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Lifting in Besov spaces

Petru Mironescu^{*†} Emmanuel Russ[‡] Yannick Sire[§]

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Abstract

Let Ω be a smooth bounded (simply connected) domain in \mathbb{R}^n and let u be a complex-valued measurable function on Ω such that $|u(x)| = 1$ a.e. Assume that u belongs to a Besov space $B_{p,q}^s(\Omega; \mathbb{C})$. We investigate whether there exists a real-valued function $\varphi \in B_{p,q}^s(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$. This complements the corresponding study in Sobolev spaces due to Bourgain, Brezis and the first author. The microscopic parameter q turns out to play an important role in some limiting situations. The analysis of this lifting problem relies on some interesting new properties of Besov spaces, in particular a non-restriction property when $q > p$.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a simply connected domain and let $u : \Omega \rightarrow \mathbb{S}^1$ be a continuous (resp. C^k , $k \geq 1$) function; we identify u with a complex-valued function such that $|u(x)| = 1$, $\forall x$. It is a well-known fact that there exists a continuous (resp. C^k) real-valued function φ such that $u = e^{i\varphi}$. In other words, u has a continuous (resp. C^k) lifting. Moreover, φ is unique mod 2π .

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The analogous problem when Ω is a smooth bounded (simply connected) domain and u belongs to the integer or fractional order Sobolev space

$$W^{s,p}(\Omega; \mathbb{S}^1) = \{u \in W^{s,p}(\Omega; \mathbb{C}); |u(x)| = 1 \text{ a.e.}\},$$

with $s > 0$ and $1 \leq p < \infty$ was addressed by Bourgain, Brezis and the first author and received a complete answer in [4]. Further developments in the Sobolev context can be found in [1, 30, 25, 27].

In the present paper, we address the corresponding questions (existence and uniqueness mod 2π) in the framework of Besov spaces $B_{p,q}^s$. Our main interest concerns the influence of the “microscopic” parameter q on the existence and uniqueness issues. Loosely speaking, the main features of $B_{p,q}^s$ are given by s and p and one could expect that the answers to the above questions are the same for $B_{p,q}^s$ as for $W^{s,p}$. This is true “most of the time”, but not always. The analysis in Besov spaces is partly similar to the one in Sobolev spaces, as far as the results and the techniques are concerned, but some striking differences can occur and some cases remain open. Here is an example of strong influence on the lifting problem of the value of the microscopic parameter q . Assume that the space dimension is one and let $\Omega = (0, 1)$, $1 \leq p < \infty$ and $s = 1/p$. Then maps in $W^{1/p,p}((0, 1); \mathbb{S}^1)$ have a lifting in $W^{1/p,p}((0, 1); \mathbb{R})$ and this lifting is unique mod 2π [4]. We will see below that the same holds in $B_{p,q}^{1/p}((0, 1); \mathbb{S}^1)$ provided $1 \leq q < \infty$. However, in $B_{p,\infty}^{1/p}((0, 1); \mathbb{S}^1)$ we have both non-existence for a general u and, in case of existence for some specific u , non-uniqueness.

Let us now be more specific about the functional setting we consider. Given $s > 0$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$, we ask whether a map u in the space

$$B_{p,q}^s(\Omega; \mathbb{S}^1) = \{u \in B_{p,q}^s(\Omega; \mathbb{C}); |u(x)| = 1 \text{ a.e.}\},$$

can be lifted as $u = e^{i\varphi}$, with $\varphi \in B_{p,q}^s(\Omega; \mathbb{R})$. We say that $B_{p,q}^s$ has the lifting property if and only if the answer is positive for any u in this space.

A comment about the range of parameters s , p and q . We discard the case $s \leq 0$, since we want to have spaces of “genuine” maps and thus we require $B_{p,q}^s \hookrightarrow L_{loc}^1$; this does not hold when $s < 0$ and in general it does not hold when $s = 0$. (However, we will discuss an appropriate version of the lifting problem when $s = 0$.) We also discard the uninteresting case where $p = \infty$ and $s > 0$. In this case, maps are continuous and easy arguments lead to the existence and uniqueness mod 2π of a lifting in $B_{\infty,q}^s$. The restriction $q \geq 1$ is not essential: it allows us to work in a Banach spaces framework, but an inspection of our arguments shows that the case where $0 < q < 1$ could be treated using similar lines.

More is to be said about the condition $1 \leq p < \infty$. It is mainly motivated by our main interest, which is to compare the existence and uniqueness properties of $W^{s,p}$ versus $B_{p,q}^s$. This excludes from our discussion the relevant range

$0 < p < 1$ and $s > n \left(\frac{1}{p} - 1 \right)$. In this case, we do have $B_{p,q}^s \hookrightarrow L_{loc}^1$ and the lifting questions are meaningful. We don't know the answers to the existence and uniqueness of lifting questions in this range; they do not seem to follow completely by a straightforward adaptation of our techniques and it would be of interest to know them.

Let us now state our main results and compare them to the Sobolev spaces results established in [4]. For the convenience of the reader, we present them separately for $n = 1$, $n = 2$ and $n \geq 3$.

Case $n = 1$

Sobolev spaces setting. In $W^{s,p}((0,1); \mathbb{S}^1)$:

1. We have the lifting property for every s and p .
2. We have uniqueness of lifting if and only if $sp \geq 1$.

Besov spaces setting. In $B_{p,q}^s((0,1); \mathbb{S}^1)$:

1. We have the lifting property for every s , p and q *except* when $1 \leq p < \infty$, $s = 1/p$ and $q = \infty$.
2. We have uniqueness of lifting if and only if: either $sp > 1$ or [$sp = 1$ and $q < \infty$].

When $n = 1$, it is possible to adapt the Sobolev spaces techniques to Besov spaces when $sp < 1$ or $sp > 1$. New approaches are required in the limiting case where $sp = 1$.

To start with, assume that $sp = 1$ and $q = \infty$. This is a new situation compared to the one in the Sobolev setting, in the sense that we have both non-existence and non-uniqueness. In this case, our strategy consists of constructing some smooth $u \in B_{p,\infty}^{1/p}$ such that *no* lifting of u is in $B_{p,\infty}^{1/p}$. While for this u we will check by a direct calculation that its smooth liftings do not belong to $B_{p,\infty}^{1/p}$, the heart of the proof consists of proving that *no other lifting of u* belongs to $B_{p,\infty}^{1/p}$.

We now turn to the case where $sp = 1$ and $q < \infty$. In the Sobolev spaces setting, a typical argument for the existence of lifting goes as follows. Assume e.g. that $p = 3$ and let $u \in W^{1/3,3}((0,1); \mathbb{S}^1)$. Let $v \in W^{2/3,3}((0,1)^2; \mathbb{C})$ be an extension of u ; its regularity is given by the standard trace theory. A key fact is that we may construct such a v which is, in addition, \mathbb{S}^1 -valued. We next repeat the argument and construct some extension of v , $w \in W^{1,3}((0,1)^3; \mathbb{S}^1)$. It turns out that, by an argument going back to Bethuel and Zheng [2], w has a lifting ψ in $W^{1,3}((0,1)^3; \mathbb{R})$. By taking the trace of ψ first on $(0,1)^2$, next on $(0,1)$, we obtain the existence of a lifting φ of u in $W^{1/3,3}$.

A similar argument does not work in $B_{p,q}^s$, since the standard trace theory shows that only the space $B_{p,p}^s$ is the trace of some Besov space. In place of the above type of argument, we present an approach based on the trace theory of weighted Sobolev spaces and using a single extension. This approach, in the spirit of the recent work [28] of the first two authors, is new even in the Sobolev spaces setting.

Before proceeding further, let us note that the range where uniqueness mod 2π holds, i.e., $sp > 1$ or $[sp = 1$ and $q < \infty]$, is the same in any dimension, and will not be mentioned in our discussion when $n \geq 2$.

Case $n = 2$

Sobolev spaces setting. In $W^{s,p}(\Omega; \mathbb{S}^1)$:

1. We have the lifting property when $sp < 1$ or $sp \geq 2$;
2. We don't have the lifting property when $1 \leq sp < 2$.

Besov spaces setting. In $B_{p,q}^s(\Omega; \mathbb{S}^1)$:

1. We have the lifting property when $sp < 1$ or $sp > 2$ or $[sp = 2$ and $q < \infty]$;
2. We don't have the lifting property when $1 \leq sp < 2$ or $[sp = 2$ and $q = \infty]$.

Compared to the Sobolev spaces setting, the arguments of a new type rely on the trace theory of weighted Sobolev spaces (as for $n = 1$) and on the extension to higher dimensions of the counter-example obtained in $B_{p,\infty}^{1/p}((0,1); \mathbb{S}^1)$.

Things become more involved in dimension $n \geq 3$. There, unlike in the Sobolev spaces setting, we have only partial results.

Case $n \geq 3$

Sobolev spaces setting. In $W^{s,p}(\Omega; \mathbb{S}^1)$:

1. We have the lifting property when $sp < 1$ or $[s \geq 1$ and $sp \geq 2]$ or $sp \geq n$;
2. We don't have the lifting property when $1 \leq sp < 2$ or $[0 < s < 1$ and $2 \leq sp < n]$.

Besov spaces setting. In $B_{p,q}^s(\Omega; \mathbb{S}^1)$:

1. We have the lifting property when $sp < 1$ or $[s > 1$ and $sp > 2]$ or $[s > 1$ and $1 \leq q \leq p < \infty$ and $sp = 2]$ or $sp > n$ or $[sp = n$ and $q < \infty]$;
2. We don't have the lifting property when $1 \leq sp < 2$ or $[sp = 2$ and $q = \infty]$ or $[0 < s < 1$ and $2 \leq sp < n]$.

Three main new features are unveiled by the analysis of the case $n \geq 3$.

A first one is related to the strategy of solving the equation $u = e^{i\varphi}$ by differentiating it. Formally, we have

$$u = e^{i\varphi} \implies \nabla u = iu \nabla \varphi \implies \nabla \varphi = \frac{\nabla u}{iu} = -i\bar{u} \nabla u := F. \quad (1.1)$$

Let $u \in B_{p,q}^s$. Assuming that F has the expected regularity, i.e., that $F \in B_{p,q}^{s-1}$, we may hopefully find some $\varphi \in B_{p,q}^s$ solving $\nabla \varphi = F$, and then φ is a good candidate for a lifting of u . In addition to the regularity issues on F , this strategy requires that $\operatorname{curl} F = 0$ (in order to have $F = \nabla \varphi$ for some φ). The necessary condition $\operatorname{curl} F = 0$ holds indeed if $u \in W^{s,p}$ with $s \geq 1$ and $sp \geq 2$ [4]; this can be proved using a simple argument. In our case, we are not aware of a similar proof covering the limiting case $sp = 2$. Instead, we have devised a proof relying on a new result, the *disintegration of Jacobians* of \mathbb{S}^1 -valued maps (Lemma 3.11). This result is interesting in its own right. It will be straightforward from its proof that the disintegration formula can be extended to more general target manifolds, making it potentially useful in other situations. In the setting of \mathbb{S}^1 -valued maps, it allows us to prove, in dimension $n \geq 3$, that $\operatorname{curl} F = 0$ provided either $sp > 2$ or $[1 \leq q \leq p$ and $sp = 2]$. This holds even when $0 < s < 1$, although F does not even seem to be defined in this range.

A second new feature is related to the seemingly strange condition $1 \leq q \leq p$ that appears above. We don't know whether this condition is relevant for the existence of lifting in the Besov spaces setting, but we do know that it is related to a limitation of our methods. To be more specific, when $1 \leq p < q \leq \infty$ and $s > 0$, there exists some $f \in B_{p,q}^s(\mathbb{R}^3)$ such that for a.e. $x \in [0, 1]$ we have $f(x, \cdot) \notin B_{p,q}^s(\mathbb{R}^2)$. [We will come back to this striking "non-restriction" phenomenon at the end of the introduction.] Since we are unable to bypass this non-restriction property, we don't know whether, when $N \geq 3$, $sp = 2$ and $1 \leq p < q < \infty$, the vector field F defined above satisfies $\operatorname{curl} F = 0$. As a consequence, we are unaware whether the Besov spaces $B_{p,q}^s(\Omega; \mathbb{S}^1)$ with $n \geq 3$, $s > 1$, $sp = 2$ and $1 \leq p < q < \infty$ do have the lifting property.

A third new feature is related to product estimates. To start with, let us assume that $u \in W^{s,p}(\Omega; \mathbb{S}^1)$, with $s \geq 1$. Then the vector-field F defined in (1.1) satisfies $F \in W^{s-1,p}$ [4]. This is straightforward e.g. when $s = 1$, since in that case we clearly have $F \in L^\infty \cdot L^p = L^p$. However, when $s = 1$ a similar property is wrong, *in general*, for Besov spaces. Indeed, in [29] the first author and Van Schaftingen construct an example of smooth map $u : \Omega \rightarrow \mathbb{S}^1$ (with Ω a smooth bounded domain in \mathbb{R}^2) such that $u \in B_{1,1}^1$ but u has no phase in $B_{1,1}^1$. One can prove that for this u the corresponding F does not belong to $B_{1,1}^0$. Again, we could not overcome the difficulties arising from this pathological property of Besov spaces and we don't know whether the Besov spaces $B_{p,q}^1(\Omega; \mathbb{S}^1)$ with $n \geq 3$, $[2 \leq p < n$ and $1 \leq q < \infty]$ or $[2 < p \leq n$ and $q = \infty]$ do have the lifting property.

After this dimension-dependent discussion, let us gather, for the convenience of the reader, the existence results presented above in a “positive” and a “negative” statement.

Theorem 1.1. Let $s > 0$, $1 \leq p < \infty$, $1 \leq q \leq \infty$. The lifting problem has a positive answer in the following cases:

1. $s > 0$, $1 \leq q \leq \infty$ and $sp > n$.
2. $0 < s < 1$, $1 \leq q \leq \infty$ and $sp < 1$.
3. $0 < s \leq 1$, $1 \leq q < \infty$ and $sp = n$.
4. $s > 1$, $1 \leq q < \infty$, $n = 2$ and $sp = 2$.
5. $s > 1$, $1 \leq q \leq p$, $n \geq 3$ and $sp = 2$.
6. $s > 1$, $1 \leq q \leq \infty$, $n \geq 2$ and $sp > 2$.

Theorem 1.2. Let $s > 0$, $1 \leq p < \infty$, $1 \leq q \leq \infty$. The lifting problem has a negative answer in the following cases:

1. $0 < s < 1$, $1 \leq q < \infty$, $n \geq 2$ and $1 \leq sp < n$.
2. $0 < s < 1$, $q = \infty$, $n \geq 2$ and $1 < sp < n$.
3. $s > 0$, $1 \leq q < \infty$, $n \geq 2$ and $1 \leq sp < 2$.
4. $s > 0$, $q = \infty$, $n \geq 2$ and $1 < sp \leq 2$.
5. $0 < s \leq 1$, $q = \infty$ and $sp = 1$.

As already mentioned, Theorems 1.1 and 1.2 do not cover the full range of n , s , p and q . We will come back to the open cases at the end of introduction, and also in Section 6.

Outline of the proofs and organization of the paper. For the convenience of the reader, we regroup the proofs of our results and discussions on lifting in three sections, 4 to 6, containing respectively the analysis of “positive” cases (where we have existence of lifting), of “negative” cases (non-existence of lifting) and of “open” (at least to us) cases. These “cases” correspond to ranges of n , s , p and q where the same arguments apply.

Let us now describe more precisely our methods. When $sp > n$, functions in $B_{p,q}^s$ are continuous, which readily implies that $B_{p,q}^s$ has the lifting property (Case 1).

In the case where $sp < 1$, we rely on a characterization of $B_{p,q}^s$ in terms of the Haar basis [3, Théorème 5] in order to prove that $B_{p,q}^s$ has the lifting property (Case 2).

Assume now that $[0 < s \leq 1, sp = n$ and $q < \infty]$. Let $u \in B_{p,q}^s(\Omega; \mathbb{S}^1)$ and let $F(x, \varepsilon) := u * \rho_\varepsilon$, where ρ is a standard mollifier. Since $B_{p,q}^s \hookrightarrow \text{VMO}$, for all ε sufficiently small and all $x \in \Omega$ with $\text{dist}(x, \partial\Omega) > \varepsilon$ we have $1/2 < |F(x, \varepsilon)| \leq 1$. Writing $F(x, \varepsilon)/|F(x, \varepsilon)| = e^{i\psi_\varepsilon}$, where ψ_ε is C^∞ , and relying on a slight modification of the trace theory for weighted Sobolev spaces as revisited in [28], we conclude, letting ε tend to 0, that $u = e^{i\psi_0}$, where $\psi_0 = \lim_{\varepsilon \rightarrow 0} \psi_\varepsilon \in B_{p,q}^s$, and therefore $B_{p,q}^s$ has the lifting property (Cases 3 and 4).

Consider now the range $[s > 1$ and $sp \geq 2]$. Arguing as in [4, Section 3], it is easily seen that the lifting property for $B_{p,q}^s$ will follow from the following property: given $u \in B_{p,q}^s(\Omega; \mathbb{S}^1)$, if $F := -i\bar{u}\nabla u \in L^p(\Omega; \mathbb{R}^n)$, then $(*) \text{ curl } F = 0$. The proof of $(*)$ is much more involved than the corresponding one for $W^{s,p}$ spaces [4, Section 3]. It relies on a disintegration argument for the Jacobians, more generally applicable in $W^{1/p,p}$. In order to conclude, we combine disintegration with the fact that $\text{curl } F = 0$ when $[u \in \text{VMO}$ and $n = 2]$ and a slicing argument. The slicing argument is not needed when $n = 2$ and is trivial when $sp > 2$. In the limiting case $sp = 2$, it relies on a restriction property for Besov spaces, namely the fact that, for $[s > 0$ and $1 \leq q \leq p < \infty]$, for all $f \in B_{p,q}^s$, the partial maps of f belong a.e. to $B_{p,p}^s$ (Lemma 3.1). All this leads to the following: when $[s > 1$ and $1 \leq p < \infty]$, $B_{p,q}^s$ does have the lifting property when $[1 \leq q < \infty, n = 2, \text{ and } sp = 2]$, or $[1 \leq q \leq p, n \geq 3, \text{ and } sp = 2]$, or $[1 \leq q \leq \infty, n \geq 2, \text{ and } sp > 2]$ (Case 5).

One can improve the conclusion of Lemma 3.1 as follows. For $[s > 0, 1 \leq p < \infty$ and $1 \leq q \leq p]$, for all $f \in B_{p,q}^s$, the partial maps of f belong a.e. to $B_{p,q}^s$ (Proposition 3.4). This is reminiscent of the well-known fact that, if $f \in W^{s,p}(\mathbb{R}^2)$, then for a.e. $x \in \mathbb{R}$ we have $f(x, \cdot) \in W^{s,p}(\mathbb{R})$. It turns out that a similar conclusion holds in $B_{p,q}^s(\mathbb{R}^2)$ *precisely under the assumption* $q \leq p$. The sufficiency of the condition $q \leq p$ follows from Proposition 3.4. In the opposite direction, when $q > p$ and $s > 0$, we construct a compactly supported function $f \in B_{p,q}^s(\mathbb{R}^2)$ such that, for almost every $x \in [0, 1]$, $f(x, \cdot) \notin B_{p,\infty}^s(\mathbb{R})$, and in particular $f(x, \cdot) \notin B_{p,q}^s(\mathbb{R})$ (Proposition 3.5). [It is quite easy to adapt this construction to higher dimensions n and to a.e. $x \in \mathbb{R}^{n-1}$.] This phenomenon, which has not been noticed before, shows a picture strikingly different not only from the one for $W^{s,p}$, but more generally for the one in the scale of Triebel-Lizorkin spaces.

Following a suggestion of the first author, Brasseur investigated in higher generality this “non-restriction” property. In [10] (which is independent of the present work), he obtains the same result in the full range $0 < p < q \leq \infty$; his construction is somewhat similar to ours. [10] also contains an interesting positive result: it exhibits function spaces X intermediate between $B_{p,q}^s(\mathbb{R})$ and $\bigcup_{\varepsilon > 0} B_{p,q}^{s-\varepsilon}(\mathbb{R})$ such that, if $f \in B_{p,q}^s(\mathbb{R}^2)$, then for a.e. $x \in \mathbb{R}$ we have $f(x, \cdot) \in X$.

Let us return to the case where $[0 < s < 1, 1 \leq p < \infty$ and $n \geq 2]$. This time, we assume that $1 \leq q < \infty$ and $1 \leq sp < n$ or $[q = \infty$ and $1 < sp < n]$. In this case, we show that $B_{p,q}^s$ does not have the lifting property (Case 6).

The argument uses embedding theorems and the following fact, for which we provide a proof: let $[s_i > 0, 1 \leq p_i < \infty]$, and $[s_j p_j = 1 \text{ and } 1 \leq q_j < \infty]$ or $[s_j p_j > 1 \text{ and } 1 \leq q_j \leq \infty]$, $i = 1, 2$. If $f_i \in B_{p_i, q_i}^{s_i}$, $i = 1, 2$, and the function $f = f_1 + f_2$ assumes only integer values, then f is constant (Lemma 7.14).

Assume next that $[0 < s < \infty, 1 \leq p < \infty, n \geq 2]$, and $[1 \leq q < \infty \text{ and } 1 \leq sp < 2]$ or $[q = \infty \text{ and } 1 \leq sp \leq 2]$. In this case, $B_{p, q}^s$ does not have the lifting property either. We provide a counterexample of topological nature, inspired by [4, Section 4]: namely, the function $u(x) = \frac{(x_1, x_2)}{(x_1^2 + x_2^2)^{1/2}}$ belongs to $B_{p, q}^s$ but has no lifting in $B_{p, q}^s$ (Case 7).

In the case where $[q = \infty \text{ and } sp = 1]$, we show that $B_{p, q}^s$ does not have the lifting property (Case 8). More specifically, we first construct a function $\psi \in C^\infty(\mathbb{R})$ which does not belong to $B_{p, \infty}^{1/p}$ such that $u := e^{i\psi}$ does belong to $B_{p, \infty}^{1/p}$, and prove that there is no $\varphi \in B_{p, \infty}^{1/p}$ such that $u = e^{i\varphi}$. This relies, in particular, on the fact that integer-valued functions in $B_{p, \infty}^{1/p}(\mathbb{R})$ are step functions. We next use this ψ to prove that the lifting property does not hold in $B_{p, \infty}^{1/p}(\mathbb{R}^n)$ with $n \geq 2$ neither.

