Boundary singularities of solutions to semilinear fractional equations

Phuoc-Tai Nguyen, Laurent Veron

BOUNDARY SINGULARITIES OF SOLUTIONS TO SEMILINEAR FRACTIONAL EQUATIONS

PHUOC-TAI NGUYEN AND LAURENT VÉRON

Abstract. We prove the existence of a solution of $(-\Delta)^s u + f(u) = 0$ in a smooth bounded domain Ω with a prescribed boundary value μ in the class of Radon measures for a large class of continuous functions f satisfying a weak singularity condition expressed under an integral form. We study the existence of a boundary trace for positive moderate solutions. In the particular case where $f(u) = u^p$ and μ is a Dirac mass, we prove the existence of several critical exponents p. We also prove the existence of several types of separable solutions of the problem $(-\Delta)^s u + u^p = 0$ in $\mathbb{R}^N_+$.

2010 Mathematics Subject Classification. 35J62, 35J66, 35J67.
Key words: s-harmonic functions, semilinear fractional equations, boundary trace.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary and $s \in (0,1)$. Define the s-fractional Laplacian as

$$(-\Delta)^s u(x) := \lim_{\varepsilon \to 0} (-\Delta)^s_\varepsilon u(x)$$

where

$$(-\Delta)^s_\varepsilon u(x) := a_{N,s} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad a_{N,s} := \frac{\Gamma(N/2+s)}{\pi^{N/2}\Gamma(2-s)} s(1-s).$$

We denote by G^Ω_s and M^Ω_s the Green kernel and the Martin kernel of $(-\Delta)^s$ in Ω respectively. Denote by $\mathbb{G}^\Omega_s$ and $\mathbb{M}^\Omega_s$ the Green operator and the Martin operator (see section 2 for more details). Further, for $\phi \geq 0$, denote by $\mathfrak{M}(\Omega, \phi)$ the space of Radon measures τ on Ω satisfying $\int_\Omega \phi d|\tau| < \infty$ and by $\mathfrak{M}(\partial\Omega)$ the space of bounded Radon measures on $\partial\Omega$. Let $\rho(x)$ be the distance from x to $\partial\Omega$. For $\beta > 0$, set

$$\Omega_\beta := \{x \in \Omega : \rho(x) < \beta\},\ D_\beta := \{x \in \Omega : \rho(x) > \beta\},\ \Sigma_\beta := \{x \in \Omega : \rho(x) = \beta\}.$$

Definition 1.1. *We say that a function $u \in L^1_{loc}(\Omega)$ possesses a s-boundary trace on $\partial\Omega$ if there exists a measure $\mu \in \mathfrak{M}(\partial\Omega)$ such that*

$$\lim_{\beta\to 0} \beta^{1-s} \int_{\Sigma_\beta} |u - \mathbb{M}^\Omega_s[\mu]| dS = 0. \tag{1.1}$$

The s-boundary trace of u is denoted by $\mathrm{tr}_s(u)$.

Let $\tau \in \mathfrak{M}(\Omega, \rho^s)$, $\mu \in \mathfrak{M}(\partial\Omega)$ and $f \in C(\mathbb{R})$ be a nondecreasing function with $f(0) = 0$. In this paper, we study boundary singularity problem for semilinear fractional equation of the form

$$\begin{cases} (-\Delta)^s u + f(u) = \tau & \text{in } \Omega \\ \qquad\qquad \mathrm{tr}_s(u) = \mu & \\ \qquad\qquad u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.2}$$

We denote by $\mathbb{X}_s(\Omega)$ the space of test functions ξ satisfying

(i) $\mathrm{supp}\,(\xi) \subset \bar{\Omega}$,

(ii) $(-\Delta)^s\xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^s\xi(x)| \le C$ for some $C > 0$,

(iii) there exists $\varphi \in L^1(\Omega, \rho^s dx)$ and $\epsilon_0 > 0$ such that $|(-\Delta)^s_\epsilon \xi| \le \varphi$ a.e. in Ω, for all $\epsilon \in (0, \epsilon_0]$.

Definition 1.2. *Let $\tau \in \mathfrak{M}(\Omega, \rho^s)$ and $\mu \in \mathfrak{M}(\partial\Omega)$. A function u is called a weak solution of (1.2) if $u \in L^1(\Omega)$, $f(u) \in L^1(\Omega, \rho^s)$ and*

$$\int_\Omega (u(-\Delta)^s\xi + f(u)\xi)\, dx = \int_\Omega \xi d\tau + \int_\Omega \mathbb{M}^\Omega_s[\mu](-\Delta)^s\xi\, dx, \quad \forall \xi \in \mathbb{X}_s(\Omega). \tag{1.3}$$

The boundary value problem with measure data for semilinear elliptic equations

$$\begin{cases} -\Delta u + f(u) = 0 & \text{in } \Omega \\ \qquad\quad u = \mu & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

was first studied by A. Gmira and L. Véron in [18] and then the typical model, i.e. problem (1.4) with $f(u) = u^p$ ($p > 1$), has been intensively by numerous authours (see [20–24] and references therein). They proved that if f is a continuous, nondecreasing function satisfying

$$\int_1^\infty [f(t) - f(-t)]s^{-1-p_c} dt < \infty, \tag{1.5}$$

where $p_c := \frac{N+1}{N-1}$, then problem (1.4) admits a unique weak solution. In particular, when $f(u) = u^p$ with $1 < p < p_c$ and $\mu = k\delta_0$ with $0 \in \partial\Omega$ and $k > 0$, there exists a unique solution u_k of (1.4). It was showed [20, 24] that the sequence $\{u_k\}$ is increasing and converges to a function u_∞ which is a solution of the equation in (1.4).

To our knowledge, few papers concerning boundary singularity problem for nonlinear fractional elliptic equation have been published in the literature. The earlies works in this direction are the papers [10, 17] by P. Felmer et al. which deal with the existence, nonexistence and asymptotic behavior of large solutions for equations involving fractional Laplacian. Afterwards, N. Abatangelo [1] presented a suitable setting for the study of fractional Laplacian equations in a measure framework and provided a fairly comprehensive description of large solutions which improve the results in [10, 17]. Recently, H. Chen et al. [9] investigated semilinear elliptic equations involving measures concentrated on the boundary by employing approximate method.

In the present paper, we aim to establish the existence and uniqueness of weak solutions of (1.2). To this end, we develop a theory for linear equations associated to (1.2)

$$\begin{cases} (-\Delta)^s u = \tau & \text{in } \Omega \\ \mathrm{tr}_s(u) = \mu & \\ \qquad u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.6}$$

Existence and uniqueness result for (1.6) is stated in the following proposition.

Proposition A. *Assume $s \in (\frac{1}{2}, 1)$. Let $\tau \in \mathfrak{M}(\Omega, \rho^s)$ and $\mu \in \mathfrak{M}(\partial\Omega)$. Then problem (1.6) admits a unique weak solution. The solution is given by*

$$u = \mathbb{G}_s^\Omega[\tau] + \mathbb{M}_s^\Omega[\mu]. \tag{1.7}$$

Moreover, there exists a positive constant $c = c(N, s, \Omega)$ such that

$$\|u\|_{L^1(\Omega)} \le c(\|\tau\|_{\mathfrak{M}(\Omega,\rho^s)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}). \tag{1.8}$$

This proposition allows to study semilinear equation (1.2). We first deal with the case of L^1 data.

Theorem B. *Assume $s \in (\frac{1}{2}, 1)$. Let $f \in C(\mathbb{R})$ be a nondecreasing function satisfying $tf(t) \ge 0$ for every $t \in \mathbb{R}$.*

I. Existence and uniqueness. *For every $\tau \in L^1(\Omega, \rho^s)$ and $\mu \in L^1(\partial\Omega)$, problem (1.2) admits a unique weak solution u. Moreover,*

$$u = \mathbb{G}_s^\Omega[\tau - f(u)] + \mathbb{M}_s^\Omega[\mu] \quad \text{in } \Omega, \tag{1.9}$$

$$-\mathbb{G}_s^\Omega[\tau^-] - \mathbb{M}_s^\Omega[\mu^-] \le u \le \mathbb{G}_s^\Omega[\tau^+] + \mathbb{M}_s^\Omega[\mu^+] \quad \text{in } \Omega. \tag{1.10}$$

II. Monotonicity. *The mapping $(\tau, \mu) \mapsto u$ is nondecreasing.*

Remark. The restriction $s \in (\frac{1}{2}, 1)$ is due to the fact that in this range of s, $\operatorname{tr}_s(\mathbb{G}[\tau]) = 0$ for every $\tau \in \mathfrak{M}(\Omega, \rho^s)$ (see Proposition 2.11). We conjecture that this still holds if $s \in (0, \frac{1}{2}]$.

We reveal that, in measures framework, because of the the interplay between the nonlocal operator $(-\Delta)^s$ and the nonlinearity term $f(u)$, the analysis is much more intricate and there are 3 critical exponents

$$p_1^* := \frac{N+2s}{N}, \quad p_2^* := \frac{N+s}{N-s}, \quad p_3^* := \frac{N}{N-2s}.$$

This yields substantial new difficulties and leads to disclose new types of results. The new aspects are both on the technical side and on the one of the new phenomena observed.

Theorem C. *Assume $s \in (\frac{1}{2}, 1)$. Let $f \in C(\mathbb{R})$ be a nondecreasing function, $tf(t) \ge 0$ for every $t \in \mathbb{R}$ and*

$$\int_1^\infty [f(s) - f(-s)] s^{-1-p_2^*} ds < \infty. \tag{1.11}$$

I. Existence and Uniqueness. *For every $\tau \in \mathfrak{M}(\Omega, \rho^s)$ and $\mu \in \mathfrak{M}(\partial\Omega)$ there exists a unique weak solution of (1.2). This solution satisfies (1.9) and (1.10). Moreover, the mapping $(\tau, \mu) \mapsto u$ is nondecreasing.*

II. Stability. *Assume $\{\tau_n\} \subset \mathfrak{M}(\Omega, \rho^s)$ converges weakly to $\tau \in \mathfrak{M}(\Omega, \rho^s)$ and $\{\mu_n\} \subset \mathfrak{M}(\partial\Omega)$ converges weakly to $\mu \in \mathfrak{M}(\partial\Omega)$. Let u and u_n be the unique weak solutions of (1.2) with data (τ, μ) and (τ_n, μ_n) respectively. Then $u_n \to u$ in $L^1(\Omega)$ and $f(u_n) \to f(u)$ in $L^p(\Omega, \rho^s)$.*

If μ is a Dirac mass concentrated at a point on $\partial\Omega$, we obtain the behavior of the solution near that boundary point.

Theorem D. *Assume $s \in (\frac{1}{2}, 1)$ and f is a continuous nondecreasing function on $\mathbb{R}$ satisfying $f(0) = 0$ and (1.11). Let $z \in \partial\Omega$ and $k > 0$. Let $u_{z,k}^\Omega$ be the unique weak solution of*

$$\begin{cases} (-\Delta)^s u + f(u) = 0 & \text{in } \Omega \\ \operatorname{tr}_s(u) = k\delta_z & \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.12}$$

Then

$$\lim_{\Omega \ni x \to z} \frac{u_{z,k}^\Omega(x)}{M_s^\Omega(x, z)} = k. \tag{1.13}$$

We next assume that $0 \in \partial\Omega$. Let $0 < p < p_2^*$ and denote by u_k^Ω the unique weak solution of

$$\begin{cases} (-\Delta)^s u + u^p &= 0 \qquad \text{in } \Omega \\ \operatorname{tr}_s(u) &= k\delta_0 \\ u &= 0 \qquad \text{in } \Omega^c. \end{cases} \tag{1.14}$$

By Theorem C, $u_k^\Omega \le kM_s^\Omega(\cdot,0)$ and $k \mapsto u_k^\Omega$ is increasing. Therefore, it is natural to investigate $\lim_{k\to\infty} u_k^\Omega$. This is accomplishable thanks to the study of separable solutions of

$$\begin{cases} (-\Delta)^s u + u^p &= 0 \qquad \text{in } \mathbb{R}^N_+ \\ u &= 0 \qquad \text{in } \overline{\mathbb{R}^N_-}. \end{cases} \tag{1.15}$$

when $p > 1$. Denote by

$$S^{N-1} := \left\{\sigma = (\cos\phi\,\sigma', \sin\phi) : \sigma' \in S^{N-2}, -\tfrac{\pi}{2} \le \phi \le \tfrac{\pi}{2}\right\}$$

the unit sphere in $\mathbb{R}^N$ and by $S^{N-1}_+ := S^{N-1} \cap \mathbb{R}^N_+$ the upper hemisphere. Writing separable solution under the form $u(x) = u(r,\sigma) = r^{-\frac{2s}{p-1}}\omega(\sigma)$, with $r > 0$ and $\sigma \in S^{N-1}_+$, we obtain that ω satisfies

$$\begin{cases} \mathcal{A}_s\omega - \mathcal{L}_{s,\frac{2s}{p-1}}\omega + \omega^p = 0 \qquad \text{in } S^{N-1}_+ \\ \omega = 0 \qquad \text{in } \overline{S^{N-1}_-}, \end{cases} \tag{1.16}$$

where $\mathcal{A}_s$ is a nonlocal operator naturally associated to the s-fractional Laplace-Beltrami operator and $\mathcal{L}_{s,\frac{2s}{p-1}}$ is a linear integral operator with kernel. In analyzing the spectral properties of $\mathcal{A}_s$ we prove

Theorem E. *Let $N \ge 2$, $s \in (0,1)$ and $p > p_1^*$.*
I- If $p_2^* \le p < p_3^*$ there exists no positive solution of (1.16) belonging to $W_0^{s,2}(S^{N-1}_+)$.
II- If $p_1^* < p < p_2^*$ there exists a unique positive solution $\omega^* \in W_0^{s,2}(S^{N-1}_+)$ of (1.16).

As a consequence of this result we obtain the behavior of u_k^Ω when $k \to \infty$.

Theorem F *Assume $s \in (\frac{1}{2}, 1)$. Let $\Omega = \mathbb{R}^N_+$ or Ω be a bounded domain with C^2 boundary containing 0.*
I- If $p \in (p_1^*, p_2^*)$ then $u_\infty^\Omega := \lim_{k\to 0} u_k^\Omega$ is a positive solution of

$$\begin{cases} (-\Delta)^s u + u^p &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{in } \Omega^c. \end{cases} \tag{1.17}$$

(i) If $\Omega = \mathbb{R}^N_+$ then

$$u_\infty^{\mathbb{R}^N_+}(x) = |x|^{-\frac{2s}{p-1}}\omega^*(\sigma), \quad \text{with } \sigma = \frac{x}{|x|} \qquad \forall x \in \mathbb{R}^N_+.$$

(ii) If Ω is a bounded C^2 domain with $\partial\Omega$ containing 0 then

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} = \sigma \in S^{N-1}_+}} |x|^{\frac{2s}{p-1}} u_\infty^\Omega(x) = \omega^*(\sigma), \tag{1.18}$$

locally uniformly on S^{N-1}_+. In particular, there exists a positive constant c depending on N, s, p and the C^2 norm of $\partial\Omega$ such that

$$c^{-1}\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}} \le u_\infty^\Omega(x) \le c\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}} \quad \forall x \in \Omega. \tag{1.19}$$

II- Assume $p \in (0, p_1^*]$. Then $\lim_{k\to\infty} u_k^\Omega = \infty$ in Ω.

The main ingredients of the present study: estimates on Green kernel and Martin kernel, theory for linear fractional equations in connection with the notion $s-$boundary trace as mentioned above, similarity transformation and the study of equation (1.16).

The paper is organized as follows. In Section 2, we present important properties of s-boundary trace and prove Proposition A. Theorems B,C,D and F are obtained in Section 3. Finally, in Appendix, we discuss separable solutions of (1.15) and demonstrate Theorem E.

2. Linear problems

Throughout the present paper, we denote by $c, c', c_1, c_2, C, ...$ positive constants that may vary from line to line. If necessary, the dependence of these constants will be made precise.

2.1. **s-harmonic functions.** We first recall the definition of *s-harmonic functions* (see [3, page 46], [4, page 230], [6, page 20]). Denote by (X_t, P^x) the standard rotation invariant $2s$-stable Lévy process in $\mathbb{R}^N$ (i.e. homogeneous with independent increments) with characteristic function

$$E^0 e^{i\xi X_t} = e^{-t|\xi|^{2s}}, \quad \xi \in \mathbb{R}^N, t \geq 0.$$

Denote by E^x the expectation with respect to the distribution P^x of the process starting from $x \in \mathbb{R}^N$. We assume that sample paths of X_t are right-continuous and have left-hand limits a.s. The process (X_t) is Markov with transition probabilities given by

$$P_t(x, A) = P^x(X_t \in A) = \mu_t(A - x),$$

where μ_t is the one-dimensional distribution of X_t with respect to P^0. It is well known that $(-\Delta)^s$ is the generator of the process (X_t, P^x).

For each Borel set $D \subset \mathbb{R}^N$, set $t_D := \inf\{t \geq 0 : X_t \notin D\}$, i.e. t_D is the first exit time from D. If D is bounded then $t_D < \infty$ a.s. Denote

$$E^x u(X_{t_D}) = E^x\{u(X_{t_D}) : t_D < \infty\}.$$

Definition 2.1. *Let u be a Borel measurable function in $\mathbb{R}^N$. We say that u is s-harmonic in Ω if for every bounded open set $D \Subset \Omega$,*

$$u(x) = E^x u(X_{t_D}), \quad x \in D.$$

We say that u is singular s-harmonic in Ω if u is s-harmonic and $u = 0$ in Ω^c.

Put

$$\mathcal{D}_s := \left\{ u : \mathbb{R}^N \mapsto \mathbb{R} : \text{Borel measurable such that} \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}} \right\}.$$

The following result follows from [5, Corollary 3.10 and Theorem 3.12] and [6, page 20].

