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On explicit error estimates for the elastic wave propagation in heterogeneous media

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Abstract — This work deals with explicit a posteriori error estimates for elastic wave propagation in heterogeneous media. Based on some previous works on acoustic problems, the proposed explicit error estimator is derived in a non-natural L^∞ norm by using several reconstructions of numerically calculated fields. The effectivity of the estimator on uniform meshes and adaptive meshes is studied numerically. The results indicate that the error estimator gives an upper bound to the true error. Improvements for the reconstruction in time and in space are proposed.

Keywords — A posteriori error estimate, wave equation, L^∞ norm, effectivity

1. A posteriori error estimate for wave equation

1.1. Elastic wave propagation model problem

For $t \in (0, T)$ and $x \in \Omega$, we consider the elastic wave propagation problem in an elastic solid Ω : finding $(u(x, t) \times v(x, t)) \in X^2 \times N^2$ with $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f \in N^2$ such that

$$\begin{cases} \rho \partial_t v(x, t) - \nabla \cdot \sigma(u) = f & \forall (x, t) \in \Omega \times (0, T) \\ \text{Div}(\sigma(\partial_t u(x, t) - v(x, t))) = 0 & \forall (x, t) \in \Omega \times (0, T) \\ u = 0 & \forall (x, t) \in \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) & \forall x \in \Omega \\ v(x, 0) = v_0(x) & \forall x \in \Omega \end{cases} \quad (1)$$

We define the space of functions $F(0, T; E) = \{u \in E; \text{ for } \forall t \in (0, T) \text{ and } t \rightarrow u(x, \cdot) \text{ is in } F\}$. For example, here we denote: $N^2 = L^2(0, T; L^2(\Omega))$, $X^2 = L^2(0, T; V(\Omega))$. Notice that the second-order elastodynamic equation is transformed here to a first-order hyperbolic system (1) and the equivalence of $\partial_t u$ and v is imposed through the equilibrium of internal forces.

1.2. Space and time reconstructions

To be solved, the system (1) is discretized in space by the finite element method and in time by a finite difference scheme. The so-obtained numerical solutions are denoted by $(u_{h\tau}, v_{h\tau})$. With respect to different regularities required by corresponding ingredients of the estimator, we take a time-continuous reconstruction $(\bar{u}_{h\tau}, \bar{v}_{h\tau})$ with Hermitian interpolation:

$$\begin{cases} \bar{u}_{h\tau}(x, t) := \frac{(t - t^{n+1})(2t + t^{n+1} - 3t^n)}{\tau^n} u_{h\tau}^n(x) + \frac{(t - t^n)(-2t - t^{n+1} + 3t^n)}{\tau^n} u_{h\tau}^{n+1}(x) \\ \quad + \frac{(t - t^n)(t^{n+1} - t)^2}{\tau^n} v_{h\tau}^n(x) + \frac{(t - t^{n+1})(t^n - t)^2}{\tau^n} v_{h\tau}^{n+1}(x) \\ \bar{v}_{h\tau}(x, t) := \frac{t^{n+1} - t}{\tau^n} v_{h\tau}^n(x) + \frac{t - t^n}{\tau^n} v_{h\tau}^{n+1}(x) - \frac{(t - t^n)(t^{n+1} - t)^2}{\tau^n} \frac{v_{h\tau}^{n+2} - 2v_{h\tau}^{n+1} + v_{h\tau}^n}{(\tau^n)^2}(x) \end{cases} \quad (2)$$

and two field reconstructions in space $(\tilde{\sigma}_{h\tau}, \tilde{\delta}_{h\tau})$:

$$\begin{aligned} \tilde{\sigma}_{h\tau}(\cdot, t) &\in H(\text{div}, \Omega), \\ (\partial_t \bar{v}_{h\tau} - \nabla \cdot \tilde{\sigma}_{h\tau}^n, 1)_K &= (f, 1)_K \quad \text{in } I_n \quad \forall t \in I_n, \forall 1 \leq n \leq N \end{aligned} \quad (3)$$

$$\begin{aligned} \tilde{\delta}_{h\tau}(\cdot, t) &\in H(\text{div}, \Omega), \\ (\partial_{tt} \bar{v}_{h\tau} - \nabla \cdot \tilde{\delta}_{h\tau}, 1)_K &= (\partial_t f, 1)_K \quad \text{in } I_n \quad \forall t \in I_n, \forall 1 \leq n \leq N \end{aligned} \quad (4)$$

1.3. Explicit error bound

The residual method of a posteriori error estimates is exploited with all these reconstructions mentioned. We define the indicator as follows:

