^0, v_0) - \psi(0, x^0) \leq \min_{w\in V} \varphi_*(0, x^0) - \psi(0, x^0) \leq \min_{w\in V} \varphi_*(t, x) - \psi(t, x) \leq \varphi_*(t, x, v) - \psi(t, x),$$

and thus $(0, x^0, v^0)$ is a local minimum of $\varphi_* - \psi$, strict in the (t, x) variables. By the definition of the lower semi-limit, there exists a sequence $(t^\varepsilon_\delta, x^\varepsilon_\delta, v^\varepsilon_\delta)$ of minimum points of $\varphi^\varepsilon - \psi^\varepsilon$ satisfying $(t^\varepsilon_\delta, x^\varepsilon_\delta) \to (0, x^0)$. We first claim that there exists a subsequence $(t_{\varepsilon_k}, x_{\varepsilon_k}, v_{\varepsilon_k})_k$ of the above sequence, with $\varepsilon_k \to 0$ as $k \to \infty$, such that $t_{\varepsilon_k} > 0$, for all k.

Suppose that this is not true. Then, take a sequence $(x^{\varepsilon_{k'}}_\delta, v^{\varepsilon_{k'}}_\delta)_{k'}$ such that $(\varepsilon_{k'}, x^{\varepsilon_{k'}}_\delta) \to (0, x^0)$ and that $\varphi^{\varepsilon_{k'}} - \psi^{\varepsilon_{k'}}$ has a local minimum at $\left(0, x^{\varepsilon_{k'}}_\delta, v^{\varepsilon_{k'}}_\delta\right)$. It follows that, for all (t, x, v) in some neighborhood of $(0, x^{\varepsilon_{k'}}_\delta, v^{\varepsilon_{k'}}_\delta)$, we have

$$\begin{aligned}
\min_{v\in V} \varphi_0(x^{\varepsilon_{k'}}_\delta, v) - \psi^{\varepsilon_{k'}}\left(0, x^{\varepsilon_{k'}}_\delta, v^{\varepsilon_{k'}}_\delta\right) &\leq \varphi_0(x^{\varepsilon_{k'}}_\delta, v^{\varepsilon_{k'}}_\delta) - \psi^{\varepsilon_{k'}}\left(0, x^{\varepsilon_{k'}}_\delta, v^{\varepsilon_{k'}}_\delta\right) \\
&\leq \varphi^{\varepsilon_{k'}}\left(0, x^{\varepsilon_{k'}}_\delta, v^{\varepsilon_{k'}}_\delta\right) - \psi^{\varepsilon_{k'}}\left(0, x^{\varepsilon_{k'}}_\delta, v^{\varepsilon_{k'}}_\delta\right) \\
&\leq \varphi^{\varepsilon_{k'}}(t, x, v) - \psi^{\varepsilon_{k'}}(t, x, v).
\end{aligned}$$

Taking $\liminf_{\substack{k'\to\infty \\ (t,x,v)\to(0,x^0,v_0)}}$ at the both sides of the inequality, one obtains

$$\min_{v\in V} \varphi_0(x^0, v) - \psi\left(0, x^0\right) \leq \min_{w\in V} \varphi_*(0, x^0) - \psi\left(0, x^0\right).$$

However, this is in contradiction with $\min_{w\in V} \varphi_*(0, x^0, w) < \min_{v\in V} \varphi_0(x^0, v)$. Now having in hand that this sequence of times $t_{\varepsilon_n} > 0$, one can reproduce the same argument as from the proof above along the subsequence $(t_{\varepsilon_n}, x_{\varepsilon_n}, v_{\varepsilon_n})$.

By the strong uniqueness principle satisfied by (1.12) (that is, a comparison principle for discontinuous sub- and super- solutions), we deduce that for all $(t, x, v) \in \mathbb{R}^*_+ \times \mathbb{R}^n \times V$,

$$\min_{w\in V} \varphi_*(t, x, w) \leq \varphi_*(t, x, v) \leq \varphi^*(t, x, v) = \varphi^*(t, x) \leq \min_{w\in V} \varphi_*(t, x, w)$$

We deduce that necessarily all these inequalities are equalities, and thus that φ^ε converges locally uniformly towards φ^0, independent of v, on any subcompact of $\mathbb{R}^*_+ \times \mathbb{R}^n$. □

2.3 Convergence of the macroscopic density ρ^ε

We prove a convergence result for ρ^ε in the region $\{\varphi^0 = 0\}$. Namely

Proposition 2.6. *Let φ^ε be the solution of (1.6). Then, uniformly on compact subsets of $Int\{\varphi^0 = 0\}$,*

$$\lim_{\varepsilon\to 0} \rho^\varepsilon = 1, \qquad \lim_{\varepsilon\to 0} f^\varepsilon(\cdot, v) = M(v).$$

Proof of Proposition 2.6. We develop similar arguments as in [21]. Let K be a compact set of $\{\varphi^0 = 0\}$. Note that it suffices to prove the result when K is a cylinder. Let $(t^0, x^0) \in \mathrm{Int}(K)$ and the test function

$$\forall (t,x) \in K, \qquad \psi^0(t,x) = |x - x^0|^2 + \left(t - t^0\right)^2.$$