As already mentioned, our arguments do not cover all possible situations, and we are unaware of the answer to the existence of lifting problem in some cases.

A first such case occurs when $[s > 1, 1 \leq p < \infty, p < q < \infty, n \geq 3, \text{ and } sp = 2]$ (Case 9). In this situation, since the restriction property for $B_{p, q}^s$ does not hold, the argument given in the proof of Case 5 before does not work any longer and we don't know if $B_{p, q}^s$ has the lifting property.

The case where $[s = 1, 1 \leq p < \infty, n \geq 3]$, and $[1 \leq q < \infty \text{ and } 2 \leq p < n]$ or $[q = \infty \text{ and } 2 < p \leq n]$ (Case 10) is also open (except when $s = 1$ and $p = q = 2$, since in this case, $B_{2, 2}^1 = W^{1, 2}$ has the lifting property). In a related direction, it is not known whether the map $\varphi \mapsto e^{i\varphi}$ maps $B_{p, q}^1$ into itself.

The case where $[n \geq 3, n \leq p < \infty, s = n/p \text{ and } q = \infty]$ is also open. Indeed, $B_{p, q}^s$ is not embedded into VMO in this case, and the arguments we use in Cases 3 and 4 are not applicable anymore.

The paper is organized as follows. In Section 2, we briefly recall the standard definition of Besov spaces and some classical characterizations of these spaces (by Littlewood-Paley theory and wavelets). Section 3 gathers statements and proofs of some new results about Besov spaces, which play a key role in our analysis (characterization by extensions, disintegration of the Jacobians) or are directly related to open cases (non-restriction properties) and are interesting in their own right. In Section 4 we establish Theorem 1.1, namely the cases where $B_{p, q}^s$ does have the lifting property, while Section 5 is devoted to negative cases (Theorem 1.2). In Section 6, we discuss the remaining cases, which are widely open. Finally, Section 7 contains statements and proofs of

other various results on Besov spaces (some of them being well-known) needed in the proofs of Theorems 1.1 and 1.2. For the convenience of the reader, we also provide detailed arguments for classical properties (some embeddings, Poincaré inequalities) for which we were unable to find a precise reference in the existent literature.

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Notation, framework

1. Most of our positive results are stated in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. Additional properties of Ω may be required (or not).
 - (a) When $sp < 1$, the topology of Ω plays no role. Indeed, in this range one can “glue” $B_{p,q}^s$ functions in adjacent Lipschitz domains and still obtain a $B_{p,q}^s$ function. Therefore, one obtains global existence of a lifting provided we have local existence.
 - (b) However, when $sp > 1$, or when $[sp = 1$ and $q < \infty]$, we must require that Ω has a “simple topology” in order to have existence of lifting (even for smooth u). The typical sufficient assumption is that Ω is simply connected.¹ In this range, local existence of lifting (i.e.,

¹ The assumption that Ω is simply connected can be (optimally) relaxed as follows. Let G be the commutator subgroup of the first homotopy group $\pi_1(\Omega)$ of Ω . The necessary and sufficient for the existence of a smooth lifting for every smooth \mathbb{S}^1 -valued u is

(H) the quotient $\pi_1(\Omega)/G$ is finite (see e.g. [20, Theorem 6.1, p. 45 and Theorem 7.1, p. 49]). In particular, this assumption is satisfied if Ω is simply connected, and more generally if $\pi_1(\Omega)$ is perfect. We may prove the following. Assume that $sp > 1$ or $[sp = 1$ and $q < \infty]$. Assume that n, s, p, q are such that for a ball $B \subset \mathbb{R}^n$ the space $B_{p,q}^s(B; \mathbb{S}^1)$ has the lifting property. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Then $B_{p,q}^s(\Omega; \mathbb{S}^1)$ has the lifting property if and only if (H) holds.

existence on balls or cubes) combined with uniqueness of lifting (mod 2π) implies, by a standard argument, existence of lifting in all smooth simply connected domains.

- (c) In view of the above, in all positive cases it suffices to investigate only “local” existence, i.e., existence in model domains like $\Omega = (0, 1)^n$.
2. On the other hand, it will be clear from the constructions that the non-existence results we exhibit (for specific values of n , s , p and q) are of local nature and therefore valid in all domains (with given n). Therefore, in the negative cases the topology of Ω is irrelevant.
 3. In few positive cases, proofs are simpler if we consider $(2\pi\mathbb{Z})^n$ -periodic maps $u : (0, 2\pi)^n \rightarrow \mathbb{S}^1$. [However, this assumption is made just for the sake of the simplicity of the proofs. It will be clear that the techniques we present can be adapted to smooth simply connected domains.] In this case, we denote the corresponding function spaces $B_{p,q}^s(\mathbb{T}^n; \mathbb{S}^1)$, and the question is whether a map $u \in B_{p,q}^s(\mathbb{T}^n; \mathbb{S}^1)$ has a lifting $\varphi \in B_{p,q}^s((0, 2\pi)^n; \mathbb{R})$. Note that *we do not look for a $(2\pi\mathbb{Z})^n$ -periodic phase*. Clearly, such a periodic phase need not exist, already in the smooth case.
 4. Partial derivatives are denoted ∂_j , $\partial_j\partial_k$, and so on, or ∂^α .
 5. \wedge denotes the vector product of complex numbers: $a \wedge b := a_1b_2 - a_2b_1 = \text{Im}(\bar{a}b)$. Similarly, $u \wedge \nabla v := u_1\nabla v_2 - u_2\nabla v_1$.
 6. If $u : \Omega \rightarrow \mathbb{C}$ and if ω is a k -form on Ω (with $k \in \llbracket 0, n-1 \rrbracket$, k integer), then $\omega \wedge (u \wedge \nabla u)$ denotes the $(k+1)$ -form $\omega \wedge (u_1du_2 - u_2du_1)$.
 7. We let \mathbb{R}_+^n denote the open set $\mathbb{R}^{n-1} \times (0, \infty)$.

Contents

1	Introduction	1
2	Crash course on Besov spaces	11
2.1	Preliminaries	11
2.2	Definitions of Besov spaces	12
2.3	Besov spaces on \mathbb{T}^n	13
2.4	Characterization by differences	14
2.5	Lizorkin type characterizations	14
2.6	Characterization by the Haar system	15
2.7	Characterization via smooth wavelets	17
2.8	Nikolskiĭ type decompositions	18

3	Analysis in Besov spaces	19
3.1	Restrictions	19
3.2	(Non-)restrictions	20
3.3	Characterization of $B_{p,q}^s$ via extensions	24
3.4	Disintegration of the Jacobians	31
3.5	Integer-valued functions in $B_{p,\infty}^{1/p} + C^1$	35
4	Positive cases	39
5	Negative cases	42
6	Open cases	47
7	Other results for Besov spaces	48
7.1	Embeddings	48
7.2	Poincaré type inequalities	50
7.3	Product estimates	53
7.4	Superposition operators	54
7.5	Integer-valued functions in sums of Besov spaces	54

2 Crash course on Besov spaces

We briefly recall here the basic properties of the Besov spaces in \mathbb{R}^n , with special focus on the properties which will be instrumental for our purposes. For a complete treatment of these spaces, see [36, 18, 37, 32].

2.1 Preliminaries

In the sequel, $\mathcal{S}(\mathbb{R}^n)$ is the usual Schwartz space of rapidly decreasing C^∞ functions. Let $\mathcal{Z}(\mathbb{R}^n)$ denote the subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\partial^\alpha \varphi(0) = 0$ for every multi-index $\alpha \in \mathbb{N}^n$. Let $\mathcal{Z}'(\mathbb{R}^n)$ stand for the topological dual of $\mathcal{Z}(\mathbb{R}^n)$. It is well-known [36, Section 5.1.2] that $\mathcal{Z}'(\mathbb{R}^n)$ can be identified with the quotient space $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, where $\mathcal{P}(\mathbb{R}^n)$ denotes the space of all polynomials in \mathbb{R}^n .

We denote by \mathcal{F} the Fourier transform.

For all sequences $(f_j)_{j \geq 0}$ of measurable functions on \mathbb{R}^n , we set

$$\|(f_j)\|_{l^q(L^p)} := \left(\sum_{j \geq 0} \left(\int_{\mathbb{R}^n} |f_j(x)|^p dx \right)^{q/p} \right)^{1/q},$$

with the usual modification when $p = \infty$ and/or $q = \infty$. If (f_j) is labelled by \mathbb{Z} , then $\|(f_j)\|_{l^q(L^p)}$ is defined analogously with $\sum_{j \geq 0}$ replaced by $\sum_{j \in \mathbb{Z}}$.

Finally, we fix some notation for translation and finite order differences. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $f : \Omega \rightarrow \mathbb{R}$. For all integers $M \geq 0$, all $t > 0$ and all $x, h \in \mathbb{R}^n$, set

$$\Delta_h^M f(x) = \begin{cases} \sum_{l=0}^M \binom{M}{l} (-1)^{M-l} f(x+lh), & \text{if } x, x+h, \dots, x+Mh \in \Omega \\ 0, & \text{otherwise} \end{cases}. \quad (2.1)$$

Define also $\tau_h f(x) := f(x+h)$ whenever $x, x+h \in \Omega$.

2.2 Definitions of Besov spaces

We first focus on inhomogeneous Besov spaces. Fix a sequence of functions $(\varphi_j)_{j \geq 0} \in \mathcal{S}(\mathbb{R}^n)$ such that:

1. $\text{supp } \varphi_0 \subset B(0, 2)$ and $\text{supp } \varphi_j \subset B(0, 2^{j+1}) \setminus B(0, 2^{j-1})$ for all $j \geq 1$.
2. For all multi-indices $\alpha \in \mathbb{N}^n$, there exists $c_\alpha > 0$ such that $|D^\alpha \varphi_j(x)| \leq c_\alpha 2^{-j|\alpha|}$, for all $x \in \mathbb{R}^n$ and all $j \geq 0$.
3. For all $x \in \mathbb{R}^n$, it holds $\sum_{j \geq 0} \varphi_j(x) = 1$.

Definition 2.1 (Definition of inhomogeneous Besov spaces). Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Define $B_{p,q}^s(\mathbb{R}^n)$ as the space of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)} := \left\| \left(2^{sj} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f(\cdot)) \right) \right\|_{l^q(L^p)} < \infty.$$

Recall [36, Section 2.3.2, Proposition 1, p. 46] that $B_{p,q}^s(\mathbb{R}^n)$ is a Banach space which does not depend on the choice of the sequence $(\varphi_j)_{j \geq 0}$, in the sense that two different choices for the sequence $(\varphi_j)_{j \geq 0}$ give rise to equivalent norms. Once the φ_j 's are fixed, we refer to the equality $f = \sum_j f_j$ in \mathcal{S}' as the Littlewood-Paley decomposition of f .

Let us now turn to the definition of homogeneous Besov spaces. Let $(\varphi_j)_{j \in \mathbb{Z}}$ be a sequence of functions satisfying:

1. $\text{supp } \varphi_j \subset B(0, 2^{j+1}) \setminus B(0, 2^{j-1})$ for all $j \in \mathbb{Z}$.
2. For all multi-indices $\alpha \in \mathbb{N}^n$, there exists $c_\alpha > 0$ such that $|D^\alpha \varphi_j(x)| \leq c_\alpha 2^{-j|\alpha|}$, for all $x \in \mathbb{R}^n$ and all $j \in \mathbb{Z}$.
3. For all $x \in \mathbb{R}^n \setminus \{0\}$, it holds $\sum_{j \in \mathbb{Z}} \varphi_j(x) = 1$.

Definition 2.2 (Definition of homogeneous Besov spaces). Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Define $\dot{B}_{p,q}^s(\mathbb{R}^n)$ as the space of $f \in \mathcal{X}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} := \left\| \left(2^{sj} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f(\cdot)) \right) \right\|_{l^q(L^p)} < \infty.$$

Note that this definition makes sense since, for all polynomials P and all $f \in \mathcal{S}'(\mathbb{R}^n)$, we have $|f|_{B_{p,q}^s(\mathbb{R}^n)} = |f + P|_{B_{p,q}^s(\mathbb{R}^n)}$.

Again, $\dot{B}_{p,q}^s(\mathbb{R}^n)$ is a Banach space which does not depend on the choice of the sequence $(\varphi_j)_{j \in \mathbb{Z}}$ [36, Section 5.1.5, Theorem, p. 240].

For all $s > 0$ and all $1 \leq p < \infty$, $1 \leq q \leq \infty$, we have [37, Section 2.3.3, Theorem], [32, Section 2.6.2, Proposition 3]

$$B_{p,q}^s(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap \dot{B}_{p,q}^s(\mathbb{R}^n) \text{ and } \|f\|_{B_{p,q}^s(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + |f|_{B_{p,q}^s(\mathbb{R}^n)}. \quad (2.2)$$

Besov spaces on domains of \mathbb{R}^n are defined as follows.

Definition 2.3 (Besov spaces on domains). Let $\Omega \subset \mathbb{R}^n$ be an open set. Then

1. $B_{p,q}^s(\Omega) := \{f \in \mathcal{D}'(\Omega); \text{ there exists } g \in B_{p,q}^s(\mathbb{R}^n) \text{ such that } f = g|_{\Omega}\}$, equipped with the norm

$$\|f\|_{B_{p,q}^s(\Omega)} := \inf \left\{ \|g\|_{B_{p,q}^s(\mathbb{R}^n)}; g|_{\Omega} = f \right\}.$$

2. $\dot{B}_{p,q}^s(\Omega) := \{f \in \mathcal{D}'(\Omega); \text{ there exists } g \in \dot{B}_{p,q}^s(\mathbb{R}^n) \text{ such that } f = g|_{\Omega}\}$, equipped with the semi-norm

$$\|f\|_{\dot{B}_{p,q}^s(\Omega)} := \inf \left\{ |g|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}; g|_{\Omega} = f \right\}.$$

Local Besov spaces are defined in the usual way: $f \in B_{p,q}^s$ near a point x if for some cutoff φ which equals 1 near x we have $\varphi f \in B_{p,q}^s$. If f belongs to $B_{p,q}^s$ near each point, then we write $f \in (B_{p,q}^s)_{loc}$.

The following is straightforward.

Lemma 2.4. Let $f : \Omega \rightarrow \mathbb{R}$. If, for each $x \in \bar{\Omega}$, $f \in B_{p,q}^s(B(x,r) \cap \Omega)$ for some $r = r(x) > 0$, then $f \in B_{p,q}^s$.

2.3 Besov spaces on \mathbb{T}^n

Let $\varphi_0 \in \mathcal{D}(\mathbb{R}^n)$ be such that

$$\varphi_0(x) = 1 \text{ for all } |x| < 1 \text{ and } \varphi_0(x) = 0 \text{ for all } |x| \geq \frac{3}{2}.$$

For all $k \geq 1$ and all $x \in \mathbb{R}^n$, define

$$\varphi_k(x) := \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x).$$

Definition 2.5. Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Define $B_{p,q}^s(\mathbb{T}^n)$ as the space of distributions $f \in \mathcal{D}'(\mathbb{T}^n)$ whose Fourier coefficients $(a_m)_{m \in \mathbb{Z}^n}$ satisfy

$$\|f\|_{B_{p,q}^s(\mathbb{T}^n)} := \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| x \mapsto \sum_{m \in \mathbb{Z}^n} a_m \varphi_j(2\pi m) e^{2i\pi m \cdot x} \right\|_{L^p(\mathbb{T}^n)}^q \right)^{1/q} < \infty$$

(with the usual modification when $q = \infty$). Again, the choice of the system $(\varphi_j)_{j \geq 0}$ is irrelevant, and the equality $f = \sum f_j$, with $f_j := \sum_m a_m \varphi_j(2\pi m) e^{2i\pi m \cdot x}$, is the Littlewood-Paley decomposition of f .

Alternatively, we have $f \in B_{p,q}^s(\mathbb{T}^n)$ if and only if f can be identified with a \mathbb{Z}^n -periodic distribution in \mathbb{R}^n , still denoted f , which belongs to $(B_{p,q}^s)_{loc}(\mathbb{R}^n)$ [33, Section 3.5.4, pp. 167-169].

2.4 Characterization by differences

Among the various characterizations of Besov spaces, we recall here the ones involving differences [36, Section 5.2.3], [32, Theorem, p. 41], [38, Section 1.11.9, Theorem 1.118, p. 74].

Proposition 2.6. Let $s > 0$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Let $M > s$ be an integer. Then, with the usual modification when $q = \infty$:

1. In the space $\dot{B}_{p,q}^s(\mathbb{R}^n)$ we have the equivalence of semi-norms

$$\begin{aligned} \|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} &\sim \left(\int_{\mathbb{R}^n} |h|^{-sq} \left\| \Delta_h^M f \right\|_{L^p(\mathbb{R}^n)}^q \frac{dh}{|h|^n} \right)^{1/q} \\ &\sim \sum_{j=1}^n \left(\int_{\mathbb{R}} |h|^{-sq} \left\| \Delta_{he_j}^M f \right\|_{L^p(\mathbb{R}^n)}^q \frac{dh}{|h|} \right)^{1/q}. \end{aligned} \quad (2.3)$$

2. The full $B_{p,q}^s$ norm satisfies, for all $\delta > 0$,

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_{|h| \leq \delta} |h|^{-sq} \left\| \Delta_h^M f \right\|_{L^p(\mathbb{R}^n)}^q \frac{dh}{|h|^n} \right)^{1/q}.$$

2.5 Lizorkin type characterizations

Such characterizations involve restrictions of the Fourier transform on cubes or corridors; see e.g. [36, Section 2.5.4, pp. 85-86] or [33, Section 3.5.3, pp. 166-167]. The following special case [33, Section 3.5.3, Theorem, p. 167] will be useful later.

Proposition 2.7. Let $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q \leq \infty$. Set $K_0 := \{0\} \subset \mathbb{Z}^n$ and, for $j \geq 1$, let $K_j := \{m \in \mathbb{Z}^n; 2^{j-1} \leq |m| < 2^j\}$.² Let $f \in \mathcal{D}'(\mathbb{T}^n)$ have the Fourier series expansion $f = \sum_{m \in \mathbb{Z}^n} a_m e^{2i\pi m \cdot x}$. We set $f_j := \sum_{m \in K_j} a_m e^{2i\pi m \cdot x}$. Then we have the norm equivalence

$$\|f\|_{B_{p,q}^s(\mathbb{T}^n)} \sim \left(\sum_{j=0}^{\infty} 2^{jsq} \|f_j\|_{L^p(\mathbb{T}^n)}^q \right)^{1/q}$$

(with the usual modification when $q = \infty$).

2.6 Characterization by the Haar system

Besov spaces can also be described via the size of their wavelet coefficients. To illustrate this, we start with low smoothness Besov spaces, which can be described using the Haar basis. (The next section is devoted to smoother spaces and bases.) For the results of this section, see e.g. [17, Corollary 5.3], [3, Section 7], [38, Theorem 1.58], [39, Theorem 2.21].

Let

$$\psi_M(x) := \begin{cases} 1, & \text{if } 0 \leq x < 1/2 \\ -1, & \text{if } 1/2 \leq x \leq 1, \text{ and } \psi_F(x) := |\psi_M(x)|. \\ 0, & \text{if } x \notin [0, 1] \end{cases} \quad (2.4)$$

When $j \in \mathbb{N}$, we let

$$G^j := \begin{cases} \{F, M\}^n, & \text{if } j = 0 \\ \{F, M\}^n \setminus \{(F, F, \dots, F)\}, & \text{if } j > 0 \end{cases} \quad (2.5)$$

For all $m \in \mathbb{Z}^n$, all $x \in \mathbb{R}^n$ and all $G \in \{F, M\}^n$, define

$$\Psi_m^G(x) := \prod_{r=1}^n \psi_{G_r}(x_r - m_r). \quad (2.6)$$

Finally, for all $m \in \mathbb{Z}^n$, all $j \in \mathbb{N}$, all $G \in G^j$ and all $x \in \mathbb{R}^n$, let

$$\Psi_m^{j,G}(x) := 2^{nj/2} \Psi_m^G(2^j x). \quad (2.7)$$

Recall that the family $(\Psi_m^{j,G})$, called the Haar system, is an orthonormal basis of $L^2(\mathbb{R}^n)$ [38, Proposition 1.53]. Moreover, we have the following result [39, Theorem 2.21].

² Here, $|m| := \max_{l=1}^n |m_l|$.

Proposition 2.8. Let $s > 0$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$ be such that $sp < 1$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B_{p,q}^s(\mathbb{R}^n)$ if and only if there exists a sequence $(\mu_m^{j,G})_{j \geq 0, G \in G^j, m \in \mathbb{Z}^n}$ such that

$$\sum_{j=0}^{\infty} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} |\mu_m^{j,G}|^p \right)^{q/p} < \infty \quad (2.8)$$

(obvious modification when $q = \infty$) and

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \mu_m^{j,G} 2^{-j(s-n/p)} 2^{-nj/2} \Psi_m^{j,G}. \quad (2.9)$$

Here, the series in (2.9) converges unconditionally in $B_{p,q}^s(\mathbb{R}^n)$ when $q < \infty$. Moreover,

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)} \sim \left(\sum_{j=0}^{\infty} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} |\mu_m^{j,G}|^p \right)^{q/p} \right)^{1/q} \quad (2.10)$$

(obvious modification when $q = \infty$).

Equivalently, Proposition 2.8 can be reformulated as follows. Consider the partition of \mathbb{R}^n into standard dyadic cubes Q of side 2^{-j} .³ For all $x \in \mathbb{R}^n$, denote by $Q_j(x)$ the unique dyadic cube of side 2^{-j} containing x . If $f \in L_{loc}^1(\mathbb{R}^n)$, define $E_j(f)(x) := \int_{Q_j(x)} f$ for all $j \geq 0$. We also set $E_{-1}(f) := 0$. We have the following results (see [3, Theorem 5 with $m = 0$] in \mathbb{R}^n ; see also [4, Appendix A] in the framework of Sobolev spaces on \mathbb{T}^n).

Proposition 2.9. Let $s > 0$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$ be such that $sp < 1$. Let $f \in L_{loc}^1(\mathbb{R}^n)$. Then

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)}^q \sim \sum_{j \geq 0} 2^{sjq} \|E_j(f) - E_{j-1}(f)\|_{L^p}^q$$

(obvious modification when $q = \infty$).

