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# Nonstandard analysis: New way and criticism of the previous approaches

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**Keyword.** Nonstandard analysis, hyperreals, Internal set theory.

## Abstract

In this paper we propose a new approach to nonstandard analysis without using the ultrafilters, this method is very simple in practice. Moreover, we construct explicitly the total order relation in the new field of the infinitesimal numbers. To illustrate the importance of this work, we suggest a few applications of the new approach, and we propose to compare it with the previous methods.

## 1 Introduction

In 1961 Abraham Robinson [1] showed how infinitely large and infinitesimal numbers can be rigorously defined and used to develop the field of non-standard analysis. To understand his theory nonconstructively, it's necessary to use the essential properties given by the model theory and mathematical logic.

After the birth of this new theory, more mathematicians have found the important applications [5, 3] of the nonstandard analysis in physics [2, 4, 6], numerical analysis and variational methods.

In 1977 a new axiomatic representation of hyperreals is given by Edward Nelson [7], in the sense to simplify the Robinson's method, he proposed to add three axioms on the set theory and obtained a new theory called internal set theory [7, 13].

Another axiomatic method was published in 2003, this method called Alpha-theory [10] is simple compared to that of Nelson. Despite that, we think that we are obliged to answer a few questions about its effectiveness in practice as an axiomatic approach.

From the construction of Robinson we can see every hyperreal as an element of  $\mathbb{R}^{\mathbb{N}}$  modulo a maximal ideal  $M$ , the ideal  $M$  is defined with a non principal ultrafilter  $\mathcal{U}$ , the existence of  $\mathcal{U}$  is proved by the axiom of choice, by using the ultrafilter  $\mathcal{U}$  we define the order relation in the field of hyperreals. Unfortunately, we can't determine exactly this order relation because the ultrafilter is unknown.

Our aim in this article is to give a new field which contains the infinite and infinitesimal numbers without using the properties of model theory, without using the ultrafilters and without adding the new axioms to ZFC (Zermelo-Frankel+Axiom of choice). To be or not to be a specialist in mathematical logic, is not very important to understand this new theory, only the classical results of analysis and the properties of the analytic functions are sufficient in construction. The new approach is very simple and we can determine precisely the order relation defined on the new field.

In this article, we propose to present the plan as follows:

Firstly, we begin by some notions to define the infinite and infinitesimal numbers, after we propose to present the previous approaches (Robinson's approach, Internal Set Theory and Alpha-Theory). To give the criticism of the classical approaches, we find in section 8 some examples to prove that the choice of the ring  $\mathbb{R}^{\mathbb{N}}$  in construction of the hyperreal numbers is too broad to be effective in practice. For example, we propose the example of a hyperreal which can be equal to zero, but it is impossible to determine its value. In the section 9, we begin the proposed method by the construction of a proper subset  $\Delta$  of  $\mathbb{R}^{\mathbb{N}}$ , this new set is a unitary ring of  $\mathbb{R}^{\mathbb{N}}$ . By using a maximal ideal of  $\Delta$ , we obtain a new field called the field

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of Omicran-reals which is a totally ordered field and an extension of the set of real numbers  $\mathbb{R}$ . To illustrate the importance of the new approach, we propose some applications as:

- For the logarithmic function: We prove the following equalities for every real  $x > 0$ :

$$\ln(x) = \lim_{\alpha \rightarrow 0} \frac{x^\alpha - 1}{\alpha},$$

while  $x \neq 1$ , we obtain:

$$\frac{x-1}{\ln(x)} = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} x^{\frac{k}{n}}.$$

- Prime numbers: Let  $\mathcal{P}$  be the set of prime numbers. As  $x \rightarrow +\infty$ , we get

$$\pi(x) \sim \frac{1}{x^{\frac{1}{x}} - 1},$$

where  $\pi(x) = \#\{p \leq x : p \in \mathcal{P}\}$ .

In addition, we can prove that:

$$p_n \sim n^2(\sqrt[n]{n} - 1), \text{ while } n \rightarrow +\infty,$$

where  $(p_n)$  is the sequence of the prime numbers.

- The length of a curve: We define the length of the arc  $\widetilde{AB}$  and we determine the conditions of rectifiability from the new approach. We calculate easily the length, and we obtain:

$$l(\widetilde{AB}) = \int_a^b \sqrt{1 + f'^2(x)} dx,$$

where  $l(\widetilde{AB})$  is the length of the arc defined by the curve of the function  $f$  between  $A(a, f(a))$  and  $B(b, f(b))$ .

- We calculate the limit by using a new notion called the exact limit.
- We calculate the finite sum by using the exact limit of a series.
- To calculate the exact limit of a series we define a new matrix called the black magic matrix, this beautiful matrix admits twelve magical properties and we can determine the Bernoulli numbers by using it.
- We can obtain the standard Euler-Maclaurin formula applied to the zeta function  $\zeta(s)$  by using the coefficients of the above matrix.

Finally, we determine in the last section of this paper the relationship between the hyperreal numbers and the Omicran-reals, and we prove that Any property that is true for every hyperreal number is also true for every Omicran.

## 2 Preliminary results

In this section, we find a few definitions and results that are applied in this work.

1. The binomial coefficient is defined as:  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ .



- (ii) In addition if  $(E, +)$  is a commutative group, we consider  $|\alpha| = \max(\alpha, -\alpha) = \begin{cases} \alpha, & \text{when } -\alpha \prec \alpha, \\ -\alpha, & \text{when } \alpha \prec -\alpha. \end{cases}$
- (iii) We note  $x \prec y$  while:  $x \preceq y$  and  $x \neq y$ .
- (iv) Let the set  $I_E$  be defined as follows  $I_E = \{\alpha \in E / 0 \prec |\alpha| \prec \varepsilon \forall \varepsilon \in \mathbb{R}^{+*}\}$ .  
 $I_E$  is a set of infinitesimals.

**Remark 1.** If it has not ambiguity we replace the symbol  $\preceq$  by  $\leq$ , and  $\prec$  by  $<$ .

To construct the new extension of  $\mathbb{R}$  which contains the infinite and infinitesimal numbers, it's sufficient to prove the following theorem:

**Theorem 3.** *There exist a extension field  $(E, +, \cdot)$  of  $(\mathbb{R}, +, \cdot)$ , and partial order  $\leq$  such that:  $(E, \leq)$  is a order  $\mathbb{R}$ -extension and  $I_E \neq \emptyset$ .*

**Remark 2.** An element  $\delta$  of  $I_E \neq \emptyset$  is called infinitesimal.

**Notation.**  $\mathbb{N} = \{1, 2, 3, \dots\}$ .

## 4 Previous methods

### 4.1 Robinson's approach

From the works of Abraham Robinson, we know that the heuristic idea of infinite and infinitesimal numbers has obtained a formal rigor, he proved that the field of real numbers  $\mathbb{R}$  can be considered as a proper subset of a new field,  ${}^*\mathbb{R}$ , which is called field of hyperreal [1] numbers and contains the infinite and infinitesimal numbers. From the approach of Robinson we can represent every hyperreal by a sequence of  $\mathbb{R}^{\mathbb{N}}$  modulo a maximal ideal  $\mathcal{I}$ , this ideal is defined by using an ultrafilter  $\mathcal{U}$ . Unfortunately, the Ultrafilter  $\mathcal{U}$  and the order relation defined on  ${}^*\mathbb{R}$  are unknown, only the existence can be proved by the axiom of choice.

### 4.2 Nelson's approach

In 1977, Edward Nelson expands the language of set theory by adding a new basic predicate  $\text{st}(x)$ . We obtain a new axiomatic representation of the nonstandard analysis by using the above predicate. To explain the behavior of this unary predicate symbol  $\text{st}(x)$ , Nelson proposes to add three axioms [7]:

- (a) Idealization. (I)
- (b) Standarization. (S)
- (c) Transfer principle.(T)

### 4.3 Alpha-theory

This axiomatic approach published in 2003 is based on the existence of a new element called  $\alpha$ . In this method, we need five axioms to justify the behavior of this new mathematical object. In the following section, we restart by the construction of the hyperreals, then, we propose to study the previous axiomatic approaches.

## 5 Construction of the hyperreal numbers

Let  $I$  be a nonempty set, and  $\mathcal{P}(I)$  the power set of  $I$ .

**Definition 2.** An ultrafilter  $\mathcal{U}$  is a proper subset of  $\mathcal{P}(I)$ , such that:

(i) *Intersections:* if  $A, B \in \mathcal{U}$ , then  $A \cap B \in \mathcal{U}$ .

(ii) *Supersets:* if  $A \subseteq B \subseteq I$ , then  $B \in \mathcal{U}$ .

(iii) For any  $A \subseteq I$ , either  $A \in \mathcal{U}$  or  $A^c \in \mathcal{U}$ .

**Example 1.** 1.  $\mathcal{F}^i = \{A \subseteq I : i \in A\}$  is an ultrafilter, called the principal ultrafilter generated by  $i$ .

2.  $\mathcal{F}^{co} = \{A \subseteq I : I - A \text{ is finite}\}$  is the cofinite (or Frchet), filter on  $I$ .  $\mathcal{F}^{co}$  is not an ultrafilter.

To construct the field of hyperreal numbers, we use the unitary ring  $\mathbb{R}^{\mathbb{N}}$  as follow:

(a)  $\mathbb{R} \subseteq \mathbb{R}^{\mathbb{N}}$ : We can identify every sequence  $u = (l, l, \dots, l, \dots)$  by the real number  $l$ .

(b) We define in  $\mathbb{R}^{\mathbb{N}}$  the total order relation  $\leq$  by:

$$u = (u_1, u_2, \dots, u_n, \dots) \leq v = (v_1, v_2, \dots, v_n, \dots) \iff \{i : u_i \leq v_i\} \in \mathcal{U},$$

where  $\mathcal{U}$  is a nonprincipal ultrafilter of  $\mathbb{N}$ .

To show the existence of the above ultrafilter, we use the axiom of choice.

(c)  $(\mathbb{R}^{\mathbb{N}}, +, \cdot)$  is a commutative ring with unity  $(1, 1, \dots, 1, \dots)$ , but it is not a field, since

$$(1, 0, 1, 0, \dots)(0, 1, 0, 1, \dots) = 0_{\mathbb{R}^{\mathbb{N}}}.$$

We construct the field of hyperreal numbers by using the following maximal ideal [1, 12] of  $\mathbb{R}^{\mathbb{N}}$ :

$$I = \{u \in \mathbb{R}^{\mathbb{N}} : \{i : u_i = 0\} \in \mathcal{U}\}.$$

Finally, we deduce that the new field of the hyperreal numbers is given by:  ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/I$ .

**Notation.** For every hyperreal  $u$  defined by the sequence  $(u_i)$ , we note:

$$u = \langle u_1, u_2, \dots, u_n, \dots \rangle, \text{ or } u = \langle u_i \rangle.$$

(d) We can verify that the hyperreal  $\delta = \langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle$  is infinitesimal.

## 6 Internal Set Theory

From the new theory called Internal Set Theory, Nelson's view is different from that of Robinson. In fact, in IST, we can find in the set of real numbers noted  ${}^*\mathbb{R}$  the infinite and infinitesimal numbers. In addition, the classical families of real numbers  $\mathbb{R} = \{st(x), x \in {}^*\mathbb{R}\}$  and natural numbers  $\mathbb{N} = \{st(x), x \in {}^*\mathbb{N}\}$  are not sets in IST. To clarify this point, we propose to study the properties of a set  $A$  by using the axioms added by Nelson. We start by the following abbreviations:

$$\forall^{st}x \text{ for } \forall x(x \text{ standard}) \Rightarrow, \exists^{st}x \text{ for } \exists x(x \text{ standard}) \wedge.$$

We call a formula of IST internal in case it does not involve the new predicate "standard", otherwise we call it external.

A set  $x$  is finite if there is no bijection of  $x$  with a proper subset of itself.