Proposition 2.2. *Let $u \in \mathcal{D}_s$.*

(i) *u is s-harmonic in Ω if and only if $(-\Delta)^s u = 0$ in Ω in the sense of distributions.*

(ii) *u is singular s-harmonic in Ω if and only if u is s-harmonic in Ω and $u = 0$ in Ω^c.*

2.2. **Green kernel, Poisson kernel and Martin kernel.** In what follows the notation $f \sim g$ means: there exists a positive constant c such that $c^{-1} f < g < cf$ in the domain of the two functions or in a specified subset of this domain.

Denote by G_s^Ω the Green kernel of $(-\Delta)^s$ in Ω. Namely, for every $y \in \Omega$,

$$\begin{cases} (-\Delta)^s G_s^\Omega(\cdot, y) = \delta_y & \text{in } \Omega \\ G_s^\Omega(\cdot, y) = 0 & \text{in } \Omega^c, \end{cases}$$

where δ_y is the Dirac mass at y. By combining [1, Lemma 3.2] and [14, Corollary 1.3]), we get

Proposition 2.3. *(i) G_s^Ω is in continuous, symmetric, positive in $\{(x, y) \in \Omega \times \Omega : x \neq y\}$ and $G_s^\Omega(x, y) = 0$ if x or y belongs to Ω^c.*

(ii) $(-\Delta)^s G_s^\Omega(x, \cdot) \in L^1(\Omega^c)$ for every $x \in \Omega$ and $(-\Delta)^s G_s^\Omega(x, y) \leq 0$ for every $x \in \Omega$ and $y \in \Omega^c$.

(iii) There holds

$$G_s^\Omega(x, y) \sim \min\left\{ |x-y|^{2s-N}, \rho(x)^s \rho(y)^s |x-y|^{-N} \right\} \quad \forall (x, y) \in \Omega \times \Omega, x \neq y. \tag{2.1}$$

The similarity constant in the above estimate depends only on Ω and s.

Denote by $\mathbb{G}_s^\Omega$ the associated Green operator

$$\mathbb{G}_s^\Omega[\tau] = \int_\Omega G_s^\Omega(\cdot, y) d\tau(y) \qquad \tau \in \mathfrak{M}(\Omega, \rho^s).$$

Put

$$k_{s,\gamma} := \begin{cases} p_3^* & \text{if } \gamma \in [0, \frac{N-2s}{N} s) \\ \frac{N+s}{N-2s+\gamma} & \text{if } \gamma \in [\frac{N-2s}{N} s, s]. \end{cases} \tag{2.2}$$

H. Chen and L. Véron obtained the following estimate for Green operator [12, Proposition 2.3 and Proposition 2.6].

Lemma 2.4. *Assume $\gamma \in [0, s]$ and $k_{s,\gamma}$ be as in (2.2).*

(i) *There exists a constant $c = c(N, s, \gamma, \Omega) > 0$ such that*

$$\left\| \mathbb{G}_s^\Omega[\tau] \right\|_{M^{k_{s,\gamma}}(\Omega, \rho^s)} \le c \left\| \tau \right\|_{\mathfrak{M}(\Omega, \rho^\gamma)} \quad \forall \tau \in \mathfrak{M}(\Omega, \rho^\gamma). \tag{2.3}$$

(ii) *Assume $\{\tau_n\} \subset \mathfrak{M}(\Omega, \rho^\gamma)$ converges weakly to $\tau \in \mathfrak{M}(\Omega, \rho^\gamma)$. Then $\mathbb{G}_s^\Omega[\tau_n] \to \mathbb{G}_s^\Omega[\tau]$ in $L^p(\Omega, \rho^s)$ for any $p \in [1, k_{s,\gamma})$.*

Let P_s^Ω be the Poisson kernel of $(-\Delta)^s$ defined by (see [7])

$$P_s^\Omega(x, y) := -a_{N,-s} \int_\Omega \frac{G_s^\Omega(x, z)}{|z - y|^{N+2s}} dz, \quad \forall x \in \Omega, y \in \overline{\Omega}^c.$$

The relation between P_s^Ω and G_s^Ω is expressed in [1, Proposition 2] (see also [14, Theorem 1.4], [4, Lemma 2], [14, Theorem 1.5]).

Proposition 2.5. *(i) $P_s^\Omega(x, y) = -(-\Delta)^s G_s^\Omega(x, y)$ for every $x \in \Omega$ and $y \in \overline{\Omega}^c$. Moreover, P_s^Ω is continuous in $\Omega \times \overline{\Omega}^c$.*

(ii) There holds

$$P_s^\Omega(x, y) \sim \frac{\rho(x)^s}{\rho(y)^s (1 + \rho(y))^s} \frac{1}{|x - y|^N}, \quad \forall x \in \Omega, y \in \overline{\Omega}^c. \tag{2.4}$$

The similarity constant in the above estimate depends only on Ω and s.

Denote by $\mathbb{P}_s^\Omega$ the corresponding operator defined by

$$\mathbb{P}_s^\Omega[\nu](x) = \int_{\overline{\Omega}^c} P_s^\Omega(x, y) d\nu(y), \quad \nu \in \mathfrak{M}(\overline{\Omega}^c).$$

Fix a reference point $x_0 \in \Omega$ and denote by M_s^Ω the Martin kernel of $(-\Delta)^s$ in Ω, i.e.

$$M_s^\Omega(x, z) = \lim_{\Omega \ni y \to z} \frac{G_s^\Omega(x, y)}{G_s^\Omega(x_0, y)}, \quad \forall x \in \mathbb{R}^N, z \in \partial\Omega.$$

By [15, Theorem 3.6], the Martin boundary of Ω can be identified with the Euclidean boundary $\partial\Omega$. Denote by $\mathbb{M}_s^\Omega$ the associated Martin operator

$$\mathbb{M}_s^\Omega[\mu] = \int_{\partial\Omega} M_s^\Omega(\cdot, z) d\mu(z), \quad \mu \in \mathfrak{M}(\partial\Omega).$$

The next result [4, 15] is important in the study of s-harmonic functions, which give a unique presentation of $s-$harmonic functions in terms of Martin kernel.

Proposition 2.6. (i) *The mapping $(x, z) \mapsto M_s^\Omega(x, z)$ is continuous on $\Omega \times \partial\Omega$. For any $z \in \partial\Omega$, the function $M_s^\Omega(., z)$ is singular s-harmonic in Ω with $M_s^\Omega(x_0, z) = 1$. Moreover, if $z, z' \in \partial\Omega$, $z \ne z'$ then $\lim_{x \to z'} M_s^\Omega(x, z) = 0$.*

(ii) *There holds*

$$M_s^\Omega(x, z) \sim \rho(x)^s |x - z|^{-N} \quad \forall x \in \Omega, z \in \partial\Omega. \tag{2.5}$$

The similarity constant in the above estimate depends only on Ω and s.

(iii) *For every $\mu \in \mathfrak{M}^+(\partial\Omega)$ the function $\mathbb{M}_s^\Omega[\mu]$ is singular s-harmonic in Ω with $u(x_0) = \mu(\mathbb{R}^N)$. Conversely, if u is a nonnegative singular s-harmonic function in Ω then there exists a unique $\mu \in \mathfrak{M}^+(\partial\Omega)$ such that $u = \mathbb{M}_s^\Omega[\mu]$ in $\mathbb{R}^N$.*

(iv) *If u is a nonnegative s-harmonic function in Ω then there exists a unique $\mu \in \mathfrak{M}^+(\partial\Omega)$ such that*

$$u(x) = \mathbb{M}_s^\Omega[\mu](x) + \mathbb{P}_s^\Omega[u](x) \quad \forall x \in \Omega.$$

Lemma 2.7. (i) *There exists a constant $c = c(N, \mu, \gamma, \Omega)$ such that*

$$\left\|\mathbb{M}_s^\Omega[\mu]\right\|_{M^{\frac{N+\gamma}{N-s}}(\Omega,\rho^\gamma)} \le c\,\|\mu\|_{\mathfrak{M}(\partial\Omega)}, \quad \forall \mu \in \mathfrak{M}(\partial\Omega),\ \gamma > -s. \tag{2.6}$$

(ii) *If $\{\mu_n\} \subset \mathfrak{M}(\partial\Omega)$ converges weakly to $\mu \in \mathfrak{M}(\partial\Omega)$ then $\mathbb{M}_s^\Omega[\mu_n] \to \mathbb{M}_s^\Omega[\mu]$ in $L^p(\Omega, \rho^\gamma)$ for every $1 \le p < \frac{N+\gamma}{N-s}$.*

Proof. (i) By using (2.5) and a similar argument as in the proof of [2, Theorem 2.5], we obtain (2.6).

(ii) By combining the fact that $M_s^\Omega(x, z) = 0$ for every $x \in \Omega^c$, $z \in \partial\Omega$ and Proposition 2.6 (i) we deduce that for every $x \in \mathbb{R}^N$, $M_s^\Omega(x, \cdot) \in C(\partial\Omega)$. It follows that $\mathbb{M}_s^\Omega[\mu_n] \to \mathbb{M}_s^\Omega[\mu]$ everywhere in Ω. Due to (i) and the Holder inequality, we deduce that, for any $1 \le p \le \frac{N+\gamma}{N-s}$, $\{\mathbb{M}_s^\Omega[\mu_n]\}$ is uniformly integrable with respect to $\rho^\gamma dx$. By invoking Vitali's theorem, we obtain the convergence in $L^p(\Omega, \rho^\gamma)$. □

2.3. **Boundary trace.** We recall that, for $\beta > 0$,

$$\Omega_\beta := \{x \in \Omega : \rho(x) < \beta\},\ D_\beta := \{x \in \Omega : \rho(x) > \beta\},\ \Sigma_\beta := \{x \in \Omega : \rho(x) = \beta\}.$$

The following geometric property of C^2 domains can be found in [24].

Proposition 2.8. *There exists $\beta_0 > 0$ such that*

(i) For every point $x \in \overline{\Omega}_{\beta_0}$, there exists a unique point $z_x \in \partial\Omega$ such that $|x - z_x| = \rho(x)$. This implies $x = z_x - \rho(x)\mathbf{n}_{z_x}$.

(ii) The mappings $x \mapsto \rho(x)$ and $x \mapsto z_x$ belong to $C^2(\overline{\Omega}_{\beta_0})$ and $C^1(\overline{\Omega}_{\beta_0})$ respectively. Furthermore, $\lim_{x\to z_x} \nabla\rho(x) = -\mathbf{n}_{z_x}$.

Proposition 2.9. *Assume $s \in (0, 1)$. Then there exist positive constants $c = c(N, \Omega, s)$ such that, for every $\beta \in (0, \beta_0)$,*

$$c^{-1} \le \beta^{1-s} \int_{\Sigma_\beta} M_s^\Omega(x, y) dS(x) \le c \quad \forall y \in \partial\Omega. \tag{2.7}$$

Proof. For $r_0 > 0$ fixed, by (2.5),

$$\int_{\Sigma_\beta \setminus B_{r_0}(y)} M_s^\Omega(x, y) dS(x) \le c_1 \beta^s, \tag{2.8}$$

which implies

$$\lim_{\beta\to 0} \int_{\Sigma_\beta \setminus B_{r_0}(y)} M_s^\Omega(x, y) dS(x) = 0 \quad \forall y \in \partial\Omega. \tag{2.9}$$

Note that for r_0 fixed, the rate of convergence is independent of y.

In order to prove (2.7) we may assume that the coordinates are placed so that $y = 0$ and the tangent hyperplane to $\partial\Omega$ at 0 is $x_N = 0$ with the x_N axis pointing into the domain. For $x \in \mathbb{R}^N$ put $x' = (x_1, \cdots, x_{N-1})$. Pick $r_0 \in (0, \beta_0)$ sufficiently small (depending only on the C^2 characteristic of Ω) so that

$$\frac{1}{2}(|x'|^2 + \rho(x)^2) \le |x|^2 \quad \forall x \in \Omega \cap B_{r_0}(0).$$

Hence if $x \in \Sigma_\beta \cap B_{r_0}(0)$ then $\frac{1}{4}(|x'| + \beta) \le |x|$. Combining this inequality and (2.5) leads to

$$\begin{aligned}
\int_{\Sigma_\beta \cap B_{r_0}(0)} M_s^\Omega(x, 0) dS(x) &\le c_2 \beta^s \int_{\Sigma_{\beta,0}} (|x'| + \beta)^{-N} dS(x) \\
&\le c_2 \beta^s \int_{|x'|<r_0} (|x'| + \beta)^{-N} dx' \\
&= c_3 \beta^{s-1}.
\end{aligned}$$

Therefore, for $\beta < r_0$,

$$\beta^{1-s}\int_{\Sigma_\beta\cap B_{r_0}(0)} M_s^\Omega(x,0)dS(x)\le c_4. \tag{2.10}$$

By combining estimates (2.8) and (2.10), we obtain the second estimate in (2.7). The first estimate in (2.7) follows from (2.5). □

As a consequence, we get the following estimates.

Corollary 2.10. *Assume $s\in(0,1)$. For every $\mu\in\mathfrak{M}^+(\partial\Omega)$ and $\beta\in(0,\beta_0)$, there holds*

$$c^{-1}\left\|\mu\right\|_{\mathfrak{M}(\partial\Omega)}\le\beta^{1-s}\int_{\Sigma_\beta}\mathbb{M}_s^\Omega[\mu]dS\le c\left\|\mu\right\|_{\mathfrak{M}(\partial\Omega)}, \tag{2.11}$$

with c is as in (2.7).

Proposition 2.11. *Assume $s\in(\frac12,1)$. Then there exists a constant $c=c(s,N,\Omega)$ such that for any $\tau\in\mathfrak{M}(\Omega,\rho^s)$ and any $0<\beta<\beta_0$,*

$$\beta^{1-s}\int_{\Sigma_\beta}\mathbb{G}_s^\Omega[\tau]dS\le c\int_\Omega\rho^s d|\tau|. \tag{2.12}$$

Moreover,

$$\lim_{\beta\to0}\beta^{1-s}\int_{\Sigma_\beta}\mathbb{G}_s^\Omega[\tau]dS=0. \tag{2.13}$$

Proof. Without loss of generality, we may assume that $\tau>0$. Denote $v:=\mathbb{G}_s^\Omega[\tau]$. We first prove (2.12). By Fubini's theorem and (2.5),

$$\begin{aligned}\int_{\Sigma_\beta}v(x)dS(x)&\le c_5\Big(\int_\Omega\int_{\Sigma_\beta\cap B_{\frac\beta2}(y)}|x-y|^{2s-N}dS(x)\,d\tau(y)\\&\quad+\beta^s\int_\Omega\int_{\Sigma_\beta\setminus B_{\frac\beta2}(y)}|x-y|^{-N}dS(x)\,\rho(y)^s d\tau(y)\Big)\\&:=I_{1,\beta}+I_{2,\beta}.\end{aligned}$$

Note that, if $x\in\Sigma_\beta\cap B_{\frac\beta2}(y)$ then $\beta/2\le\rho(y)\le3\beta/2$. Therefore

$$\begin{aligned}\beta^{1-s}I_{1,\beta}&\le c_6\beta^{1-2s}\int_{\Sigma_\beta\cap B_{\frac\beta2}(y)}|x-y|^{2s-N}dS(x)\int_\Omega\rho(y)^s\,d\tau(y)\\&\le c_6\beta^{1-2s}\int_0^{\beta/2}r^{2s-N}r^{N-2}dr\int_\Omega\rho(y)^s\,d\tau(y)\\&\le c_7\int_\Omega\rho(y)^s\,d\tau(y),\end{aligned}$$

where the last inequality holds since $s>\frac12$. On the other hands, we have

$$I_{2,\beta}\le c_7\beta^s\int_{\beta/2}^\infty r^{-N}r^{N-2}dr\int_\Omega\rho(y)^s\,d\tau(y)=c_8\beta^{s-1}\int_\Omega\rho(y)^s\,d\tau(y).$$

Combining the above estimates, we obtain (2.12).

Next we demonstrate (2.13). Given $\epsilon\in(0,\|\tau\|_{\mathfrak{M}(\Omega,\rho^s)})$ and $\beta_1\in(0,\beta_0)$ put $\tau_1=\tau\chi_{\bar D_{\beta_1}}$ and $\tau_2=\tau\chi_{\Omega_{\beta_1}}$. We can choose $\beta_1=\beta_1(\epsilon)$ such that

$$\int_{\Omega_{\beta_1}}\rho(y)^s\,d\tau(y)\le\epsilon. \tag{2.14}$$

Thus the choice of β_1 depends on the rate at which $\int_{\Omega_\beta}\rho^s\,d\tau$ tends to zero as $\beta\to0$.

Put $v_i:=\mathbb{G}_s^\Omega[\tau_i]$. Then, for $0<\beta<\beta_1/2$,

$$\int_{\Sigma_\beta}v_1(x)\,dS(x)\le c_9\beta^s\beta_1^{-N}\int_\Omega\rho(y)^s d\tau_1(y),$$

which yields

$$\lim_{\beta\to 0}\beta^{1-s}\int_{\Sigma_\beta} v_1(x)\,dS(x)=0. \tag{2.15}$$

On the other hand, due to (2.12),

$$\beta^{1-s}\int_{\Sigma_\beta} v_2\,dS\le c_{10}\int_\Omega \rho^s d\tau_2\le c_{11}\epsilon\quad\forall\beta<\beta_0. \tag{2.16}$$

From (2.15) and (2.16), we obtain (2.13). □

Lemma 2.12. *Assume $s\in(\frac12,1)$. Let $u,w\in\mathcal{D}_s$ be two nonnegative functions satisfying*

$$\begin{cases}(-\Delta)^s u\le 0\le(-\Delta)^s w & \text{in }\Omega,\\ u=0 & \text{in }\Omega^c.\end{cases} \tag{2.17}$$

If $u\le w$ in $\mathbb{R}^N$ then $(-\Delta)^s u\in\mathfrak{M}(\Omega,\rho^s)$ and there exists a measure $\mu\in\mathfrak{M}^+(\partial\Omega)$ such that

$$\lim_{\beta\to 0}\beta^{1-s}\int_{\Sigma_\beta}|u-\mathbb{M}_s^\Omega[\mu]|dS=0. \tag{2.18}$$

Moreover, if $\mu=0$ then $u=0$.

