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SDEs with uniform distributions: Peacocks, Conic martingales and mean reverting uniform diffusions

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Abstract

It is known since Kellerer (1972) that for any peacock process there exist martingales with the same marginal laws. Nevertheless, there is no general method for finding such martingales that yields diffusions. Indeed, Kellerer's proof is not constructive: finding the dynamics of processes associated to a given peacock is not trivial in general. In this paper we are interested in the *uniform peacock* that is, the peacock with uniform law at all times on a generic *time-varying support* $[a(t), b(t)]$. We derive explicitly the corresponding Stochastic Differential Equations (SDEs) and prove that, under certain conditions on the boundaries $a(t)$ and $b(t)$, they admit a unique strong solution yielding the relevant diffusion process. We discuss the relationship between our result and the previous derivation of diffusion processes associated to square-root and linear time-boundaries, emphasizing the cases where our approach adds strong uniqueness, and study the local time and activity of the solution processes. We then study the peacock with uniform law at all times on a *constant support* $[-1, 1]$ and derive the SDE of an associated mean-reverting diffusion process with uniform margins that is not a martingale. For the related SDE we prove existence of a solution in $[0, T]$. Finally, we provide a numerical case study showing that these processes have the desired uniform behaviour. These results may be used to model random probabilities, random recovery rates or random correlations.

Keywords: Uniformly distributed Stochastic Differential Equation, Conic Martingales, Peacock Process, Uniformly distributed Diffusion, Uniform Martingale Diffusions, Mean Reverting Uniform Diffusion, Mean Reverting Uniform SDE, Maximum Entropy Stochastic Recovery Rates, Maximum Entropy Stochastic Correlation, Uniform SDE Simulation.

AMS classification codes: 60H10, 60J60

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1 Introduction

A *peacock* is an integrable process that is increasing in the convex order. Equivalently, a peacock is a process with (i) constant expected value and (ii) whose transform via any positive and convex function Ψ has an increasing expectation (see Definition 1.3 in [9]). More precisely, we need $t \mapsto \mathbb{E}(\Psi(X_t))$ to be increasing for any convex function Ψ such that $\mathbb{E}(|\Psi(X_t)|) < \infty$ for all t . From this equivalent representation, it is trivial to show via the law of iterated expectations and Jensen's inequality that any martingale is a peacock. Reciprocally, it is known from Kellerer [12] that for any peacock there exist martingales (called *associated martingales*) with the same marginal laws. Nevertheless, there is no guarantee that these associated martingales are diffusions. Moreover, specifying explicitly the processes associated to a given peacock is not trivial. In this paper, we provide the explicit dynamics of diffusion processes associated to the *uniform peacocks* that is, the peacocks whose marginals have a uniform distribution on a time-varying support imposing, without loss of generality, $X_0 = 0$. To that end, we introduce a new family of regular diffusion martingales evolving on the conically expanding support $t \mapsto [-b(t), b(t)]$, $b(0) = 0$ via the Stochastic Differential Equation (SDE)

$$dX_t = \left(\mathbb{I}_{\{X_t \in (-b(t), b(t))\}} \frac{\dot{b}(t)}{b(t)} (b(t)^2 - X_t^2) \right)^{1/2} dW_t, \quad X_0 = 0,$$

We will show that, under adequate conditions on the boundary, this SDE admits a unique strong solution which is associated to the uniform peacock. This extends previously known results where $b(t)$ is for example equal to t (p. 252 in [9]), adding strong uniqueness. Our result allows one to show strong existence and uniqueness for the case $b(t) = t^\alpha$, $\alpha > 1/2$. The case $b(t) = \sqrt{t}$ has to be dealt with different techniques. For cases like $b(t) = \sqrt{t}$ we use the approach in p. 253–260 of [9], and in that case we can only obtain uniqueness in law. We further show that the solution processes spend zero time at the boundaries.

The above diffusion coefficient was initially guessed by informally inverting the forward Kolmogorov (also known as Fokker-Planck) equation, when forcing the marginal density of the solution X to be uniform at all times with support $[-b, b]$ as initially sketched in the preprint [3]. This inversion technique was used in the past to construct diffusion processes with densities in exponential families [2, 4] and has been used more generally in a variety of contexts in mathematical finance. For example, [7] finds the diffusion coefficient (“local volatility”) that is consistent with a probability law extrapolated from a surface of option prices. The paper [6] deals with designing a diffusion process consistent with a mixture of distributions for volatility smile modeling, whereas [5] inverts the Kolmogorov equation to show how two stochastic processes with indistinguishable laws in a time grid under the historical measure can lead to arbitrarily different option prices.

The paper is structured as follows. In Section 2 we formulate the problem. A solution is attempted in Section 3 along the lines of the above mentioned inversion. We then study the solution rigorously in Section 4 and prove that the related SDE admits a unique strong

solution. We further prove that the solution has indeed a uniform distribution with the desired conic boundary. Being bounded on a finite horizon, the solution is thus a genuine martingale associated to the uniform peacock. In Section 5 we re-scale the conic diffusion martingale and study the related mean-reverting uniform diffusions, where now the uniform law is not conic but constant. Two special cases of interest are standard uniforms and uniforms in $[-1, 1]$, which can be used to model for example maximum entropy recovery rates or random probabilities and random correlations, respectively. We further show that the rescaled processes have zero local time at the boundaries -1 and 1 . In Section 6 we revisit the two previously known cases and hint at new choices for the boundaries. In the linear case we study the process pathwise activity, finding that the pathwise activity of the mean reverting diffusion vanishes asymptotically. The behavior of the process is illustrated based on numerical simulations.

2 Conic diffusion martingales with uniform distribution

We set out to construct a martingale diffusion process X (zero drift) with marginal at time t having a uniform distribution in an interval $[a(t), b(t)]$. The martingale condition implies that $\mathbb{E}[X_t] = \mathbb{E}[X_0]$ for all $t \geq 0$, whereas the uniform distribution requirement implies that $\mathbb{E}[X_t] = [a(t) + b(t)]/2$ for all $t \geq 0$. Thus we have $a(t) + b(t) = a(0) + b(0)$ for all $t \geq 0$. We will assume $a(0) = b(0) = 0$, taking the initial condition X_0 to be deterministic and with value zero (Dirac delta law in 0). Hence $b(t) = -a(t)$ for all $t \geq 0$.