Since $\varphi^0 = 0$ on K, the function $\varphi^0 - \psi^0$ admits a strict maximum in (t^0, x^0). The locally uniform convergence of $\varphi^\varepsilon - \psi^0$ gives a sequence $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ of maximum points with $(t^\varepsilon, x^\varepsilon) \to (t^0, x^0)$ and a bounded sequence v^ε such that at the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ one has:

$$\partial_t \psi^0 + v^\varepsilon \cdot \nabla_x \psi^0 + r \le r\rho^\varepsilon. \tag{2.24}$$

As a consequence, one has, since $r > 0$,

$$\rho^\varepsilon(t^\varepsilon, x^\varepsilon) \ge 1 + o(1), \qquad \text{as } \varepsilon \to 0, \tag{2.25}$$

and then $\lim_{\varepsilon\to 0} \rho^\varepsilon(t^\varepsilon, x^\varepsilon) = 1$ if one recalls $\rho^\varepsilon \le 1$ (which, again, is a consequence of the maximum principle).

However, we need an extra argument to get $\lim_{\varepsilon\to 0} \rho^\varepsilon(t^0, x^0) = 1$. Since $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ maximizes $\varphi^\varepsilon - \psi^0$, we deduce that for all $v \in V$, we have

$$\varphi^\varepsilon\left(t^\varepsilon, x^\varepsilon, v^\varepsilon\right) - \psi^0(t^\varepsilon, x^\varepsilon) \ge \varphi^\varepsilon\left(t^0, x^0, v\right) - \psi^0(t^0, x^0).$$

Since $\psi^0(t^\varepsilon, x^\varepsilon) \ge 0$, $\psi^0(t^0, x^0) = 0$, we find

$$f^\varepsilon(t^0, x^0, v) = M(v) e^{-\frac{\varphi^\varepsilon(t^0, x^0, v)}{\varepsilon}} \ge M(v) e^{-\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)}{\varepsilon}}. \tag{2.26}$$

We shall now prove that $\lim_{\varepsilon\to 0} \varepsilon^{-1} \varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon) = 0$. Let us rewrite (1.6) at the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ in the form

$$r\rho^\varepsilon(t^\varepsilon, x^\varepsilon)\left(e^{\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)}{\varepsilon}} - 1\right) = \left(1 - \int_V M(v') e^{\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon) - \varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v')}{\varepsilon}} dv'\right) - \left(\partial_t \psi^0 + v \cdot \nabla_x \psi^0\right)(t^\varepsilon, x^\varepsilon, v^\varepsilon)$$

We finally deduce using the maximum principle in the latter r.h.s. that

$$0 \le r\rho^\varepsilon(t^\varepsilon, x^\varepsilon)\left(e^{\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)}{\varepsilon}} - 1\right) \le o_{\varepsilon\to 0}(1)$$

and thus $\lim_{\varepsilon\to 0}\left(\varepsilon^{-1}\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)\right) = 0$. This implies $\lim_{\varepsilon\to 0} f^\varepsilon(t, x, v) = M(v)$ locally uniformly on $K \times V$.

□

2.4 Speed of expansion

To be self-contained, we recall here how to study the propagation of the front after deriving the limit variational equation, in the case $r > 0$. From Evans and Souganidis [21], we are able to identify the solution of the variational Hamilton-Jacobi equation (1.12) using the Lagrangian duality. We emphasize that, in this context, one may assume that our initial condition is well-prepared, *i.e.* $\varphi(0, x, v) = \varphi_0(x)$. We recall the equation:

$$\begin{cases} \min\left\{\partial_t\varphi + (1+r)\mathcal{H}\left(\frac{\nabla_x\varphi}{1+r}\right) + r, \varphi\right\} = 0, & \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ \varphi(0,x) = \varphi_0(x). \end{cases}$$

We recall from [8, 15] that the Hamiltonian $\mathcal{H}$ is convex. For any $e_0 \in \mathbb{S}^{n-1}$, we define the minimal speed in that direction by the formula

$$c^*(e) = \inf_{\lambda>0} c(\lambda, e), \quad c(\lambda, e) = \frac{1}{\lambda}\left((1+r)\mathcal{H}\left(\frac{\lambda e}{1+r}\right) + r\right). \tag{2.27}$$

We first discuss the speed of propagation of a front-like initial data.

Proposition 2.7. *Assume that*

$$\varphi_0(x) := \begin{cases} 0 & x \in e_0^\perp \\ +\infty & else \end{cases}.$$

Then the nullset of φ propagates at speed $c^*(e_0)$:

$$\forall t \geq 0, \qquad \{\varphi(t,\cdot) = 0\} = e_0^\perp + c^*(e_0) t\, e_0.$$