In IST, the three axioms of Nelson are defined as:

- (i) **Transfer:** If  $\phi(x, u_1, \dots, u_n)$  is an internal formula with no other free variables than those indicated, then:

$$\forall^{st} u_1, \dots, \forall^{st} u_n (\forall^{st} x \phi(x, u_1, \dots, u_n) \rightarrow \forall x \phi(x, u_1, \dots, u_n)).$$

- (ii) **Idealization:** For any internal formula  $B$  whose free variables include  $x$  and  $y$

$$\forall^{st} z (z \text{ is finite} \rightarrow \exists y \forall x \in z B(x, y)) \leftrightarrow \exists y \forall^{st} x B(x, y).$$

- (iii) **Standardization:** For every standard formula  $F(z)$  (internal or external), we have:

$$\forall^{st} x \exists^{st} y \forall^{st} z [z \in y \leftrightarrow z \in x \wedge F(z)].$$

Suppose that there exists a unique  $x$  such that  $A(x)$ , where  $A(x)$  is an internal formula whose only free variable is  $x$ . Then that  $x$  must be standard, since by transfer  $\exists x A(x) \Rightarrow \exists^{st} x A(x)$ . For example, the set  ${}^*\mathbb{N}$  of all natural numbers, the set  ${}^*\mathbb{R}$  of all real numbers, the empty set  $\emptyset$ , the real number 0, 1,  $\sqrt{\pi}$ , ... are all standard sets.

**Theorem 4.** *Let  $X$  be a set. Then every element of  $X$  is standard if and only if  $X$  is a standard finite set.*

**Proof.** We can apply the idealization principle for  $B(x, y) = [y \in X \wedge x \neq y]$  (see [7, 13] for more details).

**Corollary 1.** *Every infinite set has a nonstandard element.*

**Remark 3.** *From the above corollary, we deduce that there exists a nonstandard natural number  $\omega$ .*

**Theorem 5.** *There is a finite set  $F$  such that for all standard  $x$  we have  $x \in F$ .*

**Proof.** Just apply (I) to the formula  $x \in y \wedge y$  is finite (see [7, 13]).

**Theorem 6.** *Let  $X$  be a nonempty set. If  $X$  is a standard set, then it admits a standard element.*

**Proof.** Another version of the transfer principle is giving by:

$$\exists x \phi(x) \rightarrow \exists^{st} x \phi(x),$$

where  $\phi$  is an internal formula. We apply this version for  $x \in X$ .

**Definition 3.** 1. *Elements of the ultrapower [11] of  $\mathcal{P}(\mathbb{R})$  are the equivalence classes of sequences  $(A_i) \in \mathcal{P}(\mathbb{R})^{\mathbb{N}}$ , where the sequences  $(A_i)$  and  $(B_i)$  are defined to be equivalent if and only if we have  $\{i \in \mathbb{N} : A_i = B_i\} \in \mathcal{U}$ .*

2. *We note  $\langle A_i \rangle$  the equivalence class of  $(A_i)$ . We define the relation  $*$   $\in$  between  $x = \langle x_i \rangle \in {}^*\mathbb{R}$  and  $\langle A_i \rangle$  by:*

$$x^* \in \langle A_i \rangle \Leftrightarrow \{i : x_i \in A_i\} \in \mathcal{U}.$$

3. *With each equivalence class  $\langle A_i \rangle$  in the ultrapower of  $\mathcal{P}(\mathbb{R})$  we associate a subset  $A$  of  ${}^*\mathbb{R}$  as follows:*

$$x \in A \Leftrightarrow x^* \in \langle A_i \rangle .$$

4. *The subset  $A$  of  ${}^*\mathbb{R}$  associated with the equivalence class  $\langle A_i \rangle$  is called an internal set.*

5. The collection of all internal subsets of  ${}^*\mathbb{R}$  is denoted  ${}^*\mathcal{P}(\mathbb{R})$ .

We note:  $A = \langle A_i \rangle$  for the internal set  $A$  defined by the equivalence class  $\langle A_i \rangle$ .

**Remark 4.** A standard set  ${}^*B$  is given by the equivalence class  $\langle B, B, \dots, B, \dots \rangle$ , where  $B \in \mathcal{P}(\mathbb{R})$ .

**Example 2.** (i)  ${}^*[0, 1] = \langle [0, 1], \dots, [0, 1], \dots \rangle$ ,  ${}^*\mathbb{R} = \langle \mathbb{R}, \mathbb{R}, \dots \rangle$  and  ${}^*\mathbb{N}$  are all standard sets, then internal sets.

(ii) Let  $\omega$  the infinite number defined as  $\omega = \langle 1, 2, 3, \dots \rangle$ . The set  $\{\omega\} = \langle \{i\} \rangle$  is internal but it is not standard.

(iii) For every integer  $i \geq 1$  we put  $X_i = [\frac{1}{i+1}, \frac{1}{i}[$ , and  $X = \langle X_i \rangle$ .

The above set is internal and infinite, but we can't find any standard element in  $X$  (because there does not exist a real number  $x$  such that  $\{i : x = x_i\} \in \mathcal{U}$  for  $x_i \in X_i$ ). From the Corollary 1, we deduce that the  $X$  is not standard.

In other hand, the set  $X$  is bounded above by 1, we can check that  $X$  has a supremum in  ${}^*\mathbb{R}$ , and we have  $\sup X = \langle \frac{1}{i} \rangle$ .

**Remark 5.** • In the collection of the internal sets [7, 13], we find the standard and the nonstandard sets.

• Every nonempty internal set of hyperreals bounded above has a supremum in  ${}^*\mathbb{R}$ . In fact, since the internal set  $A = \langle A_i \rangle$  is bounded above, then there exists  $M \in \mathbb{R}$  such that the set  $J = \{i : A_i \text{ is bounded above by } M\} \in \mathcal{U}$ .

We define  $s = \langle s_i \rangle$  such that  $s_i = \sup(A_i)$  for  $i \in J$  and  $s_i = 1$  else. we can check easily that the  $s = \sup(A)$ .

• We can prove the above result for every element of  ${}^*\mathcal{P}(\mathbb{R})$  by using the transfer principle, but this property is not true for every family of hyperreals (for example, the set  $\mathbb{R}$  is bounded above by every positive infinitely large number  $L$ , but it does not have a least upper bound), then we deduce that the set  ${}^*\mathcal{P}(\mathbb{R})$  is a proper subset of  $\mathcal{P}({}^*\mathbb{R})$ . The elements of  $\mathcal{P}({}^*\mathbb{R}) \setminus {}^*\mathcal{P}(\mathbb{R})$  are called the external sets. For example, the sets  $\mathbb{R}$ ,  $\mathbb{N}$ , the infinite numbers and the infinitesimal numbers are all external sets.

## 7 Alpha-theory

This approach is based on the existence of a new mathematical object, namely  $\alpha$ . Intuitively, this new element added to  $\mathbb{N}$  is considered as a "very large" natural number.

The use of  $\alpha$  is governed by the following five axioms [10].

•  $\alpha_1$ . **Extension Axiom.** For every sequence  $\varphi$  there is a unique element  $\varphi[\alpha]$ , called the "ideal value of  $\varphi$ " or the "value of  $\varphi$  at infinity".

•  $\alpha_2$ . **Composition Axiom.** If  $\varphi$  and  $\psi$  are sequences and if  $f$  is any function such that compositions  $f \circ \varphi$  and  $f \circ \psi$  make sense, then

$$\varphi[\alpha] = \psi[\alpha] \Rightarrow (f \circ \varphi)[\alpha] = (f \circ \psi)[\alpha].$$

•  $\alpha_3$ . **Number Axiom.** If  $c_r : n \rightarrow r$  is the constant sequence with value  $r \in \mathbb{R}$ , then  $c_r[\alpha] = r$ . If  $1_{\mathbb{N}} : n \rightarrow n$  is the identity sequence on  $\mathbb{N}$ , then  $1_{\mathbb{N}}[\alpha] = \alpha \notin \mathbb{N}$ .

•  $\alpha_4$ . **Pair Axiom.** For all sequences  $\varphi$ ,  $\psi$  and  $\vartheta$ :

$$\vartheta(n) = \{\varphi(n), \psi(n)\} \text{ for all } n \Rightarrow \vartheta[\alpha] = \{\varphi[\alpha], \psi[\alpha]\}.$$

- $\alpha_5$ . **Internal Set Axiom.** If  $\psi$  is a sequence of atoms, then  $\psi[\alpha]$  is an atom. If  $c_\emptyset : n \rightarrow \emptyset$  is the sequence with constant value the empty set, then  $c_\emptyset[\alpha] = \emptyset$ . If  $\psi$  is a sequence of nonempty sets, then

$$\psi[\alpha] = \{\varphi[\alpha] / \varphi[n] \in \psi[n] \text{ for all } n\}.$$

- Proposition 1.**
1. If  $\varphi(n) = \psi(n)$  eventually (i.e. for all but finitely many  $n$ ), then  $\varphi[\alpha] = \psi[\alpha]$ .
  2. If  $\varphi(n) \neq \psi(n)$  eventually, then  $\varphi[\alpha] \neq \psi[\alpha]$ .

**Definition 4.** Let  $A$  be a nonempty set. The star-transform of  $A$  is giving by:

$$A^* = \{\varphi[\alpha] / \varphi : \mathbb{N} \rightarrow A\}.$$

In the following proposition, we verify that the star-operator preserves all basic operations of sets (except the powerset).

**Proposition 2.** For all  $A, B$ , the following hold [10]:

1.  $A = B \Leftrightarrow A^* = B^*$ ;
2.  $A \in B \Leftrightarrow A^* \in B^*$ ;
3.  $A \subseteq B \Leftrightarrow A^* \subseteq B^*$ ;
4.  $\{A, B\}^* = \{A^*, B^*\}$ ;
5.  $(A \cup B)^* = (A^* \cup B^*)$ ;
6.  $(A \cap B)^* = (A^* \cap B^*)$ ;
7.  $(A \setminus B)^* = (A^* \setminus B^*)$ ;
8.  $(A \times B)^* = (A^* \times B^*)$ .

**Definition 5.** (i) The set of hyperreal numbers is the star-transform  $\mathbb{R}^*$  of the set of real numbers:

$$\mathbb{R}^* = \{\varphi[\alpha] / \varphi : \mathbb{N} \rightarrow \mathbb{R}\}.$$

(ii) The set of hypernatural numbers is the star-transform of the set of natural numbers:

$$\mathbb{N}^* = \{\varphi[\alpha] / \varphi : \mathbb{N} \rightarrow \mathbb{N}\}.$$

(iii) We define in  $\mathbb{R}^*$  the following binary relation:

$$\xi < \zeta \Leftrightarrow (\xi, \zeta) \in \{(x, y) \in \mathbb{R} \times \mathbb{R} / x < y\}^*.$$

**Theorem 7.** The hyperreal number system  $(\mathbb{R}^*, +, \cdot, 0, 1, <)$  is an ordered field.

**Remark 6.** • An example of an infinitesimal is given by  $\frac{1}{\alpha}$ , the ideal value of the sequence  $(\frac{1}{n})_{n \geq 1}$ . Other examples of infinitesimals are the following:

$$-\sin\left(\frac{1}{\alpha}\right), \frac{\alpha}{3 + \alpha^2}, \log\left(1 - \frac{1}{\alpha}\right).$$

- For the infinite numbers, we propose the following examples:

$$\alpha^2 + 1, 3 + \sqrt{\alpha}, \log(7\alpha - 3).$$

## 8 Criticism of the classical approaches to non-standard analysis

In this section we propose some examples, to see clearly the difficulties that can be encountered in practice by working on the classical approaches of the non-standard analysis. Firstly, we begin by the study of the Robinson's approach, afterwards, we proceed to the study of axiomatic approaches. Finally, we conclude with a small discussion as an introduction to the new approach.