With such preliminaries in mind, we state the following

Problem 1 (Designing conic martingale diffusions with given uniform law). *Consider the diffusion process*

$$dX_t = \sigma(X_t, t)dW_t, \quad X_0 = 0. \quad (1)$$

Find a diffusion coefficient $\sigma(x, t)$ such that

1. *The SDE (1) has a unique strong solution;*
2. *The solution of (1) at time $t > 0$ is uniformly distributed in $[-b(t), b(t)]$ for a non-negative strictly increasing continuous function $t \mapsto b(t)$ with $b(0) = 0$.*

In other terms, our aim is to build a diffusion martingale X as in (1) such that the process X has a density $p(x, t)$ at time $t > 0$ at the point x given by the uniform density

$$p(x, t) = \rho(x, t) := \mathbb{I}_{\{x \in [-b(t), b(t)]\}} / (2b(t)). \quad (2)$$

In Problem 1, b is restricted to be strictly increasing in time. The reason is that the tight upper (resp. lower) bound of any bounded martingale must be a non-decreasing (resp. non-increasing) function ([16]). Hence, X is a *conic martingale*; it is a martingale that exhibits a conic behavior. We will need strict monotonicity in the following derivation, so we assumed b to be strictly increasing in Problem 1.

3 Deriving the candidate SDE for a uniformly distributed martingale

Let us now guess a candidate solution σ for Problem 1. To do this, we write the forward Kolmogorov (or Fokker Planck) equation for the density p of (1), impose ρ to be a solution and derive the resulting σ . The derivation is informal but it is given full mathematical rigor by showing later that the resulting SDE (1) has a unique strong solution and confirming further, via moments analysis, that the density is indeed uniform.

The forward Kolmogorov eq. for (1) with ρ plugged in as a solution reads

$$\frac{\partial \rho(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma(x, t)^2 \rho(x, t)), \quad \rho(x, 0) = \delta_0(x). \quad (3)$$

Now we integrate twice both sides of (3) with respect to x and assume we can switch integration with respect to x and differentiation with respect to t (one can check *a posteriori* that the solution we find has a continuous partial derivative with respect to t so that Leibniz's rule can be used). We obtain

$$\frac{\partial}{\partial t} \left(\int_{-\infty}^x \left(\int_{-\infty}^y \rho(z, t) dz \right) dy \right) = \frac{1}{2} \sigma(x, t)^2 \rho(x, t), \quad (4)$$

assuming the relevant first and second derivatives with respect to x on the right hand side vanish fast enough at minus infinity. Compute for $t > 0$, substituting from (2),

$$\varphi(x, t) := \int_{-\infty}^x \left(\int_{-\infty}^y \rho(z, t) dz \right) dy = \begin{cases} 0, & \text{if } x < -b(t) \\ \frac{(x+b(t))^2}{4b(t)}, & \text{if } x \in [-b(t), b(t)] \\ x, & \text{if } x > b(t) \end{cases}$$

and note that φ is continuous in x . Equivalently,

$$\varphi(x, t) = \frac{(x+b(t))^2}{4b(t)} \mathbb{I}_{\{x \in [-b(t), b(t)]\}} + x \mathbb{I}_{\{x > b(t)\}}. \quad (5)$$

Thus, rewriting (4) as

$$\frac{\partial \varphi(x, t)}{\partial t} = \frac{1}{2} \sigma(x, t)^2 \rho(x, t), \quad (6)$$

and substituting (5) we are done. To do this, we need to differentiate φ with respect to time. The calculations are all standard but one has to pay attention when differentiating terms in (5) such as

$$\mathbb{I}_{\{x \in [-b(t), b(t)]\}} = \mathbb{I}_{\{x \geq -b(t)\}} - \mathbb{I}_{\{x > b(t)\}}$$

which can be differentiated in the sense of distributions,

$$\frac{d}{dt} \mathbb{I}_{\{x > b(t)\}} = \frac{d}{dt} \mathbb{I}_{\{t < b^{-1}(x)\}} = -\delta_{b^{-1}(x)}(t)$$

where the index in δ denotes the point where the Dirac delta distribution is centered. One can check that all terms involving δ 's either offset each other or are multiplied by a function that vanishes at the point of evaluation.

Assuming b is differentiable, omitting time arguments and denoting differentiation with respect to time with a dot one gets:

$$\frac{\partial \varphi(x, t)}{\partial t} = -\frac{-\dot{b}(2b)(x+b) + (2\dot{b})(x+b)^2}{2(2b)^2} \mathbb{I}_{\{x \in [-b, b]\}}.$$

We notice that \dot{b} appears only in ratios \dot{b}/b , so that this quantity may be extended to time $t = 0$ by continuity if needed provided that the limit exists.

The above quantity is the left hand side of (6). We can substitute ρ on the right hand side and we have that

$$-\frac{-\dot{b}(2b)(x+b) + (2\dot{b})(x+b)^2}{2(2b)^2} \mathbb{I}_{\{x \in (-b, b)\}} = \frac{1}{2} \frac{\sigma(x, t)^2}{2b} \mathbb{I}_{\{x \in [-b, b]\}}.$$

After some algebra, one obtains

$$\sigma^2(x, t) = \mathbb{I}_{\{x \in [-b(t), b(t)]\}} \frac{\dot{b}(t)}{b(t)} (b(t)^2 - x^2).$$

From the above development, we expect the diffusion coefficient $\sigma(x, t)$ defined as

$$\sigma(x, t) := \mathbb{I}_{\{x \in [-b(t), b(t)]\}} \sqrt{\frac{\dot{b}(t)}{b(t)} (b(t)^2 - x^2)}. \quad (7)$$

to be a valid candidate for the solution X of (1) to be a martingale with marginals having a uniform law in $[-b, b]$. In order to rigorously show that, we prove in the next section that, under suitable regularity condition on the boundaries $t \mapsto b(t)$, the SDE (1) with diffusion coefficient (7) admits a unique strong solution and that this solution has indeed a uniform law at all times. **In the more general case where regularity of the boundary is relaxed we prove that the solution is unique in law.**

4 Analysis of the SDE: solutions and distributions

Theorem 1 (Existence and Uniqueness of Solution for candidate SDE solving Problem 1). *Let $t \mapsto b(t)$ be a strictly increasing function defined on $[0, T]$, continuous in $[0, T]$ and continuously differentiable in $]0, T]$. Assume $b(0) = 0$ and $T > 0$. Assume **$\dot{b}(t)$ to be bounded in $[0, T]$** . The stochastic differential equation*

$$dX_t = \left(\mathbb{I}_{\{X_t \in [-b(t), b(t)]\}} \frac{\dot{b}(t)}{b(t)} (b(t)^2 - X_t^2) \right)^{1/2} dW_t, \quad X_0 = 0, \quad (8)$$

whose diffusion coefficient is extended to $t = 0$ by continuity, if needed, via

$$\sigma(x, 0) := 0 \text{ for all } x,$$

admits a unique strong solution and its solution X is distributed at every point in time as a uniform distribution concentrated in $[-b(t), b(t)]$. We thus have a conic diffusion martingale with the cone expansion controlled by the time function $b(t)$. *If moreover $\dot{b}(t)b(t)$ admits a finite limit for $t \downarrow 0$, then one can show that the solution processes spend zero time at the boundaries $-b(t)$ and $b(t)$.*

Proof. By continuity of diffusion paths, the solution X to the SDE (8), if it exists, belongs to $[-b(t), b(t)]$ almost surely. Indeed, the diffusion coefficient $\sigma(t, x)$ vanishes at the boundaries $\{-b(t), b(t)\}$. Because $b(t)$ is increasing, the process cannot exit the cone $[-b(t), b(t)]$.