Proof. By the assumption, there exists a nonnegative Radon measure τ on Ω such that $(-\Delta)^s u=-\tau$.

We first prove that $\tau\in\mathfrak{M}^+(\Omega,\rho^s)$. Define

$$\tilde{M}_s^\Omega(x,z):=\lim_{\Omega\ni y\to z}\frac{G_s^\Omega(x,y)}{\rho(y)^s}. \tag{2.19}$$

By [1, page 5547], there holds is a positive constant $c=c(\Omega,s)$ such that

$$\tilde{M}_s^\Omega(x,z)\sim\rho(x)^s|x-z|^{-N},\quad\forall x\in\Omega,z\in\partial\Omega, \tag{2.20}$$

where the similarity constant depends only on Ω and s. This follows

$$\begin{aligned}c_{12}^{-1}<c_{13}^{-1}\int_{\partial\Omega}\rho(x)|x-z|^{-N}dS(z)&\\ \le\rho(x)^{1-s}\int_{\partial\Omega}\tilde{M}_s^\Omega(x,z)dS(z)&\\ \le c_{13}\int_{\partial\Omega}\rho(x)|x-z|^{-N}dS(z)<c_{12}\quad\forall x\in\Omega.&\end{aligned} \tag{2.21}$$

We define

$$\mathbb{E}_s^\Omega[u](z):=\lim_{\Omega\ni x\to z}\frac{u(x)}{\int_{\partial\Omega}\tilde{M}_s^\Omega(x,y)dS(y)}\quad z\in\partial\Omega.$$

For any $\beta\in(0,\beta_0)$, denote by τ_β the restriction of τ to D_β and by v_β the restriction of u on Σ_β. By [1, Theorem 1.4], there exists a unique solution v_β of

$$\begin{cases}(-\Delta)^s v_\beta=-\tau_\beta & \text{in } D_\beta\\ \mathbb{E}_s^{D_\beta}[v_\beta]=0 & \text{on }\Sigma_\beta\\ v_\beta=u|_{D_\beta^c} & \text{in } D_\beta^c.\end{cases}$$

Moreover, the solution can be written as

$$v_\beta+\mathbb{G}_s^{D_\beta}[\tau_\beta]=\mathbb{P}_s^{D_\beta}[u|_{D_\beta^c}]\quad\text{in } D_\beta. \tag{2.22}$$

By the maximum principle [1, Lemma 3.9], $v_\beta=u$ and $\mathbb{P}_s^{D_\beta}[u|_{D_\beta^c}]\le w$ a.e. in $\mathbb{R}^N$. This, together with (2.22), implies that $\mathbb{G}_s^{D_\beta}[\tau_\beta]\le w$ in D_β. Letting $\beta\to 0$ yields $\mathbb{G}_s^\Omega[\tau]<\infty$. For fixed $x_0\in\Omega$, by (2.1), $G_s^\Omega(x_0,y)>c\rho(y)^s$ for every $y\in\Omega$. Hence the finiteness of $\mathbb{G}_s^\Omega[\tau]$ implies that $\tau\in\mathfrak{M}^+(\Omega,\rho^s)$.

We next show that there exists a measure $\mu \in \mathfrak{M}^+(\partial\Omega)$ such that (2.18) holds. Put $v = u + \mathbb{G}_s^\Omega[\tau]$ then v is a nonnegative singular s-harmonic in Ω due to the fact that $\mathbb{G}_s^\Omega[\tau] = 0$ in Ω^c. By Proposition 2.2 and Proposition 2.6 (iii), there exists $\mu \in \mathfrak{M}^+(\partial\Omega)$ such that $v = \mathbb{M}_s^\Omega[\mu]$ in $\mathbb{R}^N$. By Proposition 2.11, we obtain (2.18). If $\mu = 0$ then $v = 0$ and thus $u = 0$. □

Definition 2.13. *A function u possesses a s-boundary trace on $\partial\Omega$ if there exists a measure $\mu \in \mathfrak{M}(\partial\Omega)$ such that*

$$\lim_{\beta\to 0} \beta^{1-s} \int_{\Sigma_\beta} |u - \mathbb{M}_s^\Omega[\mu]| dS = 0. \tag{2.23}$$

The s-boundary trace of u is denoted noted by $\mathrm{tr}_s(u)$.

Remark (i) The notation of s-boundary trace is well defined. Indeed, suppose that μ and μ' satisfy (2.23). Put $v = (\mathbb{M}_s^\Omega[\mu - \mu'])^+$. Clearly $v \le \mathbb{M}_s^\Omega[|\mu| + |\mu'|]$, $v = 0$ in Ω^c and $\lim_{\beta\to 0} \beta^{1-s} \int_{\Sigma_\beta} |v| dS = 0$. By Kato's inequality [8, Theorem 1.2], $(-\Delta)^s v \le 0$ in Ω. Therefore, we deduce $v \equiv 0$ from Lemma 2.12. This implies $\mathbb{M}_s^\Omega[\mu - \mu'] \le 0$. By permuting the role of μ and μ', we obtain $\mathbb{M}_s^\Omega[\mu - \mu'] \ge 0$. Thus $\mu = \mu'$.

(ii) It is clear that for every $\mu \in \mathfrak{M}(\partial\Omega)$, $\mathrm{tr}_s(\mathbb{M}_s^\Omega[\mu]) = \mu$. Moreover, if $s > \frac{1}{2}$, by Proposition 2.11, for every $\tau \in \mathfrak{M}(\Omega, \rho^s)$, $\mathrm{tr}_s(\mathbb{G}_s^\Omega[\tau]) = 0$.

2.4. Weak solutions of linear problems.

Definition 2.14. *Let $\tau \in \mathfrak{M}(\Omega)$ and $\mu \in \mathfrak{M}(\partial\Omega)$. A function u is called a weak solution of (1.6) if $u \in L^1(\Omega)$ and*

$$\int_\Omega u(-\Delta)^s \xi \, dx = \int_\Omega \xi d\tau + \int_\Omega \mathbb{M}_s^\Omega[\mu](-\Delta)^s \xi \, dx, \quad \forall \xi \in \mathbb{X}_s(\Omega). \tag{2.24}$$

Proof of Proposition A. The uniqueness follows from [12, Proposition 2.4]. Let u be as in (1.7). By [12],

$$\int_\Omega (u - \mathbb{M}_s^\Omega[\mu])(-\Delta)^s \xi \, dx = \int_\Omega \mathbb{G}_s^\Omega[\tau](-\Delta)^s \xi \, dx = \int_\Omega \xi d\tau \quad \forall \xi \in \mathbb{X}_s(\Omega).$$

This implies (2.24) and therefore u is the unique solution of (1.6). Since $s \in (\frac{1}{2}, 1)$, by Proposition 2.11, $\mathrm{tr}_s(u) = \mathrm{tr}_s(\mathbb{M}_s^\Omega[\mu]) = \mu$. Finally, estimate (1.8) follows from Lemma 2.4 and Lemma 2.7. □

3. Nonlinear problems

In this section, we study the nonlinear problem (1.2). The definition of weak solutions of (1.2) is given in Definition 1.2.

3.1. Subcritical absorption. Proof of Theorem B.

Monotonicity. Let $\tau, \tau' \in L^1(\Omega, \rho^s)$, $\mu, \mu' \in L^1(\partial\Omega)$ and u and u' be the weak solutions of (1.2) with data (τ, μ) and (τ', μ') respectively. We will show that if $\tau \le \tau'$ and $\mu \le \mu'$ then $u \le u'$ in Ω. Indeed, put $v := (u - u')^+$, it is sufficient to prove that $v \equiv 0$. Since (1.9) holds, it follows

$$|u| \le \mathbb{G}_s^\Omega[|\tau| + |f(u)|] + \mathbb{M}_s^\Omega[|\mu|] \quad \text{in } \Omega.$$

Similarly

$$|u'| \le \mathbb{G}_s^\Omega[|\tau'| + |f(u')|] + \mathbb{M}_s^\Omega[|\mu'|] \quad \text{in } \Omega.$$

Therefore

$$0 \le v \le |u| + |u'| \le \mathbb{G}_s^\Omega[|\tau| + |\tau'| + |f(u)| + |f(u')|] + \mathbb{M}_s^\Omega[|\mu| + |\mu'|] := w.$$

By Kato inequality, the assumption $\tau \le \tau'$ and the monotonicity of f, we obtain

$$(-\Delta)^s v \le \mathrm{sign}^+(u - u')(\tau - \tau') - \mathrm{sign}^+(u - u')(f(u) - f(u')) \le 0.$$

Therefore

$$(-\Delta)^s v \le 0 \le (-\Delta)^s w \quad \text{in } \Omega.$$

Since $\mu \le \mu'$, it follows that $\mathrm{tr}_s(v) = 0$. By Lemma 2.12, $v = 0$ and thus $u \le u'$.

EXISTENCE.

Step 1: Assume that $\tau \in L^\infty(\Omega)$ and $\mu \in L^\infty(\partial\Omega)$.

Put $\hat f(t) := f(t + \mathbb{M}_s^\Omega[\mu]) - f(\mathbb{M}_s^\Omega[\mu])$ and $\hat\tau := \tau - f(\mathbb{M}_s^\Omega[\mu])$. Then $\hat f$ is nondecreasing and $t\hat f(t) \geq 0$ for every $t \in \mathbb{R}$ and $\hat\tau \in L^1(\Omega, \rho^s)$. Consider the problem

$$\begin{cases} (-\Delta)^s v + \hat f(v) = \hat\tau & \text{in } \Omega \\ v = 0 & \text{in } \Omega^c. \end{cases} \tag{3.1}$$

By [11, Proposition 3.1] there exists a unique weak solution v of (3.1). It means that $v \in L^1(\Omega)$, $\hat f(v) \in L^1(\Omega, \rho^s)$ and

$$\int_\Omega (v(-\Delta)^s\xi + \hat f(v)\xi)\, dx = \int_\Omega \xi\hat\tau dx, \quad \forall \xi \in \mathbb{X}_s(\Omega). \tag{3.2}$$

Put $u := v + \mathbb{M}_s^\Omega[\mu]$ then $u \in L^1(\Omega)$ and $f(u) \in L^1(\Omega, \rho^s)$. By (3.2) u satisfies (1.3).

Step 2: Assume that $0 \leq \tau \in L^1(\Omega, \rho^s)$ and $0 \leq \mu \in L^1(\partial\Omega)$.

Let $\{\tau_n\} \subset C^1(\overline\Omega)$ be a nondecreasing sequence converging to τ in $L^1(\Omega, \rho^s)$ and $\{\mu_n\} \subset C^1(\partial\Omega)$ be a nondecreasing sequence converging to μ in $L^1(\partial\Omega)$. Then $\{\mathbb{M}_s^\Omega[\mu_n]\}$ is nondecreasing and by Lemma 2.7 (ii) it converges to $\mathbb{M}_s^\Omega[\mu]$ a.e. in Ω and in $L^p(\Omega, \rho^s)$ for every $1 \leq p < p_2^*$. Let u_n be the unique solution of (1.2) with τ and μ replaced by τ_n and μ_n respectively. By step 1 and the monotonicity of f, we derive that $\{u_n\}$ and $\{f(u_n)\}$ are nondecreasing. Moreover

$$\int_\Omega (u_n(-\Delta)^s\xi + f(u_n)\xi)\, dx = \int_\Omega \xi d\tau_n + \int_\Omega \mathbb{M}_s^\Omega[\mu_n](-\Delta)^s\xi\, dx \qquad \forall \xi \in \mathbb{X}_s(\Omega). \tag{3.3}$$

Let $\eta \in C(\overline\Omega)$ be the solution of

$$\begin{cases} (-\Delta)^s\eta = 1 & \text{in } \Omega \\ \eta = 0 & \text{in } \Omega^c, \end{cases} \tag{3.4}$$

then $c^{-1}\rho^s < \eta < c\rho^s$ in Ω for some $c > 1$. By choosing $\xi = \eta$ in (3.3), we get

$$\begin{aligned} \|u_n\|_{L^1(\Omega)} + \|f(u_n)\|_{L^1(\Omega,\rho^s)} &\leq c(\|\tau_n\|_{L^1(\Omega,\rho^s)} + \|\mu_n\|_{L^1(\partial\Omega)}) \\ &\leq c'(\|\tau\|_{L^1(\Omega,\rho^s)} + \|\mu\|_{L^1(\partial\Omega)}). \end{aligned} \tag{3.5}$$

Hence $\{u_n\}$ and $\{f(u_n)\}$ are uniformly bounded in $L^1(\Omega)$ and $L^1(\Omega, \rho^s)$ respectively. By the monotone convergence theorem, there exists $u \in L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and $f(u_n) \to f(u)$ in $L^1(\Omega, \rho^s)$. By letting $n \to \infty$ in (3.3), we deduce that u satisfies (1.3), namely u is a weak solution of (1.2).

The uniqueness follows from the monotonicity.

Step 3: Assume that $\tau \in L^1(\Omega, \rho^s)$ and $\mu \in L^1(\partial\Omega)$.

Let $\{\tau_n\} \subset C^1(\overline\Omega)$ be a sequence such that $\{\tau_n^+\}$ and $\{\tau_n^-\}$ are nondecreasing and $\tau_n^\pm \to \tau^\pm$ in $L^1(\Omega, \rho^s)$. Let $\{\mu_n\} \subset C^1(\partial\Omega)$ be a sequence such that $\{\mu_n^+\}$ and $\{\mu_n^-\}$ are nondecreasing and $\mu_n^\pm \to \mu^\pm$ in $L^1(\partial\Omega)$. Let u_n be the unique weak solution of (1.2) with data (τ_n, μ_n), then

$$u_n = \mathbb{G}_s^\Omega[\tau_n - f(u_n)] + \mathbb{M}_s^\Omega[\mu_n]. \tag{3.6}$$

Let $w_{1,n}$ and $w_{2,n}$ be the unique weak solutions of (1.2) with data (τ_n^+, μ_n^+) and $(-\tau_n^-, -\mu_n^-)$ respectively. Then

$$\|w_{i,n}\|_{L^1(\Omega)} + \|f(w_{i,n})\|_{L^1(\Omega,\rho^s)} \leq c'(\|\tau\|_{L^1(\Omega,\rho^s)} + \|\mu\|_{L^1(\partial\Omega)}), \quad i = 1, 2. \tag{3.7}$$

Moreover, for any $n \in \mathbb{N}$, $w_{2,n} \leq 0 \leq w_{1,n}$ and

$$-\mathbb{G}_s^\Omega[\tau_n^-] - \mathbb{M}_s^\Omega[\mu_n^-] \leq w_{2,n} \leq u_n \leq w_{1,n} \leq \mathbb{G}_s^\Omega[\tau_n^+] + \mathbb{M}_s^\Omega[\mu_n^+]. \tag{3.8}$$

It follows that

$$|u_n| \leq w_{1,n} - w_{2,n} \quad \text{and} |f(u_n)| \leq f(w_{1,n}) - f(w_{2,n}). \tag{3.9}$$

This, together with (3.7), implies

$$\|u_n\|_{L^1(\Omega)} + \|f(u_n)\|_{L^1(\Omega,\rho^s)} \leq c''(\|\tau\|_{L^1(\Omega,\rho^s)} + \|\mu\|_{L^1(\partial\Omega)}). \tag{3.10}$$

Put $v_n := \mathbb{G}_s^\Omega[\tau_n - f(u_n)]$. By (3.10), the sequence $\{\tau_n - f(u_n\}$ is uniformly bounded in $L^1(\Omega,\rho^s)$. Hence by [12, Proposition 2.6], the sequence $\{v_n\}$ is relatively compact in $L^q(\Omega)$ for $1 \le q < \frac{N}{N-s}$. Consequently, up to a subsequence, $\{v_n\}$ converges in $L^q(\Omega)$ and a.e. in Ω to a function v. On the other hand, by Lemma 2.7 ii), up to a subsequence, $\{\mathbb{M}_s^\Omega[\mu_n]\}$ converges in $L^q(\Omega,\rho^s)$ for $1 \le q < p_2^*$ and a.e. in Ω to $\mathbb{M}_s^\Omega[\mu]$. Due to (3.6), we deduce that $\{u_n\}$ converges a.e. in Ω to $u = v + \mathbb{M}_s^\Omega[\mu]$. Since f is continuous, $\{f(u_n)\}$ converges a.e. in Ω to $f(u)$.

By step 2, the sequences $\{w_{1,n}\}$, $\{f(w_{1,n})\}$, $\{-w_{2,n}\}$ and $\{-f(w_{2,n})\}$ are increasing and converge to w_1 in $L^1(\Omega)$, $f(w_1)$ in $L^1(\Omega,\rho^s)$, $-w_2$ in $L^1(\Omega)$ and $-f(w_2)$ in $L^1(\Omega,\rho^s)$ respectively. In light of (3.9) and the generalized dominated convergence theorem, we obtain that $\{u_n\}$ and $\{f(u_n)\}$ converge to u and $f(u)$ in $L^1(\Omega)$ and $L^1(\Omega,\rho^s)$ respectively. By passing to the limit in (3.3), we derive that u satisfies (1.3).

The uniqueness follows from the monotonicity. □

Define

$$C(\overline{\Omega},\rho^{-s}) := \{\zeta \in C(\overline{\Omega}) : \rho^{-s}\zeta \in C(\overline{\Omega})\}.$$

This space is endowed with the norm

$$\|\zeta\|_{C(\overline{\Omega},\rho^{-s})} = \left\|\rho^{-s}\zeta\right\|_{C(\overline{\Omega})}.$$

We say that a sequence $\{\tau_n\} \subset \mathfrak{M}(\Omega,\rho^s)$ converges weakly to a measure $\tau \in \mathfrak{M}(\Omega,\rho^s)$ if

$$\lim_{n\to\infty}\int_\Omega \zeta d\tau_n = \int_\Omega \zeta d\tau \quad \forall \zeta \in C(\overline{\Omega},\rho^{-s}).$$

Proof of Theorem C.