Similar results hold when \mathbb{R}^n is replaced by $(0,1)^n$ or \mathbb{T}^n ; it suffices to consider only dyadic cubes contained in $[0,1)^n$.

Corollary 2.10. Let $s > 0$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$ be such that $sp < 1$. Let $f \in L_{loc}^1(\mathbb{R}^n)$. Then

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)}^q \sim \sum_{j \geq 0} 2^{sjq} \|f - E_j(f)\|_{L^p}^q$$

(obvious modification when $q = \infty$).

Similar results hold when \mathbb{R}^n is replaced by $(0,1)^n$ or \mathbb{T}^n .

³ Thus the Q 's are of the form $Q = 2^{-j} \prod_{k=1}^n [m_k, m_k + 1)$, with $m_k \in \mathbb{Z}$.

Corollary 2.11. Let $s > 0$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$ be such that $sp < 1$. Let $(\varphi_j)_{j \geq 0}$ be a sequence of functions on $(0, 1)^n$ such that: for any j , φ_j is constant on each dyadic cube Q of side-length 2^{-j} . Assume that $\sum_{j \geq 1} 2^{sjq} \|\varphi_j - \varphi_{j-1}\|_{L^p}^q < \infty$. Then (φ_j) converges in L^p to some $\varphi \in B_{p,q}^s$, and we have

$$\|\varphi\|_{B_{p,q}^s((0,1)^n)} \lesssim \left(\sum_{j \geq 0} 2^{sjq} \|\varphi_j - \varphi_{j-1}\|_{L^p}^q \right)^{1/q}$$

(with the convention $\varphi_{-1} := 0$ and with the usual modification when $q = \infty$).

In the framework of Sobolev spaces, Corollaries 2.10 and 2.11 are easy consequences of Proposition 2.9; see [4, Appendix A, Theorem A.1] and [4, Appendix A, Corollary A.1]. The arguments in [4] apply with no changes to Besov spaces. Details are left to the reader.

2.7 Characterization via smooth wavelets

Proposition 2.8 has a counterpart when $sp \geq 1$; this requires smoother “mother wavelet” ψ_M and “father wavelet” ψ_F . Given ψ_F and ψ_M two real functions, define $\psi_m^{j,G}$ as in (2.5)–(2.7). Then [23, Chapter 6], [38, Section 1.7.3] for every integer $k > 0$ we may find some $\psi_F \in C_c^k(\mathbb{R})$ and $\psi_M \in C_c^k(\mathbb{R})$ such that the following result holds.

Proposition 2.12. Let $s > 0$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$ be such that $s < k$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B_{p,q}^s(\mathbb{R}^n)$ if and only if there exists a sequence $(\mu_m^{j,G})_{j \geq 0, G \in G^j, m \in \mathbb{Z}^n}$ such that

$$\sum_{j=0}^{\infty} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} |\mu_m^{j,G}|^p \right)^{q/p} < \infty \quad (2.11)$$

(obvious modification when $q = \infty$) and

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \mu_m^{j,G} 2^{-j(s-n/p)} 2^{-nj/2} \Psi_m^{j,G}. \quad (2.12)$$

Here, the series in (2.9) converges unconditionally in $B_{p,q}^s(\mathbb{R}^n)$ when $q < \infty$. Moreover,

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)} \sim \left(\sum_{j=0}^{\infty} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} |\mu_m^{j,G}|^p \right)^{q/p} \right)^{1/q} \quad (2.13)$$

(obvious modification when $q = \infty$).

For further use, let us note that, if $f \in B_{p,q}^s(\mathbb{R}^n)$ for some $s > 0$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$, then we have

$$\mu_m^{j,G} = \mu_m^{j,G}(f) = 2^{j(s-n/p+n/2)} \int_{\mathbb{R}^n} f(x) \Psi_m^{j,G}(x) dx. \quad (2.14)$$

This immediately leads to the following consequence of Proposition 2.12, the proof of which is left to the reader.

Corollary 2.13. Let $s > 0$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$ be such that $s < k$. Assume that $f \in L^p(\mathbb{R}^n)$ is such that the coefficients $\mu_m^{j,G}$ given by (2.14) satisfy

$$\sum_{j=0}^{\infty} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} \left| \mu_m^{j,G} \right|^p \right)^{q/p} = \infty \quad (2.15)$$

(obvious modification when $q = \infty$). Then $f \notin B_{p,q}^s(\mathbb{R}^n)$.

2.8 Nikolskiĭ type decompositions

In practice, we often do not know the Littlewood-Paley decomposition of some given f , but only a Nikolskiĭ representation (or decomposition) of f . More specifically, set $\mathcal{C}_j := B(0, 2^{j+2})$, with $j \in \mathbb{N}$. Let $f^j \in \mathcal{S}'$ satisfy

$$\text{supp } \mathcal{F} f^j \subset \mathcal{C}_j, \quad \forall j \in \mathbb{N}, \quad \text{and } f = \sum_j f^j \text{ in } \mathcal{S}'; \quad (2.16)$$

the decomposition $f = \sum_j f^j$ is a Nikolskiĭ decomposition of f . Note that the Littlewood-Paley decomposition is a special Nikolskiĭ decomposition.

We have the following result.

Proposition 2.14. Let $s > 0$, $1 \leq p < \infty$, $1 \leq q \leq \infty$. Assume that (2.16) holds. Then we have

$$\left\| \sum_j f^j \right\|_{B_{p,q}^s} \lesssim \left(\sum_j 2^{sqj} \|f^j\|_{L^p}^q \right)^{1/q}, \quad (2.17)$$

with the usual modification when $q = \infty$.

The above was proved in [13, Lemma 1] (see also [41]) in the framework of Triebel-Lizorkin spaces $F_{p,q}^s$; the proof applies with no change to Besov spaces and will be omitted here. For related results in the framework of Besov spaces, see [36, Section 2.5.2, pp. 79-80] and [33, Section 2.3.2, Theorem, p. 105].

3 Analysis in Besov spaces

3.1 Restrictions

Captatio benevolentiae. Let $f \in L^1(\mathbb{R}^2)$. Then, for a.e., $y \in \mathbb{R}$, the restriction $f(\cdot, y)$ of f to the line $\mathbb{R} \times \{y\}$ belongs to L^1 . In this section and the next one, we examine some analogues of this property in the framework of Besov spaces.

For this purpose, we first introduce some notation for partial functions. Let $\alpha \subset \{1, \dots, n\}$ and set $\bar{\alpha} := \{1, \dots, n\} \setminus \alpha$. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then we identify x with the couple $(x_\alpha, x_{\bar{\alpha}})$, where $x_\alpha := (x_j)_{j \in \alpha}$ and $x_{\bar{\alpha}} := (x_j)_{j \in \bar{\alpha}}$. Given a function $f = f(x_1, \dots, x_n)$, we let $f_\alpha = f_\alpha(x_\alpha)$ denote the partial function $x_{\bar{\alpha}} \mapsto f(x)$. Another useful notation: given an integer m such that $1 \leq m \leq n$, set

$$I(n-m, n) := \{\alpha \subset \{1, \dots, n\}; \#\alpha = n-m\}.$$

Thus, when $\alpha \in I(n-m, n)$, $f_\alpha(x_\alpha)$ is a function of m variables.

When $q = p$, we have the following result.

Lemma 3.1. Let $1 \leq m < n$. Let $s > 0$ and $1 \leq p < \infty$. Let $f \in B_{p,p}^s(\mathbb{R}^n)$.

1. Let $\alpha \in I(n-m, n)$. Then, for a.e. $x_\alpha \in \mathbb{R}^{n-m}$, we have $f_\alpha(x_\alpha) \in B_{p,p}^s(\mathbb{R}^m)$.
2. We have

$$\|f\|_{B_{p,p}^s(\mathbb{R}^n)}^p \sim \sum_{\alpha \in I(n-m, n)} \int_{\mathbb{R}^{n-m}} \|f_\alpha(x_\alpha)\|_{B_{p,p}^s(\mathbb{R}^m)}^p dx_\alpha.$$

Proof. For the case where $m = 1$, see [36, Section 2.5.13, Theorem, (i), p. 115]. The general case is obtained by a straightforward induction on m . \square

Lemma 3.2. Let $s > 0$, $1 \leq p < \infty$ and $1 \leq q \leq p$. Let $1 \leq m < n$ be an integer. Assume that $sp \geq m$ and let $f \in B_{p,q}^s(\mathbb{T}^n)$. Then, for every $\alpha \in I(n-m, n)$ and for a.e. $x_\alpha \in \mathbb{T}^{n-m}$, the partial map $f_\alpha(x_\alpha)$ belongs to $\text{VMO}(\mathbb{T}^m)$.

Same conclusion if $s > 0$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$, and we have $sp > m$.

Similar conclusions when $\Omega = \mathbb{R}^n$ or $(0, 1)^n$.

Proof. In view of the Besov embeddings (Lemma 7.1), we may assume that $sp = m$ and $q = p$. By Lemma 3.1 and Lemma 7.5, for a.e. x_α we have $f_\alpha(x_\alpha) \in B_{p,p}^s(\mathbb{T}^m) \hookrightarrow \text{VMO}(\mathbb{T}^m)$. \square

Lemma 3.3. Let $s > 0$, $1 \leq p < \infty$ and $1 \leq q < \infty$. Let $M > s$ be an integer. Let $f \in B_{p,q}^s$. For $x' \in \mathbb{T}^{n-1}$, consider the partial map $v(x_n) = v_{x'}(x_n) := f(x', x_n)$, with $x_n \in \mathbb{T}$. Then there exists a sequence $(t_l) \subset (0, \infty)$ such that $t_l \rightarrow 0$ and for a.e. $x' \in \mathbb{T}^{n-1}$, we have

$$\lim_{l \rightarrow \infty} \frac{\left\| \Delta_{t_l e_n}^M v \right\|_{L^p(\mathbb{T})}}{t_l^s} = 0. \tag{3.1}$$

More generally, given a finite number of functions $f_j \in B_{p_j, q_j}^{s_j}$, with $s_j > 0$, $1 \leq p_j < \infty$ and $1 \leq q_j < \infty$, and given an integer $M > \max_j s_j$, we may choose a common set A of full measure in \mathbb{T}^{n-1} and a sequence (t_l) of positive numbers converging to 0 such that the analog of (3.1), i.e.,

$$\lim_{l \rightarrow \infty} \frac{\left\| \Delta_{t_l e_n}^M f_j(x', \cdot) \right\|_{L^{p_j}(\mathbb{T})}}{t_l^{s_j}} = 0, \quad (3.2)$$

holds simultaneously for all j and all $x' \in A$.

Proof. We treat the case of a single function; the general case is similar.

Set $g_t := \left\| \Delta_{t e_n}^M f \right\|_{L^p}$. By (2.3), we have $\int_0^1 t^{-sq-1} g_t^q dt < \infty$, which is equivalent to $\int_{1/2}^1 \sum_{m \geq 0} 2^{msq} g_{2^{-m}\sigma}^q d\sigma < \infty$. Therefore, there exists some $\sigma \in (1/2, 1)$ such that

$$\sum_{m \geq 0} 2^{msq} g_{2^{-m}\sigma}^q < \infty. \quad (3.3)$$

By (3.3), we find that

$$\lim_{m \rightarrow \infty} \frac{g_{2^{-m}\sigma}}{(2^{-m}\sigma)^s} = 0. \quad (3.4)$$

Using (3.4) we find that, along a subsequence (m_l) , we have

$$\lim_{l \rightarrow \infty} \frac{\left\| \Delta_{2^{-m_l}\sigma} f \right\|_{L^p}}{(2^{-m_l}\sigma)^s} = 0 \quad \text{for a.e. } x' \in \mathbb{T}^{n-1}.$$

This implies (3.1) with $t_l := 2^{-m_l}\sigma$. \square

3.2 (Non-)restrictions

We now address the question whether, given $f \in B_{p, q}^s(\mathbb{R}^2)$, we have $f(x, \cdot) \in B_{p, q}^s(\mathbb{R})$ for a.e. $x \in \mathbb{R}$. This kind of questions can also be asked in higher dimensions. The answer crucially depends on the sign of $q - p$.

We start with a simple result.

Proposition 3.4. Let $s > 0$ and $1 \leq q \leq p < \infty$. Let $f \in B_{p, q}^s(\mathbb{R}^2)$. Then for a.e. $x \in \mathbb{R}$ we have $f(x, \cdot) \in B_{p, q}^s(\mathbb{R})$.

Proof. Let $f \in B_{p, q}^s(\mathbb{R}^2)$. Using (2.3) (part 2) and Hölder's inequality, we find that for every finite interval $[a, b] \subset \mathbb{R}$ and $M > s$ we have

$$\begin{aligned} \int_a^b |f(x, \cdot)|_{B_{p, q}^s(\mathbb{R})}^q dx &\sim \int_a^b \int_{\mathbb{R}} \frac{1}{|h|^{sq+1}} \left(\int_{\mathbb{R}} |\Delta_{he_2}^M f(x, y)|^p dy \right)^{q/p} dh dx \\ &\leq (b-a)^{(p-q)/p} \int_{\mathbb{R}} \frac{1}{|h|^{sq+1}} \left(\int_{[a, b] \times \mathbb{R}} |\Delta_{he_2}^M f(x, y)|^p dx dy \right)^{q/p} dh \\ &\lesssim |f|_{B_{p, q}^s(\mathbb{R}^2)}^q < \infty \end{aligned}$$

whence the conclusion. \square

When $q > p$, a striking phenomenon occurs.

Proposition 3.5. Let $s > 0$ and $1 \leq p < q \leq \infty$. Then there exists some compactly supported $f \in B_{p,q}^s(\mathbb{R}^2)$ such that for a.e. $x \in (0, 1)$ we have $f(x, \cdot) \notin B_{p,\infty}^s(\mathbb{R})$.

In particular, for any $1 \leq r < \infty$ and a.e. $x \in (0, 1)$ we have $f(x, \cdot) \notin B_{p,r}^s(\mathbb{R})$.

Before proceeding to the proof, let us note that if $f \in B_{p,q}^s(\mathbb{R}^2)$ then $f \in L^p(\mathbb{R}^2)$, and thus the partial function $f(x, \cdot)$ is a well-defined element of $L^p(\mathbb{R})$ for a.e. x .

Proof. Since $B_{p,q}^s(\mathbb{R}^2) \subset B_{p,\infty}^s(\mathbb{R}^2)$, $\forall q$, we may assume that $q < \infty$. We rely on the characterization of Besov spaces in terms of smooth wavelets, as in Section 2.7.

We start by explaining the construction of f . Let ψ_F and ψ_M be as in Section 2.7. With no loss of generality, we may assume that $\text{supp } \psi_M \subset [0, a]$ with $a \in \mathbb{N}$. Consider $(\alpha, \beta) \subset (0, a)$ and $\gamma > 0$ such that $\psi_M \geq \gamma$ in $[\alpha, \beta]$.

Set $\delta := \beta - \alpha > 0$ and consider some integer N such that $[0, 1] \subset [\alpha - N\delta, \beta + N\delta]$. We look for an f of the form

$$f = \sum_{\ell=-N}^N \underbrace{\sum_{j \geq j_0} g_j^\ell}_{f^\ell} := \sum_{\ell=-N}^N f^\ell, \quad (3.5)$$

with each g_j^ℓ of the form

$$g_j^\ell(x, y) = \mu_j 2^{-j(s-2/p)} \sum_{m_1 \in I_j} \psi_M(2^j x - m_1 - \ell \delta) \\ \times \psi_M(2^j y - m_1 - 2^{j+1} \ell a - \ell \delta). \quad (3.6)$$

Here, the set I_j satisfying

$$I_j \subset \{0, 1, \dots, 2^j\}, \quad (3.7)$$

the integer j_0 and the coefficients $\mu_j > 0$ will be defined later.

We consider the partial sums $f_J^\ell := \sum_{j=j_0}^J g_j^\ell$. Clearly, we have $f_J^\ell \in C^k$ and, provided j_0 is sufficiently large,

$$\text{supp } f_J^\ell \subset K_\ell := [-N\delta, 5/4] \times [2\ell a - 1/4, (2\ell + 1)a + 1/4].$$

We next note that $(K_\ell)_{\ell=-N}^N$ is a fixed family of mutually disjoint compacts. Combining this with Proposition 2.6 item 2, we easily find that

$$\left\| \sum_{\ell=-N}^N f_J^\ell \right\|_{B_{p,q}^s(\mathbb{R}^2)}^q \sim \sum_{\ell=-N}^N \left\| f_J^\ell \right\|_{B_{p,q}^s(\mathbb{R}^2)}^q. \quad (3.8)$$

On the other hand, if ψ_M and ψ_F are wavelets such that Proposition 2.12 holds, then so are $\psi_F(\cdot - \lambda)$ and $\psi_M(\cdot - \lambda)$, $\forall \lambda \in \mathbb{R}$ [38, Theorem 1.61 (ii), Theorem 1.64]. Combining this fact with (3.8), we find that

$$\left\| \sum_{\ell=-N}^N f_J^\ell \right\|_{B_{p,q}^s(\mathbb{R}^2)}^q \sim \sum_{j=j_0}^J (\#I_j(\mu_j)^p)^{q/p}. \quad (3.9)$$

We now make the size assumption

$$\sum_{j=j_0}^{\infty} (\#I_j(\mu_j)^p)^{q/p} < \infty. \quad (3.10)$$

By (3.9) and (3.10), we see that the formal series in (3.5) defines a compactly supported $f \in B_{p,q}^s(\mathbb{R}^2)$, with $\sum_{\ell=-N}^N f_J^\ell \rightarrow f$ in $B_{p,q}^s(\mathbb{R}^2)$ (and therefore in $L^p(\mathbb{R}^2)$) as $J \rightarrow \infty$.

We next investigate the $B_{p,\infty}^s$ norm of the restrictions $f_J^\ell(x, \cdot)$. As in (3.8), we have

$$\left\| \sum_{\ell=-N}^N f_J^\ell(x, \cdot) \right\|_{B_{p,\infty}^s(\mathbb{R})} \sim \sum_{\ell=-N}^N \|f_J^\ell(x, \cdot)\|_{B_{p,\infty}^s(\mathbb{R})}. \quad (3.11)$$

Rewriting (3.6) as

$$\begin{aligned} g_j^\ell(x, y) &= \mu_j 2^{-j(s-1/p)} 2^{j/p} \sum_{m_1 \in I_j} \psi_M(2^j x - m_1 - \ell \delta) \\ &\quad \times \psi_M(2^j y - m_1 - 2^{j+1} \ell a - \ell \delta), \end{aligned} \quad (3.12)$$

we obtain

$$\|f_J^\ell(x, \cdot)\|_{B_{p,\infty}^s(\mathbb{R})}^p \sim \sup_{j_0 \leq j \leq J} 2^j (\mu_j)^p \sum_{m_1 \in I_j} |\psi_M(2^j x - m_1 - \ell \delta)|^p. \quad (3.13)$$

We next make the size assumption

$$\sum_{\ell=-N}^N \sup_{j \geq j_0} 2^j (\mu_j)^p \sum_{m_1 \in I_j} |\psi_M(2^j x - m_1 - \ell \delta)|^p = \infty, \quad \forall x \in [0, 1]. \quad (3.14)$$

Then we claim that for a.e. $x \in (0, 1)$ we have

$$f(x, \cdot) \notin B_{p,\infty}^s(\mathbb{R}). \quad (3.15)$$

Indeed, since $\sum_{\ell=-N}^N f_J^\ell \rightarrow f$ in $L^p(\mathbb{R}^2)$, for a.e. $x \in \mathbb{R}$ we have

$$\sum_{\ell=-N}^{\ell} f_J^\ell(x, \cdot) \rightarrow f(x, \cdot) \text{ in } L^p(\mathbb{R}). \quad (3.16)$$

We claim that for every $x \in [0, 1]$ such that (3.16) holds, we have $f(x, \cdot) \notin B_{p, \infty}^s(\mathbb{R})$. Indeed, on the one hand (3.14) implies that for some ℓ we have $\lim_{J \rightarrow \infty} \|f_J^\ell(x, \cdot)\|_{B_{p, \infty}^s(\mathbb{R})} = \infty$. We assume e.g. that this holds when $\ell = 0$. Thus

$$\sup_{j \geq j_0} 2^j (\mu_j)^p \sum_{m_1 \in I_j} |\psi_M(2^j x - m_1)|^p = \infty. \quad (3.17)$$

On the other hand, assume by contradiction that $f(x, \cdot) \in B_{p, \infty}^s(\mathbb{R})$. As in (3.8), we have $f^0(x, \cdot) \in B_{p, \infty}^s(\mathbb{R})$. Then we may write $f^0(x, \cdot)$ as in (2.12), with coefficients as in (2.14). In particular, taking into account the explicit formula of g_j^0 and the fact that $f_j^0(x, \cdot) \rightarrow f(x, \cdot)$ in $L^p(\mathbb{R})$, we find that for $k \geq j_0$ and $m_1 \in I_j$ we have

$$\begin{aligned} \mu_{m_1}^{k, \{M\}}(f^0(x, \cdot)) &= \mu_{m_1}^{k, \{M\}} \left(\sum_{j=j_0}^J g_j^0(x, \cdot) \right) = \mu_{m_1}^{k, \{M\}}(g_k^0(x, \cdot)) \\ &= 2^{k/p} \mu_k \psi_M(2^k x - m_1), \quad \forall J \geq k. \end{aligned} \quad (3.18)$$

We obtain a contradiction combining (3.17), (3.18) and Corollary 2.13.

It remains to construct I_j and μ_j satisfying (3.7), (3.10) and (3.14). We will let $I_j = \llbracket s_j, t_j \rrbracket$, with $0 \leq s_j \leq t_j \leq 2^j$ integers to be determined later. Set $t := q/p \in (1, \infty)$ and

$$\mu_j := \left(\frac{1}{(t_j - s_j + 1) j^{1/t} \ln j} \right)^{1/p}.$$

Clearly, (3.7) and (3.10) hold. It remains to define I_j in order to have (3.14). Consider the dyadic segment $L_j := [s_j/2^j, t_j/2^j]$. We claim that

$$\sum_{\ell=-N}^N \sum_{m_1 \in I_j} |\psi_M(2^j x - m_1 - \ell \delta)|^p \geq \gamma^p, \quad \forall x \in L_j. \quad (3.19)$$

Indeed, let $m_1 \in [s_j, t_j]$ be the integer part of $2^j x$. By the definition of δ and by choice of N , there exists some $\ell \in \llbracket -N, N \rrbracket$ such that $\alpha \leq 2^j x - m_1 - \ell \delta \leq \beta$, whence the conclusion.