Next, to explain our point of view about the Robinson's approach, we propose some examples in the following subsection:

### 8.1 For Robinson's approach

- For the infinitesimal number  $\delta = \langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle$ , we can't imagine intuitively his nature, because it is defined by the sequence  $(\frac{1}{n})_{n \geq 1}$  modulo the unknown ideal  $I$ .
- Let  $u$  be a hyperreal number defined as  $u = \langle -1, 1, -1, 1, \dots \rangle$ . Despite that the field  $({}^*\mathbb{R}, \leq)$  is totally ordered set, but we can't determine the sign of  $u$ . In other hand, we have two cases:
  1. If  $u \geq 0$ , then there exists an element  $F \in \mathcal{U}$  such that:  $F = \{i : u_i \geq 0\} = \{i : u_i = 1\}$ .  
In this case, we deduce that the set  $F \subseteq 2\mathbb{N} \in \mathcal{U}$ , and we find  $u = 1$ .
  2. If  $u \leq 0$ , we find  $2\mathbb{N} + 1 \in \mathcal{U}$ , and  $u = -1$ .

Now, to complicate this problem, we put the following question:

Where is the sign of the hyperreal number defined as:

$$\zeta = \langle \sin(1), \sin(2), \sin(3), \dots \rangle .$$

Since the total order in the hyperreal numbers is not defined explicitly, then we deduce that the Robinson's approach is very complicated to give us a simple property as the sign of an element of  ${}^*\mathbb{R}$ . Moreover, the sign of the hyperreal  $u$  which is defined by the sequence  $(u_i)$  is not sufficient to know the sign of this sequence at infinity which can be invariant (is not stable from a certain rank).

- Let  $v$  the hyperreal defined as:

$$v = \langle 1, 10^1, \frac{1}{2}, 10^{10^2}, \frac{1}{3}, 10^{10^3}, \dots, \frac{1}{i}, 10^{10^i}, \dots \rangle .$$

Where is the nature of this number? it is infinite or infinitesimal?

If  $2\mathbb{N} \in \mathcal{U}$  then  $v$  is infinite else it is infinitesimal. The determination of the nature of an hyperreal is not easy and evident in general, we can find other case very complicated than the above example. In addition, if we put  $(v_i)$  the sequence which defines the hyperreal  $v$ , and  $w$  the hyperreal defined by the sequence  $(v_{i+1})$ , then:

$$w = \langle 10^1, \frac{1}{2}, 10^{10^2}, \frac{1}{3}, 10^{10^3}, \dots, \frac{1}{i}, 10^{10^i}, \dots \rangle .$$

If  $2\mathbb{N} \in \mathcal{U}$  then  $v$  is infinite, and  $w$  is infinitesimal. Thus, we can find two hyperreals do not have the same nature, the first is defined by a sequence  $(v_i)$ , the second by its subsequence  $(v_{\phi(i)})$ . In the above example, the translation of the indices of the sequence  $(u_i)$  which defines the infinitesimal number  $\langle u_i \rangle$ , is sufficient to transform it to an infinite number. This is not well to be effective in practice, for example, if  $\langle u_i \rangle = 1$ , (in general) we can't know every thing about the value of the hyperreal  $v = \langle u_{3i+1} \rangle$ ,  $v$  can be zero, infinite, infinitesimal number...

Next, we propose an example of an hyperreal number  $\langle u_i \rangle$  which can be zero or an integer  $1 \leq i \leq 9$ , but it is impossible to determine his value.

- For every real number  $x$ , let  $(x_i)$  the sequence defined by the decimal representation of  $x$ , we have:

$$x = x_1, x_2 x_3 x_4 x_5 x_6 \dots$$

Let  $\tilde{x}$  the hyperreal defined as:  $\tilde{x} = \langle x_i \rangle$ . For the number  $\pi$ , we get:

$$\pi = 3.1415926535897932385\dots$$

Then,  $\tilde{\pi} = \langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, \dots \rangle$ . We attempt to determine the value of this hyperreal, for that, we propose to prove the following lemma:

**Lemma 1.** *Let  $A$  be a finite subset of  $\mathbb{R}$ .*

*For every element  $u = (u_i)$  of  $A^{\mathbb{N}}$ , the hyperreal number  $\langle u_i \rangle$  is an element of  $A$ .*

**Proof.** We put  $A = \{a_1, a_2, \dots, a_n\}$ , and  $F_x = \{i : u_i = x\}$  for every  $x \in A$ .

Let  $\mathcal{U}$  the ultrafilter defined in Robinson's approach.

If there exists  $1 \leq i_0 \leq n - 1$  such that  $F_{a_{i_0}} \in \mathcal{U}$ .

Then,  $\langle u_i \rangle = a_{i_0}$ , else  $F_{a_1}^c, F_{a_2}^c, \dots, F_{a_{n-1}}^c \in \mathcal{U}$ .

Then:  $F_{a_1}^c \cap F_{a_2}^c \cap \dots \cap F_{a_{n-1}}^c \in \mathcal{U}$ .

We deduce that:  $(F_{a_1} \cup F_{a_2} \cup \dots \cup F_{a_{n-1}})^c = F_{a_n} \in \mathcal{U}$ .

Which implies that  $\langle u_i \rangle = a_n$ .

From the above lemma, we deduce that  $\tilde{\pi} \in \{0, 1, 2, \dots, 9\}$  (then can be non invertible). Unfortunately, we haven't any way to determine his value. Let  $x$  be the natural number in  $\{0, 1, 2, \dots, 9\}$  such that  $\tilde{\pi} = x$ . Consider the hyperreal  $\tilde{\alpha} = \langle \alpha_i \rangle$  defined as:

$$\alpha_i = \begin{cases} \frac{1}{i}, & \text{when } x = \pi_i; \\ 10^{10^i} & \text{else.} \end{cases}$$

The nature of the number  $\tilde{\alpha}$  is not compatible with the behavior of the sequence used to define it. In fact, the values taken by the sequence  $(\alpha_i)$  are very "large" in an infinity of indices. The set  $\{i : \alpha_i = \frac{1}{i}\}$  is very small compared to  $\{i : \alpha_i = 10^{10^i}\}$ . Despite that, this number  $\tilde{\alpha}$  is infinitesimal. In addition, we do not have any rule to determine in general the set of indices  $i$  that gives us the nature of an hyperreal  $\langle u_i \rangle$ .

## 8.2 For Nelson's approach and Alpha-Theory

Despite the importance of the axiomatic methods, which allows us to give and explain rigorously the behavior of each new defined notion, but in practice, is not very effective, especially if the notions of the proposed theory are not defined explicitly. In this paper we propose a constructive approach to nonstandard analysis without adding any axiom, only the properties of the classical analysis are sufficient to construct the new field. In addition, we define an explicit total order relation in the new set called the field of Omicron-reals.

## 8.3 Discussion

Abraham Robinson succeeded to show the existence of a total order relation on  $\mathbb{R}^{\mathbb{N}}$ , but the explicit determination of this relation is very difficult. The judgment of the scientific work of Robinson begins with the study of the choice of the  $\mathbb{R}^{\mathbb{N}}$ . To find or not to find an explicit total order is another question that can be asked after the determination of the initial set in construction. Now, the question we might ask is the following: why we need the ring  $\mathbb{R}^{\mathbb{N}}$  to define the field of nonstandard analysis? From the incompatibility between his nature and the behavior of the sequence which defined it, the hyperreal like

$\tilde{\alpha}, \tilde{\pi}, \langle 1, -1, 1, \dots \rangle$ , or  $\langle \sin(1), \sin(2), \dots \rangle$  does not matter in practice, then, the choice of the ring  $\mathbb{R}^{\mathbb{N}}$  is too broad to be effective. In the following section, we attempt to give the answer of the following question: can we construct the field of the infinitesimal numbers by using a proper subset of  $\mathbb{R}^{\mathbb{N}}$  with an explicit total order?

## 9 The proposed method

### 9.1 The metallic map

Let  $D(0, 1)$  (resp.  $D'(0, 1)$ ) the open (resp. closed) disk of radius 1 and center 0.

**Definition 6.** Let  $u$  be a map from  $]0, 1]$  to  $\mathbb{R}$ , such that:

- (i) There exists a map  $\tilde{u}$  is defined on  $D'(0, 1)$ , and holomorphic in a neighborhood of 0.
- (ii) There exists  $\varepsilon > 0$ , such that  $\forall x \in ]0, \varepsilon[$  we have:  $\tilde{u}(x) = u(x)$ .

The  $u$  is called a metallic map, and  $\tilde{u}$  is a metallic extension of  $u$ .

**Example 3.** If  $f$  is defined in an interval  $]0, 1]$  as:  $f(x) = \begin{cases} 2x + 1 & \text{when } x \in ]\varepsilon, 1], \\ 1 - 3x^2 & \text{when } x \in ]0, \varepsilon] \end{cases}$

then a metallic extension  $\tilde{f}$  is given by:  $\tilde{f}(z) = 1 - 3z^2$  in the disk  $D'(0, 1)$ .

**Remark 7.** If  $u$  is metallic, then the two metallic extension  $\tilde{u}$  and  $\hat{u}$  of  $u$  are identic in a disk  $D(0, \varepsilon)$  from the theorem 2.

**Definition 7.** We note  $\Delta_1 = \{ u, u \text{ is a metallic map} \}$ , and we have the following definitions:

- $\Delta_1(\frac{1}{n}) = \{ u(\frac{1}{n})_{n \geq 1}, u \text{ is a metallic map} \}$ .
- $H_0 =$  the set of maps  $\tilde{u}$  defined on the disk  $D'(0, 1)$  and holomorphic in a neighborhood of 0.
- $(H_0, +)$  is a commutative group, let  $\mathcal{O}_0$  be a subgroup of  $H_0$  containing the maps defined on the disk  $D'(0, 1)$  and are zero in a neighborhood of 0.
- Let  $\theta_0$  be a map defined as :

$$\theta_0 : \begin{array}{ccc} \Delta_1(\frac{1}{n}) & \longrightarrow & H_0/\mathcal{O}_0 \\ u(\frac{1}{n})_{n \geq 1} & \longrightarrow & C(\tilde{u}) \end{array}$$

which  $C(\tilde{u})$  is the equivalence class of  $\tilde{u}$  modulo  $\mathcal{O}_0$ . The map  $\theta_0$  is a well-defined from the unicity of  $C(\tilde{u})$ .

- We consider the surjective map  $\theta_1$  defined as:

$$\theta_1 : \begin{array}{ccc} \Delta_1(\frac{1}{n}) & \longrightarrow & \theta_0(\Delta_1(\frac{1}{n})) \\ u(\frac{1}{n})_{n \geq 1} & \longrightarrow & C(\tilde{u}) \end{array}$$

and the set:  $\overline{\Delta_1(\frac{1}{n})} = \{ \theta_1^{-1}(C(\tilde{u})), C(\tilde{u}) \in \theta_0(\Delta_1(\frac{1}{n})) \}$ .

- We define on the set  $\Delta_1(\frac{1}{n})$  the following equivalence relation  $\sim$ :

$$u(\frac{1}{n})_{n \geq 1} \sim v(\frac{1}{n})_{n \geq 1} \iff \exists n_0, \forall n \geq n_0, u(\frac{1}{n}) = v(\frac{1}{n}).$$

- $\overline{u(\frac{1}{n})_{n \geq 1}}$  is the equivalence class of  $u(\frac{1}{n})_{n \geq 1}$  modulo  $\sim$ .

**Remark 8.** 1. We can check the equality:  $\overline{u(\frac{1}{n})_{n \geq 1}} = \theta_1^{-1}(C(\tilde{u}))$ .

Then :

$$\overline{\Delta_1(\frac{1}{n})} = \left\{ \overline{u(\frac{1}{n})_{n \geq 1}}, u \in \Delta_1 \right\}.$$

2. The sets  $\Delta_1$  and  $\overline{\Delta_1(\frac{1}{n})}$  are commutative groups.

3. The map defined as:

$$\overline{\theta_1} : \begin{array}{ccc} \overline{\Delta_1(\frac{1}{n})} & \longrightarrow & E_1 = \theta_0(\Delta_1(\frac{1}{n})) \\ u(\frac{1}{n})_{n \geq 1} & \longrightarrow & C(\tilde{u}) \end{array}$$

is an isomorphism between two groups.