- the residual and flux indicator:

$$\|\eta_R + \eta_{DF}\|_{N^2} = \left\{ \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h} (\eta_{R,K}^n + \eta_{DF,K}^n)^2 dt \right\}^{\frac{1}{2}} \quad (5)$$

with $\eta_{R,K}^n = \frac{C_P h_K C_K}{\sqrt{\lambda_{\min}}} \|f - \partial_t \bar{v}_{h\tau} + \nabla \cdot \tilde{\sigma}_{h\tau}^n\|_{0,K}$ in I_n (which represents dynamic volume residues in the interior of each element), $\eta_{DF,K}^n = \frac{1}{\sqrt{\lambda_{\min}}} \|\tilde{\sigma}_{h\tau}^n - \sigma(\bar{u}_{h\tau})\|_{0,K}$ in I_n (which represents constraints vectors' jumps of solutions before reconstruction); where $I_n = (t^n, t^{n+1}]$, λ_{\min} is the smallest eigenvalue of the fourth order tensor \mathbb{C} , C_P is the Poincaré constant and C_K is the Korn constant;

- the derived residual and flux indicator:

$$\|\eta_{DE} + \eta_{GV}\|_{N^2} = \left\{ \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}_h} (\eta_{DE,K}^n + \eta_{GV,K}^n)^2 dt \right\}^{\frac{1}{2}} \quad (6)$$

with $\eta_{DE,K}^n = \frac{C_P h_K C_K}{\sqrt{\lambda_{\min}}} \|-\partial_t f|_{I_n} + \partial_{tt} \bar{v}_{h\tau} - \nabla \cdot \tilde{\delta}_{h\tau}^n\|_{0,K}$ in I_n (which represents velocity of dynamic volume residues) and $\eta_{GV,K}^n = \frac{1}{\sqrt{\lambda_{\min}}} \|-\tilde{\delta}_{h\tau}^n + \sigma(\partial_t \bar{u}_{h\tau})\|_{0,K}$ (which represents velocity of constraints vectors' jumps);

- the nonconformity indicator:

$$\|\eta_{VN}\|_{X^2} = \|\partial_t \bar{u}_{h\tau} - \bar{v}_{h\tau}\|_{X^2}; \quad (7)$$

- the two initial condition indicators:

$$\eta_{CI,1} = \frac{1}{\sqrt{2}} \|v_0 - \bar{v}_{h\tau}^0\|_{0,\Omega}; \quad (8)$$

$$\eta_{CI,2} = \frac{1}{\sqrt{2}} \|u_0 - \bar{u}_{h\tau}^0\|_{a,\Omega}.$$

We can derive the error estimator with these indicators:

$$\begin{aligned}
& \sqrt{\|v - \bar{v}_{h\tau}\|_{N^\infty}^2 + \|u - \bar{u}_{h\tau}\|_{X^\infty}^2} \tag{9} \\
& \leq 2 \left\{ \sup_n \{\|\eta_R^n + \eta_{DF}^n\|_{0,\Omega}\} + \|\eta_R^0 + \eta_{DF}^0\|_{0,\Omega} + \sqrt{T}(\|\eta_{VN}\|_{X^2} + \|\eta_{DE} + \eta_{GV}\|_{N^2}) \right\} \\
& \quad + \sqrt{2} \{ \|\eta_R + \eta_{DF}\|_{N^2} \|\eta_{VN}\|_{X^2} + \eta_{CI,1}^2 + \eta_{CI,2}^2 \}^{\frac{1}{2}}
\end{aligned}$$

We notice that all the indicators are fully computable from $(u_{h\tau}, v_{h\tau})$ and we can distinguish and estimate separately the different error components.

2. Numerical results

A one-dimensional elastic rod is considered as a numerical example. The left end of the rod is submitted to a pressure loading that evolves in time as a truncated Ricker-type signal while its right end is left free. See Figure 1.