Monotonicity. The monotonicity can be proved by using a similar argument as in the proof of Theorem B.

Existence. Let $\{\tau_n\} \subset C^1(\Omega)$ and $\{\mu_n\} \subset C^1(\partial\Omega)$ such that $\tau_n^\pm \to \tau^\pm$ weakly and $\mu_n^\pm \to \mu^\pm$ weakly. Then there is a positive constant c independent of n such that

$$\|\tau_n\|_{\mathfrak{M}(\Omega,\rho^s)} \le c\,\|\tau\|_{\mathfrak{M}(\Omega,\rho^s)} \quad \text{and} \quad \|\mu_n\|_{\mathfrak{M}(\partial\Omega)} \le c\,\|\mu\|_{\mathfrak{M}(\partial\Omega)}. \tag{3.11}$$

Let u_n, $w_{1,n}$ and $w_{2,n}$ as in the proof of Theorem B. Then

$$|u_n| \le \max(w_{1,n}, -w_{2,n}) \le \mathbb{G}_s^\Omega[|\tau_n|] + \mathbb{M}_s^\Omega[|\mu_n|]. \tag{3.12}$$

This, together with (2.3), (2.6) and (3.11), implies that

$$\|u_n\|_{M^{p_2^*}(\Omega,\rho^s)} \le c(\|\tau_n\|_{\mathfrak{M}(\Omega,\rho^s)} + \|\mu_n\|_{\mathfrak{M}(\partial\Omega)}) \le c'(\|\tau\|_{\mathfrak{M}(\Omega,\rho^s)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}). \tag{3.13}$$

We have

$$\begin{aligned}
&\int_\Omega (w_{1,n}(-\Delta)^s\xi + f(w_{1,n})\xi)\,dx = \int_\Omega \xi d\tau_n^+ + \int_\Omega \mathbb{M}_s^\Omega[\mu_n^+](-\Delta)^s\xi\,dx,\\
&\int_\Omega (w_{2,n}(-\Delta)^s\xi + f(w_{2,n})\xi)\,dx = -\int_\Omega \xi d\tau_n^- - \int_\Omega \mathbb{M}_s^\Omega[\mu_n^-](-\Delta)^s\xi\,dx \quad \forall \xi \in \mathbb{X}_s(\Omega).
\end{aligned} \tag{3.14}$$

From this it follows

$$\int_\Omega [(w_{1,n} - w_{2,n}) + (f(w_{1,n}) - f(w_{2,n})\eta]\,dx = \int_\Omega \eta\, d|\tau_n| + \int_\Omega \mathbb{M}_s^\Omega[|\mu_n|]\,dx. \tag{3.15}$$

We infer from (3.9) and the estimate $c^{-1}\rho^s \le \eta \le c\rho^s$ that

$$\begin{aligned}
\|u_n\|_{L^1(\Omega)} + \|f(u_n)\|_{L^1(\Omega;\rho^s)} &\le c(\|\tau_n\|_{L^1(\Omega,\rho^s)} + \|\mu_n\|_{\mathfrak{M}(\partial\Omega)})\\
&\le c'(\|\tau\|_{\mathfrak{M}(\Omega,\rho^s)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}).
\end{aligned} \tag{3.16}$$

This implies that $\{u_n\}$ and $\{f(u_n)\}$ are uniformly bounded in $L^1(\Omega)$ and $L^1(\Omega,\rho^s)$ respectively. By a similar argument as in step 3 of the proof of Theorem B, we deduce that, up to a subsequence, $\{u_n\}$ converges a.e. in Ω to a function u and $\{f(u_n)\}$ converges a.e. in Ω to $f(u)$. By Hölder inequality, we infer that $\{u_n\}$ is uniformly integrable in $L^1(\Omega)$.

Next we prove that $\{f \circ u_n\}$ is uniformly integrable in $L^1(\Omega, \rho^s)$. Define $\tilde{f}(s) := f(|s|) - f(-|s|)$, $s \in \mathbb{R}$. Then $\tilde{f}$ is nondecreasing in $\mathbb{R}$ and $|f(s)| \le \tilde{f}(s)$ for every $s \in \mathbb{R}$. For $\ell > 0$ and $n \in \mathbb{N}$, set

$$A_n(\ell) := \{x \in \Omega : |u_n(x)| > \ell\}, \qquad a_n(\ell) := \int_{A_n(\ell)} \rho^s \, dx.$$

We take an arbitrary Borel set $D \subset \Omega$ and estimate

$$\begin{aligned} \int_D |f(u_n)|\rho^s dx &= \int_{D\cap A_n(\ell)} |f(u_n)|\rho^s dx + \int_{D\setminus A_n(\ell)} |f(u_n)|\rho^s dx \\ &\le \int_{A_n(\ell)} \tilde{f}(u_n)\rho^s dx + \tilde{f}(\ell)\int_D \rho^s dx. \end{aligned} \tag{3.17}$$

On one hand, we have

$$\int_{A_n(\ell)} \tilde{f}(u_n)\rho^s dx = a_n(\ell)\tilde{f}(\ell) + \int_\ell^\infty a_n(s) d\tilde{f}(s).$$

From (3.13), we infer $a_n(s) \le \tilde{c}\, s^{-p_2^*}$ where $\tilde{c}$ is a positive constant independent of n. Hence, for any $l > \ell$,

$$\begin{aligned} a_n(\ell)\tilde{f}(\ell) + \int_\ell^l a_n(s) d\tilde{f}(s) &\le \tilde{c}\,\ell^{-p_2^*}\tilde{f}(\ell) + \tilde{c}\int_\ell^l s^{-p_2^*} d\tilde{f}(s) \\ &\le \tilde{c}\, l^{-p_2^*}\tilde{f}(l) + \frac{\tilde{c}}{p_2^*+1}\int_\ell^l s^{-1-p_2^*}\tilde{f}(s) ds. \end{aligned} \tag{3.18}$$

By assumption (1.11), there exists a sequence $\{l_k\}$ such that $l_k \to \infty$ and $l_k^{-p_2^*}\tilde{f}(l_k) \to 0$ as $k \to \infty$. Taking $l = l_k$ in (3.18) and then letting $k \to \infty$, we obtain

$$a_n(\ell)\tilde{f}(\ell) + \int_\ell^\infty a_n(s) d\tilde{f}(s) \le \frac{\tilde{c}}{p_2^*+1}\int_\ell^\infty s^{-1-p_2^*}\tilde{f}(s) ds. \tag{3.19}$$

From assumption (1.11), we see that the right hand-side of (3.19) tends to 0 as $\ell \to \infty$. Therefore, for any $\epsilon > 0$, one can choose $\ell > 0$ such that the right hand-side of (3.19) is smaller than $\epsilon/2$. Fix such ℓ, one then can choose $\delta > 0$ small such that if $\int_D \rho^s dx < \delta$ then $\tilde{f}(\ell)\int_D \rho^s dx < \epsilon/2$. Therefore, from (3.17), we derive

$$\int_D \rho^s dx < \delta \Longrightarrow \int_D |f(u_n)|\rho^s dx < \epsilon.$$

This means $\{f \circ u_n\}$ is uniformly integrable in $L^1(\Omega, \rho^s)$.

By Vitali convergence theorem, we deduce that, up to a subsequence, $u_n \to u$ in $L^1(\Omega)$ and $f(u_n) \to f(u)$ in $L^1(\Omega, \rho^s)$. Since u_n satisfies (3.3), by passing to the limit, we deduce that u is a weak solution of (1.2).

Stability. Assume $\{\tau_n\} \subset \mathfrak{M}(\Omega, \rho^s)$ converges weakly to $\tau \in \mathfrak{M}(\Omega, \rho^s)$ and $\{\mu_n\} \subset \mathfrak{M}(\partial\Omega)$ converges weakly to $\mu \in \mathfrak{M}(\partial\Omega)$. Let u and u_n be the unique weak solution of (1.2) with data (τ, μ) and (τ_n, μ_n) respectively. Then by a similar argument as in Existence part, we deduce that $u_n \to u$ in $L^1(\Omega)$ and $f(u_n) \to f(u)$ in $L^p(\Omega, \rho^s)$. □.

Proposition 3.1. *Assume f is a continuous nondecreasing function on $\mathbb{R}$ satisfying $f(0) = 0$ and (1.11). Then for every $z \in \partial\Omega$,*

$$\lim_{\Omega \ni x \to z} \frac{\mathbb{G}_s^\Omega[f(M_s^\Omega(\cdot, z))](x)}{M_s^\Omega(x, z)} = 0. \tag{3.20}$$

Proof. By (2.1),

$$G_s^\Omega(x, y) \le c_{14}\rho(x)^s |x - y|^{-N} \min\{\rho(y)^s, |x - y|^s\} \qquad \forall x \ne y.$$

Hence

$$\frac{\mathbb{G}_s^\Omega[f(M_s^\Omega(\cdot, z))](x)}{M_s^\Omega(x, z)} \le c_{15}|x - z|^N \int_\Omega |x - y|^{-N} \min\{|x - y|^s, |y - z|^s\} f(|y - z|^{s-N}) dy. \tag{3.21}$$

Put

$$\mathcal{D}_1 := \Omega \cap B(x, |x-z|/2), \quad \mathcal{D}_2 := \Omega \cap B(z, |x-z|/2), \quad \mathcal{D}_3 := \Omega \setminus (\mathcal{D}_1 \cup \mathcal{D}_2), \tag{3.22}$$

$$I_i := |x-z|^N \int_{\mathcal{D}_i} |x-y|^{-N} \min\{|x-y|^s, |y-z|^s\} f(|y-z|^{s-N}) dy, \quad i = 1, 2, 3.$$

For every $y \in \mathcal{D}_1$, $|x-z| \leq 2|y-z|$, therefore

$$I_1 \leq c_{16}|x-z|^N f(|x-z|^{s-N}) \int_{\mathcal{D}_1} |x-y|^{s-N} dy \leq c_{17}|x-z|^{N+s} f(|x-z|^{s-N}).$$

Hence

$$\lim_{x\to z} I_1 \leq c_{17} \lim_{x\to z} |x-z|^{N+s} f(|x-z|^{s-N}) = 0. \tag{3.23}$$

We next estimate I_2. For every $y \in \mathcal{D}_2$, $|x-z| \leq 2|x-y|$, hence

$$I_2 \leq c_{18} \int_{\mathcal{D}_2} |y-z|^s f(|y-z|^{s-N}) dy \leq c_{37} \int_{|x-z|^{s-N}}^{\infty} t^{-1-p_2^*} f(t) dt.$$

Therefore, by (1.11),

$$\lim_{x\to z} I_2 \leq c_{19} \lim_{x\to z} \int_{|x-z|^{s-N}}^{\infty} t^{-1-p_2^*} f(s) ds = 0. \tag{3.24}$$

Finally, we estimate I_3. For every $y \in \mathcal{D}_3$, $|y-z| \leq 3|x-y|$, therefore

$$I_3 \leq c_{20}|x-z|^N \int_{\mathcal{D}_3} |y-z|^{s-N} f(|y-z|^{s-N}) dy \leq c_{21}|x-z|^N \int_0^{|x-z|^{s-N}} t^{-\frac{N}{N-s}} f(t) dt. \tag{3.25}$$

Put

$$g_1(r) = \int_0^{r^{s-N}} t^{-\frac{N}{N-s}} f(t) dt, \quad g_2(r) = r^{-N}.$$

If $\lim_{r\to 0} g_1(r) < \infty$, then $\lim_{x\to z} I_3 = 0$ by (3.25). Otherwise, $\lim_{r\to 0} g_1(r) = \infty = \lim_{r\to 0} g_2(r)$. Therefore, by L' Hôpital's rule,

$$\lim_{r\to 0} \frac{g_1(r)}{g_2(r)} = \lim_{r\to 0} \frac{g_1'(r)}{g_2'(r)} = \lim_{r\to 0} \frac{N-s}{N} r^{N+s} f(r^{s-N}) = 0. \tag{3.26}$$

By combining (3.25) and (3.26) we obtain

$$\lim_{x\to z} I_3 \leq c_{22} \lim_{x\to z} |x-z|^N \int_0^{|x-z|^{s-N}} t^{-\frac{N}{N-s}} f(t) dt = 0. \tag{3.27}$$

We deduce (3.21) by gathering (3.23), (3.24) and (3.27). □

Proof of Theorem D. From Theorem C we get

$$kM_s^\Omega(x,z) - \mathbb{G}_s^\Omega[f(M_s^\Omega(\cdot,z))](x) \leq u_{k,z}^\Omega(x) \leq kM_s^\Omega(x,z), \tag{3.28}$$

which implies

$$k - \frac{\mathbb{G}_s^\Omega[f(M_s^\Omega(\cdot,z))](x)}{M_s^\Omega(x,z)} \leq \frac{u_{k,z}^\Omega(x)}{M_s^\Omega(x,z)} \leq k.$$

We derive (1.13) due to Proposition 3.1. □

3.2. **Power absorption.** In this subsection we assume that $0 \in \partial\Omega$. Let $0 < p < p_2^*$ and denote by u_k^Ω the unique solution of (1.14). By Theorem C, $u_k^\Omega \le kM_s^\Omega(\cdot,0)$ and $k \mapsto u_k^\Omega$ is increasing. Therefore, it is natural to investigate $\lim_{k\to\infty} u_k^\Omega$.

For any $\ell > 0$, put

$$T_\ell[u](y) := \ell^{\frac{2s}{p-1}} u(\ell y), \quad y \in \Omega_\ell := \ell^{-1}\Omega.$$

If u is a solution of (1.17) in Ω then $T_\ell[u]$ is a solution of (1.17) in Ω_ℓ.

By Corollary A.9, the function

$$x \mapsto U(x) = \ell_{s,p}|x|^{-\frac{2s}{p-1}}, \quad x \neq 0, \tag{3.29}$$

where $\ell_{s,p}$ is a positive constant, is a radial singular solution of

$$(-\Delta)^s u + u^p = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{3.30}$$

Lemma 3.2. *Assume $p \in (p_1^*, p_2^*)$. Then there exists a positive constant C depending on N, s, p and the C^2 characteristic of Ω such that the following holds. If u is a positive solution of (1.17) satisfying $u \le U$ in Ω then there holds*

$$u(x) \le C\rho(x)^s |x|^{-\frac{(p+1)s}{p-1}} \quad \forall x \in \Omega. \tag{3.31}$$

Proof. Let $P \in (\partial\Omega \setminus \{0\}) \cap B_1(0)$ and put

$$d = d(P) := \frac{1}{2}|P| < \frac{1}{2}.$$

Put

$$u_d(y) = T_d[u](y), \quad y \in \Omega_d := d^{-1}\Omega.$$

Then u_d is a solution of

$$\begin{cases} (-\Delta)^s u + u^p = 0 & \text{in } \Omega_d \\ u = 0 & \text{in } (\Omega_d)^c. \end{cases} \tag{3.32}$$

Moreover

$$u_d(y) \le T_d[U](y) = d^{\frac{2s}{p-1}} U(dy) = \ell_{s,p}|y|^{-\frac{2s}{p-1}} = U(y).$$

Put $P_d = d^{-1}P$ and let β_0 be the constant in Proposition 2.8. We may assume $\beta_0 \le \frac{1}{4}$. Let $\zeta_P \in C^\infty(\mathbb{R}^N)$ such that $0 \le \zeta \le 1$ in $\mathbb{R}^N$, $\zeta = 0$ in $B_{\beta_0}(P_d)$ and $\zeta = 1$ in $\mathbb{R}^N \setminus B_{2\beta_0}(P_d)$. Let $\eta_d \in C(\overline{\Omega}_d)$ be the solution of (3.4) with Ω replaced by Ω_d. For $l > 0$, denote

$$V_{d,l} := \zeta_P\, U + l\, \eta_d.$$

We will compare u_d with $V_{d,l}$.

Step 1: We show that $V_{d,l}$ is a super solution of (3.32) for l large enough.

For $y \in \Omega_d \setminus B_{4\beta_0}(P_d)$, $\zeta_P(y) = 1$ and hence

$$\begin{aligned}
(-\Delta)^s(\zeta_P U)(y) &= \lim_{\epsilon\to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(y)} \frac{U(y) - \zeta_P(z)U(z)}{|y-z|^{N+2s}} dz \\
&= (-\Delta)^s U(y) + \lim_{\epsilon\to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(y)} \frac{U(z) - \zeta_P(z)U(z)}{|y-z|^{N+2s}} dz \\
&\ge (-\Delta)^s U(y) - \int_{B_{\frac{1}{2}}(P_d)} \frac{U(z)}{|y-z|^{N+2s}} dz \\
&\ge (-\Delta)^s U(y) - c_{26},
\end{aligned}$$

where $c_{26} = c_{26}(N, s, p, \beta_0)$. Since $(\Omega_d \cap B_{2\beta_0}(0)) \subset (\Omega_d \setminus B_{4\beta_0}(P_d))$, it follows that, for any $y \in \Omega_d \cap B_{2\beta_0}(0) \setminus \{0\}$,

$$\begin{aligned}
(-\Delta)^s V_{d,l}(y) + (V_{d,l}(y))^p &= (-\Delta)^s(\zeta_P U)(y) + l(-\Delta)^s \eta_d(y) + (\zeta_P(y)U(y) + l\eta_d(y))^p \\
&\ge (-\Delta)^s U(y) - c_{26} + l + U(y)^p.
\end{aligned}$$

Therefore if we choose $l \ge c_{26}$ then

$$(-\Delta)^s V_{d,l} + (V_{d,l})^p \ge 0 \quad \text{in } \Omega_d \cap B_{2\beta_0}(0) \setminus \{0\}. \tag{3.33}$$

Next we see that there exists $c_{27} > 0$ such that

$$|(-\Delta)^s(\zeta_P U)| \le c_{27} \quad \text{in } \Omega_d \setminus B_{2\beta_0}(0).$$

Consequently,

$$\begin{aligned}(-\Delta)^s V_{d,l} &= (-\Delta)^s(\zeta_P U) + l(-\Delta)^s\eta_d \\ &\ge -c_{27} + l.\end{aligned}$$

Therefore if we choose $l \ge c_{27}$ then

$$(-\Delta)^s V_{d,l} \ge 0 \quad \text{in } \Omega_d \setminus B_{2\beta_0}(0). \tag{3.34}$$

By combining (3.33) and (3.34), for $l \ge \max\{c_{26}, c_{27}\}$, we deduce that $V_{d,l}$ is a super solution of (3.32).