Proof of Proposition 2.7. We first notice that since the initial data is invariant under any translation in $e_0^\perp$, and the the equation (1.12) invariant by translation, the solution φ depends only on $x \cdot e_0$. That is $\varphi(t,x) = \varphi(t,(x\cdot e_0)e_0) = \overline{\varphi}(t, x\cdot e_0)$. The function $\overline{\varphi}$ satisfies

$$\begin{cases} \min\left\{\partial_t\overline{\varphi} + (1+r)\mathcal{H}\left(\frac{\partial_\xi\overline{\varphi}}{1+r}e_0\right) + r, \varphi\right\} = 0, & \forall (t,\xi) \in \mathbb{R}_+^* \times \mathbb{R}, \\ \overline{\varphi}(0,\xi) = \overline{\varphi}_0(\xi). \end{cases}$$

where

$$\overline{\varphi}_0(\xi) := \begin{cases} 0 & \xi = 0, \\ +\infty & \text{else} \end{cases}.$$

The Lagrangian associated to the latter Hamilton-Jacobi equation is by definition

$$\begin{aligned} \mathcal{L}(p) &= \sup_{q\in\mathbb{R}}\left(pq - (1+r)\mathcal{H}\left(\frac{q}{1+r}e_0\right) - r\right), \\ &= \sup_{q\in\mathbb{R}}\left(pq - (1+r)\mathcal{H}\left(\frac{|q|}{1+r}e_0\right) - r\right), \\ &= \sup_{q\in\mathbb{R}}\left(pq - |q|c(|q|, e_0)\right), \end{aligned}$$

To solve the variational Hamilton-Jacobi equation, let us define

$$J(x,t) = \inf_{x\in X}\left\{\int_0^t [\mathcal{L}(\dot{x})]\,ds \,\middle|\, x(0)=x, x(t)=0\right\}$$

the minimizer of the action associated to the Lagrangian. Thanks to the so-called Freidlin condition, see [21, 24] we deduce that the solution of (1.12) is

$$\overline{\varphi}(t,\xi) = \max\left(J(\xi,t),0\right).$$

The Hopf-Lax formula gives $J(\xi,t) = t\mathcal{L}\left(t^{-1}\xi\right)$ thanks to the assumption on the initial condition. Hence,

$$\begin{aligned}
\xi \in \{\overline{\varphi}(t,\cdot)=0\} \Longleftrightarrow \mathcal{L}\left(t^{-1}\xi\right) \le 0 \quad &\Longleftrightarrow \quad \sup_{q\in\mathbb{R}}\left(q\xi - |q|c(|q|,e_0)t\right) \le 0,\\
&\Longleftrightarrow \quad \forall q\in\mathbb{R}, \quad q\xi - |q|c(|q|,e_0)t \le 0,\\
&\Longleftrightarrow \quad |\xi| \le c^*(e_0)t.
\end{aligned}$$

We deduce the result for φ by changing the variables back. □

For a compactly supported initial data, the issue of the speed of propagation in general is more involved, since different directions may have different speeds of propagation. Namely, the following Freidlin-Gärtner formula holds:

Proposition 2.8. *Assume that*

$$\varphi_0(x) := \left\{\begin{array}{ll} 0 & x=0\\ +\infty & else\end{array}\right. .$$

Define

$$w^*(e_0) = \min_{\substack{e\in\mathbb{S}^{n-1}\\ e_0\cdot e>0}}\left(\frac{c^*(e)}{e_0\cdot e}\right).$$

Then the nullset of φ propagates at speed $w^*(e_0)$ in the direction e_0 :

$$\forall t\ge 0, \qquad \{x\in\mathbb{R},\ \varphi(t,x\,e_0)=0\} = \{x\in\mathbb{R},\ |x|\le w^*(e_0)t\}.$$

Proof of Proposition 2.8. The Lagrangian is by definition

$$\begin{aligned}
\mathcal{L}(p) &= \sup_{q\in\mathbb{R}^n}\left(p\cdot q - (1+r)\mathcal{H}\left(\frac{q}{1+r}\right) - r\right),\\
&= \sup_{e\in\mathbb{S}^{n-1}}\sup_{\lambda\in\mathbb{R}^+}\left(\lambda p\cdot e - \left[(1+r)\mathcal{H}\left(\frac{\lambda e}{1+r}\right)+r\right]\right),\\
&= \sup_{e\in\mathbb{S}^{n-1}}\sup_{\lambda\in\mathbb{R}^+}\left(\lambda\left[p\cdot e - c(\lambda,e)\right]\right),
\end{aligned}$$

To solve the variational Hamilton-Jacobi equation, let us define

$$J(x,t) = \inf_{x\in X}\left\{\int_0^t [\mathcal{L}(\dot{x})]\,ds \,\middle|\, x(0)=x, x(t)=0\right\}$$

the minimizer of the action associated to the Lagrangian. Thanks to the so-called Freidlin condition, see [21, 24] we deduce that the solution of (1.12) is

$$\varphi(x,t) = \max\left(J(x,t), 0\right).$$

The Lax formula gives

$$J(x,t) = \min_{y\in\mathbb{R}^n}\left\{t\mathcal{L}\left(\frac{x-y}{t}\right) + \varphi_0(y)\right\} = t\mathcal{L}\left(\frac{x}{t}\right)$$