**Definition 8.** •  $\mathcal{A}_2 = \left\{ \frac{1}{u}, u \in \Delta_1 \forall x \in ]0, 1[ \ u(x) \neq 0 \text{ and } \lim u(\frac{1}{n}) = 0 \right\}$ .

•  $\Delta_2 = \left\{ v / v : ]0, 1[ \longrightarrow \mathbb{R}, \text{ there exists } 1/u \in \mathcal{A}_2 \text{ and } \varepsilon > 0 \text{ such that } v|_{]0, \varepsilon[} = (\frac{1}{u})|_{]0, \varepsilon[} \right\}$ .

•  $\Delta_2(\frac{1}{n}) = \left\{ (v(\frac{1}{n}))_{n \geq 1}, v \in \Delta_2 \right\}$ .

## 9.2 Construction of a unitary ring

**Lemma 2.** Let  $\Delta = \Delta_1 \cup \Delta_2$ ,  $(\Delta, +, \cdot)$  is a unitary ring.

**Proof.**

• The stability of the sum: The set  $\Delta$  is non-empty set, because  $\Delta_1 \neq \emptyset$  and  $\mathbb{R} \subseteq \Delta_1$  (we identify the constant functions by the real numbers). Show that:  $\forall g \in \Delta$ , and  $h \in \Delta$  then  $g + h \in \Delta$ .

**First case :** If  $(g, h) = (u, v) \in \Delta_1^2$ , we verify that the function  $s = f + g$  is matalic map, and:  $\tilde{s} = \tilde{u} + \tilde{v}$ .

**Second case :** If  $(g, h) \in \Delta_2^2$ , there exists  $(u, v) \in \Delta_1^2$  such that  $\lim u(\frac{1}{n}) = \lim v(\frac{1}{n}) = 0$  and  $u(x)v(x) \neq 0$  for every  $x \in ]0, \varepsilon[$ , for  $\varepsilon > 0$  small enough, in addition:  $g|_{]0, \varepsilon[} = \frac{1}{u}|_{]0, \varepsilon[}$  and  $h|_{]0, \varepsilon[} = \frac{1}{v}|_{]0, \varepsilon[}$ .

Since  $\tilde{u}$  and  $\tilde{v}$  are holomorphic functions on a neighborhood of 0, there exists  $(m, n, l) \in \mathbb{N}^3$  such that:

$$\tilde{u}(z) = z^m b_1(z), \tilde{v}(z) = z^n b_2(z) \text{ and } \tilde{u}(z) + \tilde{v}(z) = z^l b_3(z),$$

where  $b_1, b_2, b_3$  are 3 holomorphic functions on a neighborhood of 0, such that:  $b_1(0)b_2(0)b_3(0) \neq 0$ .

Let :

$$\psi(x) = \frac{u(x)v(x)}{u(x) + v(x)}.$$

The map  $g + h$  is defined in the interval  $]0, 1[$ , for  $\varepsilon > 0$  small enough, and  $b_1(z)b_2(z)b_3(z) \neq 0$  in the disk  $D(0, \varepsilon)$ , we have :  $g(x) + h(x) = \frac{1}{u(x)} + \frac{1}{v(x)} = \frac{1}{\psi(x)} = x^{l-m-n} \frac{b_3(x)}{b_1(x)b_2(x)}$ , for every  $x \in ]0, \varepsilon[$ .

✓ If  $l - m - n \geq 0$ , The map defined as  $\tilde{\phi}(z) = \begin{cases} z^{l-m-n} \frac{b_3(z)}{b_1(z)b_2(z)} & \text{in } D(0, \varepsilon) \\ 1 & \text{if not} \end{cases}$  is a metallic extension of  $g + h$ , then  $g + h$  is a element of  $\Delta_1$ .

✓ If  $m + n - l < 0$ , the map defined as:  $\tilde{\psi}(z) = \begin{cases} z^{m+n-l} \frac{b_1(z)b_2(z)}{b_3(z)} & \text{in } D(0, \varepsilon) \\ 1 & \text{if not} \end{cases}$  is a metallic extension of  $\psi$ , in addition  $\lim \psi(\frac{1}{n}) = 0$ , we deduce that the  $g + h$  is an element of  $\Delta_2$ .

**Third case:** If  $(g, h) \in \Delta_1 \times \Delta_2$ , there exists  $(u, v) \in \Delta_1^2$ , such that :  $g = u$ ,  $h_{/]0, \varepsilon]} = \frac{1}{v}_{/]0, \varepsilon]}$ ,  
 $\lim v(\frac{1}{n}) = 0$ .

Let  $k(z) = \frac{\tilde{v}(z)}{\tilde{u}(z)\tilde{v}(z) + 1}$ .

Since  $\tilde{v}(0) = 0$ , we have  $k(0) = 0$ , and  $k$  is a holomorphic function in a disk  $D(0, \varepsilon)$ , for  $\varepsilon > 0$  small enough.

The map  $k$  is nonzero and holomorphic at  $z = 0$ , we can choose the  $\varepsilon$  so that  $k(x) \neq 0$  for every  $x \in ]0, \varepsilon]$ .

Let  $\phi$  the function defined in  $]0, 1]$  as:

$$\phi(x) = \begin{cases} k(x) & \text{if } x \in ]0, \varepsilon] \\ 1 & \text{if not} \end{cases} .$$

We verify that:  $g + h_{/]0, \varepsilon]} = (\frac{1}{\phi})_{/]0, \varepsilon]}$ ,  $\phi \in \Delta_1$  then  $g + h \in \Delta_2 \subseteq \Delta$ .

Finally, we deduce that  $(\Delta, +)$  is a commutative group.

- Now, we can show the stability of the law  $(.)$  in  $\Delta$ , for that, we distinguish tree cases:

1. We can easily verify that the product of two metallic functions is a metallic function.  
 ( if  $g \in \Delta_1$  and  $h \in \Delta_1$  then  $gh \in \Delta_1 \subseteq \Delta$  )

2. In this case, we assume that  $g \in \Delta_1$  and  $h \in \Delta_2$ , we can show that  $gh \in \Delta$ , in fact, there exists  $(u, v) \in \Delta_1^2$ , such that:  $g = u$ ,  $h_{/]0, \varepsilon]} = \frac{1}{v}_{/]0, \varepsilon]}$ ,  $\lim v(\frac{1}{n}) = 0$ .

- if  $\lim u(\frac{1}{n}) \neq 0$ , then  $\frac{\tilde{v}}{u}$  is holomorphic in the disk  $D(0, \varepsilon)$ , which implies that:  $\frac{u}{v} \in \Delta_2 \subseteq \Delta$ .

- if  $\lim u(\frac{1}{n}) = 0$ , then  $\lim \tilde{u}(\frac{1}{n}) = 0$  and  $\tilde{u}(0) = 0$ , we deduce that:  $\tilde{u}(z) = z^k b_1(z)$  in  $D(0, \varepsilon)$ , and  $\tilde{v}(z) = z^{k'} b_2(z)$ , where  $b_i(z) \in H(D(0, \varepsilon))$ , for  $i \in \{0, 1\}$  and  $b_i(0) \neq 0$ .

We get:

$$\frac{\tilde{u}(z)}{\tilde{v}(z)} = z^{k-k'} \frac{b_1(z)}{b_2(z)}$$

-First case : if  $k = k'$  then the function  $\frac{\tilde{u}}{\tilde{v}}$  is holomorphic in  $D(0, \varepsilon)$ , which implies that  $\frac{u}{v} \in \Delta_1$ .

-Second case : if  $k > k'$  then  $\lim_{z \rightarrow 0} \frac{\tilde{u}}{\tilde{v}}(z) = 0$ , and  $\frac{\tilde{u}}{\tilde{v}}$  is a holomorphic function in the disk  $D(0, \varepsilon)$ .

Then:  $\frac{u}{v} \in \Delta_1$ .

-Third case : if  $k < k'$  then  $\lim_{z \rightarrow 0} \frac{\tilde{u}}{\tilde{v}}(z) = 0$ , which implies that  $\frac{u}{v} \in \Delta_2$ .

3. In the case of  $g \in \Delta_2$  and  $h \in \Delta_2$ , we verify easily the stability of the law  $(.)$ .

Finally, we deduce that  $(\Delta, +, .)$  is a commutative and unitary ring, where the constant function  $1_\Delta$  is a multiplicative identity of  $\Delta$ .

### 9.3 Construction of the new field

Let  $\mathcal{I}_0$  the set defined as:

$$\mathcal{I}_0 = \{u_{/]0, 1]} / u \in \mathcal{O}_0 \text{ and } u(]0, 1]) \subset \mathbb{R}\}$$

it is a set of maps defined in  $]0, 1]$ , and zero on  $]0, \varepsilon[$  ( for  $0 < \varepsilon \leq 1$  ).

Thus, we have demonstrated the following proposition.

**Proposition 3.**  $\mathcal{I}_0$  is a maximal ideal of  $\Delta$ .

**Proof.**

- We can prove easily that the  $\mathcal{I}_0$  is a additive subgroup of  $\Delta$ .
- $\mathcal{I}_0$  is an ideal of  $\Delta$ . In fact, if  $\theta$  is an element of  $\mathcal{I}_0$ , then  $\theta_{/]0, \varepsilon[} = 0$  for  $\varepsilon > 0$ . For every  $u \in \Delta$ , we have  $(\theta u)_{/]0, \varepsilon[} = 0$  then  $\theta u \in \mathcal{I}_0$ .
- Let  $I$  be an ideal of  $\Delta$  such that  $\mathcal{I}_0 \subseteq I$ . We assume that this inclusion is strict, then there exists  $u \in I \setminus \mathcal{I}_0$ . Since  $u$  is an element of  $\Delta$ , we can distinguish two cases:
  1. First case  $u \in \Delta_1$ : If  $u$  admits an infinity of zeros in  $]0, \varepsilon[$  for every  $\varepsilon > 0$ , then  $\tilde{u} = 0$  and we deduce that  $u \in \mathcal{I}_0$ , which is absurd. Then there exists  $\varepsilon > 0$  such that  $u(x) \neq 0$  for every  $x \in ]0, \varepsilon[$ . Let  $v$  be a function defined in  $]0, 1]$  by  $v(x) = \frac{1}{u(x)}$  in  $]0, \varepsilon[$  and  $v(x) = 1$  while  $x \in [\varepsilon, 1]$ . We have  $u(x)v(x) = 1$  in  $]0, \varepsilon[$ , then  $1 - uv \in \mathcal{I}_0$ . Consider  $i \in \mathcal{I}_0 \subseteq I$  such that  $1 - uv = i$ .  
Then  $1 = i + uv$  and we deduce that  $1 \in I$  which implies that  $I = \Delta$ .
  2. Second case  $u \in \Delta_2$ : In this case there exists  $\varepsilon > 0$ ,  $v \in \Delta_1$  such that  $u(x) = \frac{1}{v(x)}$  in  $]0, \varepsilon[$ .  
Then  $1 - uv \in \mathcal{I}_0$  and we deduce that  $I = \Delta$ .

From (1) and (2) we deduce that the ideal  $\mathcal{I}_0$  is an ideal maximal of  $\Delta$ .

**Theorem 8.** The ring  $(\Delta/\mathcal{I}_0, +, \cdot)$  is a field.

**Proof.** From the above proposition, the ideal  $\mathcal{I}_0$  is maximal, so we deduce that the ring  $(\Delta/\mathcal{I}_0, +, \cdot)$  is a field.

#### 9.4 The field of Omicran-reals

We consider the set defined as  $\Delta(\frac{1}{n}) = \{ h(\frac{1}{n})_{n \geq 1}, h \in \Delta \}$ .