Step 2: We show that $u_d \le V_{d,l}$ in Ω_d. By contradiction, we assume that there exists $x_0 \in \Omega_d$ such that

$$(u_d - V_{d,l})(x_0) = \max_{x\in\Omega_d}(u_d - V_{d,l}) > 0.$$

Then $(-\Delta)^s(u_d - V_{d,l})(x_0) \ge 0$. It follows that

$$0 \le (-\Delta)^s(u_d - V_{d,l})(x_0) \le -(u_d(x_0)^p - V_{d,l}(x_0)^p) < 0.$$

This contradiction implies that $u_d \le V_{d,l}$ in Ω_d.

Step 3: End of proof. From step 2, we deduce that

$$u_d \le l\eta_d \quad \text{in } \Omega_d \cap B_{\beta_0}(P_d).$$

We note that $\eta_d(y) \le c\text{dist}\,(y, \partial\Omega_d)^s$ for every $y \in \Omega_d$. Here the constant c depends on N, s and the C^2 characteristic of Ω_d. Since $d < \frac{1}{2}$, a C^2 characteristic of Ω_d can be taken as a C^2 characteristic of Ω. Therefore the constant c can be taken independently of P. Consequently,

$$u_d(y) \le lc\,\text{dist}\,(y, \partial\Omega_d)^s \quad \forall y \in \Omega_d \cap B_{\beta_0}(P_d).$$

This implies

$$u(x) \le c'\rho(x)^s d^{-\frac{(p+1)s}{p-1}} \quad \forall x \in \Omega \cap B_{d\beta_0}(P). \tag{3.35}$$

Put

$$\mathcal{F}_1 := \Omega_{\beta_0} \cap B_{\frac{1}{1+\beta_0}}(0) \cap \{x : \rho(x) \le \beta_0|x|\}, \quad \mathcal{F}_2 := \Omega_{\beta_0} \cap B_{\frac{1}{1+\beta_0}}(0) \cap \{x : \rho(x) > \beta_0|x|\}.$$

If $x \in \mathcal{F}_1$ then let $P \in \partial\Omega \setminus \{0\}$ such that $\rho(x) = |x - P|$. It follows that

$$\frac{1}{2}(1-\beta_0)|x| < d = \frac{1}{2}|P| \le \frac{1}{2}(1+\beta_0)|x| < \frac{1}{2}. \tag{3.36}$$

By combining (3.35) and (3.36), we get

$$u(x) \le c'(1-\beta_0)^{-\frac{(p+1)s}{p-1}}\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}}.$$

If $x \in \mathcal{F}_2$ then (3.31) follows from the assumption $u \le U$. Thus (3.31) holds for every $x \in \Omega_{\beta_0} \cap B_{\frac{1}{1+\beta_0}}(0)$. If $x \in \Omega \setminus B_{\frac{1}{1+\beta_0}}(0)$ then by a similar argument as in Step 1 and Step 2 without similarity transformation, we deduce that there exist constants c and $\tilde\beta \in (0, \frac{1}{2(1+\beta_0)})$ depending on N, s, p and the C^2 characteristic of Ω such that (3.31) holds in $B_{\tilde\beta}(P) \cap \Omega$ for every $P \in \partial\Omega \setminus B_{\frac{1}{1+\beta_0}}(0)$. Finally, since $u \le U$, (3.31) holds in $D_{\frac{\tilde\beta}{2}} = \{x \in \Omega : \rho(x) > \frac{\tilde\beta}{2}\}$. Thus (3.31) holds in Ω. □

Lemma 3.3. *Let $p \in (0, p_2^*)$. There exists a constant $c = c(N, s, p, \Omega) > 0$ such that for any $x \in \Omega$ and $z \in \partial\Omega$, there holds*

$$\mathbb{G}_s^\Omega[M_s^\Omega(\cdot, z)^p](x) \le \begin{cases} c\rho(x)^s|x-z|^{s-(N-s)p} & \text{if } \dfrac{s}{N-s} < p < p_2^* \\ -c\rho(x)^s \ln|x-z| & \text{if } p = \dfrac{s}{N-s} \\ c\rho(x)^s & \text{if } 0 < p < \dfrac{s}{N-s}. \end{cases} \tag{3.37}$$

Proof. We use a similar argument as in the proof of Proposition 3.1. It is easy to see that for every $x \in \Omega$ and $z \in \partial\Omega$,

$$\mathbb{G}_s^\Omega[M_s^\Omega(\cdot,z)^p](x) \le c_{23}\rho(x)^s \int_\Omega |x-y|^{-N}|y-z|^{(s-N)p}\min\{|x-y|^s,|y-z|^s\}dy \tag{3.38}$$

Let $\mathcal{D}_i$, $i=1,2,3$ be as in (3.22) and put

$$J_i := \rho(x)^s \int_{\mathcal{D}_i} |x-y|^{-N}|y-z|^{(s-N)p}\min\{|x-y|^s,|y-z|^s\}dy.$$

By proceeding as in the proof of Proposition 3.1 we deduce easily that there is positive constants $c_{24} = c_{24}(N,s,p,\Omega)$ such that

$$J_i \le c_{24}\rho(x)^s|x-z|^{s-(N-s)p}, \quad i=1,2, \tag{3.39}$$

and

$$J_3 \le c_{24}\rho(x)^s \int_{|x-z|/2}^{diam(\Omega)} r^{s-1-(N-s)p}dr \le \begin{cases} c_{25}\rho(x)^s|x-z|^{s-(N-s)p} & \text{if} \quad \dfrac{s}{N-s} < p < p_2^* \\ -c_{25}\rho(x)^s \ln|x-z| & \text{if} \quad p = \dfrac{s}{N-s} \\ c_{25}\rho(x)^s & \text{if} \quad 0 < p < \dfrac{s}{N-s}. \end{cases} \tag{3.40}$$

Combining (3.39) and (3.40) implies (3.37). □

Proposition 3.4. *Assume $p \in (p_1^*, p_2^*)$. Then $u_\infty^\Omega := \lim_{k\to 0} u_k^\Omega$ is a positive solution of (1.17). Moreover, there exists $c = c(N,s,p,\Omega) > 0$ such that*

$$c^{-1}\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}} \le u_\infty^\Omega(x) \le c\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}} \quad \forall x \in \Omega. \tag{3.41}$$

Proof. We first claim that for any $k > 0$,

$$u_k^\Omega \le U \quad \text{in } \Omega. \tag{3.42}$$

Indeed, by (2.5),

$$u_k^\Omega(x) \le kM_s^\Omega(x,0) \le c_{28}k\rho(x)^s|x|^{-N} \le c_{28}k|x|^{s-N} \quad \forall x \in \Omega.$$

Since $p < p_2^*$, it follows that

$$\lim_{\Omega \ni x \to 0} \frac{u_k^\Omega(x)}{U(x)} = 0.$$

By proceeding as in Step 2 of the proof of Lemma 3.2, we deduce that $u_k^\Omega \le U$ in Ω.

Consequently, $u_\infty^\Omega := \lim_{k\to\infty} u_k^\Omega$ is a solution of (1.17) vanishing on $\partial\Omega \setminus \{0\}$ and satisfying $u_\infty^\Omega \le U$ in Ω. In light of Lemma 3.2, we obtain the upper bound in (3.41).

Next we prove the lower bound in (3.41). By (2.5) and Lemma 3.3, for any $k > 0$ and $x \in \Omega$, we have

$$\begin{aligned} u_k^\Omega(x) &\ge kM_s^\Omega(x,0) - k^p\mathbb{G}_s^\Omega[M_s^\Omega(\cdot,0)^p](x) \\ &\ge c_{29}^{-1}k\rho(x)^s|x|^{-N}(1 - c_{29}c_{30}k^{p-1}|x|^{N+s-(N-s)p}). \end{aligned}$$

For $x \in \Omega$, one can choose $r > 0$ such that $x \in \Omega \cap (B_{2r}(0) \setminus B_r(0))$. Choose $k = ar^{-\frac{N+s-(N-s)p}{p-1}}$, where $a > 0$ will be made precise later on, then

$$u_k^\Omega(x) \ge c_{31}a\,\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}}(1 - c_{29}c_{30}a^{p-1}).$$

By choosing $a = (2c_{29}c_{30})^{-\frac{1}{p-1}}$, we deduce for any $x \in \Omega$ there exists $k > 0$ depending on $|x|$ such that

$$u_k^\Omega(x) \ge c_{32}\rho(x)^s|x|^{-\frac{(p+1)s}{p-1}}.$$

Since $u_\infty^\Omega \ge u_k^\Omega$ in Ω we obtain the first inequality in (3.41). □

Proposition 3.5. *Assume $0 < p \le p_1^*$. There exist $k_0 = k_0(N, s, p)$ and $c = c(N, s, p, \Omega)$ such that the following holds. There exists a decreasing sequence of positive numbers $\{r_k\}$ such that $\lim_{k\to\infty} r_k = 0$ and for any $k > k_0$,*

$$u_k^\Omega(x) \ge \begin{cases} c\rho(x)^s|x|^{-N-s} & \text{if } 0 < p < p_1^*, \\ \\ c\rho(x)^s|x|^{-N-s}(-\ln|x|)^{-1} & \text{if } p = p_1^*, \end{cases} \quad \forall x \in \Omega \setminus B_{r_k}(0). \tag{3.43}$$

Proof. For any $\ell > 0$, we have

$$u_\ell^\Omega(x) \ge \ell M_s^\Omega(x, 0) - \ell^p \mathbb{G}_s^\Omega[M_s^\Omega(\cdot, 0)^p](x) \quad \forall x \in \Omega. \tag{3.44}$$

Case 1: $p \in (\frac{s}{N-s}, p_1^*)$. Put $k_1 := (2c_{29}c_{30})^{\frac{s}{N+2s-Np}}$ and take $k > k_1$. For $\ell > 0$, put $r_\ell = \ell^{-\frac{1}{s}}$, then $\ell = r_\ell^{-s}$. Take arbitrarily $x \in \Omega \setminus B_{r_k}(0)$ then one can choose $\ell \in (\max(2^{-s}k, k_1), k)$ such that $x \in \Omega \cap (B_{r_\ell}(0) \setminus B_{\frac{r_\ell}{2}}(0))$. From (3.44), (2.5) and (3.37), we get

$$\begin{aligned} u_\ell^\Omega(x) &\ge c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1 - c_{29}c_{30}\ell^{p-1}|x|^{N+s-(N-s)p}) \\ &\ge c_{29}^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s}(1 - c_{29}c_{30}r_\ell^{N+2s-Np}) \\ &\ge (2c_{29})^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s} \\ &\ge c_{33}\rho(x)^s|x|^{-N-s}. \end{aligned}$$

Here thefirst estimate holds since $\frac{N}{N-s} < p < p_2^*$ and the third estimate holds since $p < p_1^*$ and $\ell > k_1$. Since $k > \ell$, we deduce that

$$u_k^\Omega(x) \ge c_{33}\rho(x)^s|x|^{-N-s}, \quad \forall x \in \Omega \setminus B_{r_k}(0). \tag{3.45}$$

Case 2: $p = \frac{s}{N-s}$. Put $k_2 = (\frac{2c_{29}c_{30}(1+s)}{s})^{\frac{s}{N-sp}}$ and take $k > k_2$. For $\ell > 0$, put $r_\ell = \ell^{-\frac{1}{s}}$, then $\ell = r_\ell^{-s}$. Take arbitrarily $x \in \Omega \setminus B_{r_k}(0)$ then one can choose $\ell \in (\max(2^{-s}k, k_2), k)$ such that $x \in \Omega \cap (B_{r_\ell}(0) \setminus B_{\frac{r_\ell}{2}}(0))$. From (3.44), (2.5) and Lemma 3.3, we get

$$\begin{aligned} u_\ell^\Omega(x) &\ge c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1 + c_{29}c_{30}\ell^{p-1}|x|^N \ln|x|) \\ &\ge c_{29}^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s}(1 + c_{29}c_{30}r_\ell^{N+s-sp}\ln(\frac{r_\ell}{2})) \\ &\ge (2c_{29})^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s} \\ &\ge c_{33}\rho(x)^s|x|^{-N-s}. \end{aligned}$$

Here the third estimate holds since $\ell > k_2$ and $N - sp > 0$. Therefore (3.45) holds.

Case 3: $p \in (0, \frac{s}{N-s})$. Put $k_3 = (2c_{29}c_{30})^{\frac{s}{N+s-sp}}$ and take $k > k_3$. For $\ell > 0$, put $r_\ell = \ell^{-\frac{1}{s}}$, then $\ell = r_\ell^{-s}$. Take arbitrarily $x \in \Omega \setminus B_{r_k}(0)$ then one can choose $\ell \in (\max(2^{-s}k, k_3), k)$ such that $x \in \Omega \cap (B_{r_\ell}(0) \setminus B_{\frac{r_\ell}{2}}(0))$. From (3.44), (2.5) and (3.37), we get

$$\begin{aligned} u_\ell^\Omega(x) &\ge c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1 - c_{29}c_{30}\ell^{p-1}|x|^N) \\ &\ge c_{29}^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s}(1 - c_{29}c_{30}r_\ell^{N+s-sp}) \\ &\ge (2c_{29})^{-1}\rho(x)^s|x|^{-N}r_\ell^{-s} \\ &\ge c_{33}\rho(x)^s|x|^{-N-s}. \end{aligned}$$

Here the third estimate holds since $\ell > k_3$ and $N + s - sp > 0$. Therefore (3.45) holds.

Case 4: $p = p_1^*$. Put $k_4 = \exp((2c_{29}c_{30})^{\frac{s}{N+s-(N-s)p}})$ and take $k > k_4$. For $\ell > 0$, put $r_\ell = (\ell\ln(\ell))^{-\frac{1}{s}}$, then $\ell\ln(\ell) = r_\ell^{-s}$ and $\ell < r_\ell^{-s}$ when $\ell > 3$. Take arbitrarily $x \in \Omega \setminus B_{r_k}(0)$ then one can choose $\ell \in (\max(2^{-s}k, k_4), k)$ such that $x \in \Omega \cap (B_{r_\ell}(0) \setminus B_{\frac{r_\ell}{2}}(0))$. From (3.44), (2.5)

and (3.37), we get

$$\begin{aligned} u_\ell^\Omega(x) &\geq c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1-c_{29}c_{30}\ell^{p-1}|x|^{N+s-(N-s)p}) \\ &\geq c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1-c_{29}c_{30}\ell^{p-1}(\ell\ln(\ell))^{-\frac{N+s-(N-s)p}{s}}) \\ &= c_{29}^{-1}\ell\rho(x)^s|x|^{-N}(1-c_{29}c_{30}\ln(\ell)^{-\frac{N+s-(N-s)p}{s}}) \\ &\geq (2c_{29})^{-1}\ell\rho(x)^s|x|^{-N} \\ &\geq c_{34}\rho(x)^s|x|^{-N-s}(-\ln|x|)^{-1}. \end{aligned}$$

Here the inequality holds since $p = p_1^*$ and the last estimate follows from the following estimate

$$\ell = \frac{r_\ell^{-s}}{\ln(\ell)} > \frac{|x|^{-s}}{-s2^s\ln|x|}.$$

Since $u_k^\Omega(x) \geq u_\ell^\Omega(x)$, we derive

$$u_k^\Omega(x) \geq c_{34}\rho(x)^s|x|^{-N-s}(-\ln|x|)^{-1}.$$

By putting $k_0 := \max(k_1, k_2, k_3, k_4)$, we obtain (3.43). □

Proposition 3.6. *Assume $0 < p \leq p_1^*$. Then $\lim_{k\to\infty} u_k^\Omega(x) = \infty$ for every $x \in \Omega$.*

Proof. The proposition can be obtained by adapting the argument in the proof of [**?**, Theorem 1.2]. Let $r_0 > 0$ and put

$$\theta_k := \int_{B_{r_0}(0)} u_k^\Omega(x)dx.$$

Then

$$\theta_k \geq c\int_{(B_{r_0}\cap\Omega)\setminus B_{r_k}(0)} \rho(x)^s|x|^{N-s}(-\ln|x|)^{-1}dx,$$

which implies

$$\lim_{k\to\infty}\theta_k = \infty. \tag{3.46}$$

Fix $y_0 \in \Omega \setminus \overline{B}_{r_0}(0)$ and set $\delta := \frac{1}{2}\min\{\rho(y_0), |y_0| - r_0\}$. By [13, Lemma 2.4] there exists a unique classical solution w_k of the following problem

$$\begin{cases} (-\Delta)^s w_k + w_k^p = 0 \quad \text{in } B_\delta(y_0), \\ \qquad\qquad w_k = 0 \quad \text{in } \mathbb{R}^N \setminus (B_\delta(y_0)\cup B_{r_0}(0)), \\ \qquad\qquad w_k = u_k^\Omega \text{ in } B_{r_0}(0). \end{cases} \tag{3.47}$$