thanks to the assumption on the initial condition. Hence,

$$\begin{aligned}
\varphi(t, xe_0) = 0 \Longleftrightarrow \mathcal{L}\left(\frac{x}{t}e_0\right) \leq 0 \quad &\Longleftrightarrow \quad \sup_{e\in\mathbb{S}^{n-1}}\sup_{\lambda\in\mathbb{R}^+}\left(\lambda\left[x(e_0\cdot e) - c_e(\lambda)t\right]\right) \leq 0,\\
&\Longleftrightarrow \quad \forall\lambda\in\mathbb{R}^+, \forall e\in\mathbb{S}^{n-1}, \quad \lambda\left[(x\cdot e_0)(e_0\cdot e) - c_e(\lambda)t\right] \leq 0,\\
&\Longleftrightarrow \quad \forall e\in\mathbb{S}^{n-1}, \quad x(e_0\cdot e) \leq c^*(e)t\\
&\Longleftrightarrow \quad |x| \leq \min_{\substack{e\in\mathbb{S}^{n-1}\\ e_0\cdot e>0}}\left(\frac{c^*(e)}{e_0\cdot e}\right)t = w^*(e_0)t.
\end{aligned}$$

□

3 Existence of travelling waves and spreading result

In this Section, we now explain how to construct travelling wave solutions to (1.1). We will follow closely the construction in [11]. As is classical in this type of Fisher-KPP problems, the speeds of propagation are given by studying the linearized problem at infinity. As we will see later on, the main difference that has motivated this paper is the possible singularity of $c(\lambda, e)$ at $\lambda^*(e)$.

3.1 Proof of Theorem 1.4 : Travelling wave solutions

Given a direction $e\in\mathbb{S}^{n-1}$, looking for exponential solutions to the linearized problem of the form $e^{-\lambda(x\cdot e - c(\lambda,e)t)}F_{\lambda,e}(v)$ for any positive λ is exactly looking for solutions to

$$\left[1+\lambda(c(\lambda,e) - v\cdot e)\right]F_{\lambda,e}(v) = (1+r)M(v)\int_V F_{\lambda,e}(v')dv', \qquad v\in V.$$

In view of earlier computations, it boils down to setting $c(\lambda,e)$ as in (2.27) and $F_{\lambda,e} := \tilde{Q}_{\frac{\lambda e}{1+r}}$ as in (2.20).

Recall that $\frac{\lambda e}{1+r}\in \mathrm{Sing}(M)$ if and only if $l(e)\leq\frac{\lambda}{1+r}$, that is $\lambda\geq\tilde{\lambda}(e) := (1+r)l(e)$. Thus, for $\lambda\leq\tilde{\lambda}(e)$, the function $c(\lambda,e)$ is convex and regular, and the profile is explicitly given by

$$F_{\lambda,e}(v) = \frac{(1+r)M(v)}{1+\lambda(c(\lambda,e) - v\cdot e)} > 0.$$

For $\lambda\geq\tilde{\lambda}(e)$, that is to say $\frac{\lambda e}{1+r}\in\mathrm{Sing}(M)$ one has $c(\lambda,e) = \overline{v}(e) - \frac{1}{\lambda}$ which is concave and increasing. As such, the infimum of $\lambda\mapsto c(\lambda,e)$ is attained for a $\lambda\leq\tilde{\lambda}(e)$, which we denote $\lambda^*(e)$.

As a consequence, the minimal speed $c^*(e)$ is always associated to an integrable eigenvector, since if $\lambda^*(e) = \tilde{\lambda}(e)$, one has

$$F_{\tilde{\lambda}(e),e}(v) = \frac{(1+r)M(v)}{\tilde{\lambda}(e)\left(\bar{v}(e) - v\cdot e\right)},$$

with $\int_V F_{\tilde{\lambda}(e),e}(v)\,dv = 1$ thanks to the definition of $\tilde{\lambda}(e)$.

Given a direction $e \in \mathbb{S}^{n-1}$, we shall now discuss the type of functions $\lambda \mapsto c(\lambda, e)$ that may arise from this problem. Qualitatively, four situations may happen. The first possibility is the one already appearing in [11] in the mono-dimensional case, that is $\tilde{\lambda}(e) = +\infty$ and thus $\mathrm{Sing}(M) = \emptyset$. We plot an exemple of this case in Figure 3.1, case 1. If $\tilde{\lambda}(e) < +\infty$, three supplementary situations can occur. Either the infimum of $\lambda \mapsto c(\lambda, e)$ is attained for $\lambda < \tilde{\lambda}(e)$, as shown in Figure 3.1, case 2, either it is attained for $\lambda = \tilde{\lambda}(e)$. In the latter case, the infimum can either be attained at a point where the left derivative of $c(\lambda, e)$ is zero (Figure 3.1, case 3), or where it is negative (Figure 3.1, case 4).