Let  $\sim$  the equivalence relation defined on the set  $\Delta(\frac{1}{n})$  as:

$$g(\frac{1}{n})_{n \geq 1} \sim h(\frac{1}{n})_{n \geq 1} \Leftrightarrow \exists n_0 / \forall n \geq n_0, h(\frac{1}{n}) = g(\frac{1}{n}).$$

The equivalence class is given by:  $\overline{g(\frac{1}{n})_{n \geq 1}} = \{ h(\frac{1}{n})_{n \geq 1}, \text{ for } h \in \Delta \text{ and } \exists n_0 \in \mathbb{N} / \forall n \geq n_0, h(\frac{1}{n}) = g(\frac{1}{n}) \}$ .

The map

$$\theta : \left( \overline{\Delta(\frac{1}{n})}, +, \cdot \right) \longrightarrow (\Delta/\mathcal{I}_0, +, \cdot)$$

$$g(\frac{1}{n})_{n \geq 1} \longrightarrow C(g) = \bar{g}$$

is well defined, in addition we have:

1.  $(\overline{\Delta(\frac{1}{n})}, +, \cdot)$  is a field.
2.  $\theta$  is an isomorphism.

**Theorem 9.** There exists a set  $\mathcal{O}$  and a total order  $\leq$  such that:

- (i)  $(\mathcal{O}, +, \cdot)$  is an extension field of  $(\mathbb{R}, +, \cdot)$ .

(ii)  $(\mathcal{O}, \leq)$  is a ordered  $\mathbb{R}$ -extension.

(iii)  $I_{\mathcal{O}} \neq \emptyset$ .

**Proof.** The set  $\Delta/\mathcal{I}_{\mathcal{O}}$  is noted  $\Delta_0$ , we replace  $C(g) = \bar{g}$  by  $g$  ( if it has not ambiguity ).  
From what precedes, we deduce that the map:

$$\theta : \begin{array}{l} \overline{(\Delta(\frac{1}{n}), +, \cdot)} \longrightarrow (\Delta_0, +, \cdot) \\ g(\frac{1}{n})_{n \geq 1} \longrightarrow g \end{array}$$

is a ring isomorphism.

Let  $\delta$  be an indeterminate, and  $\mathcal{O} = \Delta_0(\delta) = \{ \tilde{g}(\delta), g \in \Delta_0 \}$ .

If it has not ambiguity, we replace  $\tilde{g}(\delta)$  by  $g(\delta)$ .

The map defined as:

$$\delta^* : \begin{array}{l} \Delta_0 \longrightarrow \mathcal{O} \\ g \longrightarrow g(\delta) \end{array}$$

is a ring isomorphism.

Then  $(\mathcal{O}, +, \cdot)$  is a field, in addition, the map  $\varphi = \delta^* \circ \theta$  defined as :

$$\varphi : \begin{array}{l} \overline{(\Delta(\frac{1}{n}), +, \cdot)} \longrightarrow (\mathcal{O}, +, \cdot) \\ (g(\frac{1}{n}))_{n \geq 1} \longrightarrow g(\delta) \end{array}$$

is a isomorphism.

- for  $g = cste = l$ , we note  $\bar{l} = \overline{(g(\frac{1}{n}))_{n \geq 1}}$ , we identify  $l$  by the image of  $\bar{l}$  by  $\varphi$ , and we find  $\varphi(\bar{l}) = l$ . by using the identification, we deduce that:  $\mathbb{R} \subseteq \mathcal{O}$ .
- We can define on  $\mathcal{O}$  the following relation  $\leq$ :  
 $g(\delta) \leq h(\delta)$  if and only if there exists an integer  $n_0$ , such that:  $g(\frac{1}{n}) \leq h(\frac{1}{n})$  for every  $n \geq n_0$ .  
It is easy to check that  $\leq$  is reflexive, transitive and antisymmetric, then it is a partial order.
- To show that set  $(\mathcal{O}, \leq)$  is ordered  $\mathbb{R}$ -extension, we must show that relation  $\leq$  is total.  
For that we consider  $g, h \in \Delta_0$ , we assume that these propositions are true ( *not*  $g(\delta) \leq h(\delta)$  ) and ( *not*  $h(\delta) \leq g(\delta)$  ).  
Next, we need to find a contradiction.  
If the above propositions are true, then:  $\forall k \in \mathbb{N}, \exists n_k > k$  and  $\exists n'_k > k / g(\frac{1}{n_k}) > h(\frac{1}{n_k})$  and  $g(\frac{1}{n'_k}) < h(\frac{1}{n'_k})$ .

1. We assume that  $g, h \in \Delta_1$ . From the Intermediate value theorem we deduce that there exists  $\beta_k \in ]\frac{1}{n_k}, \frac{1}{n'_k}[$  such that  $(g - h)(\beta_k) = 0$ , (we can choose the  $\beta_k$  so that the sequence  $(\beta_k)$  is strictly decreasing), then the holomorphic function  $\tilde{g} - \tilde{h}$  has an infinite number of roots in neighborhood of 0, from the theorem2, we deduce that the function  $\tilde{g} - \tilde{h}$  is the zero function, then  $g = h$ , which is absurd.
2. Now, we suppose that  $g, h \in \Delta_2$ . There exists  $(u, v) \in \Delta_1^2$ , such that:  
 $u_{/]0, \varepsilon[} = (\frac{1}{g})_{/]0, \varepsilon[}$  and  $v_{/]0, \varepsilon[} = (\frac{1}{h})_{/]0, \varepsilon[}$ .  
from the above results, we deduce that:  $u(\delta) \leq v(\delta)$  or  $v(\delta) \leq u(\delta)$ .  
which implies that:  $g(\delta)$  and  $h(\delta)$  are comparable.
3. In the case of  $g \in \Delta_1$  and  $h \in \Delta_2$ . There exists  $h_1 \in \Delta_1$  and  $\varepsilon > 0$ , such that:  
 $h_{/]0, \varepsilon[} = (\frac{1}{h_1})_{/]0, \varepsilon[}$ .  
Since  $h_1$  is a metallic function, then the sequence  $(h_1(\frac{1}{n}))$  keeps a constant sign from a certain

rank, in fact:

if it's not the case, then  $\forall k \in \mathbb{N}$  there exists  $n_k > k$  et  $n'_k > k$  such that:  $h_1(\frac{1}{n_k}) > 0$  and  $h_1(\frac{1}{n'_k}) < 0$ .

From the Intermediate value theorem we deduce that there exists  $\beta_k \in ]\frac{1}{n_k}, \frac{1}{n'_k}[$  and  $(h_1)(\beta_k) = 0$ , (we can choose the  $\beta_k$  such that the sequence  $(\beta_k)$  is strictly decreasing), from the theorem2,  $h_1$  is the zero function on the neighborhood of 0, which absurd( because we suppose  $(h_1(\frac{1}{n}))$  does not keep a constant sign from a certain rank)

Finally, we deduce that the sequence  $(h_1(\frac{1}{n}))$  keeps a constant sign, since  $\lim h_1(\frac{1}{n}) = 0$ , then  $\lim \frac{1}{h_1(\frac{1}{n})}$  exists, and  $\lim \frac{1}{h_1(\frac{1}{n})} = \pm\infty$ , which implies  $\lim h(\frac{1}{n}) = \pm\infty$ .

- if  $\lim h(\frac{1}{n}) = +\infty$ , then  $g(\delta) \leq h(\delta)$  ( because  $g(\frac{1}{n}) \leq h(\frac{1}{n})$ , from a certain rank ).
- if  $\lim h(\frac{1}{n}) = -\infty$ , then  $h(\delta) \leq g(\delta)$  ( because  $h(\frac{1}{n}) \leq g(\frac{1}{n})$ , from a certain rank ).

- Now, it remains to show that:  $I_{\mathcal{O}} \neq \emptyset$ .

For that, it is necessary to find a element  $\delta \in \mathcal{O}$  which infinitesimal.

for  $u : x \rightarrow x$ , we have  $u \in \Delta$  (more precisely  $\Delta_1$ ) and  $\delta = u(\delta) = \overline{u(\frac{1}{n})_{n \geq 1}}$ .

In addition, we have:  $0 < \delta < \varepsilon$  for every real strictly positive  $\varepsilon$ , because there exists  $p \in \mathbb{N}$  such that:  $0 < u(\frac{1}{n}) < \varepsilon$ , for every integer  $n > p$ . Then  $\delta$  is infinitesimal.

**Conclusions 1.** Finally, we deduce that:

(i)  $(\mathcal{O}, +, \cdot)$  is a extension field of  $(\mathbb{R}, +, \cdot)$ .

(ii)  $(\mathcal{O}, \leq)$  is a ordered  $\mathbb{R}$ -extension, which contains the infinitesimal element  $\delta$ .

The field  $(\mathcal{O}, +, \cdot)$  is called the field of **Omicran-reals**.

An element of  $\mathcal{O}$  is called an **Omicran** (or an **Omicran-real**).

## 10 The applications of the field of Omicran-reals

### 10.1 The exact limit

**Proposition 4.** The map  $\varphi$  defined as :

$$\varphi : \begin{array}{l} \overline{(\Delta(\frac{1}{n}), +, \cdot)} \longrightarrow (\mathcal{O}, +, \cdot) \\ \overline{(g(\frac{1}{n}))_{n \geq 1}} \longrightarrow g(\delta) \end{array}$$

is isomorphism.

If we want to define a new concept more precise than the limit that allows to give the value taken by the sequence  $(f(\frac{1}{n}))_{n \geq 1}$  on ad infinitum, then this concept (called exact limit) is dependent on the values taken by  $(f(\frac{1}{n}))_{n \geq 1}$  from a certain rank  $n_0$ . Intuitively, the equivalence class  $\overline{(f(\frac{1}{n}))_{n \geq 1}}$  is a only concept can give these values independently  $n_0$ . On other hand, if  $f$  is an element of  $\Delta$ , then we can identify the equivalence class  $\overline{(f(\frac{1}{n}))_{n \geq 1}}$  by  $f(\delta)$  from the above proposition, so, we deduce that we can define the new concept as follow:

**Definition 9.** Let  $f \in \Delta$ .

The Omicran  $\varphi(\overline{(f(\frac{1}{n}))_{n \geq 1}}) = f(\delta)$  is called the exact limit of the sequence  $(f(\frac{1}{n}))_{n \geq 1}$ .

We note:

$$\lim_{exact} f(\frac{1}{n}) = f(\delta).$$

**Remark 9.** We remark that  $\lim_{exact} = \varphi \circ s$ , where  $s$  is a canonical surjection defined as:

$$s : \begin{aligned} (\Delta(\frac{1}{n}), +, \cdot) &\longrightarrow \overline{(\Delta(\frac{1}{n}), +, \cdot)} \\ (f(\frac{1}{n}))_{n \geq 1} &\longrightarrow (f(\frac{1}{n}))_{n \geq 1} \end{aligned} .$$

**Example 4.** •  $\lim_{exact} \frac{1}{n} = \delta$ .

•  $\lim_{exact} \sin(\frac{1}{n}) = \sin(\delta)$ .

•  $\lim_{exact} \frac{1}{n+1} = \frac{\delta}{\delta+1}$ .

•  $\lim_{exact} \sin(2\pi n) = 0$ , the writing  $\sin(\frac{2\pi}{\delta})$  doesn't make sense, because the function  $z \longrightarrow \sin(\frac{2\pi}{z})$  is not element of  $\Delta$ .

• We can verify that there does not exist an element  $f \in \Delta$ , such that  $f(\frac{1}{n}) = (-1)^n$  from a certain rank, then we can't define the exact limit  $\lim_{exact} (-1)^n$ .

• Generally, from the proprieties of the elements of  $\Delta$ , we can show that if  $(x_n)$  does not keep a constant sign from a certain rank, then this sequence not admits a exact limit, for example, if  $x_n = \frac{(-1)^n}{n}$ , then  $\lim x_n = 0$ , but we can't define the exact limit of  $(x_n)$ .