This will be further confirmed by our local time calculation in Theorem 4 below.

It remains to prove that the solution X to (8) exists and is unique. To that end, it is enough to show that $\sigma(x, t)$ satisfies the linear growth bound and is Holder-1/2 for all $t \in [0, T]$ [11].

Clearly, $\sigma(x, t)$ in (7) satisfies the linear growth bound since it is uniformly bounded on $[0, T]$. To see this, notice that

$$0 \leq \sigma^2(x, t) = \mathbb{I}_{\{-b(t) \leq x \leq b(t)\}} (\dot{b}(t)/b(t)) (b^2(t) - x^2) \leq \dot{b}(t)b(t) \text{ for all } x,$$

and that $\dot{b}(t)b(t)$ is bounded on $[0, T]$ by assumption, with zero limit when $t \downarrow 0$. This allows us to conclude that

$$\lim_{t \downarrow 0} \sigma^2(x, t) = 0 \text{ for all } x.$$

Since $\sigma(x, t)^2$ is continuous and bounded on $(0, T]$, it admits a continuous extension at $t = 0$ taking value zero. The extended $\sigma(x, t)$ is unique and uniformly bounded on $[0, T]$.

We now proceed with the Holder continuity of σ . Of course, $f(x) = \sqrt{|x|}$ is Holder-1/2 on \mathbb{R} since $|\sqrt{|x|} - \sqrt{|y|}| \leq \sqrt{|x - y|}$ for all x, y . We now check that $\sigma(t, x)$ is Holder-1/2 uniformly in $t > 0$ ($t = 0$ is not a problem given the above extension by continuity). See also [10]).

Define $I(t) := [-b(t), b(t)]$. We check the possible cases.

1. If $x, y \notin I(t)$, the diffusion coefficient vanishes and one gets $|\sigma(t, x) - \sigma(t, y)| = 0$
2. If $x, y \in I(t)$, using the Holder-1/2 continuity of $\sqrt{|x|}$:

$$\begin{aligned} |\sigma(t, x) - \sigma(t, y)| &= \sqrt{\frac{\dot{b}(t)}{b(t)}} |\sqrt{b^2(t) - x^2} - \sqrt{b^2(t) - y^2}| \\ &\leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{|(b^2(t) - x^2) - (b^2(t) - y^2)|} \\ &= \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{|y^2 - x^2|} \leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{|y + x|} \sqrt{|y - x|} \leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{2b(t)} \sqrt{|x - y|} \\ &= \sqrt{2\dot{b}(t)} \sqrt{|x - y|} \end{aligned} \tag{9}$$

and we are done since \dot{b} is assumed to be bounded in $[0, T]$.

3. If $x \in I(t)$, $y > b(t)$:

$$\begin{aligned} |\sigma(t, x) - \sigma(t, y)| &= |\sigma(t, x)| = \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{b^2(t) - x^2} = \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{b(t) + x} \sqrt{b(t) - x} \leq \\ &\leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{2b(t)} \sqrt{b(t) - x} \leq \sqrt{2\dot{b}(t)} \sqrt{|x - y|} \end{aligned}$$

and again we are done since \dot{b} is bounded in t .

4. If $x \in I(t)$, $y < -b(t)$ (so that $-y > b(t)$) :

$$\begin{aligned} |\sigma(t, x) - \sigma(t, y)| &= |\sigma(t, x)| \leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{b(t) - x} \sqrt{b(t) + x} \leq \\ &\leq \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{2b(t)} \sqrt{x + b(t)} \leq \sqrt{2\dot{b}(t)} \sqrt{x - y} \end{aligned}$$

5. The case $x \notin I(t)$, $y \in I(t)$ is similar to steps 3 and 4.

Hence, the solution X to (8) exists and is unique. Because it is bounded and evolves between $-b(t)$ and $b(t)$, it is a conic $[-b(t), b(t)]$ -martingale. \square

Remark 1 (Indicator function in the diffusion coefficient). We notice that the diffusion coefficient vanishes for $x = \pm b(t)$, that diffusion paths are continuous and that the boundary is expanding. It follows that even if we omit the indicator in the diffusion coefficient expression, the related SDE will not leave the cone $[-b, b]$. Therefore, one could omit the indicator whenever the diffusion coefficient is featured inside a SDE.

We have proven that the SDE (8) has a unique strong solution. The SDE itself has been obtained by inverting the Kolmogorov equation for a uniform marginal density at time t in $[-b(t), b(t)]$, so we expect the density of the solution to be that uniform distribution. However, we haven't proven that the forward Kolmogorov equation for the density of (8) has a unique solution. To prove that our SDE (8) has the desired uniform distribution, one resorts to a characterization of the uniform distribution by its moments, showing that the moments of the solution of (8) are the same as the moments of the desired uniform law, and showing that this characterizes the uniform law. The latter is clearly related to Carleman's theorem, as it is well known that having uniformly bounded moments, the continuous uniform distribution on an interval $[a, b]$ with finite $a, b \in \mathbb{R}$ is determined by its moments, see for example Chapter 30 of [1]. This proof is straightforward but we include it in Appendix A for completeness. A different approach is using Theorem 2 below, since that is enough to guarantee a uniform distribution.

The special case $b(t) = kt$ gives us a conic martingale with uniform distribution where the boundaries grow symmetrically and linearly in time. This example was considered originally in [3] and is also in [9] (see for instance ex. 6.5 p.253 with $\varphi(x) = x$ and $f(z) = 1/2 \mathbb{I}_{\{-1 \leq z \leq 1\}}$). More generally, our result allows to treat the case $b(t) = t^\alpha$, for $\alpha \geq 1$. Staying in the class of boundaries t^α , we see that the case $\alpha < 1$ violates our assumptions, since in that case \dot{b} is not bounded in 0, and has to be dealt with differently. For $1/2 \leq \alpha < 1$, and with the square root case in mind in particular, we now introduce a different approach to prove existence (but not uniqueness) of the SDE solution, as done in the peacock processes literature [9].