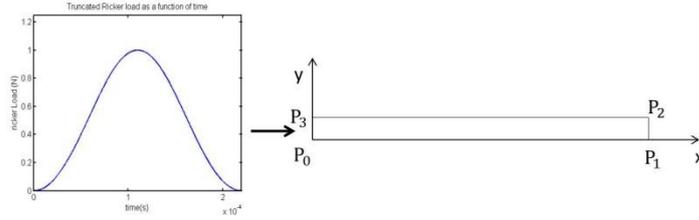
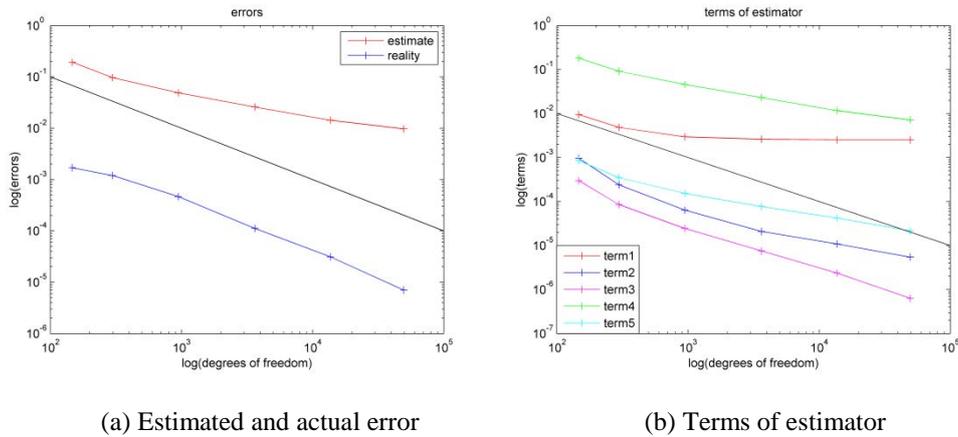


Figure 1 – One-dimensional rod example

We consider two cases: homogeneous material with uniform mesh and heterogeneous material with uniform and adaptive meshes; then we compare the estimated and actual errors in heterogeneous case and we analyze the different terms of estimator: $\text{term1} = 2 \sup_n \{\|\eta_R^n + \eta_{DF}^n\|_{0,\Omega}\}$; $\text{term2} = 2 \|\eta_R^0 + \eta_{DF}^0\|_{0,\Omega}$; $\text{term3} = 2\sqrt{T} \|\eta_{VN}\|_{X^2}$; $\text{term4} = 2\sqrt{T} \|\eta_{DE} + \eta_{GV}\|_{N^2}$; $\text{term5} = \sqrt{2} \|\eta_R + \eta_{DF}\|_{N^2} \|\eta_{VN}\|_{X^2}$.



(a) Estimated and actual error

(b) Terms of estimator

Figure 2.1 – Heterogeneous case with uniform mesh

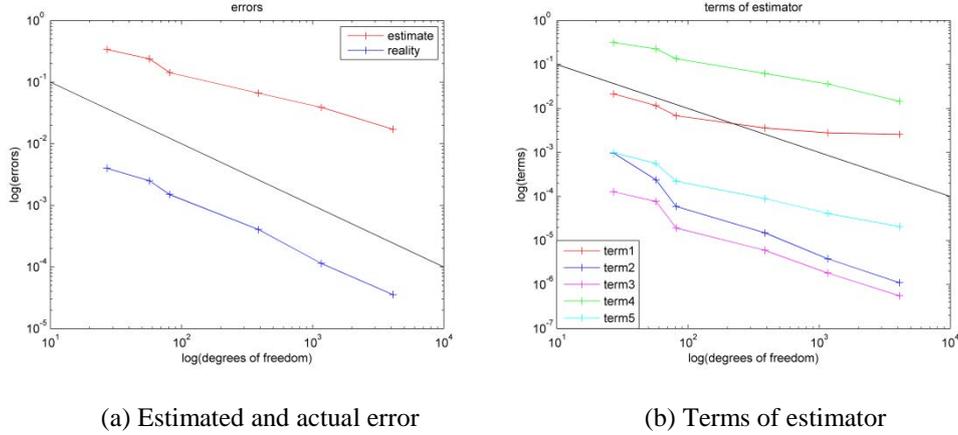


Figure 2.2 – Heterogeneous case with adaptive mesh

In Figure 3(a), we notice that the estimated error gives a computable upper bound on the unknown error to actual error in all cases; but obviously, the effectivity index, as the ratio of the estimated and actual error, is too large and it cannot converge to 1 as the computational effort grows. In Figure 3(b), we find that the first term $\|\eta_{DE} + \eta_{GV}\|_{N^2}$ and the third term $\sup\{\|\eta_R^n + \eta_{DF}^n\|_{0,\Omega}\}$ contribute the most to the estimated error and their performance is determined by the reconstructions in time and in space.

3. Conclusion

We develop an explicit a posteriori error estimator for elastic wave propagation in heterogeneous media. Further research works are needed by taking account higher order reconstructions in time $(\bar{u}_{h\tau}, \bar{v}_{h\tau})$ and in space $(\tilde{\sigma}_{h\tau}, \tilde{\delta}_{h\tau})$, which are two key points to improve the numerical results.

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