By the above, (3.14) holds provided we have

$$\sup_{j \geq j_0} 2^j (\mu_j)^p \mathbb{1}_{L_j(x)} = \infty, \quad \forall x \in [0, 1]. \quad (3.20)$$

We next note that

$$2^j (\mu_j)^p \sim \frac{1}{|L_j| j^{1/t} \ln j} = \frac{u_j}{|L_j|}, \quad (3.21)$$

where $u_j := 1/(j^{1/t} \ln j)$ satisfies

$$\sum_{j \geq j_0} u_j = \infty. \quad (3.22)$$

In view of (3.21) and (3.22), existence of I_j satisfying (3.20) is a consequence of Lemma 3.6 below. The proof of Proposition 3.5 is complete. \square

Lemma 3.6. Consider a sequence (u_j) of positive numbers such that $\sum_{j \geq j_0} u_j = \infty$. Then there exists a sequence (L_j) of dyadic intervals $L_j = [s_j/2^j, t_j/2^j]$, such that:

1. $s_j, t_j \in \mathbb{N}$, $0 \leq s_j < 2^j$.
2. $|L_j| = o(u_j)$ as $j \rightarrow \infty$.
3. Every $x \in [0, 1]$ belongs to infinitely many L_j 's.

Proof. Consider a sequence (v_j) of positive numbers such that $\sum_{j \geq j_0} v_j u_j = \infty$ and $v_j \rightarrow 0$. Let L_{j_0} be the largest dyadic interval of the form $[0, t_{j_0}/2^{j_0}]$ of length $\leq v_{j_0} u_{j_0}$. This defines $s_{j_0} = 0$ and t_{j_0} .

Assuming $L_j = [s_j/2^j, t_j/2^j] = [a_j, b_j]$ constructed for some $j \geq j_0$, one of the following two occurs. Either $b_j < 1$ and then we let L_{j+1} be the largest dyadic interval of the form $[2t_j/2^{j+1}, t_{j+1}/2^{j+1}]$ such that $|L_{j+1}| \leq v_{j+1} u_{j+1}$. Or $b_j \geq 1$, and then we let L_{j+1} be the largest dyadic interval of the form $[0, t_{j+1}/2^{j+1}]$ such that $|L_{j+1}| \leq v_{j+1} u_{j+1}$.

Using the assumption $\sum_{j \geq j_0} v_j u_j = \infty$ and the fact that $|L_j| \geq v_j u_j - 2^{-j}$, we easily find that for every $j \geq j_0$ there exists some $k > j$ such that $L_k = [a_k, b_k]$ satisfies $b_k \geq 1$, and thus the intervals L_j cover each point $x \in [0, 1]$ infinitely many times. \square

3.3 Characterization of $B_{p,q}^s$ via extensions

The type of results we present in this section are classical for functions defined on the whole \mathbb{R}^n and for the harmonic extension. Such results were obtained by Uspenskiĭ in the early sixties [40]. For further developments, see [36, Section 2.12.2, Theorem, p. 184]. When the harmonic extension is replaced by other extensions by regularization, the kind of results we present below were known to experts at least for maps defined on \mathbb{R}^n ; see [22, Section 10.1.1, Theorem 1, p. 512] and also [28] for a systematic treatment of extensions by smoothing. The local variants (involving extensions by averages in domains) we present below could be obtained by adapting the arguments we developed in a more general setting in [28], and which are quite involved. However, we present here a more elementary approach, inspired by [22], sufficient to our purpose. In what follows, we let $|\cdot|$ denote the $\|\cdot\|_\infty$ norm in \mathbb{R}^n .

For simplicity, we state our results when $\Omega = \mathbb{T}^n$, but they can be easily adapted to arbitrary Ω .

Lemma 3.7. Let $0 < s < 1$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, and $\delta \in (0, 1]$. Set $V_\delta := \mathbb{T}^n \times (0, \delta)$.

1. Let $F \in C^\infty(\overline{V_\delta})$. If

$$\left(\int_0^{\delta/2} \varepsilon^{q-sq} \|(\nabla F)(\cdot, \varepsilon)\|_{L^p}^q \frac{d\varepsilon}{\varepsilon} \right)^{1/q} < \infty \quad (3.23)$$

(with the obvious modification when $q = \infty$), then F has a trace $f \in B_{p,q}^s(\mathbb{T}^n)$, satisfying

$$|f|_{B_{p,q}^s} \lesssim \left(\int_0^{\delta/2} \varepsilon^{q-sq} \|(\nabla F)(\cdot, \varepsilon)\|_{L^p}^q \frac{d\varepsilon}{\varepsilon} \right)^{1/q}. \quad (3.24)$$

2. Conversely, let $f \in B_{p,q}^s(\mathbb{T}^n)$. Let $\rho \in C^\infty$ be a mollifier supported in $\{|x| \leq 1\}$ and set $F(x, \varepsilon) := f * \rho_\varepsilon(x)$, $x \in \mathbb{T}^n$, $0 < \varepsilon < \delta$. Then

$$\left(\int_0^\delta \varepsilon^{q-sq} \|(\nabla F)(\cdot, \varepsilon)\|_{L^p}^q \frac{d\varepsilon}{\varepsilon} \right)^{1/q} \lesssim |f|_{B_{p,q}^s}. \quad (3.25)$$

A word about the existence of the trace in item 1 above. We will prove below that for every $0 < \lambda < \delta/4$ we have

$$|F|_{\mathbb{T}^n \times \{\lambda\}}|_{B_{p,q}^s} \lesssim \left(\int_0^{\delta/2} \varepsilon^{q-sq} \|(\nabla F)(\cdot, \varepsilon)\|_{L^p}^q \frac{d\varepsilon}{\varepsilon} \right)^{1/q}. \quad (3.26)$$

By Lemma 7.7 and a standard argument, this leads to the existence, in $B_{p,q}^s$, of the limit $\lim_{\varepsilon \rightarrow 0} F(\cdot, \varepsilon)$. This limit is the trace of F on \mathbb{T}^n and clearly satisfies (3.24).

Proof. For simplicity, we treat only the case where $q < \infty$; the case where $q = \infty$ is somewhat simpler and is left to the reader.

We claim that in item 1 we may assume that $F \in C^\infty(\overline{V_\delta})$. Indeed, assume that (3.24) holds (with $\text{tr} F = F(\cdot, 0)$) for such F . By Lemma 7.7, we have the stronger inequality $\|\text{tr} F - f\|_{B_{p,q}^s} \lesssim I(F)$, where $I(F)$ is the integral in (3.23). Then, by a standard approximation argument, we find that (3.24) holds for every F .

So let $F \in C^\infty(\overline{V_\delta})$, and set $f(x) := F(x, 0)$, $\forall x \in \mathbb{T}^n$. Denote by $I(F)$ the quantity in (3.23). We have to prove that f satisfies

$$|f|_{B_{p,q}^s} \lesssim I(F). \quad (3.27)$$

If $|h| \leq \delta$, then

$$|\Delta_h f(x)| \leq |f(x+h) - F(x+h/2, |h|/2)| + |f(x) - F(x+h/2, |h|/2)|. \quad (3.28)$$

By symmetry and (3.28), the estimate (3.27) will follow from

$$\left(\int_{|h| \leq \delta} |h|^{-sq} \|f - F(\cdot + h/2, |h|/2)\|_{L^p}^q \frac{dh}{|h|^n} \right)^{1/q} \lesssim I(F). \quad (3.29)$$

In order to prove (3.29), we start from

$$\begin{aligned} |F(x+h/2, |h|/2) - f(x)| &= \left| \int_0^1 (\nabla F)(x+th/2, t|h|/2) \cdot (h/2, |h|/2) dt \right| \\ &\leq |h| \int_0^1 |\nabla F(x+th/2, t|h|/2)| dt. \end{aligned} \quad (3.30)$$

Let $J(F)$ denote the left-hand side of (3.29). Using (3.30) and setting $r := |h|/2$, we obtain

$$\begin{aligned} [J(F)]^q &\leq \int_{|h| \leq \delta} |h|^{q-sq} \left(\int_0^1 \|\nabla F(\cdot + th/2, t|h|/2)\|_{L^p} dt \right)^q \frac{dh}{|h|^n} \\ &= \int_{|h| \leq \delta} |h|^{q-sq} \left(\int_0^1 \|\nabla F(\cdot, t|h|/2)\|_{L^p} dt \right)^q \frac{dh}{|h|^n} \\ &\sim \int_0^{\delta/2} r^{q-sq-1} \left(\int_0^1 \|\nabla F(\cdot, tr)\|_{L^p} dt \right)^q dr \\ &\sim \int_0^{\delta/2} r^{-sq-1} \left(\int_0^r \|\nabla F(\cdot, \sigma)\|_{L^p} d\sigma \right)^q dr \lesssim [I(F)]^q. \end{aligned} \quad (3.31)$$

The last inequality is a special case of Hardy's inequality [34, Chapter 5, Lemma 3.14], that we recall here when $\delta = \infty$.⁴ Let $1 \leq q < \infty$ and $1 < \rho < \infty$. If $G \in W_{loc}^{1,1}([0, \infty))$, then

$$\int_0^\infty \frac{|G(r) - G(0)|^q}{r^\rho} dr \leq \left(\frac{q}{\rho - 1} \right)^q \int_0^\infty \frac{|G'(r)|^q}{r^{\rho-q}} dr. \quad (3.32)$$

We obtain (3.31) by applying (3.32) with $G'(r) := \|\nabla F(\cdot, r)\|_{L^p}$ and $\rho := sq + 1$. The proof of item 1 is complete.

We next turn to item 2. We have

$$\nabla F(x, \varepsilon) = \frac{1}{\varepsilon} f * \eta_\varepsilon(x), \quad (3.33)$$

where ∇ stands for $(\partial_1, \dots, \partial_n, \partial_\varepsilon)$. Here, $\eta = (\eta^1, \dots, \eta^{n+1}) \in C^\infty(\mathbb{T}^n; \mathbb{R}^{n+1})$ is supported in $\{|x| \leq 1\}$ and is given in coordinates by

$$\eta^j = \partial_j \rho, \quad \forall j \in \llbracket 1, n \rrbracket, \quad \eta^{n+1} = -\operatorname{div}(x\rho). \quad (3.34)$$

Noting that $\int \eta = 0$, we find that

$$\begin{aligned} |\nabla F(x, \varepsilon)| &= \frac{1}{\varepsilon} \left| \int_{|y| \leq \varepsilon} (f(x-y) - f(x)) \eta_\varepsilon(y) dy \right| \\ &\lesssim \frac{1}{\varepsilon^{n+1}} \int_{|h| \leq \varepsilon} |f(x+h) - f(x)| dh. \end{aligned} \quad (3.35)$$

⁴ But the argument adapts to a finite δ ; see e.g. [9, Proof of Corollary 7.2].

Integrating (3.35) and using Minkowski's inequality, we obtain

$$\|\nabla F(\cdot, \varepsilon)\|_{L^p} \lesssim \frac{1}{\varepsilon^{n+1}} \int_{|h| \leq \varepsilon} \|\Delta_h f\|_{L^p} dh. \quad (3.36)$$

Let $L(F)$ be the quantity in the left-hand side of (3.25). Combining (3.36) with Hölder's inequality, we find that

$$\begin{aligned} [L(F)]^q &\lesssim \int_0^\delta \frac{1}{\varepsilon^{nq+sq+1}} \left(\int_{|h| \leq \varepsilon} \|\Delta_h f\|_{L^p} dh \right)^q d\varepsilon \\ &\lesssim \int_0^\delta \frac{1}{\varepsilon^{nq+sq+1}} \varepsilon^{n(q-1)} \int_{|h| \leq \varepsilon} \|\Delta_h f\|_{L^p}^q dh d\varepsilon \\ &\lesssim \int_{|h| \leq \delta} |h|^{-sq} \|\Delta_h f\|_{L^p}^q \frac{dh}{|h|^n} = |f|_{B_{p,q,\delta}^s}^q, \end{aligned} \quad (3.37)$$

i.e, (3.25) holds. \square

In the same vein, we have the following result, involving the semi-norm appearing in Proposition 2.6, more specifically the quantity

$$|f|_{B_{p,q,\delta}^1} := \left(\int_{|h| \leq \delta} |h|^{-q} \|\Delta_h^2 f\|_{L^p}^q \frac{dh}{|h|^n} \right)^{1/q} \quad (3.38)$$

when $q < \infty$, with the obvious modification when $q = \infty$. We first introduce a notation. Given $F \in C^2(V_\delta)$, we let $D_{\#}^2 F$ denote the collection of the second order derivatives of F which are either completely horizontal (that is of the form $\partial_j \partial_k F$, with $j, k \in \llbracket 1, n \rrbracket$), or completely vertical (that is $\partial_{n+1} \partial_{n+1} F$).

Lemma 3.8. Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Let $F \in C^\infty(V_\delta)$ and set

$$M(F) := \left(\int_0^\delta \varepsilon^q \|\nabla F(\cdot, \varepsilon)\|_{L^{2p}}^{2q} \frac{d\varepsilon}{\varepsilon} \right)^{1/q}$$

and

$$N(F) := \left(\int_0^\delta \varepsilon^q \|(D_{\#}^2 F)(\cdot, \varepsilon)\|_{L^p}^q \frac{d\varepsilon}{\varepsilon} \right)^{1/q}$$

(with the obvious modification when $q = \infty$).

1. If $M(F) < \infty$ and $N(F) < \infty$, then F has a trace $f \in B_{p,q}^1(\mathbb{T}^n)$, satisfying

$$\left\| f - \int f \right\|_{L^p} \lesssim M(F)^{\frac{1}{2}} \quad (3.39)$$

and

$$|f|_{B_{p,q,\delta}^1} \lesssim N(F). \quad (3.40)$$

2. Conversely, let $f \in B_{p,q}^1(\mathbb{T}^n; \mathbb{S}^1)$. Let $\rho \in C^\infty$ be an even mollifier supported in $\{|x| \leq 1\}$ and set $F(x, \varepsilon) := f * \rho_\varepsilon(x)$, $x \in \mathbb{T}^n$, $0 < \varepsilon < \delta$. Then

$$M(F) + N(F) \lesssim |f|_{B_{p,q,\delta}^1}. \quad (3.41)$$

The above result is inspired by the proof of [22, Section 10.1.1, Theorem 1, p. 512]. The arguments we present also lead to a (slightly different) proof of Lemma 3.7.

We start by establishing some preliminary estimates. We call $H \in \mathbb{R}^n \times \mathbb{R}$ “pure” if H is either horizontal, or vertical, i.e., either $H \in \mathbb{R}^n \times \{0\}$ or $H \in \{0\} \times \mathbb{R}$. For further use, let us note the following fact, valid for $X \in V_\delta$ and $H \in \mathbb{R}^{n+1}$

$$H \text{ pure} \implies |D^2 F(X) \cdot (H, H)| \lesssim |D_\#^2 F(X)| |H|^2. \quad (3.42)$$

Lemma 3.9. Let X, H be such that $[X, X + 2H] \subset \overline{V_\delta}$. Let $F \in C^2(\overline{V_\delta})$. Then

$$|\Delta_H^2 F(X)| \leq \int_0^2 \tau |D^2 F(X + \tau H) \cdot (H, H)| d\tau. \quad (3.43)$$

In particular, if H is pure and we write $H = |H|K$, then

$$|\Delta_H^2 F(X)| \lesssim \int_0^{2|H|} t |D_\#^2 F(X + tK)| dt. \quad (3.44)$$

Proof. Set

$$G(s) := F(X + (1-s)H) + F(X + (1+s)H), \quad s \in [0, 1],$$

so that $G \in C^2$ and in addition we have

$$G'(0) = 0, \quad G''(s) = [D^2 F(X + (1-s)H) + D^2 F(X + (1+s)H)] \cdot (H, H), \quad (3.45)$$

and

$$\int_0^1 (1-s) G''(s) ds = G(1) - G(0) - G'(0) = \Delta_H^2 F(X). \quad (3.46)$$

Estimate (3.43) is a consequence of (3.45) and (3.46) (using the changes of variable $\tau := 1 \pm s$). In the special case where H is pure, we rely on (3.42) and (3.43) and obtain (3.44) via the change of variable $t := \tau|H|$. \square

If we combine (3.44) (applied first with $H = (h, 0)$, $h \in \mathbb{R}^n$, next with $H = (0, t)$, $t \in [0, \delta/2]$) with Minkowski's inequality, we obtain the two following consequences⁵

$$[h \in \mathbb{R}^n, 0 \leq \varepsilon \leq \delta] \implies \|\Delta_h^2 F(\cdot, \varepsilon)\|_{L^p} \lesssim |h|^2 \|D_{\#}^2 F(\cdot, \varepsilon)\|_{L^p}, \quad (3.47)$$

and⁶

$$[t, \varepsilon \geq 0, \varepsilon + 2t \leq \delta] \implies \|\Delta_{te_{n+1}}^2 F(\cdot, \varepsilon)\|_{L^p} \lesssim \int_0^{2t} r \|D_{\#}^2 F(\cdot, \varepsilon + r)\|_{L^p} dr. \quad (3.48)$$

Proof of Lemma 3.8. We start by proving (3.39). By Lemma 3.7 (applied with $s = 1/2$ and with $2p$ (respectively $2q$) instead of p (respectively q)), F has, on \mathbb{T}^n , a trace $\text{tr} F \in B_{2p, 2q}^{1/2}$. By Lemma 3.7, item 1, and Lemma 7.8, we have

$$\left\| \text{tr} F - \int \text{tr} F \right\|_{L^p} \lesssim \left\| \text{tr} F - \int \text{tr} F \right\|_{L^{2p}} \lesssim M(F)^{1/2}$$

i.e., (3.39) holds.

We next establish (3.40). Arguing as at the beginning of the proof of Lemma 3.7, one concludes that it suffices to prove (3.40) when $F \in C^\infty(\overline{V_\delta})$.

So let us consider some $F \in C^\infty(\overline{V_\delta})$. We set $f(x) = F(x, 0)$, $\forall x \in \mathbb{T}^n$. Then (3.40) is equivalent to

$$|f|_{B_{p,q,\delta}^1} \lesssim N(F). \quad (3.49)$$

We treat only the case where $q < \infty$; the case where $q = \infty$ is slightly simpler and is left to the reader.

The starting point is the following identity, valid when $|h| \leq \delta$ and with $t := |h|$

$$\begin{aligned} \Delta_h^2 f &= \Delta_{te_{n+1}/2}^2 F(\cdot + 2h, 0) - 2\Delta_{te_{n+1}/2}^2 F(\cdot + h, 0) + \Delta_{te_{n+1}/2}^2 F(\cdot, 0) \\ &\quad + 2\Delta_h^2 F(\cdot, t/2) - \Delta_h^2 F(\cdot, t). \end{aligned} \quad (3.50)$$

By (3.47), (3.48) and (3.50), we find that

$$\begin{aligned} \|\Delta_h^2 f\|_{L^p} &\lesssim \int_0^{|h|} r \|D_{\#}^2 F(\cdot, r)\|_{L^p} dr + |h|^2 \|D_{\#}^2 F(\cdot, |h|/2)\|_{L^p} \\ &\quad + |h|^2 \|D_{\#}^2 F(\cdot, |h|)\|_{L^p}. \end{aligned} \quad (3.51)$$

Finally, (3.51) combined with Hardy's inequality (3.32) (applied to the integral \int_0^δ and with $G'(r) := r \|D_{\#}^2 F(\cdot, r)\|_{L^p}$ and $\rho := q + 1$) yields

$$\begin{aligned} |f|_{B_{p,q,\delta}^1}^q &\lesssim \int_{|h| \leq \delta} \frac{1}{|h|^q} \left(\int_0^{|h|} r \|D_{\#}^2 F(\cdot, r)\|_{L^p} dr \right)^q \frac{dh}{|h|^n} + [N(F)]^q \\ &\lesssim [N(F)]^q. \end{aligned} \quad (3.52)$$

⁵ In (3.47), we let $\Delta_h^2 F(\cdot, \varepsilon) := F(\cdot + 2h, \varepsilon) - 2F(\cdot + h, \varepsilon) + F(\cdot, \varepsilon)$.

⁶ With the slight abuse of notation $\Delta_{te_{n+1}}^2 F(\cdot, \varepsilon) := F(\cdot, \varepsilon + 2t) - 2F(\cdot, \varepsilon + t) + F(\cdot, \varepsilon)$.

This implies (3.49) and completes the proof of item 1.

We now turn to item 2. We claim that

$$|f|_{B_{2p,2q,\delta}^{1/2}} \lesssim |f|_{B_{p,q,\delta}^1}. \quad (3.53)$$

Indeed, it suffices to note the fact that $|\Delta_h^2 f|^{2p} \lesssim |\Delta_h^2 f|^p$ (since $|f| = 1$). By combining (3.53) with Lemma 3.7, we find that

$$M(F) = \left(\int_0^\delta \varepsilon^q \|(\nabla F)(\cdot, \varepsilon)\|_{L^{2p}}^{2q} \frac{d\varepsilon}{\varepsilon} \right)^{1/q} \lesssim |f|_{B_{p,q,\delta}^1}. \quad (3.54)$$

Thus, in order to complete the proof of (3.41), it suffices to combine (3.54) with the following estimate

$$N(F) \lesssim |f|_{B_{p,q,\delta}^1}, \quad (3.55)$$

that we now establish. The key argument for proving (3.55) is the following second order analog of (3.35):

$$|D_{\#}^2 F(x, \varepsilon)| \lesssim \frac{1}{\varepsilon^{n+2}} \int_{|h| \leq \varepsilon} |\Delta_h^2 f(x-h)| dh. \quad (3.56)$$

The proof of (3.56) appears in [22, p. 514]. For the sake of completeness, we reproduce below the argument. First, differentiating the expression defining F , we have

$$\partial_j \partial_k F(x, \varepsilon) = \frac{1}{\varepsilon^2} f * (\partial_j \partial_k \rho)_\varepsilon, \quad \forall j, k \in \llbracket 1, n \rrbracket. \quad (3.57)$$

Using (3.57) and the fact that $\partial_j \partial_k \rho$ is even and has zero average, we obtain the identity

$$\partial_j \partial_k F(x, \varepsilon) = \frac{1}{2\varepsilon^{n+2}} \int_{|h| \leq \varepsilon} \partial_j \partial_k \rho(h/\varepsilon) \Delta_h^2 f(x-h) dh,$$

and thus (3.56) holds for the derivatives $\partial_j \partial_k F$, with $j, k \in \llbracket 1, n \rrbracket$.