Remark 3.1. *One can get a criterion to check which case holds. The dispersion relation defining $c(\lambda, e)$ on $(0, \tilde{\lambda}(e))$ is*

$$\mathcal{I}(\lambda, c(\lambda, e), e) = 1,$$

where

$$\mathcal{I}(\lambda, c, e) := \int_V \frac{(1+r)M(v)}{1 + \lambda(c - v\cdot e)}dv. \tag{3.28}$$

Differentiating with respect to λ, we find

$$\int_V \frac{\lambda c'(\lambda, e)M(v)}{\left[1 + \lambda(c(\lambda, e) - v\cdot e)\right]^2}dv + \int_V \frac{(c(\lambda, e) - v\cdot e)M(v)}{\left[1 + \lambda(c(\lambda, e) - v\cdot e)\right]^2}dv = 0$$

Recalling $\int_V \frac{M(v)}{1+\lambda(c(\lambda,e)-v\cdot e)}dv = (1+r)^{-1}$ and defining

$$\mathcal{J}(\lambda, e) = \int_V \frac{M(v)}{\left[1 + \lambda(c(\lambda, e) - v\cdot e)\right]^2}dv,$$

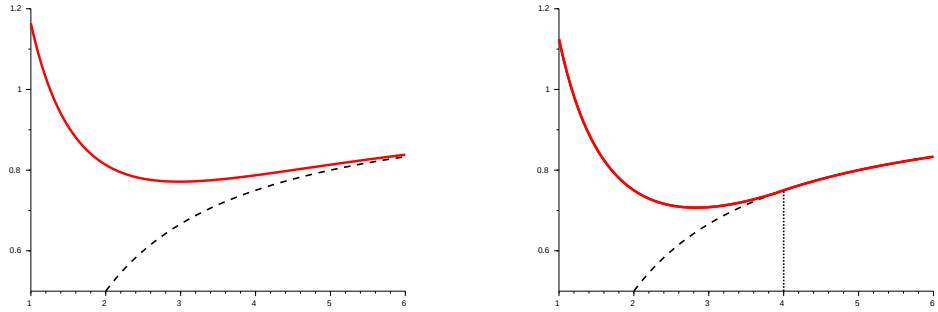
we get

$$c'(\lambda, e) = \left(1 - \frac{(1+r)^{-1}}{J(\lambda, e)}\right)\frac{1}{\lambda^2}.$$

As such, computing the value of $\lim_{\lambda\to\tilde{\lambda}(e)^-}\mathcal{J}(\lambda, e)$ allows to know in which case one falls. Indeed, the function $\lambda \mapsto c(\lambda)$ attains its minimum at $\tilde{\lambda}(e)$ if and only if $c'\left(\tilde{\lambda}^-(e)\right) \le 0$, which is equivalent to $\mathcal{J}(\tilde{\lambda}(e)) \le (1+r)^{-1}$ which is in turn equivalent to

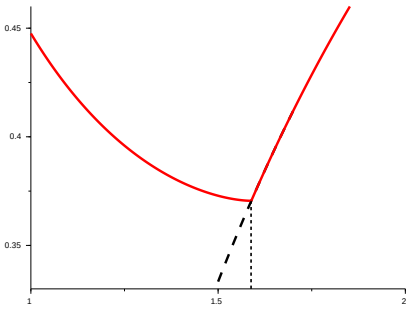
$$\int_V \frac{M(v)}{\left(\bar{v}(e) - v\cdot e\right)^2}dv \le (1+r)l(e)^2, \tag{3.29}$$

which can be checked case by case. Note that one has always, given the Cauchy-Schwarz inequality, $l(e)^2 \le \int_V \frac{M(v)}{(\bar{v}(e)-v\cdot e)^2}dv$.

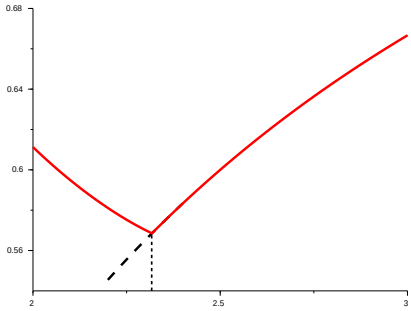


(a) Case 1

(b) Case 2



(c) Case 3



(d) Case 4

Figure 1: Various cases of speed functions $\lambda \mapsto c(\lambda, e)$. Red plain line: $\lambda \mapsto c(\lambda, e)$. Black dotted line: $\lambda \mapsto \bar{v}(e) - \frac{1}{\lambda}$. (a) $n = 1$, $V = [-1, 1]$, $e = 1$, $M \equiv \frac{1}{2}$ and $r = 1$. In this case, $\mathrm{Sing}(M) = \emptyset$ so that the function $\lambda \mapsto c(\lambda, 1)$ is regular. This is the case discussed in [11]. (b) $n = 2$, $V = D(0, 1)$, $M \equiv \frac{1}{\pi}$ and $r = 1$. In this case, $\mathrm{Sing}(M) \neq \emptyset$ but the minimum of $c(\lambda, e_1)$ is attained for $\lambda < \tilde{\lambda}(e_1) = 4$. (c) $n = 1$, $V = [-1, 1]$, $M(v) = \frac{3}{2}(1 - |v|)^2$ and $r = -1 + l(1)^{-2} \int_{-1}^{1} \frac{M(v)}{(1-v)^2} dv \approx 0.37$. In this case, the minimum of $c(\lambda, e_1)$ is attained for $\lambda = \tilde{\lambda}(e_1)$, with a zero left derivative. Numerically $\tilde{\lambda}(1) \approx 1.58$. (d) $n = 1$, $V = [-1, 1]$, $M(v) = \frac{3}{2}(1 - |v|)^2$ and $r = 1$. In this case, the minimum of $c(\lambda, 1)$ is attained for $\lambda = \tilde{\lambda}(1)$, with a negative left derivative. Numerically, $\tilde{\lambda}(1) \approx 2.31$.