By [13, Lemma 2.2],

$$u_k^\Omega \geq w_k \quad \text{in } B_\delta(y_0). \tag{3.48}$$

Next put $\tilde{w}_k := w_k - \chi_{B_{r_0}(0)}u_k$ then $\tilde{w}_k = w_k$ in $B_\delta(y_0)$. Moreover, for $x \in B_\delta(y_0)$

$$\begin{aligned} (-\Delta)^s\tilde{w}_k(x) &= \lim_{\epsilon\to0}\int_{B_\delta(y_0)\setminus B_\epsilon(x)} \frac{w_k(x)-w_k(z)}{|z-x|^{N+2s}}dz + \lim_{\epsilon\to0}\int_{B_\delta^c(y_0)\setminus B_\epsilon(x)} \frac{w_k(x)}{|z-x|^{N+2s}}dz \\ &= \lim_{\epsilon\to0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{w_k(x)-w_k(z)}{|z-x|^{N+2s}}dz + \int_{B_{r_0}(0)} \frac{u_k^\Omega(z)}{|z-x|^{N+2s}}dz \\ &\geq (-\Delta)^s w_k(x) + A\theta_k \end{aligned} \tag{3.49}$$

where $A = (|y_0| + r_0)^{-N-2s}$. It follows that, for $x \in B_\delta(y_0)$,

$$(-\Delta)^s\tilde{w}_k(x) + \tilde{w}_k^p(x) \geq (-\Delta)^s w_k(x) + w_k^p(x) + A\theta_k = A\theta_k. \tag{3.50}$$

Therefore $\tilde{w}_k \in C(\overline{B_\delta(y_0)})$ is a supersolution of

$$\begin{cases} (-\Delta)^s w + w^p = A\theta_k \quad \text{in } B_\delta(y_0), \\ \qquad\qquad w = 0 \qquad \text{in } \mathbb{R}^N \setminus B_\delta(y_0). \end{cases} \tag{3.51}$$

Let $\eta_0 \in C(\overline{B_\delta(y_0)})$ be the unique solution of

$$\begin{cases} (-\Delta)^s \eta_0 = 1 & \text{in } B_\delta(y_0), \\ \eta_0 = 0 & \text{in } \mathbb{R}^N \setminus B_\delta(y_0). \end{cases} \tag{3.52}$$

We can choose k large enough so that the function

$$\frac{\eta_0 (A\theta_k)^{\frac{1}{p}}}{2 \max_{\mathbb{R}^N} \eta_0}$$

is a subsolution of (3.51). By [13, Lemma 2.2] we obtain

$$\tilde{w}_k(x) \geq \frac{\eta_0 (A\theta_k)^{\frac{1}{p}}}{2 \max_{\mathbb{R}^N} \eta_0} \quad \forall x \in B_\delta(y_0). \tag{3.53}$$

Put

$$\underline{c} := \min_{x \in B_\delta(y_0)} \frac{\eta_0}{2 \max_{\mathbb{R}^N} \eta_0}$$

then we derive from (3.53) that

$$w_k(x) \geq \underline{c} (A\theta_k)^{\frac{1}{p}} \quad \forall x \in B_\delta(y_0). \tag{3.54}$$

By combining (3.46), (3.48) and (3.54), we deduce that

$$\lim_{k \to \infty} u_k^\Omega(x) = \infty \quad \forall x \in B_{\frac{\delta}{2}}(y_0).$$

This implies

$$\lim_{k \to \infty} u_k^\Omega(x) = \infty \quad \forall x \in \Omega.$$

□

Theorem 3.7. *Assume $p \in (1, p_2^*)$ and either $\Omega = \mathbb{R}^N_+ := \{x = (x', x_N) : x_N > 0\}$ or $\partial\Omega$ is compact with $0 \in \partial\Omega$. Then, for any $k > 0$, there exists a unique solution solution u_k^Ω of problem (1.14) satisfying $u_k^\Omega \leq k M_s^\Omega(\cdot, 0)$ in Ω and*

$$\lim_{|x| \to 0} \frac{u_k^\Omega(x)}{M_s^\Omega(x, 0)} = k. \tag{3.55}$$

Moreover, the map $k \mapsto u_k^\Omega$ is increasing.

Proof. Step 1: Existence. For $R > 0$ we set $\Omega_R = \Omega \cap B_R$ and let $u := u_k^{\Omega_R}$ be the unique solution of

$$\begin{cases} (-\Delta)^s u + u^p = 0 & \text{in } \Omega_R \\ \operatorname{tr}_s(u) = k\delta_0 & \\ u = 0 & \text{on } \Omega_R^c. \end{cases} \tag{3.56}$$

Then

$$u_k^{\Omega_R}(x) \leq k M_s^{\Omega_R}(x, 0) \qquad \forall x \in \Omega_R. \tag{3.57}$$

Since $R \mapsto M_s^{\Omega_R}(., 0)$ is increasing, it follows from (1.13) that $R \mapsto u_k^{\Omega_R}$ is increasing too with the limit u^* and there holds

$$u^*(x) \leq k M_s^\Omega(x, 0) \qquad \forall x \in \Omega. \tag{3.58}$$

From (3.57), we deduce that

$$u_k^{\Omega_R}(x) \leq ck|x|^{s-N} \qquad \forall x \in \Omega_R$$

where c depends only on N, s and the C^2 characteristic of Ω. Hence by the regularity up to the boundary [25], $\{u_k^{\Omega_R}\}$ is uniformly bounded in $C^s_{loc}(\overline{\Omega} \setminus B_\epsilon)$ and in $C^{2s+\alpha}_{loc}(\Omega \setminus B_\epsilon)$ for any $\epsilon > 0$. Therefore, $\{u_k^{\Omega_R}\}$ converges locally uniformly, as $R \to \infty$, to $u^* \in C(\overline{\Omega} \setminus \{0\}) \cap C^{2s+\alpha}(\Omega)$.

Thus u^* is a positive solution of (1.17). Moreover by combining (1.13), (3.57), the fact that $M_s^{\Omega_R} \uparrow M_s^\Omega$ and $u_k^{\Omega_R} \uparrow u_k^\Omega$, we deduce that $\mathrm{tr}_s(u^*) = k\delta_0$ and

$$\lim_{\Omega \ni x \to 0} \frac{u^*(x)}{M_s^\Omega(x,0)} = k.$$

Step 2: Uniqueness. Suppose u and u' are two weak solutions of (1.17) satisfying $\max\{u, u'\} \le kM_s^\Omega(\cdot,0)$ in Ω and

$$\lim_{\Omega \ni x \to 0} \frac{u(x)}{M_s^\Omega(x,0)} = \lim_{\Omega \ni x \to 0} \frac{u'(x)}{M_s^\Omega(x,0)} = k. \tag{3.59}$$

Take $\epsilon > 0$ and put $u_\epsilon := (1+\epsilon)u' + \epsilon$, $v := (u - u_\epsilon)_+$. Then by (3.59) there exists a smooth bounded domain $G \subset \Omega$ such that $v = 0$ in G^c and $\mathrm{tr}_s^G(v) = 0$. In light of Kato's inequality, we derive $(-\Delta)^s v \le 0$ in G. Moreover, $v \le kM_s^\Omega(\cdot,0)$ in G. By Lemma 2.12 we obtain $v = 0$ in G and therefore $u \le (1+\epsilon)u' + \epsilon$ in Ω. Letting $\epsilon \to 0$ yields $u \le u'$ in Ω. By permuting the role of u and u', we derive $u = u'$ in Ω.

By a similar argument as in step 2, we can show that $k \mapsto u_k^\Omega$ is increasing. □

Proof of Theorem F. (i) *Case 1:* $p_1^* < p < p_2^*$.

Since $\partial\Omega \in C^2$, there exist two open balls B and B' such that $B \subset \Omega \subset B'^c$ and $\partial B \cap \partial B' = \{0\}$. Since $M_s^B(x,0) \le M_s^\Omega(x,0) \le M_s^{B'^c}(x,0)$ it follows from Theorem 3.7 that

$$u_k^B \le u_k^\Omega \le u_k^{B'^c} \tag{3.60}$$

where the first inequality holds in B and the second inequality holds in Ω.

Let $\mathcal{O}$ be B, Ω or B'^c. Because of uniqueness, we have

$$T_\ell[u_k^{\mathcal{O}}] = u_{k\ell^{\frac{2s}{p-1}+1-N}}^{\mathcal{O}_\ell} \qquad \forall \ell > 0, \tag{3.61}$$

with $\mathcal{O}_\ell = \ell^{-1}\mathcal{O}$. By Theorem 3.7, the sequence $\{u_k^{\mathcal{O}}\}$ is increasing and by (3.42), $u_k^{\mathcal{O}} \le U$. It follows that $\{u_k^{\mathcal{O}}\}$ converges to a function $u_\infty^{\mathcal{O}}$ which is a positive solution of (1.17) with Ω replaced by $\mathcal{O}$.

Step 1: $\mathcal{O} := \mathbb{R}^N_+$. Then $\mathcal{O}_\ell = \mathbb{R}^N_+$. Letting $k \to \infty$ in (3.61) yields to

$$T_\ell[u_\infty^{\mathbb{R}^N_+}] = u_\infty^{\mathbb{R}^N_+} \qquad \forall \ell > 0. \tag{3.62}$$

Therefore $u_\infty^{\mathbb{R}^N_+}$ is self-similar and thus it can be written in the separable form

$$u_\infty^{\mathbb{R}^N_+}(x) = u_\infty^{\mathbb{R}^N_+}(r,\sigma) = r^{-\frac{2s}{p-1}}\omega(\sigma)$$

where $r = |x|$, $\sigma = \frac{x}{|x|} \in S^{N-1}$ and ω satisfies (1.16). Since $p_1^* < p < p_2^*$, it follows from Theorem E that $\omega = \omega^*$, the unique positive solution of (1.16). This means

$$u_\infty^{\mathbb{R}^N_+}(x) = r^{-\frac{2s}{p-1}}\omega^*(\sigma). \tag{3.63}$$

This implies (3.41).

Step 2: $\mathcal{O} := B$ *or* B'^c. In accordance with our previous notations, we set $B_\ell = \ell^{-1}B$ and $(B'^c)_\ell = \ell^{-1}B'^c$ for $\ell > 0$ and we have,

$$T_\ell[u_\infty^B] = u_\infty^{B_\ell} \text{ and } T_\ell[u_\infty^{B'^c}] = u_\infty^{(B'^c)_\ell} \tag{3.64}$$

and

$$u_\infty^{B_{\ell'}} \le u_\infty^{B_\ell} \le u_\infty^{\mathbb{R}^N_+} \le u_\infty^{(B'^c)_\ell} \le u_\infty^{(B'^c)_{\ell''}} \qquad 0 < \ell \le \ell', \ell'' \le 1. \tag{3.65}$$

When $\ell \to 0$, $u_\infty^{B_\ell} \uparrow \underline{u}_\infty^{\mathbb{R}^N_+}$ and $u_\infty^{(B'^c)_\ell} \downarrow \overline{u}_\infty^{\mathbb{R}^N_+}$ where $\underline{u}_\infty^{\mathbb{R}^N_+}$ and $\overline{u}_\infty^{\mathbb{R}^N_+}$ are positive solutions of (3.42) in $\mathbb{R}^N_+$ such that

$$u_\infty^{B_\ell} \le \underline{u}_\infty^{\mathbb{R}^N_+} \le u_\infty^{\mathbb{R}^N_+} \le \overline{u}_\infty^{\mathbb{R}^N_+} \le u_\infty^{(B'^c)_\ell} \qquad 0 < \ell \le 1. \tag{3.66}$$

Furthermore there also holds for $\ell, \ell' > 0$,

$$T_{\ell'\ell}[u_\infty^B] = T_{\ell'}[T_\ell[u_\infty^B]] = u_\infty^{B_{\ell\ell'}} \text{ and } T_{\ell'\ell}[u_\infty^{B'^c}] = T_{\ell'}[T_\ell[u_\infty^{B'^c}]] = u_\infty^{(B'_c)_{\ell\ell'}}. \tag{3.67}$$

Letting $\ell \to 0$ and using (3.64) and the above convergence, we obtain

$$\underline{u}_\infty^{\mathbb{R}^N_+} = T_{\ell'}[\underline{u}_\infty^{\mathbb{R}^N_+}] \text{ and } \overline{u}_\infty^{\mathbb{R}^N_+} = T_{\ell'}[\overline{u}_\infty^{\mathbb{R}^N_+}] \qquad \ell' > 0. \tag{3.68}$$

Again this implies that $\underline{u}_\infty^{\mathbb{R}^N_+}$ and $\overline{u}_\infty^{\mathbb{R}^N_+}$ are separable solutions of (1.15). Since $p_1^* < p < p_2^*$, by Theorem E,

$$\underline{u}_\infty^{\mathbb{R}^N_+}(x) = \overline{u}_\infty^{\mathbb{R}^N_+}(x) = u_\infty^{\mathbb{R}^N_+}(x) = r^{-\frac{2s}{p-1}}\omega^*(\sigma) \quad \text{with } r = |x|,\ \ \sigma = \frac{x}{|x|}, \quad x \neq 0.$$

Step 3: End of the proof. From (3.60) and (3.64) there holds

$$u_\infty^{B_\ell} \le T_\ell[u_\infty^\Omega] \le u_\infty^{(B'^c)_\ell} \qquad 0 < \ell \le 1. \tag{3.69}$$

Since the left-hand side and the right-hand side of (3.69) converge to the same function $u_\infty^{\mathbb{R}^N_+}$, we obtain

$$\lim_{\ell\to 0} \ell^{\frac{2s}{p-1}} u_\infty^\Omega(\ell x) = |x|^{-\frac{2s}{p-1}}\omega^*(\frac{x}{|x|}) \tag{3.70}$$

and this convergence holds in any compact subset of Ω. Take $|x| = 1$, we derive (1.18). Estimate (3.41) follows from Proposition 3.4.

(ii) *Case 2:* $0 < p \le p_1^*$. Then by Proposition 3.6, $\lim_{k\to\infty} u_k^\Omega(x) = \infty$ for every $x \in \Omega$. □

Appendix A. Appendix - Separable solutions

A.1. **Separable s-harmonic functions.** We denote by $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N$, consider the following parametric representation of the unit sphere

$$S^{N-1} = \left\{\sigma = (\cos\phi\,\sigma', \sin\phi) : \sigma' \in S^{N-2}, -\tfrac{\pi}{2} \le \phi \le \tfrac{\pi}{2}\right\}, \tag{A.1}$$

hence $x_N = r\sin\phi$. We define the spherical fractional Laplace-Beltrami operator $\mathcal{A}_s$ by

$$\mathcal{A}_s\omega(\sigma) := \lim_{\epsilon\to 0} \mathcal{A}_{s,\epsilon}\omega(\sigma) \tag{A.2}$$

with

$$\mathcal{A}_{s,\epsilon}\omega(\sigma) := a_{N,s}\iint_{\mathbb{R}_+\times S^{N-1}\setminus B_\epsilon(\overrightarrow{\sigma})} \frac{(\omega(\sigma) - \omega(\eta))\tau^{N-1}}{(1 + \tau^2 - 2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}} dS(\eta)d\tau \tag{A.3}$$

where $\overrightarrow{\sigma} = (1, \sigma)$. If $u : (r, \sigma) \mapsto u(r, \sigma) = r^{-\beta}\omega(\sigma)$ is s-harmonic in $\mathbb{R}^N \setminus \{0\}$, it satisfies, at least formally,

$$\mathcal{A}_s\omega - \mathcal{L}_{s,\beta}\omega = 0 \qquad \text{on } S^{N-1} \tag{A.4}$$

where $\mathcal{L}_{s,\beta}$ is the integral operator

$$\mathcal{L}_{s,\beta}\omega(\sigma) := a_{N,s}\int_0^\infty\int_{S^{N-1}} \frac{(\tau^{-\beta} - 1)\tau^{N-1}}{(1 + \tau^2 - 2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\omega(\eta)dS(\eta)d\tau, \tag{A.5}$$

whenever this integral is defined. We will see in the next two lemmas that the role of the exponent $\beta_0 = N$ is fundamental for the definition of $\mathcal{L}_{s,\beta}\omega$ since we have

Lemma A.1. *If $N \ge 2$, $s \in (0, 1)$, $\beta < N$ and $(\sigma, \eta) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$ such that $\langle\sigma,\eta\rangle \neq 1$, we define*

$$B_{s,\beta}(\sigma, \eta) := \int_0^\infty \frac{(\tau^{-\beta} - 1)\tau^{N-1}}{(1 + \tau^2 - 2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}} d\tau. \tag{A.6}$$

Then

(i) $B_{s,\beta}(\sigma, \eta) < 0 \iff \beta < N - 2s$,

(ii) $B_{s,\beta}(\sigma, \eta) = 0 \iff \beta = N - 2s$,

(iii) $B_{s,\beta}(\sigma, \eta) > 0 \iff \beta > N - 2s$.

Proof. Since $\beta < N$, the integral in (A.7) is absolutely convergent. We write

$$\begin{aligned} B_{s,\beta}(\sigma,\eta) &= \int_0^1 \frac{(\tau^{-\beta}-1)\tau^{N-1}}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}d\tau + \int_1^\infty \frac{(\tau^{-\beta}-1)\tau^{N-1}}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}d\tau \\ &=: I + II. \end{aligned}$$

By the change of variable $\tau \mapsto \tau^{-1}$

$$II = -\int_0^1 \frac{(\tau^{-\beta}-1)\tau^{N-1+c_s}}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}d\tau,$$

where $c_s = \beta + 2s - N$. Since

$$B_{s,\beta}(\sigma,\eta) = \int_0^1 \frac{(\tau^{-\beta}-1)(\tau^{N-1}-\tau^{N-1+c_s})}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}d\tau, \tag{A.7}$$

the claim follows. □

As a byproduct of (A.7) we have the following monotonicity formula

Lemma A.2. *If $N \geq 2$ and $s \in (0,1)$, then for any $(\sigma,\eta) \in S^{N-1}\times S^{N-1}$ the mapping $\beta \mapsto B_{s,\beta}(\sigma,\eta)$ is continuous and increasing from $(N-2s, N)$ onto $(0,\infty)$.*

In the next result we analyze the behavior of $B_{s,\beta}(\sigma,\eta)$ when $\sigma - \eta \to 0$ on S^{N-1}.