We next note the identity

$$F(x, \varepsilon) = \frac{1}{2\varepsilon^n} \int \rho(h/\varepsilon) \Delta_h^2 f(x-h) dh + f(x), \quad (3.58)$$

which follows from the fact that ρ is even and $\int \rho = 1$.

By differentiating twice (3.58) with respect to ε , we obtain that (3.56) holds when $j = k = n + 1$. The proof of (3.56) is complete.

Using (3.56) and Minkowski's inequality, we obtain

$$\|D_{\#}^2 F(\cdot, \varepsilon)\|_{L^p} \lesssim \frac{1}{\varepsilon^{n+2}} \int_{|h| \leq \varepsilon} \|\Delta_h^2 f\|_{L^p} dh, \quad (3.59)$$

which is a second order analog of (3.36). Once (3.36) is obtained, we repeat the calculation leading to (3.37) and obtain (3.55). The details are left to the reader.

The proof of Lemma 3.8 is complete. \square

Remark 3.10. One may put Lemmas 3.7 and 3.8 in the perspective of the theory of weighted Sobolev spaces. Let us start by recalling one of the striking achievements of this theory. As it is well-known, we have $\text{tr} W^{1,1}(\mathbb{R}_+^n) = L^1(\mathbb{R}^{n-1})$, and, when $n \geq 2$, the trace operator has no linear continuous right-inverse $T : L^1(\mathbb{R}^{n-1}) \rightarrow W^{1,1}(\mathbb{R}^n)$ [19], [31]. The expected analogs of these facts for $W^{2,1}(\mathbb{R}_+^n)$ are both wrong. More specifically, we have $\text{tr} W^{2,1}(\mathbb{R}_+^n) = B_{1,1}^1(\mathbb{R}^{n-1})$ (which is a strict subspace of $W^{1,1}(\mathbb{R}^{n-1})$), and the trace operator has a linear continuous right inverse from $B_{1,1}^1(\mathbb{R}^{n-1})$ into $W^{2,1}(\mathbb{R}_+^n)$. These results are special cases of the trace theory for weighted Sobolev spaces developed by Uspenskiĭ [40]. For a modern treatment of this theory, see e.g. [28].

3.4 Disintegration of the Jacobians

The purpose of this section is to prove and generalize the following result, used in the analysis of Case 5.

Lemma 3.11. Let $s > 1$, $1 \leq p < \infty$, $1 \leq q \leq p$ and $n \geq 3$, and assume that $sp \geq 2$. Let $u \in B_{p,q}^s(\Omega; \mathbb{S}^1)$ and set $F := u \wedge \nabla u$. Then $\text{curl} F = 0$.

Same conclusion if $s > 1$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $n \geq 2$, and we have $sp > 2$.

Same conclusion if $s > 1$, $1 \leq p < \infty$, $1 \leq q < \infty$ and $n = 2$, and we have $sp = 2$.

In view of the conclusion, we may assume that $\Omega = (0, 1)^n$.

Note that in the above we have $n \geq 2$; for $n = 1$ there is nothing to prove.

Since the results we present in this section are of independent interest, we go beyond what is actually needed in Case 5.

The conclusion of (the generalization of) Lemma 3.11 relies on three ingredients. The first one is that it is possible to define, as a distribution, the product $F := u \wedge \nabla u$ for u in a low regularity Besov space; this goes back to [7] when $n = 2$, and the case where $n \geq 3$ is treated in [9]. The second one is a Fubini (disintegration) type result for the distribution $\text{curl} F$. Again, this result holds even in Besov spaces with lower regularity than the ones in Lemma 3.11; see Lemma 3.12 below. The final ingredient is the fact that when $u \in \text{VMO}((0, 1)^2; \mathbb{S}^1)$ we have $\text{curl} F = 0$; see Lemma 3.13. Lemma 3.11 is obtained by combining Lemmas 3.12 and 3.13 via a dimensional reduction (slicing) based on Lemma 3.2; a more general result is presented in Lemma 3.14.

Now let us proceed. First, following [7] and [9], we explain how to define the Jacobian $Ju := 1/2 \text{curl} F$ of low regularity unimodular maps

$$u \in W^{1/p,p}((0, 1)^n; \mathbb{S}^1),$$

with $1 \leq p < \infty$.⁷ Assume first that $n = 2$ and that u is smooth. Then, in the distributions sense, we have

$$\begin{aligned} \langle Ju, \zeta \rangle &= \frac{1}{2} \int_{(0,1)^2} \operatorname{curl} F \zeta = -\frac{1}{2} \int_{(0,1)^2} \nabla \zeta \wedge (u \wedge \nabla u) \\ &= \frac{1}{2} \int_{(0,1)^2} [(u \wedge \partial_1 u) \partial_2 \zeta - (u \wedge \partial_2 u) \partial_1 \zeta] \\ &= \frac{1}{2} \int_{(0,1)^2} (u_1 \nabla u_2 \wedge \nabla \zeta - u_2 \nabla u_1 \wedge \nabla \zeta), \quad \forall \zeta \in C_c^\infty((0,1)^2). \end{aligned} \quad (3.60)$$

In higher dimensions, it is better to identify Ju with the 2-form (or rather a 2-current) $Ju \equiv 1/2 d(u \wedge du)$.⁸ With this identification and modulo the action of the Hodge $*$ -operator, Ju acts either on $(n-2)$ -forms, or on 2-forms. The former point of view is usually adopted, and is expressed by the formula

$$\begin{aligned} \langle Ju, \zeta \rangle &= \frac{(-1)^{n-1}}{2} \int_{(0,1)^n} d\zeta \wedge (u \wedge \nabla u) \\ &= \frac{(-1)^{n-1}}{2} \int_{(0,1)^n} d\zeta \wedge (u_1 du_2 - u_2 du_1), \quad \forall \zeta \in C_c^\infty(\Lambda^{n-2}(0,1)^n).^9 \end{aligned} \quad (3.61)$$

The starting point in extending the above formula to lower regularity maps u is provided by the identity (3.62) below; when u is smooth, (3.62) is obtained by a simple integration by parts. More specifically, consider any smooth extension $U : (0,1)^n \times [0, \infty) \rightarrow \mathbb{C}$, respectively $\zeta \in C_c^\infty(\Lambda^{n-2}((0,1)^n \times [0, \infty)))$ of u , respectively of ζ .¹⁰ Then we have the identity [9, Lemma 5.5]

$$\langle Ju, \zeta \rangle = (-1)^{n-1} \int_{(0,1)^n \times (0, \infty)} d\zeta \wedge dU_1 \wedge dU_2. \quad (3.62)$$

For a low regularity u and for a well-chosen U , we take the right-hand side of (3.62) as the definition of Ju . More specifically, let $\Phi \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ be such that $\Phi(z) = z/|z|$ when $|z| \geq 1/2$, and let v be a standard extension of u by averages, i.e., $v(x, \varepsilon) = u * \rho_\varepsilon(x)$, $x \in (0,1)^n$, $\varepsilon > 0$, with ρ a standard mollifier. Set $U := \Phi(v)$. With this choice of U , the right-hand side of (3.62) does not depend on ζ (once ζ is fixed) [9, Lemma 5.4] and the map $u \mapsto Ju$ is continuous from $W^{1/p, p}((0,1)^n; \mathbb{S}^1)$ into the set of 2- (or $(n-2)$ -)currents. When $p = 1$, continuity is straightforward. For the continuity when $p > 1$, see [9, Theorem 1.1 item 2]. In addition, when u is sufficiently smooth (for example when $u \in$

⁷ In [7] and [9], maps are from \mathbb{S}^n (instead of $(0,1)^n$) into \mathbb{S}^1 , but this is not relevant for the validity of the results we present here.

⁸ We recover the two-dimensional formula (3.60) via the usual identification of 2-forms on $(0,1)^2$ with scalar functions (with the help of the Hodge $*$ -operator).

⁹ Here, $C_c^\infty(\Lambda^{n-2}(0,1)^n)$ denotes the space of smooth compactly supported $(n-2)$ -forms on $(0,1)^n$.

¹⁰ We do not claim that U is \mathbb{S}^1 -valued. When u is not smooth, existence of \mathbb{S}^1 -valued extensions is a delicate matter [26].

$W^{1,1}((0,1)^n; \mathbb{S}^1)$), Ju coincides¹¹ with $\text{curl}F$ [9, Theorem 1.1 item 1]. Finally, we have the estimate [9, Theorem 1.1 item 3]

$$|\langle Ju, \zeta \rangle| \lesssim |u|_{W^{1/p,p}}^p \|d\zeta\|_{L^\infty}, \quad \forall \zeta \in C_c^\infty(\Lambda^{n-2}(0,1)^n). \quad (3.63)$$

We are now in position to explain disintegration along two-planes. We use the notation in Section 3.1. Let $u \in W^{1/p,p}((0,1)^n; \mathbb{S}^1)$, with $n \geq 3$. Let $\alpha \in I(n-2, n)$. Then for a.e. $x_\alpha \in (0,1)^{n-2}$, the partial map $u_\alpha(x_\alpha)$ belongs to $W^{1/p,p}((0,1)^2; \mathbb{S}^1)$ (Lemma 3.1), and therefore $Ju_\alpha(x_\alpha)$ makes sense and acts on functions.¹² Let now $\zeta \in C_c^\infty(\Lambda^{n-2}(0,1)^n)$. Then we may write

$$\zeta = \sum_{\alpha \in I(n-2, n)} \zeta^\alpha dx^\alpha = \sum_{\alpha \in I(n-2, n)} (\zeta^\alpha)_\alpha(x_{\bar{\alpha}}) dx^\alpha.$$

Here, dx^α is the canonical $(n-2)$ -form induced by the coordinates x_j , $j \in \alpha$, and $(\zeta^\alpha)_\alpha(x_{\bar{\alpha}}) = \zeta^\alpha(x_\alpha, x_{\bar{\alpha}})$ belongs to $C_c^\infty((0,1)^2)$ (for fixed x_α).

We next note the following formal calculation. Fix $\alpha \in I(n-2, n)$, and let $\bar{\alpha} = \{j, k\}$, with $j < k$. Then

$$\begin{aligned} 2(-1)^{n-1} \langle Ju, \zeta^\alpha dx^\alpha \rangle &= \int_{(0,1)^n} d(\zeta^\alpha dx^\alpha) \wedge (u \wedge \nabla u) \\ &= \int_{(0,1)^n} (\partial_j \zeta^\alpha dx_j + \partial_k \zeta^\alpha dx_k) \wedge dx^\alpha \wedge u \wedge (\partial_j u dx_j + \partial_k u dx_k) \\ &= \int_{(0,1)^n} (\partial_j \zeta^\alpha u \wedge \partial_k u - \partial_k \zeta^\alpha u \wedge \partial_j u) dx_j \wedge dx^\alpha \wedge dx_k, \end{aligned}$$

that is,

$$\langle Ju, \zeta \rangle = \frac{1}{2} \sum_{\alpha \in I(n-2, n)} \varepsilon(\alpha) \int_{(0,1)^{n-2}} \langle Ju_\alpha, (\zeta^\alpha)_\alpha(x_\alpha) \rangle dx_\alpha, \quad (3.64)$$

where $\varepsilon(\alpha) \in \{-1, 1\}$ depends on α .

When $u \in W^{1,1}((0,1)^n; \mathbb{S}^1)$, it is easy to see that (3.64) is true (by Fubini's theorem). The validity of (3.64) under weaker regularity assumptions is the content of our next result.

Lemma 3.12. Let $1 \leq p < \infty$ and $n \geq 3$. Let $u \in W^{1/p,p}((0,1)^n; \mathbb{S}^1)$. Then (3.64) holds.

Proof. The case $p = 1$ being clear, we may assume that $1 < p < \infty$. We may also assume that $\zeta = \zeta^\alpha dx^\alpha$ for some fixed $\alpha \in I(n-2, n)$. A first ingredient of the proof of (3.64) is the density of $W^{1,1}((0,1)^n; \mathbb{S}^1) \cap W^{1/p,p}((0,1)^n; \mathbb{S}^1)$ into $W^{1/p,p}((0,1)^n; \mathbb{S}^1)$ [6, Lemma 23], [7, Lemma A.1]. Next, we note that the left-hand side of (3.64) is continuous with respect to the $W^{1/p,p}$ convergence of

¹¹ Up to the action of the $*$ operator.

¹² Or rather on 2-forms, in order to be consistent with our construction in dimension ≥ 3 .

unimodular maps [9, Theorem 1.1 item 2]. In addition, as we noted, (3.64) holds when $u \in W^{1,1}((0,1)^n; \mathbb{S}^1)$. Therefore, it suffices to prove that the right-hand side of (3.64) is continuous with respect to $W^{1/p,p}$ convergence of \mathbb{S}^1 -valued maps. This is proved as follows. Let $u_j, u \in W^{1/p,p}((0,1)^n; \mathbb{S}^1)$ be such $u_j \rightarrow u$ in $W^{1/p,p}$. By a standard argument, since the right-hand side of (3.64) is uniformly bounded with respect to j by (3.63), it suffices to prove that the right-hand side of (3.64) corresponding to u_j tends to the one corresponding to u possibly along a subsequence.

In turn, convergence up to a subsequence is proved as follows. Recall the following vector-valued version of the ‘‘converse’’ to the dominated convergence theorem [11, Theorem 4.9, p. 94]. If X is a Banach space, ω a measured space and $f_j \rightarrow f$ in $L^p(\omega, X)$, then (possibly along a subsequence) for a.e. $\omega \in \omega$ we have $f_j(\omega) \rightarrow f(\omega)$ in X , and in addition there exists some non-negative function $g \in L^p(\omega)$ such that $\|f_j(\omega)\|_X \leq g(\omega)$ for a.e. $\omega \in \omega$.

Using the above and Lemma 3.1 item 2 (applied with $s = 1/p$), we find that, up to a subsequence, we have

$$(u_j)_\alpha(x_\alpha) \rightarrow u_\alpha(x_\alpha) \text{ in } W^{1/p,p}((0,1)^2; \mathbb{S}^1) \text{ for a.e. } x_\alpha \in (0,1)^{n-2}, \quad (3.65)$$

and in addition we have, for some $g \in L^p((0,1)^{n-2})$,

$$\|(u_j)_\alpha(x_\alpha)\|_{W^{1/p,p}((0,1)^2)} \leq g(x_\alpha) \text{ for a.e. } x_\alpha \in (0,1)^{n-2}. \quad (3.66)$$

The continuity of the right-hand side of (3.64) (along some subsequence) is obtained by combining (3.65) and (3.66) with (3.63) (applied with $n = 2$).¹³ \square

Lemma 3.13. Let $1 \leq p < \infty$. Let $u \in W^{1/p,p} \cap \text{VMO}((0,1)^2; \mathbb{S}^1)$. Then $Ju = 0$.

Proof. Assume first that in addition we have $u \in C^\infty$. Then $u = e^{i\varphi}$ for some $\varphi \in C^\infty$, and thus $Ju = 1/2 \text{curl}(u \wedge \nabla u) = 1/2 \text{curl} \nabla \varphi = 0$.

We now turn to the general case. Let $F(x, \varepsilon) := u * \rho_\varepsilon(x)$, with ρ a standard mollifier. Since $u \in \text{VMO}((0,1)^2; \mathbb{S}^1)$, there exists some $\delta > 0$ such that $1/2 < |F(x, \varepsilon)| \leq 1$ when $0 < \varepsilon < \delta$ (see (4.2) and the discussion in Case 3). Let $\Phi \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ be such that $\Phi(z) := z/|z|$ when $|z| \geq 1/2$, and define $F_\varepsilon(x) := F(x, \varepsilon)$ and $u_\varepsilon := \Phi \circ F_\varepsilon$, $\forall 0 < \varepsilon < \delta$. Then $F_\varepsilon \rightarrow u$ in $W^{1/p,p}$ and (by Lemma 7.12 when $p > 1$, respectively by a straightforward argument when $p = 1$) we have $u_\varepsilon = \Phi(F_\varepsilon) \rightarrow \Phi(u) = u$ in $W^{1/p,p}((0,1)^2; \mathbb{S}^1)$ as $\varepsilon \rightarrow 0$. Since (by the beginning of the proof) we have $Ju_\varepsilon = 0$, we conclude via the continuity of J in $W^{1/p,p}((0,1)^2; \mathbb{S}^1)$ [9, Theorem 1.1 item 2]. \square

We may now state and prove the following generalization of Lemma 3.11.

¹³In order to be complete, we should also check that the right-hand side of (3.64) is measurable with respect to x_α . This is clear when $u \in W^{1,1}((0,1)^n; \mathbb{S}^1)$. The general case follows by density and (3.65).

Lemma 3.14. Let $s > 0$, $1 \leq p < \infty$, $1 \leq q \leq p$, $n \geq 3$, and assume that $sp \geq 2$. Let $u \in B_{p,q}^s(\Omega; \mathbb{S}^1)$. Then $Ju = 0$.

Same conclusion if $s > 0$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, $n \geq 2$, and we have $sp > 2$.

Same conclusion if $s > 0$, $1 \leq p < \infty$, $1 \leq q < \infty$, $n = 2$, and we have $sp = 2$.

Proof. We may assume that $\Omega = (0,1)^n$. By the Sobolev embeddings (Lemma 7.1), it suffices to consider the limiting case where:

1. $s > 0$, $1 \leq p < \infty$, $1 \leq q < \infty$, $n = 2$, and $sp = 2$.

Or

2. $s > 0$, $1 \leq p < \infty$, $q = p$, $n \geq 3$, and $sp = 2$.

In view of Lemmas 7.1 and 7.5, the case where $n = 2$ is covered by Lemma 3.13. Assume that $n \geq 3$. Then the desired conclusion is obtained by combining Lemmas 3.1, 3.2, 3.12 and 3.13. \square

Remark 3.15. Arguments similar to the one developed in this section lead to the conclusion that the Jacobians of maps $u \in W^{s,p}((0,1)^n; \mathbb{S}^k)$, defined when $sp \geq k$ [7], [9], disintegrate over $(k+1)$ -planes. When $s = 1$ and $p \geq k$, this assertion is implicit in [21, Proof of Proposition 2.2, pp. 701-704].

3.5 Integer-valued functions in $B_{p,\infty}^{1/p} + C^1$

Lemma 3.16. Let $p \in [1, \infty)$. For all intervals $I = (a, b) \subset \mathbb{R}$ and all functions $f \in B_{p,\infty}^{1/p}(I; \mathbb{R})$, $g \in C^1(I; \mathbb{R})$ such that $\eta := f + g : I \rightarrow \mathbb{Z}$, there exist $k \geq 1$, $a = x_0 < x_1 < \dots < x_k = b$ and integers $\alpha_1, \dots, \alpha_k$ such that $\alpha_j \neq \alpha_{j+1}$ for all $j \in \llbracket 1, k-1 \rrbracket$, and

$$\eta(x) = \alpha_j \text{ for } x \in (x_{j-1}, x_j). \quad (3.67)$$

In addition, we have

$$\sum_{j=0}^{k-1} |\alpha_{j+1} - \alpha_j| \leq \liminf_{h \rightarrow 0} \frac{\|\tau_h f - f\|_{L^p}^p}{h} \quad (3.68)$$

when $1 < p < \infty$, respectively

$$\sum_{j=0}^{k-1} |\alpha_{j+1} - \alpha_j| \leq \frac{1}{2} \liminf_{h \rightarrow 0} \frac{\|\tau_{2h} f - 2\tau_h f + f\|_{L^1}}{h} \quad (3.69)$$

when $p = 1$.

In (3.67) and in the proof that follows, equality of functions is understood a.e. Note the estimates (3.68) and (3.69): the right-hand side depends only on f , not on η as one might expect.

Proof. Let us first consider the case where $1 < p < \infty$. Since η is integer-valued, we have

$$|(\tau_h \eta - \eta)(x)| \leq |(\tau_h \eta - \eta)(x)|^p, \quad \forall h > 0, \forall x \in (a, b - h). \quad (3.70)$$

Combining (3.70) with the fact that g is C^1 and thus locally we have $|\tau_h g - g| \leq Ch$, we obtain, for every compact interval $J \subset I$:

$$\liminf_{h \rightarrow 0} \frac{\|\tau_h \eta - \eta\|_{L^1(J)}}{h} \leq \liminf_{h \rightarrow 0} \frac{\|\tau_h \eta - \eta\|_{L^p(J)}^p}{h} = \liminf_{h \rightarrow 0} \frac{\|\tau_h f - f\|_{L^p(J)}^p}{h}.$$

Therefore, the (a priori possibly infinite) total variation $|\eta'| (I)$ of η on I satisfies

$$\begin{aligned} |\eta'| (I) &= \sup \{ |\eta'| (J); J \text{ compact interval, } J \subset I \} \\ &\leq \sup \left\{ \liminf_{h \rightarrow 0} \frac{\|\tau_h \eta - \eta\|_{L^1(J)}}{h}; J \text{ compact interval, } J \subset I \right\} \\ &\leq \liminf_{h \rightarrow 0} \frac{\|\tau_h f - f\|_{L^p((a, b - h))}^p}{h} < \infty, \end{aligned} \quad (3.71)$$

which entails that $\eta \in BV(I; \mathbb{Z})$. Moreover, η has the form (3.67) and (3.68) holds.