Example 3.2. *Let us look back at Example 2.2. As was stated,* $l(1) = 3(2\ln(2) - 1)$ *and* $\int_V \frac{M(v)}{(1-v)^2} dv = 6(1 - \ln(2)) < +\infty$. *Thus, for* $r > -1 + l(1)^{-2} \int \frac{M(v)}{(1-v)^2} dv > 0$, *the condition (3.29) is satisfied so the minimum of* $\lambda \mapsto c(\lambda, e)$ *is attained at* $\tilde{\lambda}(e)$. *For* $r = -1 + l(1)^{-2} \int_V \frac{M(v)}{(1-v)^2} dv$ *the minimum has its left derivative equal to 0 (*i.e. $\lambda^*(1) = \tilde{\lambda}(1)$*). We illustrate those results in section 3.1, case 3 and 4.*

Since $c(\lambda, e)$ tends to infinity when λ tends to 0, for any $c \geq c^*(e)$ one can find $\lambda \in (0, \tilde{\lambda}(e)]$ such that $c(\lambda, e) = c$.

Fix $c \in (c^*(e), \overline{v}(e))$. Denote λ_c is the smallest solution in $(0, \tilde{\lambda}(e))$ of $c(\lambda_c, e) = c$. Notice that by construction it is possible to obtain $F_{\lambda_c, e}$ integrable and bounded (bounded since $c > c^*(e)$), the proof of [11], Section 3.2, that constructs sub and super solutions for (1.1) is unchanged. From the construction of a pair of sub- and super-solutions, we deduce the existence of travelling wave solutions exactly as in [11], by a monotonicity method when $c > c^*(e)$ and passing to the limit $c \to c^*(e)$ to get the case $c = c^*(e)$.

The main difference between the mono-dimensional case of [11] and the higher dimensional case comes here. It is rather non-standard and interesting that the function giving the speed of propagation could be singular at its minimum value.

To prove that c^* is still the minimal speed of propagation, the arguments used in [11, Lemma 3.10] are not applicable. These arguments can be summarized as follows : in the one dimensional case when $M \geq \delta > 0$, the function $\lambda \mapsto \mathcal{I}(\lambda, c, e)$ (recall (3.28)) is analytic. Thus, we can not find $\lambda > 0$ such that $\mathcal{I}(\lambda, c, e) = 1$ when $c < c^*$. However, an argument using the Rouché Theorem states that we can solve this problem in $\mathbb{C} \setminus \mathbb{R}$. Assuming that there exists a travelling wave solution f for $c < c^*$, we then can use such a $\lambda \in \mathbb{C}$ to construct a subsolution under f which dos not converge to 0 as $x \to \infty$. In our framework, the function $\lambda \mapsto \mathcal{I}(\lambda, c, e)$ might not be analytic around $\lambda^*(e)$, which prevents us from using this technique. We thus choose to use the Hamilton-Jacobi framework combined to the comparaison principle.

We now prove the following lemma.

Lemma 3.3. *Let* f *be a travelling wave solution to* (1.1) *in the direction* $e \in \mathbb{S}^{n-1}$, *with speed* c. *Then* $c \geq c^*(e)$.

Proof of Lemma 3.3. Let f be such a travelling wave solution with initial data $\tilde{f}(x, v)$, *i.e* $f(t, x, v) = \tilde{f}(x \cdot e - ct, v)$. After Proposition 2.6, we deduce that $f^\varepsilon(t, x, v) = \tilde{f}\left(\frac{1}{\varepsilon}(x \cdot e - ct), v\right)$ satisfies $\lim_{\varepsilon \to 0} f^\varepsilon = M$ on $x \cdot e - ct < 0$ and $\lim_{\varepsilon \to 0} f^\varepsilon = 0$ on $x \cdot e - ct > 0$. Take $0 < \gamma < 1$ and define $g(x, v) = \gamma M(v) \mathbf{1}_{[-1,1] \times \mathbb{R}^{n-1}}(x)$ and $g^\varepsilon(x, v) = g(x/\varepsilon, v)$. We have

$$\psi^\varepsilon(x) = -\varepsilon \ln(g(x/\varepsilon, v)/M) = \begin{cases} -\varepsilon \ln(\gamma) & x \in [-\varepsilon, \varepsilon] e_0 + e_0^\perp \\ +\infty & \text{else} \end{cases}.$$

Since $\lim_{z \to -\infty} \tilde{f}(z, v) = M$ uniformly in $v \in V$, one can shift the profile sufficiently enough so that $M \geq f \geq g \geq 0$. Thus, the comparison principle (see [11], Proposition 2.2 for a proof) yields that $f^\varepsilon \geq g^\varepsilon$. Passing to the limit $\varepsilon \to 0$, and recalling Theorem 1.3, Proposition 2.6 and Proposition 2.7, we deduce that

$$\left(e_0^\perp + c^*(e_0) t e_0\right) \cdot e_0 - ct \leq 0,$$

from which the result follows. □

From the Hamilton-Jacobi formalism, we may also deduce the following.