Theorem 2 (Existence of Solution for SDE solving Problem 1 under milder conditions on the boundary). *Let b a strictly increasing function defined on $[0, T]$ and of class C^1 in $(0, T]$, with $b(0) = 0$ and T a positive real number. Assume $b(t)\dot{b}(t)$ to be bounded in $(0, T]$. **DB: DO WE NEED A LIMIT $\lim_{t \downarrow 0} \dot{b}(t)b(t)$ FOR THE WEAK SOLUTION AND UNIQUENESS IN LAW?** The stochastic differential equation (8), namely*

$$dX_t = \mathbb{I}_{\{X_t \in [-b(t), b(t)]\}} \left(\frac{\dot{b}(t)}{b(t)} (b^2(t) - X_t^2) \right)^{1/2} dW_t, \quad t > 0, \quad X_0 = 0,$$

admits a weak solution that is unique in law and its solution X is distributed at every point in time as a uniform distribution concentrated in $(-b(t), b(t))$. We thus have a conic diffusion martingale with the cone expansion controlled by the time function $b(t)$. If moreover $\dot{b}(t)b(t)$ admits a finite limit for $t \downarrow 0$, then one can show that the solution processes spend zero time at the boundaries $-b(t)$ and $b(t)$.

Proof. By continuity of diffusion paths, the solution X to the SDE (8), if it exists, belongs to $[-b(t), b(t)]$ almost surely. The solution of (8) has to be understood in a first step as a process satisfying, for any $t \geq \epsilon > 0$

$$X_t = X_\epsilon + \int_\epsilon^t \mathbb{I}_{\{x \in [-b(s), b(s)]\}} \left(\frac{\dot{b}(s)}{b(s)} (b^2(s) - X_s^2) \right)^{1/2} dW_s$$

where X_ϵ has a uniform law in $(-b(\epsilon), b(\epsilon))$. The value of X at time 0 is defined by continuity when ϵ goes to zero (we will prove that the limit exists), and (8) can be written $X_t = \int_0^t \mathbb{I}_{\{X_s \in [-b(s), b(s)]\}} \left(\frac{\dot{b}(s)}{b(s)} (b^2(s) - X_s^2) \right)^{1/2} dW_s$ which has a meaning even if $\sigma(0, x)$ is not well defined. The diffusion coefficient $\sigma(t, x)$ vanishes at the boundaries $\{-b(t), b(t)\}$ and because b is increasing, it follows that $X_t \in [-b(t), b(t)]$ for all $t \geq 0$.

It remains to prove that a solution X to (8) exists. We follow the methodology introduced in [9], see in particular Lemma 6.8 for the case where h is the density of a uniform law on $[-1, +1]$, and a_h is defined in (6.49). In this work the authors introduce a process $Y = (Y_t)_{t \in \mathbb{R}}$ such that, for all $t \geq s$

$$Y_t = Y_s - \frac{1}{2} \int_s^t Y_u du + \frac{1}{\sqrt{2}} \int_s^t \sqrt{1 - Y_u^2} dB_u$$

with marginals having uniform distribution on $[-1, +1]$, where B is a Brownian motion on \mathbb{R} (not merely \mathbb{R}^+). Then, setting

$$X_t = b(t)Y_{\gamma(t)} \quad (10)$$

for $t > 0$, where γ is an increasing differentiable function, leads to a process with uniform marginals on $[-b(t), b(t)]$ (since by construction $Y_{\gamma(t)}$ has a uniform law). It remains to find γ making X a martingale with the prescribed dynamics. Using [14, lemma 5.1.3.], and defining $\beta(y) := \frac{1}{\sqrt{2}}\sqrt{1-y^2}$ and U as $U_t := \int_s^{\gamma(t)} \beta(Y_u)dB_u$, there exists a $\mathbb{F} = (\mathcal{F}_{\gamma(t)})_{t \geq 0}$ Brownian motion W such that

$$dU_t = \beta(Y_{\gamma(t)}) \sqrt{\dot{\gamma}(t)} dW_t.$$

It follows that

$$d_t Y_{\gamma(t)} = -\frac{1}{2} Y_{\gamma(t)} \dot{\gamma}(t) dt + \beta(Y_{\gamma(t)}) \sqrt{\dot{\gamma}(t)} dW_t \quad (11)$$

and by integration by parts

$$dX_t = b(t)\beta(Y_{\gamma(t)}) \sqrt{\dot{\gamma}(t)} dW_t \quad (12)$$

and the process X is a local martingale. Equating the diffusion coefficient of (8) to that of (12) yields to identifying $\dot{\gamma}(t) = 2\frac{\dot{b}(t)}{b(t)}$ so that a valid choice for our time-change process is $\gamma(t) = 2 \ln b(t)$. The process X is a true martingale: indeed by assumption on the boundedness of $b\dot{b}$

$$\sigma^2(x, t) = \mathbb{I}_{\{-b(t) \leq x \leq b(t)\}} (\dot{b}(t)/b(t))(b^2(t) - x^2) \leq \dot{b}(t)b(t) \leq C$$

and hence

$$\mathbb{E} \left[\left(\int_s^t \sigma(u, X_u) dW_u \right)^2 \right] = \mathbb{E} \left[\int_s^t \sigma^2(u, X_u) du \right] \leq C(t-s).$$

It remains to prove that $X_t = b(t)Y_{2 \ln b(t)}$ goes to 0 a.s. when t goes to 0 which is similar to the proof given in [9]. Again in [9] it is shown that one has uniqueness in law and the argument can be straightforwardly repeated for our process here. **Finally, the claim on the time spent at the boundaries is proven in Theorem 4.** \square

5 Mean reverting uniform diffusions with constant boundaries

Consider the SDE (8), take a small $t_0 > 0$ and consider the SDE solution for $t \geq t_0$. Define the re-scaled process

$$Z_t = X_t/b(t), \quad X_t = b(t)Z_t \quad \text{for } t \geq t_0.$$

Since X_t is uniform in $(-b(t), b(t))$, Z_t will be uniform in $[-1, 1]$ for $t \geq t_0$. We can derive the SDE for Z_t , $t \geq t_0$, using integration by parts and use that dynamics to define a new process \tilde{Z} :

$$d\tilde{Z}_t = -\frac{\dot{b}(t)}{b(t)}\tilde{Z}_t dt + \left(\frac{\dot{b}(t)}{b(t)}(1 - \tilde{Z}_t^2)\right)^{1/2} \mathbb{I}_{\{\tilde{Z}_t \in [-1, 1]\}} dW_t, \quad t \geq t_0, \quad \tilde{Z}_{t_0} := \zeta \sim U([-1, 1]).$$

Thus, with this deterministic re-scaling, we have a process Z with fixed uniform distribution and fixed boundaries. Here we assume the initial condition ζ to be independent of the driving Brownian motion.