Lemma A.3. *Assume $N \geq 2$, $s \in (0,1)$ and $\beta < N$ with $\beta \neq N-2s$, then*
I- If $N \geq 3$, there exists $c = c(N,\beta,s) > 0$ such that

$$|B_{s,\beta}(\sigma,\eta)| \leq c\,|\sigma-\eta|^{3-N-2s} \qquad \forall (\sigma,\eta) \in S^{N-1}\times S^{N-1}. \tag{A.8}$$

II- If $N = 2$,
(i) either $s > \frac{1}{2}$ and (A.8) holds with $N = 2$,
(ii) either $s = \frac{1}{2}$ and

$$|B_{s,\beta}(\sigma,\eta)| \leq c\,(-\ln|\sigma-\eta| + 1) \qquad \forall (\sigma,\eta) \in S^1\times S^1 \tag{A.9}$$

(iii) or $0 < s < \frac{1}{2}$ and

$$|B_{s,\beta}(\sigma,\eta)| \leq c \qquad \forall (\sigma,\eta) \in S^1\times S^1 \tag{A.10}$$

Proof. First, notice that the quantity

$$\int_0^{\frac{1}{2}} \frac{(\tau^{-\beta}-1)(\tau^{N-1}-\tau^{N-1+c_s})}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}d\tau$$

is uniformly bounded with respect to (σ,η). The only possible singularity in the expression given in (A.7) occurs when $\langle\sigma,\eta\rangle = 1$ and $\tau = 1$. We write $\langle\sigma,\eta\rangle = 1 - \frac{1}{2}\kappa^2$ and $t = 1-\tau$, hence

$$\begin{aligned} \big(1+\tau^2-2\tau\langle\sigma,\eta\rangle\big)^{\frac{N}{2}+s} &= \big(t^2+(1-t)\kappa^2\big)^{\frac{N}{2}+s} \\ &\approx \kappa^{N+2s}\left(1+\left(\tfrac{t}{\kappa}\right)^2\right)^{\frac{N}{2}+s} \end{aligned}$$

as $t \to 0$. Moreover

$$\begin{aligned} (\tau^{-\beta}-1)(\tau^{N-1}-\tau^{N-1+c_s}) &= ((1-t)^{-\beta}-1)((1-t)^{N-1}-(1-t)^{N-1+c_s}) \\ &= c_s\beta t^2 + O(t^3) \quad \text{as } t \to 0. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\frac{1}{2}}^1 \frac{(\tau^{-\beta}-1)(\tau^{N-1}-\tau^{N-1+c_s})}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}d\tau &= \int_0^{\frac{1}{2}} \frac{((1-t)^{-\beta}-1)((1-t)^{N-1}-(1-t)^{N-1+c_s})}{(t^2+(1-t)\kappa^2)^{\frac{N}{2}+s}}dt \\ &\approx c_s\kappa^{3-N-2s}\int_0^{\frac{1}{2\kappa}} \frac{x^2}{(1+x^2)^{\frac{N}{2}+s}}dx. \end{aligned}$$

If $N=2$ and $s<\frac{1}{2}$,

$$\left|\kappa^{1-2s}\int_0^{\frac{1}{2\kappa}}\frac{x^2}{(1+x^2)^{1+s}}dx\right|\leq M$$

for some $M>0$ independent of κ. If $N=2$ and $s=\frac{1}{2}$

$$\int_0^{\frac{1}{2\kappa}}\frac{x^2}{(1+x^2)^{1+\frac{1}{2}}}dx=\ln\left(\frac{1}{\kappa}\right)(1+o(1))$$

and if $N=3$ or $N=2$ and $s>\frac{1}{2}$,

$$\int_0^{\frac{1}{2\kappa}}\frac{x^2}{(1+x^2)^{\frac{N}{2}+s}}dx\to\int_0^{\infty}\frac{x^2}{(1+x^2)^{\frac{N}{2}+s}}dx.$$

as $\kappa\to 0$. Since $\sigma,\eta\in S^{N-1}$ there holds $\kappa^2=2(1-\langle\sigma,\eta\rangle)=|\sigma-\eta|^2$. Thus the claim follows. □

Proposition A.4. *Assume $N\geq 2$, $s\in(0,1)$ and $\beta<N$ with $\beta\neq N-2s$. Then $\omega\mapsto\mathcal{L}_{s,\beta}\omega$ is a continuous linear operator from $L^q(S^{N-1})$ into $L^r(S^{N-1})$ for any $1\leq q,r\leq\infty$ such that*

$$\frac{1}{r}>\frac{1}{q}-\frac{2(1-s)}{N-1}. \tag{A.11}$$

Furthermore, $\mathcal{L}_{s,\beta}$ is positive (resp. negative) operator if $\beta<N-2s$ (resp. $N-2s<\beta<N$).

Proof. By Lemma A.3, for any $\eta\in S^{N-1}$, $B_{s,\beta}(.,\eta)\in L^a(S^{N-1})$ for all $1<a<\frac{N-1}{N+2s-3}$ if $N\geq 3$ or $N=2$ and $s>\frac{1}{2}$; $B_{s,\beta}(.,\eta)\in\bigcap_{1\leq a<\infty}L^a(S^1)$ if $N=2$ and $s=\frac{1}{2}$ and $B_{s,\beta}(.,\eta)$ is uniformly bounded on S^1 if $N=2$ and $0<s<\frac{1}{2}$. The continuity result follows from Young's inequality and the sign assertion from Lemma A.1. □

The above calculations justifies the name of fractional Laplace-Beltrami operator given to $\mathcal{A}_s$ since we have the following relation.

Lemma A.5. *Assume $N\geq 2$ and $s\in(0,1)$, then*

$$\mathcal{A}_s\omega(\sigma)=b_{N,s}CPV\int_{S^{N-1}}\frac{(\omega(\sigma)-\omega(\eta))}{|\sigma-\eta|^{N-1+2s}}dS(\eta)+\mathcal{B}_s\omega(\sigma), \tag{A.12}$$

where $\mathcal{B}_s$ is a bounded linear operator from $L^q(S^{N-1})$ into $L^r(S^{N-1})$ for q, r satisfying (A.11) *and*

$$b_{N,s}:=2a_{N,s}\int_0^{\infty}\frac{dx}{(x^2+1)^{\frac{N}{2}+s}}. \tag{A.13}$$

Proof. If $(\sigma,\eta)\in S^{N-1}\times S^{N-1}$, we set $\langle\sigma,\eta\rangle=1-\frac{1}{2}\kappa^2$. Then

$$\int_0^{\infty}\frac{\tau^{N-1}d\tau}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}=\int_0^1\frac{\left(\tau^{N-1}+\tau^{2s-1}\right)d\tau}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}.$$

Then we put $t=1-\tau$, hence, when $t\to 0$, we have after some straightforward computation

$$\begin{aligned}\frac{\left(\tau^{N-1}+\tau^{2s-1}\right)}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}&=\frac{\left(2-(N+2s-2)t+O(t^2)\right)\left(1+\frac{(N+2s)t\kappa^2}{2(t^2+\kappa^2)}+O\left(\left(\frac{t\kappa^2}{t^2+2\kappa^2}\right)^2\right)\right)}{(t^2+\kappa^2)^{\frac{N}{2}+s}}\\&=\frac{2+2t+O(t^2)}{(t^2+\kappa^2)^{\frac{N}{2}+s}}.\end{aligned}$$

This implies

$$\begin{aligned}
&\int_0^1 \frac{\left(\tau^{N-1}+\tau^{2s-1}\right)d\tau}{\left(1+\tau^2-2\tau\langle\sigma,\eta\rangle\right)^{\frac{N}{2}+s}}\\
&= 2\kappa^{1-N-2s}\int_0^{\frac{1}{\kappa}} \frac{dx}{\left(x^2+1\right)^{\frac{N}{2}+s}} + 2\kappa^{2-N-2s}\int_0^{\frac{1}{\kappa}} \frac{xdx}{\left(x^2+1\right)^{\frac{N}{2}+s}} + O(\kappa^{3-N-s})\int_0^{\frac{1}{\kappa}} \frac{x^2dx}{\left(x^2+1\right)^{\frac{N}{2}+s}}\\
&= 2\kappa^{1-N-2s}\int_0^{\infty} \frac{dx}{\left(x^2+1\right)^{\frac{N}{2}+s}} + O(1) + O(\kappa^{3-N-s})\int_0^{\frac{1}{\kappa}} \frac{x^2dx}{\left(x^2+1\right)^{\frac{N}{2}+s}}.
\end{aligned} \tag{A.14}$$

Since $\kappa = |\sigma-\eta|$, the claim follows from Proposition A.4 and the kernel estimate in Lemma A.3. □

Lemma A.6. *Under the assumption of Lemma A.5 there holds*

$$\left|\int_{S^{N-1}} \omega\mathcal{L}_{s,\beta}\omega dS\right| \le c_{35}\int_{S^{N-1}} \omega^2 dS \qquad \forall \omega \in L^2(S^{N-1}), \tag{A.15}$$

where

$$c_{35} = \int_0^1 \left(\int_{S^{N-1}} \frac{dS(\eta)}{\left(1+\tau^2-2\tau\langle \mathbf{e}_N,\eta\rangle\right)^{\frac{N}{2}+s}}\right)\left(\tau^{-\beta}-1\right)\left|\tau^{N-1}-\tau^{N-1+c_s}\right|d\tau.$$

Proof. There holds by Cauchy-Schwarz inequality

$$\begin{aligned}
&\left|\int_{S^{N-1}} \omega\mathcal{L}_{s,\beta}\omega dS\right|\\
&\le \int_0^1 \left(\int_{S^{N-1}}\int_{S^{N-1}} \frac{|\omega(\eta)||\omega(\sigma)|dS(\eta)dS(\sigma)}{\left(1+\tau^2-2\tau\langle\sigma,\eta\rangle\right)^{\frac{N}{2}+s}}\right)\left(\tau^{-\beta}-1\right)\left|\tau^{N-1}-\tau^{N-1+c_s}\right|d\tau\\
&\le \int_0^1 \left(\int_{S^{N-1}}\int_{S^{N-1}} \frac{\omega^2(\eta)}{\left(1+\tau^2-2\tau\langle\sigma,\eta\rangle\right)^{\frac{N}{2}+s}} dS(\eta)dS(\sigma)\right)\times\left(\tau^{-\beta}-1\right)\left|\tau^{N-1}-\tau^{N-1+c_s}\right|d\tau\\
&\le \int_{S^{N-1}}\left(\int_0^1 \left(\int_{S^{N-1}} \frac{dS(\sigma)}{\left(1+\tau^2-2\tau\langle\sigma,\eta\rangle\right)^{\frac{N}{2}+s}}\right)\left(\tau^{-\beta}-1\right)\left|\tau^{N-1}-\tau^{N-1+c_s}\right|d\tau\right)\omega^2(\eta)dS(\eta).
\end{aligned}$$

Since, by invariance by rotation, we have

$$\int_{S^{N-1}} \frac{dS(\sigma)}{\left(1+\tau^2-2\tau\langle\sigma,\eta\rangle\right)^{\frac{N}{2}+s}} = \int_{S^{N-1}} \frac{dS(\sigma)}{\left(1+\tau^2-2\tau\langle\mathbf{e}_N,\sigma\rangle\right)^{\frac{N}{2}+s}},$$

we derive (A.15). □

We denote the upper hemisphere of the unit sphere in $\mathbb{R}^N$ by $S^{N-1}_+ = S^{N-1}\cap\mathbb{R}^N_+$.

Proposition A.7. *Let $N\ge 2$, $s\in(0,1)$ and $N-2s<\beta<N$. Then there exist a unique $\lambda_{s,\beta}>0$ and a unique (up to an homothety) positive $\psi_1\in W_0^{s,2}(S^{N-1}_+)$, such that*

$$\mathcal{A}_s\psi_1 = \lambda_{s,\beta}\mathcal{L}_{s,\beta}\psi_1 \qquad \text{in } S^{N-1}_+. \tag{A.16}$$

Furthermore the mapping $\beta\mapsto\lambda_{s,\beta}$ is continuous and decreasing from $(N-2s,N)$ onto $(0,\infty)$. Finally $\lambda_{s,\beta}=1$ if and only if $\beta=N-s$ and $\psi_1(\sigma)=(\sin\phi)^s$.

Proof. We first notice that

$$\int_{S^{N-1}_+} \omega\mathcal{A}_s\omega dS = \frac{1}{2}\int_{S^{N-1}_+}\int_0^\infty\int_{S^{N-1}_+} \frac{\left(\omega(\sigma)-\omega(\eta)\right)^2}{\left(1+\tau^2-2\tau\langle\sigma,\eta\rangle\right)^{\frac{N}{2}+s}}\tau^{N-1}dS(\eta)d\tau dS(\sigma), \tag{A.17}$$

for any $\omega \in C_0^1(S_+^{N-1})$. By Lemma A.5 and (A.11) with $r = q = 2$,

$$\int_{S_+^{N-1}}\int_0^\infty\int_{S_+^{N-1}} \frac{(\omega(\sigma)-\omega(\eta))^2}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\tau^{N-1}dS(\eta)d\tau dS(\sigma) \\ \leq c_{36}\left\|\omega\right\|^2_{W_0^{s,2}(S_+^{N-1})} + c_{37}\left\|\omega\right\|^2_{L^2(S_+^{N-1})},$$

where

$$\left\|\omega\right\|^2_{W_0^{s,2}(S_+^{N-1})} = \int_{S_+^{N-1}}\int_{S_+^{N-1}} \frac{(\omega(\sigma)-\omega(\eta))^2}{|\eta-\sigma|^{N-1+2s}}dS(\eta)dS(\sigma).$$

Since, by Poincaré inequality [16], there holds

$$\left\|\omega\right\|^2_{W_0^{s,2}(S_+^{N-1})} \geq c_{38}\left\|\omega\right\|^2_{L^2(S_+^{N-1})},$$

we obtain that the right-hand side of (A.17) is bounded from above by $\left(\frac{1}{2}c_{36}+\frac{c_{37}}{2c_{38}}\right)\left\|\omega\right\|^2_{W_0^{s,2}(S_+^{N-1})}$. Next we use the expansion estimates in Lemma A.5 to obtain that

$$\frac{\tau^{N-1}+\tau^{2s-1}}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}} \geq \frac{1}{(t^2+\kappa^2)^{\frac{N}{2}+s}} \qquad \forall t = 1-\tau \in (0,\epsilon_0),\ \forall(\sigma,\eta)\in S_+^{N-1}\times S_+^{N-1},$$

where $\kappa = |\sigma-\eta| \leq 2$. Hence

$$\int_0^\infty \frac{\tau^{N-1}d\tau}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}} \geq \int_0^{\epsilon_0}\frac{dt}{(t^2+\kappa^2)^{\frac{N}{2}+s}} = \kappa^{1-N-2s}\int_0^{\frac{\epsilon_0}{2}}\frac{dt}{(t^2+1)^{\frac{N}{2}+s}}.$$

Therefore,

$$\int_{S_+^{N-1}}\omega\mathcal{A}_s\omega dS \geq \int_0^{\frac{\epsilon_0}{2}}\frac{dt}{2(t^2+1)^{\frac{N}{2}+s}}\left\|\omega\right\|^2_{W_0^{s,2}(S_+^{N-1})}.$$

Finally we obtain

$$\frac{1}{c_{39}}\left\|\omega\right\|^2_{W_0^{s,2}(S_+^{N-1})} \\ \leq \int_{S_+^{N-1}}\int_0^\infty\int_{S_+^{N-1}} \frac{(\omega(\sigma)-\omega(\eta))^2}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\tau^{N-1}dS(\eta)d\tau dS(\sigma) \qquad (A.18) \\ \leq c_{39}\left\|\omega\right\|^2_{W_0^{s,2}(S_+^{N-1})}.$$

We consider the bilinear form in $W_0^{s,2}(S_+^{N-1})$

$$\mathbb{A}(\omega,\zeta) := \int_{S_+^{N-1}}\int_0^\infty\int_{S_+^{N-1}} \frac{(\omega(\sigma)-\omega(\eta))\,\zeta(\sigma)}{(1+\tau^2-2\tau\langle\sigma,\eta\rangle)^{\frac{N}{2}+s}}\tau^{N-1}dS(\eta)d\tau dS(\sigma).$$

Then $\mathbb{A}$ is symmetric and there holds

$$\mathbb{A}(\omega,\omega) = \int_{S_+^{N-1}}\omega\mathcal{A}_s\omega dS \geq \frac{1}{2c_{39}}\left\|\omega\right\|^2_{W_0^{s,2}(S_+^{N-1})},$$

and

$$|\mathbb{A}(\omega,\zeta)| \leq \left(\int_{S_+^{N-1}}\omega\mathcal{A}_s\omega dS\right)^{\frac{1}{2}}\left(\int_{S_+^{N-1}}\zeta\mathcal{A}_s\zeta dS\right)^{\frac{1}{2}} \leq \frac{c_{39}}{2}\left\|\omega\right\|_{W_0^{s,2}(S_+^{N-1})}\left\|\zeta\right\|_{W_0^{s,2}(S_+^{N-1})}.$$

By Riesz theorem, for any $L \in W^{-s,2}(S_+^{N-1})$ there exists $\omega_L \in W_0^{s,2}(S_+^{N-1})$ such that

$$\mathbb{A}(\omega_L,\zeta) = L(\zeta) \qquad \forall\zeta \in W_0^{s,2}(S_+^{N-1}).$$

We denote $\omega_L = \mathcal{A}_s^{-1}(L)$. It is clear that $\mathcal{A}_s^{-1}$ is positive and since the the embedding of $W_0^{s,2}(S_+^{N-1})$ into $L^2(S_+^{N-1})$ is compact by Rellich-Kondrachov theorem [16], $\mathcal{A}_s^{-1}$ is a compact operator. Hence the operator

$$\omega \mapsto \mathcal{A}_s^{-1}\circ\mathcal{L}_{s,\beta}\omega$$

is a compact positive operator (here we use the fact that $\beta > N-2s$ which makes $\mathcal{B}_{s,\beta}$ positive). By the Krein-Rutman theorem there exists $\mu > 0$ and $\psi_1 \in W_0^{s,2}(S_+^{N-1})$, $\psi_1 \geq 0$ such that

$$\mathcal{A}_s^{-1} \circ \mathcal{L}_{s,\beta}\psi_1 = \mu\psi_1.$$

The function ψ_1 is the unique positive eigenfunction and μ the only positive eigenvalue with positive eigenfunctions. Furthermore μ is the spectral radius of $\mathcal{A}_s^{-1} \circ \mathcal{B}_{s,\beta}$. If we set $\lambda_{s,\beta} = \mu^{-1}$, we obtain (A.16). It is also classical that $\lambda_{s,\beta}$ can be defined by

$$\lambda_{s,\beta} := \inf \left\{ \textstyle\int_{S_+^{N-1}} \omega \mathcal{A}_s \omega dS : \omega \in W_0^{s,2}(S_+^{N-1}), \omega \geq 0, \int_{S_+^{N-1}} \omega \mathcal{L}_{s,\beta}\omega dS = 1 \right\}. \tag{A.19}$$