## 10.2 The projection of an element of $\mathcal{O}$

**Definition 10.** Let  $f$  be a metallic function, and  $x \in \mathcal{O}$  such that  $x = f(\delta)$ .

If we find an element  $x^* \in \mathbb{R}$  such that  $|x - x^*| \leq |x - y|, \forall y \in \mathbb{R}$ , then the real  $x^*$  is called the projection of  $x$  onto  $\mathbb{R}$ .

**Remark 10.** The distance from  $x$  to  $\mathbb{R}$  is  $d_{\mathbb{R}}(x) = \inf_{y \in \mathbb{R}} |x - y| = |x - x^*|$ .

**Example 5.** •  $\delta^* = 0$ .

•  $(\frac{1}{\delta^2 + 1})^* = 1$ .

**Theorem 10.** Let  $f$  be a metallic function, and  $x \in \mathcal{O} / x = f(\delta)$ .

The projection  $x^*$  of  $x$  onto  $\mathbb{R}$  exist and unique, in addition, we have:  $x^* = \lim_{n \rightarrow +\infty} f(\frac{1}{n})$ .

**Proof.** Let  $x_0 = \lim f(\frac{1}{n})$ .

Then:  $\forall \varepsilon > 0 \exists n \geq n_0, |f(\frac{1}{n}) - x_0| \leq \varepsilon$ .

$\Leftrightarrow$  for every  $n \geq n_0$ , we have:  $-\varepsilon \leq f(\frac{1}{n}) - x_0 \leq \varepsilon$ .

$\Leftrightarrow \lim_{exact} f(\frac{1}{n}) - x_0 \leq \varepsilon$  and  $-\varepsilon \leq \lim_{exact} f(\frac{1}{n}) - x_0$ .

$\Leftrightarrow f(\delta) - x_0 \leq \varepsilon$  and  $-\varepsilon \leq f(\delta) - x_0$ .

$\Leftrightarrow |f(\delta) - x_0| \leq \varepsilon$  (1)

Next, we can show that:  $\forall y \in \mathbb{R}$  we have  $|f(\delta) - x_0| \leq |f(\delta) - y|$ .

If  $\exists y \in \mathbb{R}$  and  $|f(\delta) - y| \leq |f(\delta) - x_0| \leq \varepsilon$  ( $\forall \varepsilon \in \mathbb{R}^+$ )

Then, we deduce that  $|y - x_0| \leq 2\varepsilon$  ( $\forall \varepsilon \in \mathbb{R}^+$ )

Then  $y = x_0 = x^*$ .

Finally, we deduce the existence and the unicity of  $x^* \in \mathbb{R}$  such that:

$$\forall y \in \mathbb{R}, |f(\delta) - x^*| \leq |f(\delta) - y| .$$

In addition  $x^* = \lim f(\frac{1}{n})$ .

**Theorem 11.** Let  $f$  be a metallic map, and  $x = f(\delta)$ .

Then for every real  $\varepsilon > 0$  we have:

$$|x - x^*| \leq \varepsilon$$

$x^*$  is a unique element of  $\mathbb{R}$  which verify this property.

**Proof.** For every  $n \geq n_0$  we have:

$$|f(\frac{1}{n}) - x^*| \leq \varepsilon,$$

then  $\forall n \geq n_0$  :

$$x^* - \varepsilon \leq f(\frac{1}{n}) \leq x^* + \varepsilon,$$

which implies:

$$x^* - \varepsilon \leq f(\delta) \leq x^* + \varepsilon.$$

We deduce that:

$$|x - x^*| \leq \varepsilon.$$

To show the unicity of  $x^*$ , we assume that there exists an other element  $y \in \mathbb{R}$  such that:  $|x - y| \leq \varepsilon$ .

Then:  $|x^* - y| \leq 2\varepsilon$ , finally we get  $y = x^*$ .

**Theorem 12.** (of the exact limit)

If  $f \in \Delta_1$ , then the real  $\lim f(\frac{1}{n})$  is the projection of  $\lim_{exact} f(\frac{1}{n})$  onto  $\mathbb{R}$ , so we get:

$$(\lim_{exact} f(\frac{1}{n}))^* = \lim f(\frac{1}{n}).$$

### 10.3 Necessary conditions for the existence of the exact limit

- Let  $(x_n)_{n \geq 1}$  be a sequence of real numbers, we assume that the exact limit of  $(x_n)_{n \geq 1}$  exists, then there exists a function  $f \in \Delta$  such that  $\lim_{exact} x_n = f(\delta)$ .

If  $f \in \Delta_1$ , then  $f$  is a metallic function, let  $\tilde{f}$  be a metallic extension of  $f$ , we have  $f(\frac{1}{n}) = \tilde{f}(\frac{1}{n}) = x_n$  from a certain rank, since  $\tilde{f}$  is holomorphic at 0, then the limit of  $(x_n)$  exists and we have  $\lim x_n = \tilde{f}(0)$ , finally, we deduce that the existence of the exact limit implies the existence of the limit, and  $\lim x_n = \tilde{f}(0)$ . Generally, we get :

$$\lim_{exact} x_n = f(\delta) \implies \lim x_n = \begin{cases} \tilde{f}(0), & \text{while } f \in \Delta_1; \\ \pm\infty, & \text{while } f \in \Delta_2. \end{cases}$$

- The reciprocal of the above implication is not true, we can find a convergent sequence which does not have an exact limit (for example:  $x_n = \frac{(-1)^n}{n}$ ).
- If a sequence  $(x_n)$  has an exact limit, then  $(x_n)$  keeps a constant sign from a certain rank. In addition, if  $x_n > 0$  from a certain rank, then  $\lim_{exact} x_n > 0$ .
- If a sequence  $(x_n)$  has a exact limit, from the properties of the elements of  $\Delta$ , we can show that the sequence  $(x_{n+1} - x_n)$  keeps a constant sign from a certain rank.

**Theorem 13.** Let  $(a_n)$  be a real sequence, and  $f$  is a holomorphic function on  $D(0, \varepsilon) \setminus \{0\}$  such that:

- $f(]0, \varepsilon[) \subseteq \mathbb{R}$ .

- $f(\frac{1}{n}) = a_n$  from a certain rank.
- $f$  is bounded on  $D(0, \varepsilon) \setminus \{0\}$ .

Then the sequence  $(a_n)$  has an exact limit, and we have:  $\lim_{exact} a_n = f(\delta)$ .

**Proof.** 0 is an artificial singularity of  $f$ .

### 10.4 The exact derivative

**Definition 11.** Let  $f$  be a function that is differentiable at the point  $x_0 \in \mathbb{R}$ .

If the function  $h \rightarrow \frac{f(x_0+h)-f(x_0)}{h}$  is metallic, then the exact limit of  $(\frac{f(x_0+\frac{1}{n})-f(x_0)}{\frac{1}{n}})$  exist.

We note :

$$\widehat{f}(x_0) = \lim_{exact} \frac{f(x_0 + \frac{1}{n}) - f(x_0)}{\frac{1}{n}} = \frac{f(x_0 + \delta) - f(x_0)}{\delta}.$$

The Omicron  $\widehat{f}(x_0)$  is called the exact derivative of the function  $f$  at a point  $x_0$ .

**Example 6.** Consider the following function defined as:  $f : x \rightarrow x^2$ .

The exact derivative of  $f$  in  $x_0$  is given by:  $\widehat{f}(x_0) = 2x_0 + \delta$ .

**Theorem 14.** Let  $f$  be a function that is differentiable at the point  $x_0 \in \mathbb{R}$ .

If the function  $h \rightarrow \frac{f(x_0+h)-f(x_0)}{h}$  is metallic, then:

$$(\widehat{f}(x_0))^* = f'(x_0)$$

**Proof.** We can apply the theorem of the exact limit.

**Example 7.** For  $f : x \rightarrow x^2$ , the exact derivative at  $x_0$  is  $\widehat{f}(x_0) = 2x_0 + \delta$ , and the derivative at  $x_0$  is  $f'(x_0) = 2x_0$ .

We verify easily that :  $(2x_0 + \delta)^* = 2x_0$ .

**Lemma 3.** Let  $f$  be a metallic function such that for every integer  $k \in \mathbb{N}$ , the function  $t \rightarrow f(x_0 + kt)$  is metallic, then :

$$f(x_0 + N\delta) = f(x_0) + \delta(\widehat{f}(x_0) + \widehat{f}(x_0 + \delta) + \widehat{f}(x_0 + 2\delta) + \dots + \widehat{f}(x_0 + (N - 1)\delta)).$$

**Proof.** From the definition of  $\widehat{f}$ , we have:

$$\begin{aligned} f(x_0 + \delta) &= f(x_0) + \delta\widehat{f}(x_0) \\ f(x_0 + 2\delta) &= f(x_0 + \delta) + \delta\widehat{f}(x_0 + \delta) \\ &\dots \\ &\dots \\ &\dots \end{aligned}$$

$$f(x_0 + N\delta) = f(x_0 + (N - 1)\delta) + \delta\widehat{f}(x_0 + (N - 1)\delta).$$

Summing these equalities, and we find the desired result .

**Application 1.** (Calculate the sum of  $\Sigma k^n$ )

- For  $n = 1$ , if  $f(x) = x^2$ , then  $\widehat{f}(x) = 2x + \delta$ .

From the above proposition in the case of  $x_0 = 0$ , we find :

$$N^2\delta^2 = (\delta(\widehat{f}(0) + \widehat{f}(\delta) + \widehat{f}(2\delta) + \dots + \widehat{f}((N - 1)\delta))).$$

Which implies that :  $N^2\delta^2 = \delta \cdot (\sum_{k=0}^{N-1} 2k\delta + \delta)$ .

Then :  $N^2 = \sum_{k=0}^{N-1} (2k + 1) = 2 \sum_{k=0}^{N-1} k + N \Rightarrow \frac{N^2 - N}{2} = \sum_{k=0}^{N-1} k$ .

- In the case of  $n = 2$ , we choose  $f(x) = x^3$ , then  $\widehat{f}(x) = 3x^2 + 3x\delta + \delta^2$ , by using the above proposition for  $x_0 = 0$ , we find:

$$\begin{aligned} N^3\delta^3 &= \delta \cdot (\widehat{f}(0) + \widehat{f}(\delta) + \widehat{f}(2\delta) + \dots + \widehat{f}((N-1)\delta)) \\ &= \delta \cdot \sum_{k=0}^{N-1} (3k^2\delta^2 + 3k\delta\delta + \delta^2) = \delta^3 \cdot \left( \sum_{k=0}^{N-1} 3k^2 + 3k + 1 \right). \end{aligned}$$

Then:  $N^3 = 3 \sum_{k=0}^{N-1} k^2 + 3 \sum_{k=0}^{N-1} k + N$ .

We deduce that:  $\sum_{k=0}^{N-1} k^2 = \frac{N^3 - N - 3 \sum_{k=0}^{N-1} k}{3}$ .

Finally, we get:  $\sum_{k=0}^{N-1} k^2 = \frac{N(N-1)(2N-1)}{6}$ .

Similarly, we can calculate  $\sum_{k=0}^{N-1} k^3, \sum_{k=0}^{N-1} k^4, \dots$

**Application 2. (The Riemann sum)**

Let  $f, g$  be a two metallic functions, such that:  $f(\delta) = g(\delta)$ , then  $f(\frac{1}{n}) = g(\frac{1}{n})$  from a certain rank. Consider the function defined as follows:

$$f_n(x) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}}.$$

From the above lemma, we deduce that there exists a natural number  $n_0$  such that  $\forall n \geq n_0$ , we have:

$$f(x_0 + \frac{N}{n}) = f(x_0) + \frac{1}{n}(f_n(x_0) + f_n(x_0 + \frac{1}{n}) + f_n(x_0 + \frac{2}{n}) + \dots + f_n(x_0 + \frac{N-1}{n})).$$

If  $f$  is differentiable on  $\mathbb{R}$ , then  $f_n(a + \frac{k}{n}) = f'(a + \frac{k}{n}) + o(\frac{1}{n})$ .