If instead we aim to obtain a standard uniform in $[0, 1]$, we adopt a slightly different transformation:

$$\bar{Y}_t = (X_t/b(t) + 1)/2 = X_t/(2b(t)) + 1/2$$

from which

$$X_t = 2b(\bar{Y}_t - 1/2).$$

By Leibnitz's rule we have the following

Theorem 3. *Consider, for $t \geq t_0$, the SDEs*

$$d\bar{Y}_t = \frac{\dot{b}(t)}{b(t)}(1/2 - \bar{Y}_t)dt + \frac{1}{2b} \left(\mathbb{I}_{\{\bar{Y}_t \in (0, 1)\}} b(t)\dot{b}(t)(1 - 4(\bar{Y}_t - 1/2)^2) \right)^{1/2} dW_t, \quad \bar{Y}_{t_0} = \xi \sim U([0, 1])$$

and

$$d\tilde{Z}_t = -\frac{\dot{b}(t)}{b(t)}\tilde{Z}_t dt + \left(\frac{\dot{b}(t)}{b(t)}(1 - \tilde{Z}_t^2)\right)^{1/2} \mathbb{I}_{\{\tilde{Z}_t \in (-1, 1)\}} dW_t, \quad \tilde{Z}_{t_0} = \zeta \sim U([-1, 1]) \quad (13)$$

with ξ and ζ independent of W . The unique solutions of these SDEs mean-revert to $1/2$ and 0 respectively with speed \dot{b}/b and are distributed at any point in time as a standard uniform random variable and as a uniform $[-1, 1]$ random variable respectively.

Proof. The proof is immediate. For the mean reverting behaviour, taking for example \tilde{Z} , we note that $\lim_{t \uparrow +\infty} \mathbb{E}[\tilde{Z}_t] = 0$ exists and is finite, and $\lim_{t \uparrow +\infty} \text{Var}[\tilde{Z}_t] = 1/3$ also exists and is finite. Actually, we are in a special case where mean and variance are constant. Furthermore, whenever \tilde{Z}_t is above the long term mean 0 , the drift is negative, pointing back to 0 , while the variance remains bounded. A similar symmetric pattern is observed when \tilde{Z}_t is below zero.

□

\bar{Y} can be used for example to model the dynamics of recovery rates or probabilities in the case of no information (maximum entropy), whereas \tilde{Z} can be used as a model for stochastic correlation.

Remark 2. The above construction for \bar{Y} and \tilde{Z} , mean-reverting uniform diffusions with fixed boundaries based on rescaling the process X of Theorem 1, has the drawback of starting time at $t_0 > 0$, without defining the dynamics in $[0, t_0)$. This is done to avoid singularities in $t = 0$ with the rescaling. On the other hand, it has the advantage that the solution is unique **in the strong sense**. An alternative for obtaining a similar process, especially for cases like $b(t) = \sqrt{t}$, is to start from X constructed as in Theorem 2, requiring assumptions on b that are weaker than in Theorem 1. If we do so, and recalling Y in the proof of Theorem 2 **and Eq. (10) in particular**, we obviously could have $Z_t = Y_{\gamma(t)}$ where $\gamma(t) = 2 \ln b(t)$, or even $Z_t = Y_t$. Notice however that to get a diffusion with uniform law in $[-1, 1]$ we could directly define a process \hat{Z} as **$\hat{Z}_t := Y_{\alpha(t)}$ for any time change function α provided that it is increasing**. Indeed, this would not affect the marginals of \hat{Z} as Y is a diffusion with uniform marginals in $[-1, 1]$ at all times.

Remark 3. The above rescaling approach yields a diffusion associated to the uniform peacock with constant boundaries $-1, 1$. It is also obvious from (11) that defining Z as $Z_t = Y_{\gamma(t)}$ will lead to a mean-reverting diffusion. However, this is a mean-reverting diffusion process and not a diffusion martingale. Still, we know since [12] that there is a martingale associated to any peacock. Hence a natural question is: what is the diffusion martingale associated with this peacock? Looking at the forward Kolmogorov equation, the answer turns out to be: only the trivial martingale diffusions with zero drift and zero diffusion coefficients. Indeed, from an intuitive point of view, forcing $\varphi(x, t)$ to be the density of a uniform with fixed boundaries at all time implies that the left hand side of (6) vanishes, leading to $\sigma(t, x) = 0$ for all x . In other words, the only diffusion martingale associated to this peacock is the trivial martingale $Z_t = \zeta$ for all t , where $\zeta \sim U([-1, 1])$.

Finally, we are now able to discuss the behaviour of the solution \tilde{Z} of (13) at the boundaries -1 and 1 , and thus the behaviour of the original X_t , solution of (8), at the boundaries $-b(t)$ and $b(t)$.

Theorem 4. *[Local time calculation.] Given a strictly increasing function $t \mapsto b(t)$ defined in $[0, T]$, continuous, and differentiable in $(0, T]$, assume $b(0) = 0$ and $\dot{b}(t)b(t)$ to be bounded in $(0, T]$, with finite limit $\lim_{t \downarrow 0} \dot{b}(t)b(t)$ (this holds under the assumptions of Theorem 1 and is a slight reinforcement of the assumptions of Theorem 2). The local time for the process $b(t) - X_t$ (resp. $X_t + b(t)$) at point 1 (resp. -1) is zero.*

Proof. Let us introduce $U_t = b(t) - X_t$. Then $d\langle U \rangle_t = \mathbb{1}_{0 \leq U_t \leq 2b(t)} \left(\frac{\dot{b}(t)}{b(t)} \right) 2U_t(2b(t) - U_t)dt$. Then

$$\begin{aligned} t &\geq \int_0^t 2b(s) \mathbb{1}_{\{0 \leq U_s \leq 2b(s)\}} ds = \int_0^t 2b(s) \mathbb{1}_{0 \leq U_s \leq 2b(s)} \frac{b(s)}{\dot{b}(s)} \frac{1}{(2b(s) - U_s)U_s} d\langle U \rangle_s \\ &= \int_0^\infty da \int_0^t 2b(s) \mathbb{1}_{0 \leq a \leq 2b(s)} \frac{b(s)}{\dot{b}(s)} \frac{1}{(2b(s) - a)a} dL_s^a \end{aligned}$$

where the last equality comes from occupation time formula.