Proof of Proposition 1.5. We start by proving (1.16). For this, we use the the super-solution naturally provided by the linearized problem. We have

$$f(t,x,v) \leq \min\{M(v), e^{-\lambda^*(e_0)(x\cdot e_0 - c^*(e_0)t)} F_{\lambda^*(e_0),e_0}(v)\}$$

As a consequence,

$$\rho(t,x) \leq \min\{1, e^{-\lambda^*(e_0)(x\cdot e_0 - c^*(e_0)t)}\},$$

and thus one has $\lim_{t\to+\infty} \sup_{x\cdot e_0 > ct} \rho(t,x) = 0$.

For (1.17), we use the Hamilton-Jacobi results in the following way. We first notice that since the initial data is invariant under any translation in $e_0^\perp$, and the the equation (1.12) invariant by translation, the solution $f(t,x,v)$ depends only on $x\cdot e_0$. That is $f(t,x,v) = f(t,(x\cdot e_0)e_0,v) = \tilde{f}(t,x\cdot e_0,v)$. For any $c < c^*(e_0)$, recalling Theorem 1.3, Proposition 2.6 and Proposition 2.7, we have

$$\lim_{t\to\infty} f(t, e_0^\perp + cte_0, v) = \lim_{t\to\infty} \tilde{f}(t,ct,v) = \lim_{\varepsilon\to 0} \tilde{f}^\varepsilon(1,c,v) = M(v),$$

since $c < c^*(e_0)$. □

3.2 Proof of Proposition 1.6 : spreading of a compactly supported initial data

We finally prove Proposition 1.6. The spreading result (1.18) goes as for the Fisher-KPP equation in an heterogeneous media [6]. It can be found by using the super solution

$$\overline{f}(t,x,v) = \inf_{e\in\mathbb{S}^{n-1}} e^{-\lambda^*(e)(x\cdot e - c^*(e)t)} Q_{\lambda^*(e)e}(v)$$

By the comparison principle, and since the initial data is compactly supported, the function $\overline{f}$ lies above f (multiplying $\overline{f}$ by a big constant if necessary). We deduce that for any given $e_0 \in \mathbb{S}^{n-1}$, and any fixed $x \in \mathbb{R}^n$,

$$f(t,x+ce_0t,v) \leq \inf_{e\in\mathbb{S}^{n-1}} e^{-\lambda^*(e)((x+ce_0t)\cdot e - c^*(e)t)} Q_{\lambda^*(e)e}(v) = \inf_{e\in\mathbb{S}^{n-1}} e^{-\lambda^*(e)(x\cdot e + ce_0\cdot et - c^*(e)t)} Q_{\lambda^*(e)e}(v).$$

Moreover, the domain of $Q_{\lambda^*(e)e}$ contains $V\setminus\{v_{max}e\}$ and $Q_{\lambda^*(e)e}$ is bounded on all compact sets of $V\setminus\{v_{max}e\}$. Hence, for fixed $v\in V$, we can choose $e\in\mathbb{S}^{n-1}$ such that $v\in V\setminus\{v_{max}e\}$. Then, as soon as $c > w^*(e_0)$, we have $c(e\cdot e_0) > c^*(e)$ for any e, and thus $\lim_{t\to\infty} f(t,x+ce_0t,v) = 0$.

Moreover, we shall prove (1.19) as follows. For any $c < c^*(e_0)$, recalling Theorem 1.3, Proposition 2.6 and Proposition 2.8, we have

$$\lim_{t\to\infty} f(t,cte_0,v) = \lim_{t\to\infty} \tilde{f}(t,ct,v) = \lim_{\varepsilon\to 0} \tilde{f}^\varepsilon(1,ce_0,v) = M(v),$$

since $c < w^*(e_0)$.

□

References

[1] D. G Aronson and H. F Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Advances in Mathematics*, 30(1):33–76, October 1978.

[2] Guy Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*. Number 17 in Mathématiques & applications. Springer-Verlag, Paris ; New York, 1994.

[3] Guy Barles, Lawrence C. Evans, and Panagiotis E. Souganidis. Wavefront propagation for reaction-diffusion systems of PDE. *Duke Mathematical Journal*, 61(3):835–858, December 1990.

[4] Guy Barles, Sepideh Mirrahimi, and Benoit Perthame. Concentration in Lotka-Volterra Parabolic or Integral Equations: A General Convergence Result. *Methods Appl. Anal.*, 16(3):321–340, September 2009.

[5] Guy Barles and Benoît Perthame. Exit Time Problems in Optimal Control and Vanishing Viscosity Method. *SIAM Journal on Control and Optimization*, 26(5):1133–1148, September 1988.

[6] Henri Berestycki and Grégoire Nadin. Spreading speeds for one-dimensional monostable reaction-diffusion equations. *Journal of Mathematical Physics*, 53(11):115619, November 2012.

[7] Emeric Bouin. A Hamilton-Jacobi approach for front propagation in kinetic equations. *Kinetic and Related Models*, 8(2):255–280, March 2015.

[8] Emeric Bouin and Vincent Calvez. A kinetic eikonal equation. *Comptes Rendus Mathematique*, 350(5):243–248, March 2012.