Assume now that $p = 1$. Consider first the case where

$$\liminf_{h \rightarrow 0} \frac{\|\tau_{2h} f - 2\tau_h f + f\|_{L^1((a, b - 2h))}}{h} < 2. \quad (3.72)$$

Let $k \in \mathbb{Z}$ be such that $|\eta^{-1}(\{k\})| > 0$. Without loss of generality, we can assume that $k = 0$. Set $A := \eta^{-1}(2\mathbb{Z})$ and define $\bar{\eta} := \mathbb{1}_A$. Observe that $|A| > 0$. One has, for all $x \in I$ and all $h > 0$ such that $(x, x + 2h) \subset I$,

$$|\tau_{2h} \eta(x) - 2\tau_h \eta(x) + \eta(x)| \geq |\tau_{2h} \bar{\eta}(x) - \bar{\eta}(x)|. \quad (3.73)$$

Arguing as in (3.71), we find that

$$\begin{aligned} |\bar{\eta}'| (I) &= \sup \{ |\bar{\eta}'| (J); J \text{ compact interval, } J \subset I \} \\ &\leq \sup \left\{ \liminf_{h \rightarrow 0} \frac{\|\tau_{2h} \bar{\eta} - \bar{\eta}\|_{L^1(J)}}{2h}; J \text{ compact interval, } J \subset I \right\} \\ &\leq \sup \left\{ \liminf_{h \rightarrow 0} \frac{\|\tau_{2h} \eta - 2\tau_h \eta + \eta\|_{L^1(J)}}{2h}; J \text{ compact interval, } J \subset I \right\} \\ &= \sup \left\{ \liminf_{h \rightarrow 0} \frac{\|\tau_{2h} f - 2\tau_h f + f\|_{L^1(J)}}{2h}; J \text{ compact interval, } J \subset I \right\} \\ &\leq \frac{1}{2} \liminf_{h \rightarrow 0} \frac{\|\tau_{2h} f - 2\tau_h f + f\|_{L^1((a, b - 2h))}}{h} < 1; \end{aligned} \quad (3.74)$$

the next-to-the-last line follows from the fact that $g \in C^1$ and thus we locally have $|\tau_{2h}g - 2\tau_hg + g| \leq \delta(h)h$, with $\delta(h) \rightarrow 0$ as $h \rightarrow 0$. (3.74) implies that $\bar{\eta}$ is constant, and thus equal to 1 (since it equals 1 on a set of positive measure). This shows that η only takes even values. Setting $\eta_1 := \frac{1}{2}\eta$, arguing with η_1 as with η and iterating, we conclude that $\eta = 0$.¹⁴

Let us now prove, by induction on $j \geq 1$, that if

$$\liminf_{h \rightarrow 0} \frac{\|\tau_{2h}f - 2\tau_hf + f\|_{L^1((a,b-2h))}}{h} < 2j, \quad (3.75)$$

then η has the form (3.67) and we have the estimate (3.69). The case where $j = 1$ was already settled. Assume now that (3.69) holds whenever (3.75) is satisfied for $1, \dots, j-1$ for some $j \geq 2$ and let $f : I \rightarrow \mathbb{Z}$ satisfy (3.75) for this j . We may and do assume again that $\eta^{-1}(\{0\})$ has positive measure and define A , $\bar{\eta}$ and η_1 as above. Note that, by (3.73) and (3.75), we have $\bar{\eta} \in BV(I)$, and therefore $\bar{\eta}$ only has a finite number of jumps.

If $\bar{\eta} \equiv 1$, then η takes only even values. The induction hypothesis applied to η_1 shows that (3.69) holds.

Assume now that $\bar{\eta} \not\equiv 1$. Without loss of generality, we may assume that for some $c \in (a, b)$ we have

$$\lim_{t \searrow 0} \bar{\eta}(c+t) = 1 \text{ and } \lim_{t \searrow 0} \bar{\eta}(c-t) = 0.$$

For sufficiently small $\varepsilon > 0$, one has $\bar{\eta} \equiv 1$ on $(c, c+\varepsilon)$ and $\bar{\eta} \equiv 0$ on $(c-\varepsilon, c)$. This shows that, if $Q_\varepsilon := (c-\varepsilon, c+\varepsilon)$, then

$$\liminf_{h \rightarrow 0} \frac{\|\tau_{2h}\bar{\eta} - \bar{\eta}\|_{L^1(Q_\varepsilon)}}{h} \geq 2.$$

Indeed, for all $h \in (0, \varepsilon/2)$ and all $x \in (c-2h, c)$, we have $|\bar{\eta}(x+2h) - \bar{\eta}(x)| = 1$. Therefore, we also have

$$\begin{aligned} \liminf_{h \rightarrow 0} \frac{\|\tau_{2h}f - 2\tau_hf + f\|_{L^1(Q_\varepsilon)}}{h} &= \liminf_{h \rightarrow 0} \frac{\|\tau_{2h}\eta - 2\tau_h\eta + \eta\|_{L^1(Q_\varepsilon)}}{h} \\ &\geq \liminf_{h \rightarrow 0} \frac{\|\tau_{2h}\bar{\eta} - \bar{\eta}\|_{L^1(Q_\varepsilon)}}{h} \geq 2. \end{aligned}$$

This entails that, with $P_\varepsilon := (a, c-\varepsilon)$, we have

$$\liminf_{h \rightarrow 0} \frac{\|\tau_{2h}f - 2\tau_hf + f\|_{L^1(P_\varepsilon)}}{h} < 2(j-1).$$

¹⁴We will consider a similar approach in the proof of Lemma 7.15.

By the induction hypothesis, η has a finite (and independent of small ε) number of jumps on P_ε . Moreover, it follows that η satisfies (3.67) on (a, c) , for some integers k_1 and $\alpha_0, \dots, \alpha_{k_1}$ satisfying

$$\sum_{j=0}^{k_1-1} |\alpha_{j+1} - \alpha_j| \leq \frac{1}{2} \lim_{h \rightarrow 0} \frac{\|\tau_{2h}f - 2\tau_hf + f\|_{L^1((a, c-2h))}}{h}. \quad (3.76)$$

Arguing similarly, one has the analogous conclusion on (c, b) , with integers $\alpha_{k_1+1}, \dots, \alpha_{k_2}$ such that

$$\sum_{j=k_1+1}^{k_2-1} |\alpha_{j+1} - \alpha_j| \leq \frac{1}{2} \lim_{h \rightarrow 0} \frac{\|\tau_{2h}f - 2\tau_hf + f\|_{L^1((c, b-2h))}}{h}. \quad (3.77)$$

On the other hand, since for sufficiently small $h > 0$ we have

$$\eta(x) = \begin{cases} \alpha_{k_1}, & \text{on } (c-2h, c) \\ \alpha_{k_1+1}, & \text{on } (c, c+2h) \end{cases},$$

we find that

$$|\alpha_{k_1+1} - \alpha_{k_1}| = \frac{1}{2} \lim_{h \rightarrow 0} \frac{\|\tau_{2h}f - 2\tau_hf + f\|_{L^1((c-2h, c))}}{h}. \quad (3.78)$$

Finally, gathering (3.76), (3.77) and (3.78) shows that (3.69) holds. \square

Lemma 3.17. Let $n \geq 2$ and $p \in [1, \infty)$. Let $\tilde{f} \in B_{p, \infty}^{1/p}((0, 1)^n; \mathbb{R})$ and $\tilde{g} \in C^1((0, 1)^n; \mathbb{R})$ be such that $\tilde{\eta} := \tilde{f} + \tilde{g} : (0, 1)^n \rightarrow \mathbb{Z}$. Then $\tilde{\eta} \in BV((0, 1)^n)$.

Proof. Let first $1 < p < \infty$. Arguing as in the proof of (3.74), we find that for every compact $K \subset (0, 1)^n$ we have

$$\lim_{h \rightarrow 0} \frac{\|\tau_h \tilde{\eta} - \tilde{\eta}\|_{L^1(K)}}{h} \leq \lim_{h \rightarrow 0} \frac{\|\tau_h \tilde{f} - \tilde{f}\|_{L^p(K)}^p}{h} \leq C,$$

for some constant $C > 0$ independent of K . This entails that $\tilde{\eta} \in BV_{loc}((0, 1)^n)$ with $|D\tilde{\eta}|(K) \leq C$. Thus, $\tilde{\eta} \in BV((0, 1)^n)$.

Consider now the case where $p = 1$. For all $j \in \llbracket 1, n \rrbracket$ and all $x \in (0, 1)^n$, set $\hat{x}_j := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in (0, 1)^{n-1}$. Pick up a sequence $(h_k)_{k \geq 1}$ of positive numbers converging to 0 and define, for all $k \geq 1$, all $x \in (0, 1)^n$ and all $j \in \llbracket 1, n \rrbracket$,

$$F_k^j(\hat{x}_j) := \frac{\left\| \left(\tau_{2h_k e_j} \tilde{f} - 2\tau_{h_k e_j} \tilde{f} + \tilde{f} \right) (x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n) \right\|_{L^1((0, 1-2h_k))}}{h_k}.$$

One has

$$\frac{\|\tau_{2h_k e_j} \tilde{f} - 2\tau_{h_k e_j} \tilde{f} + \tilde{f}\|_{L^1}}{h_k} = \int_{(0,1)^{n-1}} F_k^j(\hat{x}_j) d(\hat{x}_j) \leq C,$$

where $C > 0$ is independent of k and j . By Fatou's lemma, if we set $G(\hat{x}_j) := \varliminf_{k \rightarrow \infty} F_k^j(\hat{x}_j)$, then we have

$$\int_{(0,1)^{n-1}} G(\hat{x}_j) d\hat{x}_j \leq C. \quad (3.79)$$

By Lemma 3.16, whenever $G(\hat{x}_j) < \infty$ (which holds for almost every $\hat{x}_j \in (0,1)^{n-1}$), we have

$$\tilde{\eta}(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n) \in BV((0,1))$$

and

$$\|\tilde{\eta}(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n)\|_{BV((0,1))} \leq \frac{1}{2} G(\hat{x}_j),$$

which, in conjunction with (3.79), yields that $\tilde{\eta} \in BV((0,1)^n)$. \square

4 Positive cases

We start with the trivial case.

Case 1. *Range.* $s > 0$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, and $sp > n$.

Conclusion. $B_{p,q}^s(\Omega; \mathbb{S}^1)$ does have the lifting property.

Proof. Since $B_{p,q}^s(\Omega) \hookrightarrow C^0(\overline{\Omega})$ (Lemma 7.2), we may write $u = e^{i\varphi}$, with φ continuous. Locally, we have $\varphi = -i \ln u$, for some smooth determination \ln of the complex logarithm. Then φ belongs to $B_{p,q}^s$ locally in $\overline{\Omega}$ (Lemma 7.13), and thus globally (Lemma 2.4). \square

Case 2. *Range.* $0 < s < 1$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, and $sp < 1$.

Conclusion. $B_{p,q}^s(\Omega; \mathbb{S}^1)$ does have the lifting property.

Proof. The argument being essentially the one in [4, Section 1], we will be sketchy. Assume for simplicity that $\Omega = (0,1)^n$. Let $u \in B_{p,q}^s(\Omega; \mathbb{S}^1)$. For all $j \in \mathbb{N}$, consider the function U_j defined by

$$U_j(x) := \begin{cases} E_j(u)(x)/|E_j(u)(x)|, & \text{if } E_j(u)(x) \neq 0 \\ 1, & \text{if } E_j(u)(x) = 0 \end{cases}$$

Since $E_j(u) \rightarrow u$ a.e., we find that $U_j \rightarrow u$ a.e. on Ω . By induction on j , for all $j \in \mathbb{N}$ we construct a phase φ_j of U_j , constant on each dyadic cube of size 2^{-j} , and satisfying the inequality

$$|\varphi_j - \varphi_{j-1}| \leq \pi |U_j - U_{j-1}| \quad \text{on } \Omega, \forall j \geq 1. \quad (4.1)$$

As in [4], (4.1) implies

$$|\varphi_j - \varphi_{j-1}| \lesssim |u - E_j(u)| + |u - E_{j-1}(u)|,$$

and thus, e.g. when $q < \infty$, we have

$$\sum_{j \geq 1} 2^{sjq} \|\varphi_j - \varphi_{j-1}\|_{L^p}^q \lesssim \sum_{j \geq 0} 2^{sjq} \|u - E_j(u)\|_{L^p}^q.$$

Applying Corollaries 2.10 and 2.11, we obtain that $\varphi_j \rightarrow \varphi$ in L^p to some $\varphi \in B_{p,q}^s(\Omega; \mathbb{R})$. Since φ_j is a phase of U_j and $U_j \rightarrow u$ a.e., we find that φ is a phase of u . In addition, we have the control $\|\varphi\|_{B_{p,q}^s} \lesssim \|u\|_{B_{p,q}^s}$. \square

Case 3. *Range.* $0 < s < 1$, $1 \leq p < \infty$, $1 \leq q < \infty$, and $sp = n$.

Conclusion. $B_{p,q}^s(\Omega; \mathbb{S}^1)$ does have the lifting property.

Proof. Here, it will be convenient to work with $\Omega = \mathbb{T}^n$. Let $|\cdot|$ denote the sup norm in \mathbb{R}^n . Let $\rho \in C^\infty$ be a mollifier supported in $\{|x| \leq 1\}$ and set $F(x, \varepsilon) := u * \rho_\varepsilon(x)$, $x \in \mathbb{T}^n$, $\varepsilon > 0$. Since $sp = n$, we have $u \in \text{VMO}(\mathbb{T}^n)$, by Lemma 7.5. Let us recall that, if $u \in \text{VMO}(\mathbb{T}^n; \mathbb{S}^1)$ then, for some $\delta > 0$ (depending on u) we have [14, Remark 3, p. 207]

$$\frac{1}{2} < |F(x, \varepsilon)| \leq 1 \quad \text{for all } x \in \mathbb{T}^n \text{ and all } \varepsilon \in (0, \delta). \quad (4.2)$$

Define

$$w(x, \varepsilon) := \frac{F(x, \varepsilon)}{|F(x, \varepsilon)|} \quad \text{for all } x \in \mathbb{T}^n \text{ and all } \varepsilon \in (0, \delta).$$

Pick up a function $\psi \in C^\infty((0, 2\pi)^n \times (0, \delta); \mathbb{R})$ such that $w = e^{i\psi}$.¹⁷ We note that we have $\nabla \psi = -i\bar{w}\nabla w$, and, for all $j \in \llbracket 1, n \rrbracket$, $\partial_j |F| = |F|^{-1}(F\partial_j \bar{F} + \bar{F}\partial_j F)/2$. Therefore, (4.2) yields

$$|\nabla \psi| = |\nabla w| \lesssim |\nabla F|. \quad (4.3)$$

In view of (4.3) and estimate (3.25) in Lemma 3.7, we find that

$$|u|_{B_{p,q}^s(\mathbb{T}^n)}^q \gtrsim \int_0^\delta \varepsilon^{q-sq} \|(\nabla F)(\cdot, \varepsilon)\|_{L^p}^q \frac{d\varepsilon}{\varepsilon} \gtrsim \int_0^\delta \varepsilon^{q-sq} \|(\nabla \psi)(\cdot, \varepsilon)\|_{L^p}^q \frac{d\varepsilon}{\varepsilon}. \quad (4.4)$$

¹⁵ Thus φ_j is the phase of U_j closest to φ_{j-1} .

¹⁶ For an explicit calculation leading to (4.2), see e.g. [24, p. 415].

¹⁷ We do not claim that ψ is $(2\pi\mathbb{Z})^n$ -periodic in x .

Combining (4.4) with the conclusion of Lemma 3.7, we obtain that the phase ψ has, on \mathbb{T}^n , a trace $\varphi \in B_{p,q}^s$, in the sense that the limit $\varphi := \lim_{\varepsilon \rightarrow 0} \psi(\cdot, \varepsilon)$ exists in $B_{p,q}^s$. In particular (using Lemma 7.4), we have that $\psi(\cdot, \varepsilon_j) \rightarrow \varphi$ a.e. along some sequence $\varepsilon_j \rightarrow 0$; this leads to $w(\cdot, \varepsilon_j) = e^{i\psi(\cdot, \varepsilon_j)} \rightarrow e^{i\varphi}$ a.e. Since, on the other hand, we have $\lim_{\varepsilon \rightarrow 0} w(\cdot, \varepsilon) = u$ a.e., we find that φ is a $B_{p,q}^s$ phase of u . \square

The next case is somewhat similar to Case 3, so that our argument is less detailed.

Case 4. *Range.* $s = 1$, $p = n$, $1 \leq q < \infty$.

Conclusion. $B_{n,q}^1(\Omega; \mathbb{S}^1)$ does have the lifting property.

Proof. We consider δ , w and ψ as in Case 3. The analog of (4.3) is the estimate

$$|\partial_j \partial_k \psi| + |\nabla \psi|^2 \lesssim |\partial_j \partial_k F| + |\nabla F|^2, \quad (4.5)$$

which is a straightforward consequence of the identities

$$\nabla \psi = -i\bar{w}\nabla w \text{ and } \partial_j \partial_k \psi = -i\bar{w}\partial_j \partial_k w + i\bar{w}^2 \partial_j w \partial_k w.$$

Combining (4.5) with the second part of Lemma 3.8, we obtain

$$|u|_{B_{n,q}^1}^q \gtrsim \int_0^\delta \varepsilon^q \left(\sum_{j,k=1}^n \|\partial_j \partial_k \psi(\cdot, \varepsilon)\|_{L^n}^q + \|\partial_\varepsilon \partial_\varepsilon \psi(\cdot, \varepsilon)\|_{L^n}^q + \|\nabla \psi(\cdot, \varepsilon)\|_{L^{2n}}^{2q} \right) \frac{d\varepsilon}{\varepsilon}. \quad (4.6)$$

By (4.6) and the first part of Lemma 3.8, we find that ψ has a trace $\varphi := \text{tr } \psi \in B_{n,q}^1(\mathbb{T}^n)$. Clearly, φ is a $B_{n,q}^1$ phase of u . \square

Case 5. *Range.* $s > 1$, $1 \leq p < \infty$, $1 \leq q < \infty$, $n = 2$, and $sp = 2$.

Or $s > 1$, $1 \leq p < \infty$, $1 \leq q \leq p$, $n \geq 3$, and $sp = 2$.

Or: $s > 1$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, $n \geq 2$, and $sp > 2$.

Conclusion. $B_{p,q}^s(\Omega; \mathbb{S}^1)$ does have the lifting property.

Note that, in the critical case where $sp = 2$, our result is weaker in dimension $n \geq 3$ (when we ask $1 \leq q \leq p$) than in dimension 2 (when we merely ask $1 \leq q < \infty$).

Proof. The general strategy is the same as in [4, Section 3, Proof of Theorem 3],¹⁸ but the key argument (validity of (4.9) below) is much more involved in our case.

It will be convenient to work in $\Omega = \mathbb{T}^n$. Let $u \in B_{p,q}^s(\mathbb{T}^n; \mathbb{S}^1)$. Assume first that we may write $u = e^{i\varphi}$, with $\varphi \in B_{p,q}^s((0,1)^n; \mathbb{R})$. Then $u, \varphi \in W^{1,p}$ (Lemma 7.4).

¹⁸ See also [15].

We are thus in position to apply chain's rule and infer that $\nabla u = \iota u \nabla \varphi$, and therefore

$$\nabla \varphi = \frac{1}{\iota u} \nabla u = F, \text{ with } F := u \wedge \nabla u \in L^p(\mathbb{T}^n; \mathbb{R}^n). \quad (4.7)$$

The assumptions on s, p, q imply that $F \in B_{p,q}^{s-1}$ (Lemma 7.11). We may now argue as follows. If φ solves (4.7), then $\nabla \varphi \in B_{p,q}^{s-1}$, and thus $\varphi \in B_{p,q}^s$ (Lemma 7.9). Next, since $u, e^{-\iota \varphi} \in W^{1,p} \cap L^\infty$, we find that

$$\nabla(u e^{-\iota \varphi}) = \nabla u e^{-\iota \varphi} - \iota u e^{-\iota \varphi} \nabla \varphi = \iota u e^{-\iota \varphi} (u \wedge \nabla u - \nabla \varphi) = 0.$$

Thus $u e^{-\iota \varphi}$ is constant, and therefore φ is, up to an appropriate additive constant, a $B_{p,q}^s$ phase of u .

There is a flaw in the above. Indeed, (4.7) need not have a solution. In \mathbb{T}^n , the necessary and sufficient conditions for the solvability of (4.7) are¹⁹

$$\int_{\mathbb{T}^n} F = \widehat{F}(0) = 0 \quad (4.8)$$

and

$$\operatorname{curl} F = 0. \quad (4.9)$$

By Lemma 3.11, (4.9) holds in the relevant range of s, p, q and n . On the hand, even if (4.8) need not hold, the auxiliary field $G = F - \widehat{F}(0)$ satisfies both (4.8) and (4.9). If ψ is a global solution of $\nabla \psi = G$, then $\varphi(x) = \psi(x) + \widehat{F}(0) \cdot x$ is, up to an additive constant, a phase of u in the space $B_{p,q}^s((0,1)^n; \mathbb{R})$. This completes Case 5. \square

Remark 4.1. We briefly discuss the lifting problem when $s \leq 0$. For such s , distributions in $B_{p,q}^s$ need not be integrable functions, and thus the meaning of the equality $u = e^{\iota \varphi}$ is unclear. We therefore address the following reasonable version of the lifting problem: let $u : \Omega \rightarrow \mathbb{S}^1$ be a measurable function such that $u \in B_{p,q}^s(\Omega)$. Is there any $\varphi \in L_{loc}^1 \cap B_{p,q}^s(\Omega; \mathbb{R})$ such that $u = e^{\iota \varphi}$?

Let us just note here that the answer is trivially positive when $s < 0$, $1 \leq p < \infty$, $1 \leq q \leq \infty$.

Indeed, let φ be any bounded measurable lifting of u . Then $\varphi \in B_{p,q}^s$, since $L^\infty \hookrightarrow B_{p,q}^s$ when $s < 0$ (see Lemma 7.3).

5 Negative cases

Case 6. Range. $0 < s < 1$, $1 \leq p < \infty$, $1 \leq q < \infty$, $n \geq 2$, and $1 \leq sp < n$.

Or $0 < s < 1$, $1 \leq p < \infty$, $q = \infty$, $n \geq 2$, and $1 < sp < n$.

Conclusion. $B_{p,q}^s(\Omega; \mathbb{S}^1)$ does not have the lifting property.

¹⁹ This is easily seen by an inspection of the Fourier coefficients.

Proof. We want to show that there exists a function $u \in B_{p,q}^s$ such that $u \neq e^{i\varphi}$ for any $\varphi \in B_{p,q}^s$.

For sufficiently small $\varepsilon > 0$, set $s_1 := s/(1 - \varepsilon)$ and $p_1 := (1 - \varepsilon)p$. By Lemma 7.1, we have $B_{p_1,q_1}^{s_1} \not\hookrightarrow B_{p,q}^s$ (for any q_1). We will use later this fact for $q_1 := (1 - \varepsilon)q$.

Let $\psi \in B_{p_1,q_1}^{s_1} \setminus B_{p,q}^s$ and set $u := e^{i\psi}$. Then $u \in B_{p_1,q_1}^{s_1} \cap L^\infty$ (Lemma 7.12) and thus $u \in B_{p,q}^s$ (Lemma 7.6).