We note that $b(s)/\dot{b}(s)$ is bounded from below by a positive constant C for all $s \geq \delta$. We

can easily see that this is indeed the case since $\dot{b}(s)b(s)$ is bounded by above in $[0, T]$ by assumption, say by a constant $K > 0$, so that $\dot{b}(s)/b(s) = \dot{b}(s)b(s)/b(s)^2 \leq K/b(\delta)^2 =: C$. This implies that $b(s)/\dot{b}(s) \geq C$ for all $t \geq \delta$.

We obtain

$$\begin{aligned} t &\geq C \int_0^\infty da \int_\delta^t \mathbb{I}_{0 \leq a \leq 2b(s)} \left(\frac{dL_s^a}{2b(s) - a} + \frac{dL_s^a}{a} \right) \\ &\geq C \int_0^\infty da \int_\delta^t \mathbb{I}_{0 \leq a \leq 2b(s)} \frac{dL_s^a}{a} = C \int_0^{2b(\delta)} \frac{L_t^a - L_\delta^a}{a} da \end{aligned}$$

which implies that $L_t^a - L_\delta^a = 0$. By continuity, L_δ^a goes to 0 when δ goes to 0. \square

Finally, with a slight abuse of notation, we will denote \bar{Y} by Y and \tilde{Z} by Z in the rest of the paper.

6 Specific choices of the boundary $b(t)$ and links with peacocks

In this section we present a number of qualitatively different choices for $b(t)$.

6.1 The square-root case $b(t) = \sqrt{t}$

As we pointed out earlier, the case $b(t) = \sqrt{t}$ for (8), which leads to

$$dX_t = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{X_t^2}{t}} \mathbb{I}_{\{X_t \in [-\sqrt{t}, \sqrt{t}]\}} dW_t, \quad X_0 = 0$$

corresponds exactly to the solution presented in [9].

6.2 The linear case $b(t) = kt$: numerical examples and activity

The case $b(t) = kt$ fits the assumption of Theorem 1 since $\dot{b}(t) = k$ is bounded on $[0, T]$ for any $T \in \mathbb{R}^+$. Notice also that $\dot{b}(t)b(t) = k^2t$ vanishes for $t \downarrow 0$. Our previous SDEs for X (8) and \tilde{Z} (13) specialize to

$$dX_t = \mathbb{I}_{\{X_t \in [-kt, kt]\}} \frac{1}{\sqrt{t}} \sqrt{(kt)^2 - X_t^2} dW_t, \quad X_0 = 0, \quad X_t \sim U([-kt, kt]) \quad \text{for all } t > 0 \quad (14)$$

and

$$dZ_t = -\frac{1}{t} Z_t dt + \mathbb{I}_{\{Z_t \in [-1, 1]\}} \frac{1}{\sqrt{t}} \sqrt{1 - Z_t^2} dW_t, \quad Z_{t_0} = \zeta \sim U(-1, 1) \quad \text{for all } t \geq t_0. \quad (15)$$

As a numerical example we implement the Euler scheme for X . We know from [8] that under our assumptions the Euler scheme converges in probability. We thus implement a

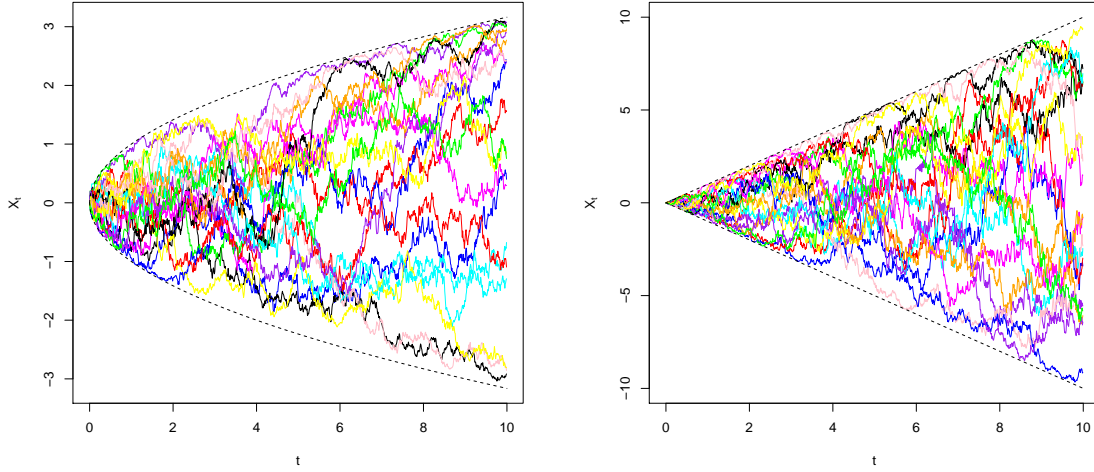


Figure 1: 20 paths of the SDE (14) at time 1y with $b(t) = \sqrt{t}$ (left) and $b(t) = t$ (right). Time step is 0.01 years. Euler Scheme.

Euler scheme for the SDE for X and then plot a histogram of the density. This is shown in Figure 2. Moreover, we show in the right panel of Figure 1 a few sample paths of the process X .

We may also apply Theorem 4 to this particular case, to see that $Z_t = X_t/(kt)$ for $t \geq t_0 > 0$ spends zero time at the boundaries -1 and 1 . As a consequence, X_t spends zero time at the boundaries $-kt$ and kt .

More qualitatively, we observe that Z in (15) mean-reverts to 0 with speed $1/t$. The speed will be very large for small time but will become almost zero when time is large. The diffusion coefficient, similarly, is divided by \sqrt{t} , so it will tend to vanish for large t . This is confirmed by the following activity calculation. We may conclude that the process will not be absorbed in the boundary and will tend to “slow down” in time, while maintaining a uniform distribution.

We show that the pathwise activity of the uniform $(-1, 1)$ process Z is vanishing for large t in the sense that the deviation of $Z_{t+\delta}(\omega)$ from $Z_t(\omega)$ collapses to zero for all $\delta > 0$, all $\omega \in \Omega$ as $t \rightarrow \infty$.