[9] Emeric Bouin, Vincent Calvez, Emmanuel Grenier, and Grégoire Nadin. Large deviations for velocity-jump processes and non-local Hamilton-Jacobi equations. *arXiv:1607.03676*, 2016.

[10] Emeric Bouin, Vincent Calvez, Nicolas Meunier, Sepideh Mirrahimi, Benoît Perthame, Gaël Raoul, and Raphaël Voituriez. Invasion fronts with variable motility: Phenotype selection, spatial sorting and wave acceleration. *Comptes Rendus Mathematique*, 350(15):761 – 766, 2012.

[11] Emeric Bouin, Vincent Calvez, and Grégoire Nadin. Propagation in a Kinetic Reaction-Transport Equation: Travelling Waves And Accelerating Fronts. *Archive for Rational Mechanics and Analysis*, 217(2):571–617, August 2015.

[12] Emeric Bouin and Sepideh Mirrahimi. A Hamilton–Jacobi approach for a model of population structured by space and trait. *Communications in Mathematical Sciences*, 13(6):1431–1452, 2015.

[13] Paul C. Bressloff and Olivier Faugeras. On the Hamiltonian structure of large deviations in stochastic hybrid systems. *arXiv:1410.2152 [math, q-bio]*, October 2014. arXiv: 1410.2152.

[14] Paul C. Bressloff and Jay M. Newby. Path integrals and large deviations in stochastic hybrid systems. *Physical Review E*, 89(4), April 2014.

[15] Nils Caillerie. Large deviations of a velocity jump process with a Hamilton–Jacobi approach. *Comptes Rendus Mathematique*, February 2017.

[16] Vincent Calvez, Pierre Gabriel, and Álvaro Mateos González. Limiting Hamilton-Jacobi equation for the large scale asymptotics of a subdiffusion jump-renewal equation. *arXiv:1609.06933 [math]*, September 2016. arXiv: 1609.06933.

[17] Jérôme Coville. Singular measure as principal eigenfunction of some nonlocal operators. *Applied Mathematics Letters*, 26(8):831–835, 2013.

[18] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27(1):1–68, November 1992.

[19] Carlota M. Cuesta, Sabine Hittmeir, and Christian Schmeiser. Traveling Waves of a Kinetic Transport Model for the KPP-Fisher Equation. *SIAM Journal on Mathematical Analysis*, 44(6):4128–4146, January 2012.

[20] Odo Diekmann, Pierre-Emmanuel Jabin, Stéphane Mischler, and Benoît Perthame. The dynamics of adaptation: An illuminating example and a Hamilton–Jacobi approach. *Theoretical Population Biology*, 67(4):257–271, June 2005.

[21] L. C. Evans and P. E. Souganidis. A PDE Approach to Geometric Optics for Certain Semilinear Parabolic Equations. *Indiana University Mathematics Journal*, 38(1):141–172, 1989.

[22] Alessandra Faggionato, Davide Gabrielli, and Marco Ribezzi Crivellari. Averaging and Large Deviation Principles for Fully-Coupled Piecewise Deterministic Markov Processes and Applications to Molecular Motors. *Markov Processes Relat. Fields*, 16(3):497–548, 2010.

[23] Ronald A. Fisher. The Wave of Advance of Advantageous Genes. *Annals of Eugenics*, 7(4):355–369, 1937.

[24] Mark Freidlin. *Functional Integration and Partial Differential Equations. (AM-109)*. Princeton University Press, 1985.

[25] Mark I. Freidlin and Jürgen Gärtner. On the propagation of concentration waves in periodic and random media. *Sov. Math. Dokl.*, 20:1282–1286, 1979.

[26] K. P. Hadeler. Hyperbolic travelling fronts. *Proceedings of the Edinburgh Mathematical Society*, 31(1):89–97, 1988.

[27] K. P. Hadeler. *Reaction transport systems in biological modelling*, pages 95–150. Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.

[28] Hélène Hivert. An asymptotic preserving scheme for front propagation in a kinetic reaction-transport equation. *hal-01522278*, 2017.

[29] Yuri Kifer. *Large deviations and adiabatic transitions for dynamical systems and Markov processes in fully coupled averaging*. Number no. 944 in Memoirs of the American Mathematical Society. American Mathematical Society, Providence, R.I, 2009. OCLC: ocn321047153.

[30] Andreï N. Kolmogorov, Ivan G. Petrovsky, and Nikolaï S. Piskunov. Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Moskow Univ. Math. Bull.*, 1:1–25, 1937.

[31] Alexander Lorz, Sepideh Mirrahimi, and Benoît Perthame. Dirac Mass Dynamics in Multidimensional Nonlocal Parabolic Equations. *Communications in Partial Differential Equations*, 36(6):1071–1098, January 2011.

[32] Benoît Perthame and Panagiotis E. Souganidis. Asymmetric potentials and motor effect: a homogenization approach. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 26(6):2055–2071, November 2009.

[33] Luca Rossi. The Freidlin-Gartner formula for general reaction terms. *arXiv:1503.09010v3*, 2015.

[34] Panagiotis E. Souganidis. Front propagation: Theory and applications. In *Viscosity Solutions and Applications*, pages 186–242. Springer Berlin Heidelberg, Berlin, Heidelberg, 1997. DOI: 10.1007/BFb0094298.