Then

$$f(a + \frac{N}{n}) = f(a) + \frac{1}{n} \cdot \sum_{k=0}^{N-1} f_n(a + \frac{k}{n}) = f(a) + \frac{1}{n} \cdot \sum_{k=0}^{N-1} (f'(a + \frac{k}{n}) + o(\frac{1}{n})).$$

We assume that  $b > a$ , we can choose  $N = E[(b-a)n]$ , then we get:

$$\begin{aligned} f(a + \frac{N}{n}) - f(a) &= \frac{1}{n} \cdot \sum_{k=0}^{N-1} f'(a + \frac{k}{n}) + \frac{1}{n} \sum_{k=0}^{N-1} o(\frac{1}{n}) \\ &= \frac{1}{n} \cdot \sum_{k=0}^{N-1} f'(a + \frac{k}{n}) + \frac{N}{n} o(\frac{1}{n}). \end{aligned}$$

Since  $N = \lfloor (b-a)n \rfloor$ , then  $b - a - \frac{1}{n} < \frac{N}{n} \leq b - a$ .

We deduce that  $\lim f(a + \frac{N}{n}) = f(b)$ , we pass to the limit and we find:

$$f(b) - f(a) = \lim \frac{1}{n} \cdot \sum_{k=0}^{\lfloor (b-a)n \rfloor - 1} f'(a + \frac{k}{n}).$$

For  $b = 1$  and  $a = 0$ , we get:

$$f(1) - f(0) = \int_0^1 f'(t)dt = \lim \frac{1}{n} \sum_{k=0}^{n-1} f'(\frac{k}{n}).$$

## 10.5 The logarithmic function

We know that:  $\forall x \in \mathbb{R}, \lim_{n \rightarrow +\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ .

Let  $x$  be a real number, the function  $f : z \rightarrow (1 + xz)^{\frac{1}{z}} = e^{\frac{1}{z} \ln(1+xz)}$  is a holomorphic function on  $D(0, \varepsilon) \setminus \{0\}$ .

In addition,  $\ln(1+xz) = xz - \frac{z^2 x^2}{2} + o(z^2 x^2)$  (for  $|z| \ll 1$ ), we deduce that:  $\lim_{z \rightarrow 0} \frac{\ln(1+xz)}{z} = x$ , and  $\lim_{z \rightarrow 0} f(z) = e^x$ , then the function  $f$  can be extended to a holomorphic function on a neighborhood of 0, which implies that  $\lim_{exact} \left(1 + \frac{x}{n}\right)^n$  exists and we have:  $\lim_{exact} \left(1 + \frac{x}{n}\right)^n = (1 + x\delta)^{\frac{1}{\delta}}$ , and  $\left((1 + x\delta)^{\frac{1}{\delta}}\right)^* = e^x$ .

From the above results we deduce that  $e^x$  is an infinitesimal approximate value of  $(1 + \delta x)^{\frac{1}{\delta}}$ .

Consider the function  $\xi(x) = (1 + \delta x)^{\frac{1}{\delta}}$ , if this function  $x \rightarrow \xi(x)$  has an inverse function, then the calculation gives us the function  $\xi^{-1}$  which is defined as:  $\xi^{-1}(x) = \frac{x^\delta - 1}{\delta}$  (if it exists !!!).

We can justify the existence of this function, for that, we consider the real number  $x \in \mathbb{R}^{*+}$ .

The map defined as  $g : z \rightarrow \frac{x^z - 1}{z} = \frac{e^{z \ln(x)} - 1}{z}$  is a holomorphic function on  $D(0, \varepsilon) \setminus \{0\}$ , and  $\lim_{z \rightarrow 0} g(z) = \ln(x)$ , then 0 is an artificial singularity of  $g$ , we deduce that there exists the exact limit of the sequence  $(n(x^{\frac{1}{n}} - 1))_{n \geq 1}$ , and his value is equal to  $\lim_{exact} n(x^{\frac{1}{n}} - 1) = \frac{x^\delta - 1}{\delta}$ .

We define the the original logarithm by:

$$\ln_o : x \rightarrow \frac{x^\delta - 1}{\delta}.$$

The function  $\xi$  is called the function of the the original exponential, we note:

$$\xi(x) = exp_o(x) = (1 + \delta x)^{\frac{1}{\delta}},$$

and we deduce that:

$$(\ln_o(x))^* = \ln(x).$$

Then:

$$\ln(x) = \lim_{\alpha \rightarrow 0} \frac{x^\alpha - 1}{\alpha}.$$

**Application 3.** From the above results, we can show the following equality:

$$\ln(x) = \lim_{\alpha \rightarrow 0} \frac{x^\alpha - x^{-\alpha}}{2\alpha}.$$

**Remark 11.**

$$\frac{\ln(x)}{\ln(y)} = \lim_{\alpha \rightarrow 0} \frac{x^\alpha - 1}{y^\alpha - 1} = \lim_{\alpha \rightarrow 0} \frac{x^\alpha - x^{-\alpha}}{y^\alpha - y^{-\alpha}}.$$

**Application 4.** By using the above results we can show the following theorem.

**Theorem 15.** For every  $0 < x$  and  $x \neq 1$ , we have :

$$\frac{x - 1}{\ln(x)} = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} x^{\frac{k}{n}}.$$

*Proof.* We have:

$$(x^{\frac{1}{n}} - 1) \left( \sum_{k=0}^{n-1} x^{\frac{k}{n}} \right) = x - 1.$$

Then:

$$n(x^{\frac{1}{n}} - 1) \left( \frac{1}{n} \sum_{k=0}^{n-1} x^{\frac{k}{n}} \right) = x - 1.$$

Since  $\lim_{\alpha \rightarrow 0} \frac{x^\alpha - 1}{\alpha} = \lim n(x^{\frac{1}{n}} - 1) = \ln(x)$ , then :

$$\frac{x - 1}{\ln(x)} = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} x^{\frac{k}{n}}.$$

**Application 5.** Consider  $\mathcal{P}$  the set of the prime numbers.

We define the prime-counting function [14] at real values of  $x$  by:

$$\pi(x) = \#\{p \leq x : p \in \mathcal{P}\}.$$

**Theorem 16.** (Hadamard and de la Valle Poussin). As  $x \rightarrow +\infty$ ,

$$\pi(x) \sim \frac{x}{\ln(x)}.$$

*Proof.* See [14].□

By using the above theorem, and the formula  $\ln(x) \approx \frac{x^\delta - 1}{\delta}$ , we can show the following theorem:

**Theorem 17.** As  $x \rightarrow +\infty$ ,

$$\pi(x) \sim \frac{1}{x^{\frac{1}{x}} - 1}.$$

*Proof.* We approach  $\ln(x)$  by  $\frac{x^\delta - 1}{\delta}$ , and  $\delta$  by  $\frac{1}{x}$ , and we deduce that an approximate value of  $\frac{x}{\ln(x)}$  is  $\frac{1}{x^{\frac{1}{x}} - 1}$ .

Next, we can verify that  $\frac{x}{\ln(x)} \sim \frac{1}{x^{\frac{1}{x}} - 1}$ , in fact: As  $x \rightarrow +\infty$ , we have:

$$\frac{1}{x^{\frac{1}{x}} - 1} \sim \frac{1}{e^{\frac{\ln(x)}{x}} - 1} = \frac{1}{\frac{e^{\frac{\ln(x)}{x}} - 1}{\frac{\ln(x)}{x}}} \frac{x}{\ln(x)}.$$

Since,  $\lim_{x \rightarrow +\infty} \frac{e^{\frac{\ln(x)}{x}} - 1}{\frac{\ln(x)}{x}} = 1$ , then:  $\frac{1}{x^{\frac{1}{x}} - 1} \sim \frac{x}{\ln(x)}$ .

Finally, we obtain:

$$\pi(x) \sim \frac{1}{x^{\frac{1}{x}} - 1}.$$

**Application 6.** Let  $(p_n)$  be the sequence of the prime numbers, we have:

**Theorem 18.**

$$p_n \sim n \ln(n), \text{ while } n \rightarrow +\infty$$

*Proof.* See [14].□

By using the above theorem and the approximations  $\ln(n) \approx \frac{n^\delta - 1}{\delta}$  and  $\delta \simeq \frac{1}{n}$ , we get the following theorem:

**Theorem 19.**

$$p_n \sim n^2 (\sqrt[n]{n} - 1), \text{ while } n \rightarrow +\infty$$

*Proof.* We can verify that  $\lim \frac{n(\sqrt[n]{n} - 1)}{\ln(n)} = 1$ .

## 10.6 The Omicran-reals in geometry

### 10.6.1 The geometric point

If  $f$  is a metallic function, and  $\tilde{f}(\delta)$  is an infinitesimal number, the function  $f$  is metallic then the sequence  $(f(\frac{1}{n}))$  keeps a constant sign from a certain rank. Assume that the above sequence is positive from a certain rank  $n_0$ . Since  $\varphi(\tilde{f}(\delta)) = \overline{f(\frac{1}{n})}_{n \geq 1}$ , then we can represent  $\tilde{f}(\delta)$  by the family of segments  $(I_n)_{n \geq 1}$ , where  $I_n = ]0, f(\frac{1}{n})]$ .

**Definition 12.** Let  $f$  and  $g$  be two metallic functions. Assume that the number  $\tilde{f}(\delta) - \tilde{g}(\delta)$  is infinitesimal and  $\tilde{f}(\delta) < \tilde{g}(\delta)$ .

An elementary Geometric point of  $\mathcal{O}$  is a segment of the type :  $[x_A, x_A + \delta[$ , where  $[x, y[ = \{z / x \leq z < y\}$ .

### 10.6.2 The length of a curve $\mathcal{C}_f$

We define the length of an elementary geometric point by:

$$l([x_A, x_A + \delta[) = \delta,$$

where  $x_A = g(\delta)$ , and  $g$  is a metallic function.

Let  $f$  be a function such that:  $x \rightarrow f(x_A^* + x)$  is metallic, and  $\tilde{f}$  the metallic extension of  $f$  on a neighborhood of  $x_A^*$ , consider  $A(x_A, f(x_A))$  and  $A'(x_A + \delta, f(x_A + \delta))$  two ordered pairs of  $\mathcal{O}^2$ , let  $\phi$  be a function defined as:

$$\phi : z \rightarrow z \sqrt{1 + \left( \frac{\tilde{f}(\tilde{g}(z) + z) - \tilde{f}(\tilde{g}(z))}{z} \right)^2}.$$

The map  $\theta : z \rightarrow \frac{\tilde{f}(\tilde{g}(z) + z) - \tilde{f}(\tilde{g}(z))}{z}$  is holomorphic on  $D(0, \varepsilon) \setminus \{0\}$ , and we have:

$$\theta(z) = \frac{\tilde{f}(\tilde{g}(z) + z) - \tilde{f}(x_A^*)}{z} - \frac{\tilde{f}(\tilde{g}(z)) - \tilde{f}(x_A^*)}{z},$$

where  $\lim_{z \rightarrow 0} \tilde{g}(z) = x_A^*$  (the projection of  $x_A$  onto  $\mathbb{R}$ ).