Lemma 1.

$$\forall \delta > 0, \text{Var}(Z_{t+\delta} - Z_t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Proof. Notice that for all $t > 0$, $\mathbb{E}(Z_t) = 0$ so that $\text{Var}(Z_t) = \mathbb{E}(Z_t^2) = v$ where $v = \sqrt{2}/12$ is the variance of a zero-mean uniform random variable distributed on $[-1, 1]$. Then,

$$\text{Var}(Z_{t+\delta} - Z_t) = \text{Var}(Z_{t+\delta}^2) + \text{Var}(Z_t^2) - 2\text{Cov}(Z_t, Z_{t+\delta}) = 2(v - \mathbb{E}(Z_t Z_{t+\delta})).$$

Since Z is bounded, one can rely on Fubini’s theorem for all $t > 0$ and exchange time-integration and expectation,

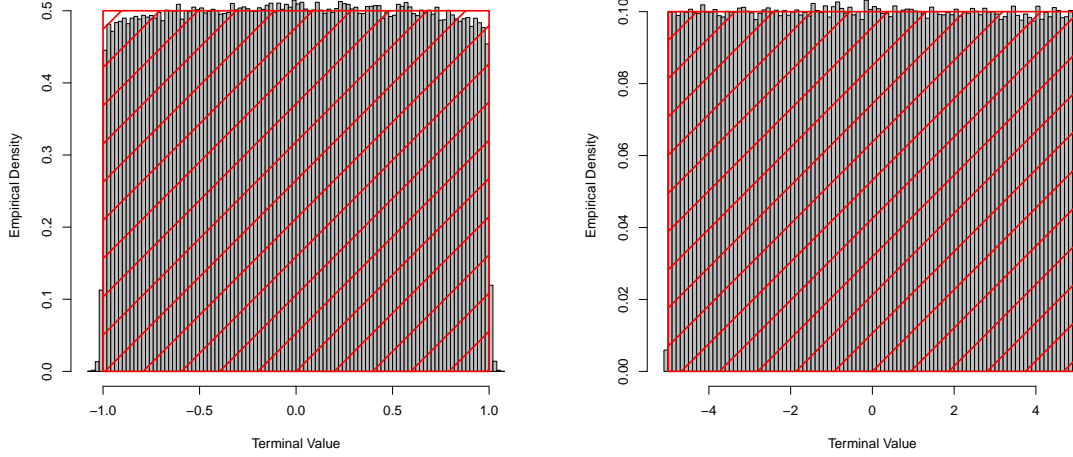


Figure 2: Histograms with 100 bins for the density of the SDE (14) at time t with $k = 1$ for 1 Million scenarios via Euler scheme: left $t = 1y$ and right $5y$. Time step is 0.01 years.

$$\begin{aligned}
 \mathbb{E}(Z_t Z_{t+\delta}) &= \mathbb{E} \left(Z_t \left(Z_t - \int_t^{t+\delta} \frac{Z_s}{s} ds + \int_t^{t+\delta} \sigma(s, Z_s) dW_s \right) \right) \\
 &= \mathbb{E}(Z_t^2) - \mathbb{E} \left(Z_t \int_t^{t+\delta} \frac{Z_s}{s} ds \right) + \mathbb{E} \left(\int_t^{t+\delta} Z_t \sigma(s, Z_s) dW_s \right) \\
 &= v - \int_t^{t+\delta} \frac{\mathbb{E}(Z_t Z_s)}{s} ds
 \end{aligned}$$

(where we have used the fact that $\frac{1}{\sqrt{s}}\sqrt{1-Z_s^2}$ is bounded) one can compute explicitly $\mathbb{E}(Z_s Z_t)$ by observing that $\mathbb{E}(Z_s Z_t) = \sqrt{\text{Var}(Z_t)\text{Var}(Z_s)}\rho(s, t) = \sigma^2 \rho(s, t)$ where $\rho(s, t)$ is the correlation between Z_t and Z_s , one gets

$$\text{Var}(Z_{t+\delta} - Z_t) = 2\sigma^2 \int_t^{t+\delta} \frac{\rho(s, t)}{s} ds \leq 2\sigma^2 \int_t^{t+\delta} \frac{1}{s} ds = 2\sigma^2 \delta \frac{\delta + 2t}{t(t + \delta)^2}.$$

The RHS of the above expression tends to zero as $t \rightarrow \infty$ showing that the variance of increments of Z over a time interval of fixed length δ collapses to zero as $t \rightarrow \infty$. \square

The activity result can be generalized to the following

Lemma 2. *Let $X_t = x_0 + \int_0^t \theta_s dW_s$ and suppose $X = (X_t)_{t \geq 0}$ is a bounded non-vanishing martingale in the sense that for all $t \geq 0$, $a \leq X_t \leq b$ and $\mathbb{P}(\theta_t = 0) < 1$. Then, the path activity of X is collapsing to zero as time passes.*

Proof. Since martingales have independent increments, the variance of increments is the increment of the variances:

$$\mathbb{V}\text{ar}(X_{t+\delta} - X_t) = \mathbb{V}\text{ar}(X_{t+\delta}) + \mathbb{V}\text{ar}(X_t) - 2\mathbb{C}\text{ov}(X_t, X_{t+\delta}) = \mathbb{V}\text{ar}(X_{t+\delta}) - \mathbb{V}\text{ar}(X_t) .$$

Because the diffusion coefficient θ_s does not vanish on $(t, t + \delta)$,

$\mathbb{V}\text{ar}(X_{t+\delta} - X_t) = \int_t^{t+\delta} \mathbb{E}(\theta_s^2) ds > 0$ showing that the variance of X_t is monotonically increasing with respect to t . But the variance of a bounded process is bounded. In particular, it is easy to see that $\mathbb{V}\text{ar}(X_t) \leq (x_0 - a)(b - x_0)$ since $\mathbb{E}(X_t) = x_0$ and the variance of any random variable Y with expectation μ_Y and taking values in $[a, b]$ is bounded from above by the variance of $a + (b - a)B$ where B is a Bernoulli random variable with parameter $\pi = (\mu_Y - a)/(b - a)$. Hence, $\mathbb{V}\text{ar}(X_t)$ and $\mathbb{V}\text{ar}(X_{t+\delta})$ are increasing to the same limit, proving that for all $\epsilon > 0$ there exists t^* such that $\mathbb{V}\text{ar}(X_{t+\delta} - X_t) < \epsilon$ for all $t > t^*$. \square

Remark 4 (Other boundaries). One could choose time-boundaries that are concave and converge asymptotically to a constant value B , e.g. $b(t) = Bt/(t + \beta)$ or $b(t) = B(1 - e^{-\beta t})$ where $B > 0, \beta > 0$. It is also possible to use convex boundaries, like e.g. $b(t) = k(e^{\beta t} - 1)$, $k > 0, \beta > 0$. Finally, as mentioned earlier, we could study boundaries of the form $b(t) = kt^\alpha$, $\alpha > 1/2$, $k > 0$, since in this case too existence and uniqueness of the SDE strong solution is guaranteed.