Using (A.7), Lemma A.2 and monotone convergence theorem, we derive that the mapping

$$\beta \mapsto \int_{S_+^{N-1}} \omega \mathcal{L}_{s,\beta}\omega dS$$

is increasing and continuous. This implies that $\beta \mapsto \lambda_{s,\beta}$ is decreasing and continuous. Since $\displaystyle\int_{S_+^{N-1}} \omega \mathcal{L}_{s,\beta}\omega dS \to \infty$ when $\beta \uparrow N$, the expression (A.19) implies that $\lambda_{s,\beta} \to 0$ when $\beta \uparrow N$. Next, if $\omega \geq 0$ is an element of $W_0^{s,2}(S_+^{N-1})$ such that $\displaystyle\int_{S_+^{N-1}} \omega \mathcal{L}_{s,\beta}\omega dS = 1$, we derive from Poincaré inequality [16] and (A.15),

$$\|\omega\|^2_{W_0^{s,2}(S_+^{N-1})} \geq c_{38} \|\omega\|^2_{L^2(S_+^{N-1})} \geq \frac{c_{38}}{c_{35}}.$$

Since $c_{35} \to 0$ when $\beta \downarrow N-2s$, we infer that $\lim_{\beta \to N-2s} \lambda_{s,\beta} = \infty$. Consequently the mapping $\beta \mapsto \lambda_{s,\beta}$ is a decreasing homeomorphism from $(N-2s, N)$ onto $(0,\infty)$ and there exists a unique $\beta_s \in (N-2s, N)$ such that $\lambda_{s,\beta_s} = 1$. The following expression of the Martin kernel in $\mathbb{R}^N_+$ is classical,

$$M_s^{\mathbb{R}^N_+}(x,y) = c_{N,s}\, x_N^s |x-y|^{-N} \quad \forall x \in \mathbb{R}^N_+, y \in \partial\mathbb{R}^N_+, \tag{A.20}$$

hence, if $y = 0$, it is a separable singular s-harmonic function expressed in spherical coordinates with $x = (r,\sigma)$ by

$$M_s^{\mathbb{R}^N_+}((r,\sigma),0) = c_{N,s} r^{s-N} (\sin\phi)^s.$$

This means that the function $\sigma \mapsto \omega(\sigma) = (\sin\phi)^s$, which vanishes on $\overline{S_-^{N-1}}$ and belongs to $W_0^{s,2}(S_+^{N-1}) \cap L^\infty(S_+^{N-1})$, satisfies

$$\mathcal{A}_s\omega - \mathcal{L}_{s,N-s}\omega = 0.$$

The uniqueness of the positive eigenfunction implies that this function is ψ_1 and $\beta = N-s$. □

A.2. **The nonlinear problem.**

A.2.1. *Separable solutions in* $\mathbb{R}^N$. If we look for separable positive solutions of

$$(-\Delta)^s u + u^p = 0 \qquad \text{in } \mathbb{R}^N, \tag{A.21}$$

under the form $u(x) = r^{-\frac{2s}{p-1}}\omega(\sigma)$ where $x = (r,\sigma) \in \mathbb{R}_+ \times S^{N-1}$, then ω satisfies

$$\mathcal{A}_s\omega - \mathcal{L}_{s,\frac{2s}{p-1}}\omega + \omega^p = 0 \qquad \text{in } S^{N-1}. \tag{A.22}$$

Proposition A.8. *Assume $N \geq 2$ and $s \in (0,1)$.*

(i) If $p \geq p_3^*$ then there exists no positive solution of (A.22).

(ii) If $p_1^* < p < p_3^*$ then the unique positive solution of (A.22) *is a constant function with value*

$$\ell_{s,p} = (c_{35})^{\frac{1}{p-1}}, \tag{A.23}$$

where c_{35} is the constant defined in Lemma A.6.

Proof. If $p \geq p_3^*$, we assume that there exists a solution $\omega \geq 0$ of (A.22). Then ω satisfies

$$\int_{S^{N-1}} \omega \mathcal{A}_s \omega dS - \int_{S^{N-1}} \omega \mathcal{L}_{s,\frac{2s}{p-1}} \omega dS + \int_{S^{N-1}} \omega^{p+1} dS = 0.$$

Since $p \geq p_3^*$, we have $c_s \leq 0$ which implies

$$\int_{S^{N-1}} \omega \mathcal{L}_{s,\frac{2s}{p-1}} \omega dS \leq 0.$$

Then $\omega = 0$ since the two other integrals are nonnegative.

Next, if $p_1^* < p < p_3^*$ it is clear that if ω is a constant nonnegative solution of (A.22) then we have

$$\omega \int_0^1 \int_{S^{N-1}} \frac{(\tau^{-\frac{2s}{p-1}} - 1)(\tau^{N-1} - \tau^{N-1+c_s})}{(1 + \tau^2 - 2\tau\langle \sigma, \eta \rangle)^{\frac{N}{2}+s}} dS(\eta) d\tau = \omega^p \qquad \forall \sigma \in S^{N-1}.$$

Using invariance by rotation of the integral term on S^{N-1}, we derive the claim. Conversely, assume ω is any bounded nonconstant positive solution, then it belongs to $C^2(S^{N-1})$ by [25]. Let $\sigma_0 \in S^{N-1}$ where ω is maximal, then $\mathcal{A}_s \omega(\sigma_0) \geq 0$ thus

$$\omega^p(\sigma_0) \leq \mathcal{L}_{s,\frac{2s}{p-1}} \omega(\sigma_0) \leq \omega(\sigma_0) \int_0^1 \int_{S^{N-1}} \frac{(\tau^{-\frac{2s}{p-1}} - 1)(\tau^{N-1} - \tau^{N-1+c_s})}{(1 + \tau^2 - 2\langle \sigma_0, \eta \rangle)^{\frac{N}{2}+s}} dS(\eta d\tau = c_{51} \omega(\sigma_0).$$

Hence $\omega(\sigma_0) < \ell_{s,p}$. Similarly $\min_{S^{N-1}} \omega > \ell_{s,p}$, contradiction. □

Corollary A.9. *Assume $N \geq 2$, $s \in (0,1)$ and $p_1^* < p < p_3^*$. Then the only positive separable solution u of* (A.21) *in $\mathbb{R}^N \setminus \{0\}$ is*

$$x \mapsto U(x) = \ell_{s,p} |x|^{-\frac{2s}{p-1}}. \tag{A.24}$$

A.2.2. *Separable solutions in $\mathbb{R}^N_+$.* If we consider separable solutions $x \mapsto u(x) = r^{-\frac{2s}{p-1}} \omega(\sigma)$ of problem (1.15) then ω satisfies (1.16).

Proof of Theorem E.

Step 1: Non-existence. Assume that such a solution $\omega \geq 0$ exists, then

$$\int_{S^{N-1}_+} \omega \mathcal{A}_s \omega dS - \int_{S^{N-1}_+} \omega \mathcal{L}_{s,\frac{2s}{p-1}} \omega dS + \int_{S^{N-1}_+} \omega^p dS = 0.$$

Hence

$$\left(\lambda_{s,\frac{2s}{p-1}} - 1\right) \int_{S^{N-1}_+} \omega \mathcal{L}_{s,\frac{2s}{p-1}} \omega dS + \int_{S^{N-1}_+} \omega^p dS \leq 0. \tag{A.25}$$

If $\lambda_{s,\frac{2s}{p-1}} \geq 1$, equivalently $p \geq p_2^*$, the only nonnegative solution is the trivial one.

Step 2: Existence. Consider the following functional with domain $W_0^{s,2}(S^{N-1}_+) \cap L^{p+1}(S^{N-1}_+)$,

$$\omega \mapsto \mathcal{J}(\omega) := \int_{S^{N-1}_+} \omega \mathcal{A}_s \omega dS + \frac{1}{p+1} \int_{S^{N-1}_+} |\omega|^{p+1} dS - \int_{S^{N-1}_+} \omega \mathcal{L}_{s,\frac{2s}{p-1}} \omega dS. \tag{A.26}$$

Because of Lemma A.6, $\mathcal{J}(\omega) \to \infty$ when $\|\omega\|_{W_0^{s,2}(S^{N-1}_+)} + \|\omega\|_{L^{p+1}(S^{N-1}_+)} \to \infty$. Furthermore, for $\epsilon > 0$, we have

$$\mathcal{J}(\epsilon \psi_1) = \epsilon^2 \left(\lambda_{s,\frac{2s}{p-1}} - 1\right) \int_{S^{N-1}_+} \psi_1 \mathcal{L}_{s,\frac{2s}{p-1}} \psi_1 dS + \frac{\epsilon^{p+1}}{p+1} \int_{S^{N-1}_+} |\psi_1|^{p+1} dS.$$

This implies that $\inf \mathcal{J}(\omega) < 0$ if $\lambda_{s,\frac{2s}{p-1}} < 1$, and thus the infimum of $\mathcal{J}$ in $W_0^{s,2}(S^{N-1}_+) \cap L^{p+1}_+(S^{N-1}_+)$ is achieved by a nontrivial nonnegative solution of (1.16).

Step 3: Uniqueness.

(i) Existence of a maximal solution. By [25] any solution ω is smooth. Hence, at its maximum σ_0, it satisfies $\mathcal{A}_s \omega(\sigma_0) \geq 0$, thus

$$\omega(\sigma_0)^p \leq \mathcal{L}_{s,\frac{2s}{p-1}} \omega(\sigma_0) \leq \omega(\sigma_0) c_{35}.$$

This implies that $\sup\omega \le \ell_{s,p}$. From the equation the set $\mathcal{E} \subset W_0^{s,2}(S_+^{N-1})$ of positive solutions of (1.16) is bounded in $W_0^{s,2}(S_+^{N-1}) \cap L^\infty(S_+^{N-1})$ and thus in $C^s(S^{N-1}) \cap C^2(S_+^{N-1})$ by [25]. We put $\overline{\omega}(\sigma) = \sup\{\omega(\sigma) : \omega \in \mathcal{E}\}$. There exists a countable dense set $\mathcal{S} := \{\sigma_n\} \subset S_+^{N-1}$ and a sequence of function $\{\omega_n\} \subset \mathcal{E}$ such that

$$\lim_{n\to\infty} \omega_n(\sigma_k) = \overline{\omega}(\sigma_k).$$

Furthermore, this sequence $\{\omega_n\}$ can be constructed such that $\{\omega_n(\sigma_k)\}$ is nondecreasing for any k. Finally by local compactness estimate, $\{\omega_n\}$ converges to $\overline{\omega}$ in $C^{s-\delta}(S^{N-1}) \cap C^2(S_+^{N-1})$ for any $\delta \in (0, s)$ and weakly in $W_0^{s,2}(S_+^{N-1})$. This implies that $\overline{\omega}$ belongs to $\mathcal{E}$. It follows from [25, Th 1.2] that any $\omega \in \mathcal{E}$ satisfies

$$\omega(\sigma) \le c_{40} \left(\mathrm{dist}\,(\sigma, \partial S_+^{N-1})\right)^s = c_{40}\phi^s \qquad \forall \sigma \in S_+^{N-1}. \tag{A.27}$$

(ii) Existence of a minimal solution. This follows from Theorem 3.7 that $u_k^{\mathbb{R}^N_+} \uparrow u_\infty^{\mathbb{R}^N_+}$ and $u_\infty^{\mathbb{R}^N_+}$ is self-similar and it is the minimal solution of (1.17) in $\mathbb{R}^N_+$ which satisfies

$$\lim_{x\to 0} \frac{u_\infty^{\mathbb{R}^N_+}(x)}{M_s^{\mathbb{R}^N_+}(x,0)} = \infty. \tag{A.28}$$

Thus $u_\infty^{\mathbb{R}^N_+}(r,\sigma) = r^{-\frac{2s}{p-1}}\underline{\omega}(\sigma)$ and $\underline{\omega}$ is the minimal positive solution of (1.16). Furthermore it follows from (3.41) that

$$\underline{\omega}(\sigma) \ge c_{41} \left(\mathrm{dist}\,(\sigma, \partial S_+^{N-1})\right)^s = c_{41}\phi^s \qquad \forall \sigma \in S_+^{N-1}, \tag{A.29}$$

if $\phi = \phi(\sigma)$ is the latitude of σ.

(iii) End of the uniqueness proof. By combining (A.27) and (A.29) we infer that there exists $K > 1$ such that

$$\overline{\omega} \le K\underline{\omega} \qquad \text{in } S_+^{N-1}. \tag{A.30}$$

Assume $\overline{\omega} \ne \underline{\omega}$, then

$$\omega_1 := \underline{\omega} - \frac{1}{2K}(\overline{\omega} - \underline{\omega})$$

is a positive supersolution (by convexity) of (1.16). Moreover

$$\omega_2 := \left(\frac{1}{2} + \frac{1}{2K}\right)\underline{\omega}$$

is a positive subsolution of (1.16) smaller than ω_1 hence also than $\underline{\omega}$. It follows by classical construction that there exists a solution $\tilde{\omega}$ of (1.16) which satisfies $\omega_2 \le \tilde{\omega} \le \omega_1$, which contradicts the minimality of $\underline{\omega}$. □

Acknowledgements. The first author is supported by Fondecyt Grant 3160207. The second author is supported by collaboration programs ECOS C14E08. The authors certify that they have no conflict of interest.

References

[1] N. Abatangelo, *Large s-harmonic functions and boundary blow-up solutions for the fractional Laplacian*, Discrete Contin. Dyn. Syst. **35** (2015), 5555-5607.

[2] M. F. Bidaut-Véron and L. Vivier, *An elliptic semilinear equation with source term involving boundary measures: the subcritical case*, Rev. Mat. Iberoamericana **16** (2000), 477-513.

[3] K. Bogdan, *The boundary Harnack principle for the fractional Laplacian*, Stud. Math. **123** (1997), 43-80.

[4] K. Bogdan, *Representation of α-harmonic functions in Lipschitz domains*, Hiroshima Math. J. **29** (1999), 227-243.

[5] K. Bogdan and T. Byczkowski, *Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains*, Studia Math. **133** (1999), 53-92.

[6] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song and Z. Vondracek, *Potential Analysis of Stable Processes and its Extensions*, Lecture Notes in Mathematics, 1980, Spring, Berlin (2009).

[7] K. Bogdan, T. Jakubowski, M. Kwaśnicki, *Estimate and structure of α-harmonic functions*, Probab. Theory Related Fields **140** (2008), 345-381.
[8] L. A. Caffarelli and Y. Sire, *On some pointwise inequalities involving nonlocal operators*, (arxiv.org/abs/1604.05665).
[9] H. Chen, S. Alhomedan, H. Hajaiej and P. Markowich, *Complete study of the existence and uniqueness of solutions for semilinear elliptic equations involving measures concentrated on boundary*, Complex Var. Elliptic Equ. (2017).
[10] H. Chen, P. Felmer and A. Quaas, *L*arge solutions to elliptic equations involving fractional Laplacian, Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015), 1199-1228.
[11] H. Chen and L. Véron, *Weakly and strongly singular solutions of semilinear fractional elliptic equations*, Asymptotic Analyisi **88** (2014), 165-184.
[12] H. Chen and L. Véron, *Semilinear fractional elliptic equations involving measures*, J. Diff. Equ. **257** (2014), 1457-1486.
[13] H. Chen and J. Yang, *Semilinear fractional elliptic equations with measures in unbounded domain*, Nonlinear Analysis: T. M. A. **145** (2016), 118-142.
[14] Z. Chen and R. Song, *Estimates on Green functions and Poisson kernels for symmetric stable process*, Math. Ann. **312** (1998), 465-501.
[15] Z. Chen and R. Song, *Martin boundary and integral representation for harmonic functions of symmetric stable processes*, J. Funct. Anal. **159** (1998), 267-294.
[16] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012) 521-573.
[17] P. Felmer and A. Quaas, *Boundary blow up solutions for fractional elliptic equations*, Asymptot. Anal. **78** (2012), 123-144.
[18] A. Gmira and L. Véron, *Boundary singularities of solutions of some nonlinear elliptic equations*, Duke Math. J. **64** (1991), 271-324.
[19] P. Graczyk, T. Jakubowski and T. Luks, *Martin representation and Relative Fatou Theorem for fractional Laplacian with a gradient perturbation*, Positivity **17** (2013), 10431070.
[20] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case*, Arch. Rational Mech. Anal. **144** (1998), 201-231.
[21] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case*, J. Math. Pures Appl. **77** (1998), 481-524.
[22] M. Marcus and L. Véron, *Removable singularities and boundary trace*, J. Math. Pures Appl. **80** (2001), 879-900.
[23] M. Marcus and L. Véron, *The boundary trace and generalized boundary value problem for semilinear elliptic equations with coercive absorption*, Comm. Pure Appl. Math. **LVI** (2003), 689-731.
[24] M. Marcus and L. Véron, *Nonlinear second order elliptic equations involving measures*, De Gruyter Series in Nonlinear Analysis and Applications, 2013.
[25] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. **101** (2014), 275-302.

Departamento de Matemática, Pontificia Universidad Católica de Chile, Avda. Vicuña Mackenna 4860, Santiago, Chile
E-mail address: nguyenphuoctai.hcmup@gmail.com

Laboratoire de Mathématiques et Physique Théorique, Université François Rabelais, Tours, France
E-mail address: veronl@univ-tours.fr