We claim that there is no $\varphi \in B_{p,q}^s$ such that $u = e^{i\varphi}$. Argue by contradiction. Since $u = e^{i\varphi} = e^{i\psi}$, the function $(\varphi - \psi)/2\pi$ belongs to $(B_{p,q}^s + B_{p_1,q_1}^{s_1})(\Omega; \mathbb{Z})$. By Lemma 7.14, this implies that $\varphi - \psi$ is constant, and thus $\psi \in B_{p,q}^s$, which is a contradiction. \square

Case 7. Range. $0 < s < \infty$, $1 \leq p < \infty$, $1 \leq q < \infty$, $n \geq 2$, and $1 \leq sp < 2$.

Or $0 < s < \infty$, $1 \leq p < \infty$, $q = \infty$, $n \geq 2$, and $1 < sp \leq 2$.

Conclusion. $B_{p,q}^s(\Omega; \mathbb{S}^1)$ does not have the lifting property.

Proof. The proof is based on the example of a topological obstruction considering the case $n = 2$. Consider the map $u(x) = \frac{x}{|x|}$, $\forall x \in \mathbb{R}^2$.

We first prove that $u \in B_{p,q}^s(\Omega)$ for any smooth bounded domain $\Omega \subset \mathbb{R}^2$. We distinguish two cases: firstly, $q \leq \infty$ and $sp < 2$ and secondly, $q = \infty$ and $sp = 2$.

In the first case, let $s_1 > s$ such that s_1 is not an integer and $1 < s_1p < 2$, which implies $W^{s_1,p} = B_{p,p}^{s_1} \hookrightarrow B_{p,q}^s$. Since $u \in W^{s_1,p}$ [4, Section 4], we find that $u \in B_{p,q}^s$.

The second case is slightly more involved. By the Gagliardo-Nirenberg inequality (Lemma 7.6 below), it suffices to prove that $u \in B_{1,\infty}^2(\Omega)$. Using Proposition 2.6, a sufficient condition for this to hold is

$$\|\Delta_h^3 u\|_{L^1(\mathbb{R}^2)} \lesssim |h|^2, \quad \forall h \in \mathbb{R}^2. \quad (5.1)$$

Since u is radially symmetric and 0-homogeneous, this amounts to checking that

$$\|\Delta_{e_1}^3 u\|_{L^1(\mathbb{R}^2)} < \infty. \quad (5.2)$$

However, by the mean-value theorem, for all $|x| \geq 1$ we have

$$|\Delta_{e_1}^3 u(x)| \lesssim 1/|x|^3, \quad (5.3)$$

while $\Delta_{e_1}^3 u$ is bounded in $B(0,1)$ since u is \mathbb{S}^1 -valued. Using this fact and estimate (5.3), we obtain (5.2).

We next claim that u has no $B_{p,q}^s$ lifting in Ω provided $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain containing the origin. Argue by contradiction, and assume that $u = e^{i\varphi}$ for some $\varphi \in B_{p,q}^s(\Omega)$. Let, as in [4, p. 50], $\theta \in C^\infty(\mathbb{R}^2 \setminus ([0, \infty) \times \{0\}))$ be such that $e^{i\theta} = u$.

Note that $\theta \in B_{p,q}^s(\omega)$ for every smooth bounded open set ω such that $\bar{\omega} \subset \mathbb{R}^2 \setminus ([0, \infty) \times \{0\})$. Since $(\varphi - \theta)/(2\pi)$ is \mathbb{Z} -valued, Lemma 7.14 yields that $\varphi - \theta$ is constant a.e. in $\Omega \setminus ([0, \infty) \times \{0\})$. Thus, $\theta \in B_{p,q}^s(\Omega)$. Similarly, $\tilde{\theta} \in B_{p,q}^s(\Omega)$, where $\tilde{\theta} \in C^\infty(\mathbb{R}^2 \setminus ((-\infty, 0] \times \{0\}))$ is such that $e^{i\tilde{\theta}} = u$. We find that $(\theta - \tilde{\theta})/(2\pi) \in B_{p,q}^s(\Omega)$. However, this is a non-constant integer-valued function. This contradicts Lemma 7.14 and proves non-existence of lifting in $B_{p,q}^s$.

When $n \geq 3$, the above arguments lead to the following. Let $u(x) = \frac{(x_1, x_2)}{|(x_1, x_2)|}$, and let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Then $u \in B_{p,q}^s(\Omega; \mathbb{S}^1)$ and, if $0 \in \Omega$, then u has no $B_{p,q}^s$ lifting. \square

Case 8. Range. $0 < s \leq 1$, $1 \leq p < \infty$, $sp = 1$, $q = \infty$, $n \geq 1$.

Conclusion. $B_{p,\infty}^{1/p}((0, 1)^n; \mathbb{S}^1)$ does not have the lifting property.

Proof. The general scheme of the proof is the following. We first construct a function $\psi \in C^\infty((0, 1)^n; \mathbb{R})$ such that ψ does not belong to $B_{p,\infty}^{1/p}((0, 1)^n)$, but $u := e^{i\psi}$ belongs to $B_{p,\infty}^{1/p}((0, 1)^n; \mathbb{S}^1)$. We next establish that, for any such ψ , there exists no function $\varphi \in B_{p,\infty}^{1/p}((0, 1)^n)$ such that $u = e^{i\varphi}$.

The main effort is devoted to the construction of ψ in one dimension. The higher dimensional case will follow via a dimensional reduction procedure. The proof being rather long, we split it, for the convenience of the reader, into several separate statements and steps.

Lemma 5.1. Let $1 \leq p < \infty$. Then the lifting problem in $B_{p,\infty}^{1/p}((0, 1); \mathbb{S}^1)$ has a negative answer.

Proof. Step 1. Construction of ψ

Fix a function $\psi_0 \in C^\infty(\mathbb{R})$ such that

$$\psi_0(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 2\pi, & \text{if } t \geq 1 \end{cases}.$$

Pick up two sequences $(a_j)_{j \geq 1} \subset (0, 1)$ and $(b_j)_{j \geq 0} \subset [0, 1)$ such that, for a suitable constant $c > 0$ (determined by the conditions 1–5 below),

1. $b_0 = 0$,
2. $a_j < b_j < a_{j+1}$ for all $j \geq 1$,
3. $b_j - a_j = \frac{c}{2^j}$ for all $j \geq 1$,
4. $a_j - b_{j-1} = \frac{c}{j^2}$ for all $j \geq 1$,
5. $\lim_{j \rightarrow \infty} a_j = \lim_{j \rightarrow \infty} b_j = 1$.

For all $j \geq 1$, define $I_j := (a_j, b_j)$, $L_j := (b_{j-1}, a_j)$ and $\varepsilon_j := |I_j| = \frac{c}{2^j}$. For all $x \in \mathbb{R}$ and all $j \geq 1$, let

$$\psi_j(x) := \psi_0\left(\frac{x - a_j}{\varepsilon_j}\right) \text{ and } \psi(x) := \sum_{j \geq 1} \psi_j(x). \quad (5.4)$$

Note that, for all $J \geq 1$ and all $x \in [0, a_J]$,

$$\psi(x) = \sum_{1 \leq j \leq J} \psi_0\left(\frac{x - a_j}{\varepsilon_j}\right),$$

which entails that $\psi \in C^\infty([0, 1])$. Define finally $u := e^{i\psi}$.

Step 2. We have $u \in B_{p,\infty}^{1/p}((0, 1))$

For all $j \geq 1$, define $u_j := e^{i\psi_j} - 1$. Note that u_j is supported in $[a_j, b_j]$. As a consequence, the supports of u_j and u_k are disjoint whenever $j \neq k$. We also note the identity $u = 1 + \sum_{j \geq 1} u_j$.

Let $h > 0$. Since $u_0 \in C_c^\infty(\mathbb{R})$, we have

$$\|\tau_h u_0 - u_0\|_{L^p(\mathbb{R})}^p \lesssim h^p \wedge 1.$$

Since, for all $j \geq 1$, we have $u_j(x) = u_0((x - a_j)/\varepsilon_j)$, we find that

$$\|\tau_h u_j - u_j\|_{L^p(\mathbb{R})}^p \lesssim \frac{h^p}{\varepsilon_j^{p-1}} \wedge \varepsilon_j.$$

For $h \in (0, c/2)$, consider the (unique) integer $J \geq 1$ such that $\frac{c}{2^{J+1}} < h \leq \frac{c}{2^J}$.

Then

$$\begin{aligned} \|\tau_h u - u\|_{L^p(\mathbb{R})} &\leq \sum_{j > J} \|\tau_h u_j - u_j\|_{L^p(\mathbb{R})} + \sum_{j \leq J} \|\tau_h u_j - u_j\|_{L^p(\mathbb{R})} \\ &\lesssim \sum_{j > J} (c/2^j)^{1/p} + \sum_{j \leq J} \frac{h}{(c/2^j)^{1-1/p}} \lesssim \frac{1}{2^{J/p}} + 2^{J(1-1/p)} h \lesssim h^{1/p}. \end{aligned} \quad (5.5)$$

It follows from (5.5) that, when $p \in (1, \infty)$,

$$\|u\|_{B_{p,\infty}^{1/p}((0,1))} \lesssim \|u\|_{L^p((0,1))} + \sup_{h < c/2} \frac{\|\tau_h u - u\|_{L^p(\mathbb{R})}}{h^{1/p}} < \infty.$$

When $p = 1$, we argue similarly, using the fact that

$$\|\tau_{2h} u_0 - 2\tau_h u_0 + u_0\|_{L^1(\mathbb{R})} \lesssim h^2 \wedge 1$$

and thus

$$\|\tau_{2h} u_j - 2\tau_h u_j + u_j\|_{L^1(\mathbb{R})} \lesssim \frac{h^2}{\varepsilon_j} \wedge \varepsilon_j. \quad ^{20}$$

Step 3. We have $\psi \notin B_{p,\infty}^{1/p}((0, 1))$. More generally, we have $\psi \notin B_{p,\infty}^{1/p}((a, 1))$, $\forall a \in (0, 1)$

Indeed, define, for all sufficiently small $h > 0$,

$$A(h) := \left\{ j \geq 1; \frac{c}{2^{j-1}} < h < \frac{2c}{(j+1)^2} \right\}.$$

²⁰Alternatively, we could have established first the fact that $u \in B_{1,\infty}^1$ and then use the Gagliardo-Nirenberg type embedding $B_{1,\infty}^1 \cap L^\infty \hookrightarrow B_{p,\infty}^{1/p}$, $1 < p < \infty$, to derive that $u \in B_{p,\infty}^{1/p}$.

For all $j \geq 1$, we have

$$j \in A(h) \iff j \in \left(\log_2 \frac{c}{h} + 1, \sqrt{\frac{2c}{h}} - 1 \right),$$

which shows that

$$\#A(h) \gtrsim \frac{1}{\sqrt{h}}. \quad (5.6)$$

Observe that, for all $j \in A(h)$,

$$a_j - b_{j-1} = \frac{c}{j^2} > \frac{c}{(j+1)^2} > \frac{h}{2}. \quad (5.7)$$

Let $j \in A(h)$ and $x \in M_j := (a_j - h/2, a_j)$. Then, by (5.7), $b_{j-1} < x < a_j$, so that

$$\psi_k(x) = \begin{cases} 0, & \text{if } k \geq j \\ 2\pi, & \text{if } k \leq j-1 \end{cases}. \quad (5.8)$$

Indeed, if $k \geq j$, then $x < a_j < a_k$, while, if $k \leq j-1$,

$$x - a_k > b_{j-1} - a_k > b_k - a_k = \varepsilon_k.$$

Similarly, we have $b_j < x + h < x + 2h < a_{j+1}$, and this implies that

$$\psi_k(x+h) = \psi_k(x+2h) = \begin{cases} 0, & \text{if } k \geq j+1 \\ 2\pi, & \text{if } k \leq j \end{cases}. \quad (5.9)$$

It follows from (5.8) and (5.9) that, for all $x \in M_j$, we have

$$|\psi(x+h) - \psi(x)| = 2\pi \quad (5.10)$$

and

$$|\psi(x+2h) - 2\psi(x+h) + \psi(x)| = 2\pi. \quad (5.11)$$

Thus, when $1 < p < \infty$, (5.10) yields, for sufficiently small $h > 0$,

$$\|\tau_h \psi - \psi\|_{L^p((0,1-h))}^p \gtrsim \sum_{j \in A(h)} |M_j| = \frac{h}{2} \#A(h) \gtrsim \sqrt{h}$$

which shows that $\psi \notin B_{p,\infty}^{1/p}$. The same conclusion holds when $p = 1$ thanks to (5.11).

Finally, since ψ is smooth on $[0,1)$ and does not belong to $B_{p,\infty}^{1/p}((0,1))$, we find that for every $a \in (0,1)$ we have $\psi \notin B_{p,\infty}^{1/p}((a,1))$.

Step 4. u has no phase in $B_{p,\infty}^{1/p}$

Assume by contradiction that $u = e^{i\varphi}$ for some $\varphi \in B_{p,\infty}^{1/p}((0,1))$. Set $\eta := \frac{\varphi - \psi}{2\pi}$, which is an integer-valued function. Lemma 3.16 ensures that η has a finite set \mathcal{J} of jump points in $(0,1)$. If $a = \max \mathcal{J}$, then η is constant on $(a,1)$, and thus up to a constant $\varphi = \psi$ on that interval. This is a contradiction, since by construction for every $a \in (0,1)$ we have $\psi \notin B_{p,\infty}^{1/p}((a,1))$. \square

Let us now turn to the n -dimensional situation.

Lemma 5.2. Let $n \geq 2$ and $1 \leq p < \infty$. Then the lifting problem in $B_{p,\infty}^{1/p}((0,1)^n; \mathbb{S}^1)$ has a negative answer.

Proof. Let ψ be as in the proof of Lemma 5.1. Define, for all $x \in (0,1)^n$,

$$\tilde{\psi}(x) := \psi(x_1) \text{ and } \tilde{u} := e^{i\tilde{\psi}}.$$

We will prove that $\tilde{u} \in B_{p,\infty}^{1/p}((0,1)^n)$ and that \tilde{u} has no $B_{p,\infty}^{1/p}((0,1)^n)$ phase.

To start with, we claim that $\tilde{u} \in B_{p,\infty}^{1/p}((0,1)^n)$. Indeed, if $u := e^{i\psi}$, then one has $u \in B_{p,\infty}^{1/p}((0,1))$. Since $\tilde{u}(x) = u(x_1)$ for all $x \in (0,1)^n$, for all $h > 0$ and all $x \in (0,1-h) \times (0,1)^{n-1}$, $(\tau_{he_1}\tilde{u} - \tilde{u})(x) = (\tau_h u - u)(x_1)$, while $(\tau_{he_j}\tilde{u} - \tilde{u}) = 0$ for all $j \in \llbracket 2, n \rrbracket$. This implies the claim when $1 < p < \infty$. When $p = 1$, we argue similarly, using $(\tau_{2he_1}\tilde{u} - 2\tau_{he_1}\tilde{u} + \tilde{u})(x) = (\tau_{2h}u - 2\tau_h u + u)(x_1)$ and $(\tau_{2he_j}\tilde{u} - 2\tau_{he_j}\tilde{u} + \tilde{u}) = 0$, $\forall j \in \llbracket 2, n \rrbracket$.

Argue by contradiction and assume that $\tilde{u} = e^{i\tilde{\varphi}}$ for some $\tilde{\varphi} \in B_{p,\infty}^{1/p}((0,1)^n)$. Set $\tilde{\eta} := \frac{\tilde{\varphi} - \tilde{\psi}}{2\pi}$, which is an integer-valued function. By Lemma 3.17, we have $\tilde{\eta} \in BV((0,1)^n)$. This leads to a contradiction as explained below.

Let $\delta \in (0, 1/2)$ and $\varepsilon > 0$ such that

$$\|\tilde{\eta}\|_{BV((1-\varepsilon,1) \times (0,1)^{n-1})} \leq \delta. \quad (5.12)$$

Define

$$A' := \left\{ x' \in (0,1)^{n-1}; \|\tilde{\eta}(\cdot, x')\|_{BV((1-\varepsilon,1))} < 2\delta \right\}.$$

The Fubini theorem and (5.12) yield $|A'| \geq 1/2$. Since $\tilde{\eta}$ is integer-valued, we find that $\tilde{\eta}(\cdot, x')$ is constant on $(1-\varepsilon, 1)$, for all $x' \in A'$. It follows that

$$\lim_{h \rightarrow 0} \frac{\|\tau_{he_1}\tilde{\varphi} - \tilde{\varphi}\|_{L^p((1-\varepsilon,1) \times A')}^p}{h} = \lim_{h \rightarrow 0} \frac{\|\tau_{he_1}\tilde{\psi} - \tilde{\psi}\|_{L^p((1-\varepsilon,1) \times A')}^p}{h} = \infty; \quad (5.13)$$

the latter equality follows from the fact that $\psi \notin B_{p,\infty}^{1/p}((1-\varepsilon, 1))$. (5.13) implies that $\tilde{\varphi} \notin B_{p,\infty}^{1/p}((0,1)^n)$. The case where $p = 1$ is similar. \square

6 Open cases

Case 9. Range. $s > 1$, $1 \leq p < \infty$, $p < q < \infty$, $n \geq 3$, and $sp = 2$.

Discussion. This case is complementary to Case 5. In the above range, we conjecture that the conclusion of Case 5 still holds, i.e., that the space $B_{p,q}^s(\Omega; \mathbb{S}^1)$ does have the lifting property. The non-restriction property (Proposition 3.5) prevents us from extending the argument used in Case 5 to Case 9.

Case 10. *Range.* $s = 1$, $1 \leq p < \infty$, $1 \leq q < \infty$, $n \geq 3$, and $2 \leq p < n$.

Or: $s = 1$, $1 \leq p < \infty$, $q = \infty$, $n \geq 3$, and $2 < p \leq n$.

Discussion. When $p = q = 2$, $B_{2,2}^1(\Omega; \mathbb{S}^1) = H^1(\Omega; \mathbb{S}^1)$ does have the lifting property [2, Lemma 1]. The remaining cases are open. The major difficulty arises from the extension of Lemma 7.11 to the range considered in Case 10.

Case 11. *Range.* $0 < s \leq 1$, $1 < p < \infty$, $q = \infty$, $n \geq 3$, and $sp = n$.

Discussion. We do not know whether $B_{p,q}^s(\Omega; \mathbb{S}^1)$ does have the lifting property. The major difficulty stems in the fact that in this range we have $B_{p,\infty}^s \not\subset \text{VMO}$, and thus we are unable to rely on the strategy used in Cases 3 and 4.

Case 12. *Range.* $s = 0$, $1 \leq p < \infty$, $1 \leq q < \infty$ (and arbitrary n).

Discussion. As explained in Remark 4.1, we consider only measurable functions $u : \Omega \rightarrow \mathbb{S}^1$. We let $B_{p,q}^0(\Omega; \mathbb{S}^1) := \{u : \Omega \rightarrow \mathbb{S}^1; u \text{ measurable and } u \in B_{p,q}^0\}$, and for u in this space we are looking for a phase $\varphi \in L_{loc}^1 \cap B_{p,q}^0$.

Note that $B_{p,\infty}^0(\Omega; \mathbb{S}^1)$ does have the lifting property. Indeed, in this case we have $L^\infty \subset B_{p,\infty}^0$ (Lemma 7.3) and then it suffices to argue as in the proof of Remark 4.1. More generally, $B_{p,q}^0(\Omega; \mathbb{S}^1)$ has the lifting property when $L^\infty \hookrightarrow B_{p,q}^0$.²¹ The remaining cases are open.

7 Other results for Besov spaces

The results we state here are valid when Ω is a smooth bounded domain in \mathbb{R}^n , or $(0,1)^n$ or \mathbb{T}^n . However, in the proofs we will consider only one of these sets, the most convenient for the proof.

7.1 Embeddings

Lemma 7.1. Let $0 < s_1 < s_0 < \infty$, $1 \leq p_0 < \infty$, $1 \leq p_1 < \infty$, $1 \leq q_0 \leq \infty$ and $1 \leq q_1 \leq \infty$. Then the following hold.

1. If $q_0 < q_1$, then $B_{p,q_0}^s \hookrightarrow B_{p,q_1}^s$.
2. If $s_0 - n/p_0 = s_1 - n/p_1$, then $B_{p_0,q_0}^{s_0} \hookrightarrow B_{p_1,q_0}^{s_1}$.
3. If $s_0 - n/p_0 > s_1 - n/p_1$, then $B_{p_0,q_0}^{s_0} \hookrightarrow B_{p_1,q_1}^{s_1}$.

²¹ A special case of this is $p = q = 2$, since $B_{2,2}^0 = L^2$. Another special case is $1 < p \leq 2 \leq q$. Indeed, in that case we have $L^\infty \hookrightarrow L^p = F_{p,2}^0 \hookrightarrow B_{p,q}^0$ [36, Section 2.3.5, p. 51], [36, Section 2.3.2, Proposition 2, p. 47].

4. If $B_{p_0, q_0}^{s_0} \hookrightarrow B_{p_1, q_1}^{s_1}$, then $s_0 - n/p_0 \geq s_1 - n/p_1$.

Consequently, when $q_0 \leq q_1$,

$$B_{p_0, q_0}^{s_0} \hookrightarrow B_{p_1, q_1}^{s_1} \iff s_0 - \frac{n}{p_0} \geq s_1 - \frac{n}{p_1}. \quad (7.1)$$

Proof. For item 1, see [36, Section 3.2.4]. For items 2 and 3, see [36, Section 3.3.1] or [32, Theorem 1, p. 82]. Item 4 follows from a scaling argument. And (7.1) is an immediate consequence of items 1–4. \square

For the next result, see e.g. [36, Section 2.7.1, Remark 2, pp. 130-131].

Lemma 7.2. Let $s > 0$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ be such that $sp > n$. Then $B_{p, q}^s(\Omega) \hookrightarrow C^0(\bar{\Omega})$.

Lemma 7.3. Let $s < 0$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then $L^\infty \hookrightarrow B_{p, q}^s$.

Similarly, if $1 \leq p \leq \infty$, then $L^\infty \hookrightarrow B_{p, \infty}^0$.