7 Conclusions and further research

We introduced a way to design Stochastic Differential Equations of diffusion type admitting a unique strong solution distributed as a uniform law with conic time-boundaries. While the result with general boundary is new and conditions for pathwise uniqueness of solutions are new, existence for the cases with square-root and linear boundaries had been dealt with previously [in the peacocks literature](#). [We further discuss our results in relation to the peacocks literature](#). We introduced also general mean-reverting diffusion processes having a constant uniform law at all times. This may be used to model random probabilities, random recovery rates or random correlations.

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Appendix

A Proof that the solution of the peacock SDE (8) has uniform law

We start with the following

Definition 1. A probability measure μ is *determined by its moments* when it is the unique probability measure having this set of moments.

Lemma 3. *The continuous uniform distribution on $[a, b]$, $-\infty < a < b < \infty$, is determined by its moments.*

Proof. Let us note $\alpha_k(p) := \int_{-\infty}^{\infty} x^k p(x) dx$ the k -th moment associated to a probability density function p . From Theorem 30.1 of [1], it is known that if all the moments $\alpha_1(p), \alpha_2(p), \dots$ are finite and are such that the series

$$S_r(p) := \sum_{k=1}^{\infty} \frac{\alpha_k(p) r^k}{k!}$$

admits a positive radius of convergence, then p is *determined by its moments*.

One concludes from this theorem that if a random variable X satisfies $\mathbb{E}(X^k) = \alpha_k(p)$ for all $k \in \mathbb{N}$, then $X \sim p$ provided that (i) $|\alpha_k(p)| < \infty$ for $k \in \mathbb{N}$ and (ii) there exists $r > 0$ such that the series $S_r(p)$ converges.

In particular, if the uniform density in $[a, b]$, $\rho(x) := \frac{1}{b-a} \mathbb{I}_{\{a \leq x \leq b\}}$, satisfies (i) and (ii), then any random variable X satisfying $\mathbb{E}(X^k) = \alpha_k(\rho)$ for all $k \in \{1, 2, \dots\}$ is uniformly distributed on $[a, b]$.

Let us show that (i) and (ii) are satisfied for the uniform density in $[a, b]$. Condition (i) is clearly met since the moments of the uniform distribution are finite. In particular, defining $c := |a| \vee |b|$ one has $|\alpha_k(\rho)| \leq c^k < \infty$. On the other hand, for $r > 0$,

$$0 \leq \frac{|\alpha_k(\rho)| r^k}{k!} \leq \frac{(cr)^k}{k!}.$$

Since the series

$$S'_r := \sum_{k=1}^{\infty} \frac{(cr)^k}{k!}$$

converges to $e^{cr} - 1$, the series $S_r(\rho)$ converges, too. This shows that both conditions (i) and (ii) are met for $p = \rho$, and completes the proof. \square

Theorem 5. *The solution X to the SDE (8) is a uniform martingale on $[-b(t), b(t)]$ in the sense that for all $t > 0$, X_t is uniformly distributed on $[-b(t), b(t)]$.*

Proof. From the above results, it is enough to show that all moments of the random variable X_t ($t > 0$) associated to eq. (8) coincide with those of the density $\mathbb{I}_{\{-b(t) \leq x \leq b(t)\}} \frac{1}{2b(t)}$.

Let X be a random variable uniformly distributed on $[a, b]$. Then,

$$\mathbb{E}(X^n) = \frac{1}{n+1} \sum_{i=0}^n (a)^i (b)^{n-i}.$$

In the special case where $a = -b$, this expression reduces to

$$\mathbb{E}(X^n) = \begin{cases} \frac{b^n}{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Let us now compute the moments of

$$X_t = \int_0^t \sigma(X_s, s) dW_s, \quad \sigma(t, x) = \mathbb{I}_{\{-b(t) \leq x \leq b(t)\}} \sqrt{\frac{\dot{b}(t)}{b(t)}} \sqrt{b^2(t) - x^2}$$

solving eq. (8). By Itô's lemma:

$$X_t^n = n \int_0^t X_s^{n-1} dX_s + \frac{1}{2} n(n-1) \int_0^t X_s^{n-2} \sigma^2(X_s, s) ds$$

and we can compute the expression for the n -th moment, $n \geq 2$ using a recursion. Using the property that Itô's integrals have zero expectation and exchanging integration and expectation operators, which is possible since $X_s^{n-2} \sigma^2(X_s, s)$ is bounded for all s and $n \geq 2$, we obtain

$$\begin{aligned} \mathbb{E}(X_t^n) &= n \mathbb{E} \left(\int_0^t X_s^{n-1} dX_s \right) + \frac{1}{2} n(n-1) \mathbb{E} \left(\int_0^t X_s^{n-2} \sigma^2(X_s, s) ds \right) \\ &= \frac{n(n-1)}{2} \left(\int_0^t b(s) \dot{b}(s) \mathbb{E}(X_s^{n-2}) ds - \int_0^t \frac{\dot{b}(s)}{b(s)} \mathbb{E}(X_s^n) ds \right) \end{aligned} \quad (16)$$

Notice that we have postulated in the last equality that the indicator $\mathbb{I}_{\{-b(s) \leq X_s \leq b(s)\}}$ in $\sigma(t, x)$ is always 1. This is a natural assumption: it says that X cannot stay on a boundary with a strict positive probability for a given period of time. This happens because in case X reaches $\pm b(t)$ at some time t , the process is locally frozen ($\sigma(t, x) = 0$) but the boundary $b(t)$ keeps on growing (see also the local time calculation in Theorem ?? for the specific case $b(t) = kt$).

Obviously, $\mathbb{E}(X_t) = X_0 = 0$ since X is a martingale and one concludes from eq. (16) that the n -th moment of X_t is zero when n odd. For n even, eq. (16) can be written as

$$f(t, n) = \frac{n(n-1)}{2} \left(\int_0^t b(s) \dot{b}(s) f(s, n-2) ds - \int_0^t \frac{\dot{b}(s)}{b(s)} f(s, n) ds \right)$$

with $f(t, n) := \mathbb{E}(X_t^n)$. This can be written as a recursive differential equation

$$\frac{\partial f(t, n)}{\partial t} = \dot{b}(t) \frac{n(n-1)}{2} \left(b(t) f(t, n-2) - \frac{1}{b(t)} f(t, n) \right)$$

with the constraint that $f(t, 0) = \mathbb{E}(X_t^0) = 1$. The solution to this equation is $f(t, n) = b^n(t)/(n+1)$. One concludes that X_t is uniform on $[-b(t), b(t)]$ since all the odd moments are zero and all the even moments are given by

$$\mathbb{E}(X_t^n) = \frac{b^n(t)}{n+1}$$

and agree with those of a random variable uniformly distributed on $[-b(t), b(t)]$. \square