Proof. We present the argument when $\Omega = \mathbb{T}^n$. Let $f \in L^\infty$, with Fourier coefficients $(a_m)_{m \in \mathbb{Z}^n}$. Consider, as in Definition 2.5, the functions

$$f_j(x) := \sum_{m \in \mathbb{Z}^n} a_m \varphi_j(2\pi m) e^{2i\pi m \cdot x}, \quad \forall j \in \mathbb{N}.$$

By the (periodic version of) the multiplier theorem [36, Section 9.2.2, Theorem, p. 267] we have

$$\|f_j\|_{L^p} \lesssim \|f\|_{L^p}, \quad \forall 1 \leq p \leq \infty, \quad \forall j \in \mathbb{N}. \quad (7.2)$$

We find that $\|f_j\|_{L^p} \lesssim \|f\|_{L^p} \leq \|f\|_{L^\infty}$, and thus (by Definition 2.5, and with the usual modification when $q = \infty$)

$$\|f\|_{B_{p, q}^s} \lesssim \left(\sum_{j \geq 0} 2^{sjq} \right)^{1/q} < \infty.$$

The second part of the lemma follows from a similar argument. The proof is left to the reader. \square

An analogous proof leads to the following result. Details are left to the reader.

Lemma 7.4. Let $s > 0$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then $B_{p, q}^s \hookrightarrow L^p$.

More generally, if $k \in \mathbb{N}$, $s > k$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$, then $B_{p, q}^s \hookrightarrow W^{k, p}$.

Lemma 7.5. Let $0 < s < \infty$, $1 \leq p < \infty$ and $1 \leq q < \infty$ be such that $sp = n$. Then $B_{p, q}^s \hookrightarrow \text{VMO}$.

Same conclusion if $0 < s < \infty$, $1 \leq p < \infty$ and $q = \infty$ are such that $sp > n$.

Proof. Assume first that $q < \infty$. Let $p_1 > \max\{n, p, q\}$ and set $s_1 := n/p_1$. By Lemma 7.1 and the fact that s_1 is not an integer, we have

$$B_{p,q}^s \hookrightarrow B_{p_1,q}^{s_1} \hookrightarrow B_{p_1,p_1}^{s_1} = W^{s_1,p_1}.$$

It then suffices to invoke the embedding

$$W^{s_1,p_1} \hookrightarrow \text{VMO} \text{ when } s_1 p_1 = n \text{ [14, Example 2, p. 210].}$$

The case where $q = \infty$ is obtained via the first part of the proof. Indeed, it suffices to choose $0 < s_1 < \infty$, $1 \leq p_1 < \infty$ and $0 < q_1 < \infty$ such that $s_1 p_1 = n$ and $B_{p,q}^s \hookrightarrow B_{p_1,q_1}^{s_1}$. Such s_1 , p_1 and q_1 do exist, by Lemma 7.1. \square

For the following special case of the Gagliardo-Nirenberg embeddings, see e.g. [32, Remark 1, pp. 39-40].

Lemma 7.6. Let $0 < s < \infty$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, and $0 < \theta < 1$. Then $B_{p,q}^s \cap L^\infty \hookrightarrow B_{p/\theta,q/\theta}^{\theta s}$.

7.2 Poincaré type inequalities

The next Poincaré type inequality for Besov spaces is certainly well-known, but we were unable to find a reference in the literature.

Lemma 7.7. Let $0 < s < 1$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$. Let $|\cdot|_{B_{p,q}^s}$ be as in (2.3). Then we have

$$\left\| f - \int f \right\|_{L^p} \lesssim |f|_{B_{p,q}^s}, \quad \forall f : \Omega \rightarrow \mathbb{R} \text{ measurable function.} \quad (7.3)$$

Proof. By (2.2), we have $\|f\|_{B_{p,q}^s} \sim \|f\|_{L^p} + |f|_{B_{p,q}^s}$. Recall that the embedding $B_{p,q}^s \hookrightarrow L^p$ is compact [35, Theorem 3.8.3, p. 296]. From this we infer that (7.3) holds for every function $f \in B_{p,q}^s$. Indeed, assume by contradiction that this is not the case. Then there exists a sequence of functions $(f_j)_{j \geq 1} \subset B_{p,q}^s$ such that, for every j ,

$$1 = \left\| f_j - \int f_j \right\|_{L^p} \geq j |f_j|_{B_{p,q}^s}.$$

Set $g_j := f_j - \int f_j$. Then, up to a subsequence, we have $g_j \rightarrow g$ in L^p , where $\|g\|_{L^p} = 1$ and $\int g = 0$. We claim that g is constant in Ω (and thus $g = 0$). Indeed, by the Fatou lemma, for every $h \in \mathbb{R}^n$ we have

$$\|\Delta_h g\|_{L^p} \leq \liminf \|\Delta_h g_j\|_{L^p} = \liminf \|\Delta_h f_j\|_{L^p}. \quad (7.4)$$

By (2.3), (7.4) and the Fatou lemma, we have

$$|g|_{B_{p,q}^s} \leq \liminf |g_j|_{B_{p,q}^s} = \liminf |f_j|_{B_{p,q}^s} = 0;$$

thus $g = 0$, as claimed. This contradicts the fact that $\|g\|_{L^p} = 1$.

Let us now establish (7.3) only assuming that $|f|_{B_{p,q}^s} < \infty$. We start by reducing the case where $q = \infty$ to the case where $q < \infty$. This reduction relies on the straightforward estimate

$$|f|_{B_{p,r}^\sigma} \lesssim |f|_{B_{p,\infty}^s}, \quad \forall 0 < \sigma < s, \quad \forall 0 < r < \infty.$$

So let us assume that $q < \infty$. For every integer $k \geq 1$, let $\Phi_k : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$\Phi_k(t) := \begin{cases} t, & \text{if } |t| \leq k \\ -k, & \text{if } t \leq -k \\ k, & \text{if } t \geq k \end{cases}.$$

Clearly, Φ_k is 1-Lipschitz, so that Proposition 2.6 leads to

$$|\Phi_k(f)|_{B_{p,q}^s} \leq |f|_{B_{p,q}^s} \tag{7.5}$$

and (by dominated convergence, using $q < \infty$ and (2.3))

$$\lim_{k \rightarrow \infty} |\Phi_k(f) - f|_{B_{p,q}^s} = 0. \tag{7.6}$$

Since $\Phi_k(f) \in L^\infty(\Omega) \subset L^p(\Omega)$, one has $\Phi_k(f) \in B_{p,q}^s$ for every k . Therefore, (7.3) and (7.5) imply

$$\|\Phi_k(f) - c_k\|_{L^p} \lesssim |\Phi_k(f)|_{B_{p,q}^s} \leq |f|_{B_{p,q}^s} \tag{7.7}$$

with $c_k := \int \Phi_k(f)$. Thanks to (7.6), we may pick up an increasing sequence of integers $(\lambda_k)_{k \geq 1}$ such that, for every k , $|\Phi_{\lambda_{k+1}}(f) - \Phi_{\lambda_k}(f)|_{B_{p,q}^s} \leq 2^{-k}$. Applying (7.3) to $\Phi_{\lambda_{k+1}}(f) - \Phi_{\lambda_k}(f)$, one therefore has

$$\|(\Phi_{\lambda_{k+1}}(f) - c_{\lambda_{k+1}}) - (\Phi_{\lambda_k}(f) - c_{\lambda_k})\|_{L^p} \lesssim |\Phi_{\lambda_{k+1}}(f) - \Phi_{\lambda_k}(f)|_{B_{p,q}^s} \leq 2^{-k},$$

which entails that $\Phi_{\lambda_k}(f) - c_{\lambda_k} \rightarrow g$ in L^p as $k \rightarrow \infty$. Up to a subsequence, one can also assume that $\Phi_{\lambda_k}(f)(x) - c_{\lambda_k} \rightarrow g(x)$ for a.e. $x \in \Omega$. Take any $x \in \Omega$ such that $\Phi_{\lambda_k}(f)(x) - c_{\lambda_k} \rightarrow g(x)$. Since $\Phi_{\lambda_k}(f)(x) \rightarrow f(x)$ as $k \rightarrow \infty$, one obtains

$$\lim_{k \rightarrow \infty} c_{\lambda_k} = c \in \mathbb{C}. \tag{7.8}$$

Finally, (7.7), (7.8) and the Fatou lemma yield $\|f - c\|_{L^p} \lesssim |f|_{B_{p,q}^s}$, from which (7.3) easily follows. \square

We next state and prove a generalization of Lemma 7.7.

Lemma 7.8. Let $0 < s < 1$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, and $\delta \in (0, 1]$. Define

$$|f|_{B_{p,q,\delta}^s} := \left(\int_{|h| \leq \delta} |h|^{-sq} \|\Delta_h f\|_{L^p}^q \frac{dh}{|h|^n} \right)^{1/q} \quad (7.9)$$

when $q < \infty$, with the obvious modifications when $q = \infty$ or \mathbb{R}^n is replaced by Ω . Then we have

$$\left\| f - \int f \right\|_{L^p} \lesssim |f|_{B_{p,q,\delta}^s}, \quad \forall f : \Omega \rightarrow \mathbb{R} \text{ measurable function.} \quad (7.10)$$

Proof. Recall that $\|f\|_{B_{p,q}^s} \sim \|f\|_{L^p} + |f|_{B_{p,q,\delta}^s}$ (Proposition 2.6). We continue as in the proof of Lemma 7.7. \square

We end with an estimate involving derivatives.

Lemma 7.9. Let $s > 0$, $1 < p < \infty$ and $1 \leq q \leq \infty$. Let $f \in \mathcal{D}'(\Omega)$ be such that $\nabla f \in B_{p,q}^{s-1}(\Omega)$. Then $f \in B_{p,q}^s(\Omega)$ and

$$\left\| f - \int f \right\|_{B_{p,q}^s} \lesssim \|\nabla f\|_{B_{p,q}^{s-1}}. \quad (7.11)$$

The above result is well-known, but we were unable to find it in the literature; for the convenience of the reader, we present the short argument when $\Omega = \mathbb{T}^n$.

Proof. We use the notation in Proposition 2.7 and the following result [16, Lemma 2.1.1, p. 16]: we have

$$\|f_j\|_{L^p} \sim 2^{-j} \|\nabla f_j\|_{L^p}, \quad \forall 1 \leq p \leq \infty, \forall j \geq 1. \quad (7.12)$$

By combining (7.12) with Proposition 2.7, we obtain, e.g. when $q < \infty$:

$$\begin{aligned} \|f - a_0\|_{B_{p,q}^s}^q &= \left\| \sum_{j \geq 1} f_j \right\|_{B_{p,q}^s}^q \sim \sum_{j \geq 1} 2^{sjq} \|f_j\|_{L^p}^q \\ &\lesssim \sum_{j \geq 1} 2^{sjq} 2^{-jq} \|\nabla f_j\|_{L^p}^q \sim \|\nabla f\|_{B_{p,q}^{s-1}}^q. \end{aligned} \quad (7.13)$$

In particular, $f \in L^1$ (Lemma 7.4), and thus $a_0 = \int f$. Therefore, (7.13) is equivalent to (7.11). \square

Remark 7.10. With more work, Lemma 7.9 can be extended to the case where $p = 1$. Although this will not be needed here, we sketch below the argument. With the notation in Section 2.3, consider the Littlewood-Paley decomposition $f = \sum f_j$, with $f_j := \sum a_m \varphi_j(2^m x) e^{2i\pi m \cdot x}$. Note that the Littlewood-Paley decomposition of ∇f is simply given by

$$\nabla f = \sum \nabla f_j. \quad (7.14)$$

In the spirit of [16, Lemma 2.1.1, p. 16] (see also [5, Proof of Lemma 1]), one may prove that we have the following analog of (7.12):

$$\|f_j\|_{L^p} \sim 2^{-j} \|\nabla f_j\|_{L^p}, \quad \forall 1 \leq p \leq \infty, \forall j \geq 1. \quad (7.15)$$

Using Definition 2.5, (7.14) and (7.15), we obtain (7.13). We conclude as in the proof of Lemma 7.9.

7.3 Product estimates

Lemma 7.11 below is a variant of [4, Lemma D.2]. Here, Ω is either smooth bounded, or $(0, 1)^n$, or \mathbb{T}^n .

Lemma 7.11. Let $s > 1$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. If $u, v \in B_{p,q}^s \cap L^\infty(\Omega)$, then $u \nabla v \in B_{p,q}^{s-1}$.

Proof. After extension to \mathbb{R}^n and cutoff, we may assume that $u, v \in B_{p,q}^s \cap L^\infty(\mathbb{R}^n)$. It thus suffices to prove that $u, v \in B_{p,q}^s \cap L^\infty(\mathbb{R}^n) \implies u \nabla v \in B_{p,q}^{s-1}(\mathbb{R}^n)$.

In order to prove the above, we argue as follows. Let $u = \sum u_j$ and $v = \sum v_j$ be the Littlewood-Paley decompositions of u and v . Set

$$f^j := \sum_{k \leq j} u_k \nabla v_j + \sum_{k < j} u_j \nabla v_k.$$

Since $\text{supp } \mathcal{F}(u_k \nabla v_j) \subset B(0, 2^{\max\{k,j\}+2})$, we find that $u \nabla v = \sum f^j$ is a Nikolskiĭ decomposition of $u \nabla v$; see Section 2.8. Assume e.g. that $q < \infty$. (The argument for $q = \infty$ is similar.) In view of Proposition 2.14, the conclusion of Lemma 7.11 follows if we prove that

$$\sum 2^{(s-1)jq} \|f^j\|_{L^p}^q < \infty. \quad (7.16)$$

In order to prove (7.16), we rely on the elementary estimates [16, Lemma 2.1.1, p. 16], [4, formulas (D.8), (D.9), p. 71]

$$\left\| \sum_{k \leq j} u_k \right\|_{L^\infty} \lesssim \|u\|_{L^\infty}, \quad \forall j \geq 0, \quad (7.17)$$

$$\left\| \sum_{k < j} \nabla v_k \right\|_{L^\infty} \lesssim 2^j \|v\|_{L^\infty}, \quad \forall j \geq 0, \quad (7.18)$$

and

$$\|\nabla v_j\|_{L^p} \lesssim 2^j \|v_j\|_{L^p}, \quad \forall j \geq 0. \quad (7.19)$$

By combining (7.17)-(7.19), we obtain

$$\begin{aligned}
\sum 2^{(s-1)jq} \|f^j\|_{L^p}^q &\lesssim \sum 2^{(s-1)jq} \left(\left\| \sum_{k \leq j} u_k \right\|_{L^\infty}^q \|\nabla v_j\|_{L^p}^q + \left\| \sum_{k < j} \nabla v_k \right\|_{L^\infty}^q \|u_j\|_{L^p}^q \right) \\
&\lesssim \|u\|_{L^\infty}^q \sum 2^{sjq} \|v_j\|_{L^p}^q + \|v\|_{L^\infty}^q \sum 2^{sjq} \|u_j\|_{L^p}^q \\
&\lesssim \|u\|_{L^\infty}^q \|v\|_{B_{p,q}^s}^q + \|v\|_{L^\infty}^q \|u\|_{B_{p,q}^s}^q,
\end{aligned}$$

and thus (7.16) holds. \square

7.4 Superposition operators

In this section, we examine the mapping properties of the operator

$$T_\Phi, \psi \xrightarrow{T_\Phi} \Phi \circ \psi.$$

We work in Ω smooth bounded, or $(0,1)^n$, or \mathbb{T}^n .

The next result is classical and straightforward; see e.g. [32, Section 5.3.6, Theorem 1].

Lemma 7.12. Let $0 < s < 1$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$. Let $\Phi: \mathbb{R}^k \rightarrow \mathbb{R}^l$ be a Lipschitz function. Then T_Φ maps $B_{p,q}^s(\Omega; \mathbb{R}^k)$ into $B_{p,q}^s(\Omega; \mathbb{R}^l)$.

Special case: $\psi \mapsto e^{i\psi}$ maps $B_{p,q}^s(\Omega; \mathbb{R})$ into $B_{p,q}^s(\Omega; \mathbb{S}^1)$.

In addition, when $q < \infty$, T_Φ is continuous.

For the next result, see [32, Section 5.3.4, Theorem 2, p. 325].

Lemma 7.13. Let $s > 0$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Let $\Phi \in C^\infty(\mathbb{R}^k; \mathbb{R}^l)$. Then T_Φ maps $(B_{p,q}^s \cap L^\infty)(\Omega; \mathbb{R}^k)$ into $(B_{p,q}^s \cap L^\infty)(\Omega; \mathbb{R}^l)$.

Special case: $\psi \mapsto e^{i\psi}$ maps $(B_{p,q}^s \cap L^\infty)(\Omega; \mathbb{R})$ into $(B_{p,q}^s \cap L^\infty)(\Omega; \mathbb{S}^1)$.

7.5 Integer-valued functions in sums of Besov spaces

The next result is a cousin of [4, Appendix B],²² but the argument in [4] does not seem to apply in our situation. Lemma 7.14 can be obtained from the results in [8], but we present below a simpler direct argument.

Lemma 7.14. Let $s > 0$, $1 \leq p < \infty$ and $1 \leq q < \infty$ be such that $sp \geq 1$. Then the functions in $B_{p,q}^s(\Omega; \mathbb{Z})$ are constant.

Same result when $s > 0$, $1 \leq p < \infty$, $q = \infty$ and $sp > 1$.

The same conclusion holds for functions in $\sum_{j=1}^k B_{p_j, q_j}^{s_j}(\Omega; \mathbb{Z})$, provided we have for all $j \in \llbracket 1, k \rrbracket$: either $s_j p_j = 1$ and $1 \leq q_j < \infty$, or $s_j p_j > 1$ and $1 \leq q_j \leq \infty$.

²² The context there is the one of the Sobolev spaces.

Proof. The case where $n = 1$ is simple. Indeed, by Lemma 7.5 we have $B_{p,q}^s \hookrightarrow \text{VMO}$ (and similarly $\sum_{j=1}^k B_{p_j,q_j}^{s_j} \hookrightarrow \text{VMO}$). The conclusion follows from the fact that $\text{VMO}((0,1);\mathbb{Z})$ functions are constant [14, Step 5, p. 229].

We next turn to the general case. Let $f = \sum_{j=1}^k f_j$, with $f_j \in B_{p_j,q_j}^{s_j}(\Omega;\mathbb{Z})$, $\forall j \in \llbracket 1, k \rrbracket$. In view of the conclusion, we may assume that $\Omega = (0,1)^n$. By the Sobolev embeddings, we may assume that for all j we have $s_j p_j = 1$ (and thus either $1 < p_j < \infty$ and $s_j = 1/p_j$, or $p_j = 1$ and $s_j = 1$) and $1 \leq q_j < \infty$. Let, as in Lemma 3.3, $A \subset (0,1)^{n-1}$ be a set of full measure such that (3.2) holds with $M = 2$. The proof of the lemma relies on the following key implication:

$$[x_1 + \dots + x_k \in \mathbb{Z}, 1 \leq p_1, \dots, p_k < \infty] \implies |x_1 + \dots + x_k| \lesssim |x_1|^{p_1} + \dots + |x_k|^{p_k}. \quad (7.20)$$

This leads to the following consequence: if $g := g_1 + \dots + g_k$ is integer-valued, then

$$\|\Delta_h^2 g\|_{L^1} \lesssim \|\Delta_h^2 g_1\|_{L^{p_1}}^{p_1} + \dots + \|\Delta_h^2 g_k\|_{L^{p_k}}^{p_k}. \quad (7.21)$$

By combining (3.2) with (7.21), we find that

$$\lim_{l \rightarrow \infty} \frac{\left\| \Delta_{t_l}^2 f(x', \cdot) \right\|_{L^1((0,1))}}{t_l} = 0, \quad \forall x' \in A, \text{ for some sequence } t_l \rightarrow 0. \quad (7.22)$$

By Lemma 7.15 below, we find that $f(x', \cdot)$ is constant, for every $x' \in A$. By a permutation of the coordinates, we find that for every $i \in \llbracket 1, n \rrbracket$, the function

$$t \mapsto f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n) \text{ is constant, } \forall i \in \llbracket 1, n \rrbracket, \text{ a.e. } \hat{x}_i \in (0,1)^{n-1}; \quad (7.23)$$

here, $\hat{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in (0,1)^{n-1}$.

We next invoke the fact that every measurable function satisfying (7.23) is constant [12, Lemma 2]. \square

Lemma 7.15. Let $g \in L^1((0,1);\mathbb{Z})$ be such that, for some sequence $t_l \rightarrow 0$, we have

$$\lim_{l \rightarrow \infty} \frac{\left\| \Delta_{t_l}^2 g \right\|_{L^1((0,1))}}{t_l} = 0. \quad (7.24)$$

Then g is constant.

Proof. In order to explain the main idea, let us first assume that $g = \mathbb{1}_B$ for some measurable set $B \subset (0,1)$. Let $h \in (0,1)$. If $x \in B$ and $x + 2h \notin B$, then $\Delta_h^2 g(x)$ is odd, and thus $|\Delta_h^2 g(x)| \geq 1$. The same holds if $x \notin B$ and $x + 2h \in B$. On the other hand, we have $|\Delta_{2h} g(x)| \leq 1$, with equality only when either $x \in B$ and $x + 2h \notin B$, or $x \notin B$ and $x + 2h \in B$. By the preceding, we obtain the inequality

$$|\Delta_h^2 g(x)| \geq |\Delta_{2h} g(x)|, \quad \forall x, \forall h. \quad (7.25)$$

Using (7.24) and (7.25), we obtain

$$g' = \lim_{l \rightarrow \infty} \frac{\Delta_{2t_l} g}{2t_l} = 0. \tag{7.26}^{23}$$

Thus either $g = 0$, or $g = 1$.

We next turn to the general case. Consider some $k \in \mathbb{Z}$ such that the measure of the set $g^{-1}(\{k\})$ is positive. We may assume that $k = 0$, and we will prove that $g = 0$. For this purpose, we set $B := g^{-1}(2\mathbb{Z})$, and we let $\bar{g} := \mathbb{1}_B$. Arguing as above, we have $|\Delta_h^2 g(x)| \geq |\Delta_{2h} \bar{g}(x)|$, $\forall x, \forall h$, and thus $\bar{g} = 0$. We find that g takes only even values. We next consider the integer-valued map $g/2$. By the above, $g/2$ takes only even values, and so on. We find that $g = 0$. \square

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²³ In (7.26), the first limit is in \mathcal{D}' , the second one in L^1 .

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