Then the  $\lim_{z \rightarrow 0} \theta(z)$  exists, and we have:

$$\lim_{z \rightarrow 0} \theta(z) = (g'(0) + 1)f'(x_A^*) - g'(0)f'(x_A^*) = f'(x_A^*) \in \mathbb{R}$$

We deduce that  $\lim_{z \rightarrow 0} \phi(z) = 0$ , and  $\phi$  is continuously extendable over 0, then the function  $\phi$  is holomorphically extendable over 0, which justify the existence of the exact limit of the sequence  $(\phi(\frac{1}{n}))$ , and we have

$$\lim_{exact} (\phi(\frac{1}{n})) = \phi(\delta) = \delta \sqrt{1 + \left( \frac{f(x_A + \delta) - f(x_A)}{\delta} \right)^2} \in \mathcal{O},$$

which is infinitesimal. We define the length of the segment  $[A, A'[$  by:

$$l([A, A'[) = \delta \sqrt{1 + \left( \frac{f(x_A + \delta) - f(x_A)}{\delta} \right)^2}.$$

We note :

$$\psi(x_A) = \delta \sqrt{1 + \left( \frac{f(x_A + \delta) - f(x_A)}{\delta} \right)^2} = \delta \sqrt{1 + \widehat{f}(x_A)^2}.$$

If  $f$  is a metallic function defined on the interval  $[0, 1]$ , Let  $A(0, f(0))$  and  $B(1, f(1))$  two points of the plane which define with  $f$  the arc  $\widetilde{AB}$ , if the exact limit of the series  $\frac{1}{n} \sum_{k=0}^{n-1} \sqrt{1 + \widehat{f}(\frac{k}{n})^2}$  exists, we note:

$$\lim_{\text{exact } n} \frac{1}{n} \sum_{k=0}^{n-1} \sqrt{1 + \widehat{f}(\frac{k}{n})^2} = \delta \sum_{k=0}^{\frac{1}{\delta}-1} \sqrt{1 + \widehat{f}(k\delta)^2}.$$

We define the exact length of the arc  $\widetilde{AB}$  by :

$$l(\widetilde{AB}) = \delta \sum_{k=0}^{\frac{1}{\delta}-1} \sqrt{1 + \widehat{f}(k\delta)^2}.$$

The length of the arc  $\widetilde{AB}$  is the real denoted by  $l^*(\widetilde{AB})$  and defined by :

$$l^*(\widetilde{AB}) = (\delta \sum_{k=0}^{\frac{1}{\delta}-1} \sqrt{1 + \widehat{f}(k\delta)^2})^* = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sqrt{1 + f_n(\frac{k}{n})^2},$$

where  $f_n(x) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}}$ .

Since  $f$  is a metallic function, then it can be extended to a function twice differentiable at 0. Assume that the function is twice differentiable on  $]0, 1[$ .

Then:

$$f_n(\frac{k}{n}) = \frac{f(\frac{k+1}{n}) - f(\frac{k}{n})}{\frac{1}{n}} = f'(\frac{k}{n}) + \frac{1}{2n} f''(\xi_k),$$

where  $1 \leq k \leq n-1$ , and  $\xi_k \in ]\frac{k}{n}, \frac{k+1}{n}[$ .

Consider  $M_1 = \sup_{]0,1[}(|f'(x)|)$  and  $M_2 = \sup_{]0,1[}(|f''(x)|)$ .

We have :

$$f_n^2(\frac{k}{n}) = (f'(\frac{k}{n}) + \frac{1}{2n} f''(\xi_k))^2 = f'^2(\frac{k}{n}) + \varepsilon_{n,k},$$

where  $\varepsilon_{n,k} = f'(\frac{k}{n}) \frac{1}{n} f''(\xi_k) + \frac{1}{4n^2} f''^2(\xi_k)$ .

If  $M_1 < +\infty$  and  $M_2 < +\infty$ , we obtain:

$$|\varepsilon_{n,k}| \leq \frac{M_1 M_2 + M_2^2}{n}.$$

Then,  $\lim_{n \rightarrow +\infty} \sup_k |\varepsilon_{n,k}| = 0$ , and we have:

$$\begin{aligned} \sqrt{1 + f_n^2(\frac{k}{n})} &= \sqrt{1 + f'^2(\frac{k}{n}) + \varepsilon_{n,k}} \\ &= \sqrt{1 + f'^2(\frac{k}{n})} + \frac{\varepsilon_{n,k}}{2\sqrt{\beta_{n,k}}}, \end{aligned}$$

where  $\beta_{n,k} \in |1 + f'^2(\frac{k}{n}), 1 + f'^2(\frac{k}{n}) + \varepsilon_{n,k}|$ .

Then:

$$\frac{1}{n} \sum_{k=0}^{n-1} \sqrt{1 + f_n^2(\frac{k}{n})} = \frac{1}{n} \sqrt{1 + f_n^2(0)} + \frac{1}{n} \sum_{k=1}^{n-1} \sqrt{1 + f'^2(\frac{k}{n})} + \frac{1}{2n} \sum_{k=1}^{n-1} \frac{\varepsilon_{n,k}}{\sqrt{\beta_{n,k}}}.$$



Then  $T$  is invertible, and we have:

$$T^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ -1 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & -1 & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & -1 & 1 & 0 \\ 0 & 0 & \cdot & 0 & -1 & 1 \end{pmatrix}$$

Then:

$$\begin{pmatrix} a_{n_0} \\ a_{n_0+1} \\ \cdot \\ \cdot \\ a_n \end{pmatrix} = \begin{pmatrix} \tilde{f}(\frac{1}{n_0}) \\ -\tilde{f}(\frac{1}{n_0}) + \tilde{f}(\frac{1}{n_0+1}) \\ \cdot \\ \cdot \\ -\tilde{f}(\frac{1}{n-1}) + \tilde{f}(\frac{1}{n}) \end{pmatrix} - \begin{pmatrix} \alpha \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

We deduce that:

$$a_n = \tilde{f}(\frac{1}{n}) - \tilde{f}(\frac{1}{n-1}), \text{ from a certain rank } n_0$$

We assume that  $\lim_{exact} a_n$  exists, in addition the sequence  $(s_n)$  is convergent, then  $(a_n)$  converges to 0. We can find a holomorphic function on a neighborhood of 0 such that  $a_n = g(\frac{1}{n})$  from a certain rank, in this case we have  $\lim_{exact} a_n = g(\delta)$ . Then  $g(0) = 0$  and there exists  $p$  such that  $\forall n \geq p$ , we have:

$$g(\frac{1}{n}) = \tilde{f}(\frac{1}{n}) - \tilde{f}(\frac{1}{n-1}).$$

Since  $f$  and  $g$  are a holomorphic functions on a neighborhood of 0, then there exists  $\varepsilon > 0$  such that:

$$g(z) = \tilde{f}(z) - \tilde{f}(\frac{z}{1-z}), \forall z \in D(0, \varepsilon).$$

On other hand, we have  $\tilde{f}(\frac{1}{n_0}) = \alpha = \sum_{k=1}^{n_0-1} a_k$ .

The above result is true for every  $p \geq n_0$ , then  $\tilde{f}(0) = \sum_{k=1}^{+\infty} a_k$ . Finally, we deduce the following theorem :

**Theorem 20.** *Let  $g$  be a metallic function, and  $(s_n)_{n \geq 1}$  the convergent series defined as  $s_n = \sum_{k=1}^n g(\frac{1}{k})$ .*

*If the exact limit of  $(s_n)_{n \geq 1}$  exists, then there exists a function  $\tilde{f}$  holomorphic at 0 such that  $\tilde{f}(\delta) = \lim_{exact} \sum_{k=1}^n g(\frac{1}{k})$ .*

*This function is given by :*

$$\begin{cases} \tilde{f}(0) = \sum_{k=1}^{+\infty} g(\frac{1}{k}), \\ g(z) = \tilde{f}(z) - \tilde{f}(\frac{z}{1-z}), \text{ on a neighborhood of } 0. \end{cases}$$

**Remark 12.** *(Calculating of the finite sum)*

*If  $\tilde{f}(\delta) = \lim_{exact} \sum_{k=1}^n a_k$ , then  $\tilde{f}(\frac{1}{n}) = \sum_{k=1}^n a_k$  from a certain rank  $n_0$ .*

**Example 8.** *We have  $\lim_{exact} \sum_{k=1}^n \frac{1}{k(k+1)} = \frac{1}{1+\delta}$ .*

*Then we deduce  $\forall n \geq 1$ ,  $\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{1}{1+\frac{1}{n}}$ .*

### 10.8 The calculation of the exact limit of $\sum a_k$

Let  $(s_n)$  be a series defined as  $s_n = \sum_{k=1}^n g(\frac{1}{k})$ , we assume that the series is convergent, and  $g$  is metallic function, then holomorphic on a neighborhood of 0. The existence of the exact limit of  $(s_n)$  implies that there exists a holomorphic function  $\tilde{f}$  on a neighborhood of 0 which verify :

$$g(z) = \tilde{f}(z) - \tilde{f}\left(\frac{z}{1-z}\right), \text{ on the disk } D(0, \varepsilon).$$

Let  $g(z) = \sum_{n=0}^{+\infty} \beta_n z^n$  and  $\tilde{f}(z) = \sum_{n=0}^{+\infty} \alpha_n z^n$ , where  $(\alpha_n)$  and  $(\beta_n)$  are a real sequences.

We have:

$$\tilde{f}(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \dots + \alpha_n z^n + o(z^n).$$

Then:

$$\begin{aligned} \tilde{f}\left(\frac{z}{1-z}\right) &= \tilde{f}(z + z^2 + \dots + z^n + o(z^n)) \\ &= \alpha_0 + \alpha_1(z + \dots + z^n + o(z^n)) + \dots + \alpha_n(z + \dots + z^n + o(z^n))^n + o(z^n) \\ &= \alpha_0 + \alpha_1 z + (\alpha_1 + \alpha_2)z^2 + (\alpha_1 + 2\alpha_2 + \alpha_3)z^3 + (\alpha_1 + 3\alpha_2 + 3\alpha_3 + \alpha_4)z^4 \\ &\quad + (\alpha_1 + 4\alpha_2 + 6\alpha_3 + 4\alpha_4 + \alpha_5)z^5 + (\alpha_1 + 5\alpha_2 + 10\alpha_3 + 10\alpha_4 + 5\alpha_5 + \alpha_6)z^6 \\ &\quad + \dots + (\alpha_1 + \binom{n-1}{1}\alpha_2 + \binom{n-1}{2}\alpha_3 + \dots + \binom{n-1}{n-2}\alpha_{n-1} + \alpha_n)z^n + o(z^n). \end{aligned}$$

Since:  $g(z) = \tilde{f}(z) - \tilde{f}\left(\frac{z}{1-z}\right)$ , we deduce that :

$$\begin{cases} \beta_0 = \beta_1 = 0 \\ \beta_k = -\alpha_1 - \binom{k-1}{1}\alpha_2 - \binom{k-1}{2}\alpha_3 - \dots - \binom{k-1}{k-2}\alpha_{k-1}, \quad \forall 2 \leq k \leq n. \end{cases}$$

**Remark 13.** Since  $\beta_0 = \beta_1 = 0$  then  $g(z) = z^2 g_1(z)$ , where  $g_1$  is a holomorphic function on a neighborhood of 0.

Now, from the above results, we deduce that:

$$\begin{cases} \beta_0 = \beta_1 = 0 \\ \beta_2 = -\alpha_1 \\ \beta_3 = -\alpha_1 - 2\alpha_2 \\ \beta_4 = -\alpha_1 - 3\alpha_2 - 3\alpha_3 \\ \dots \\ \dots \\ \beta_n = -\alpha_1 - (n-1)\alpha_2 - \dots - \binom{n-1}{k}\alpha_{k+1} - \dots - \binom{n-1}{n-2}\alpha_{n-1}. \end{cases}$$

Then:

$$\begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ -1 & -2 & 0 & \dots & 0 \\ -1 & -3 & -3 & \dots & \vdots \\ \vdots & \vdots & \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -(n-1) & \vdots & \vdots & -(n-1) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \vdots \\ \alpha_{n-1} \end{pmatrix}.$$

Consider the matrix defined as:

$$M_n = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ -1 & -2 & 0 & \dots & 0 \\ -1 & -3 & -3 & \dots & \vdots \\ \vdots & \vdots & \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -n & \vdots & \vdots & -n \end{pmatrix}$$

We have:  $\det(M_n) = (-1)^n n!$ , then  $M_n$  is invertible, and we have:

$$\begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \cdot \\ \cdot \\ \cdot \\ \beta_n \end{pmatrix} = M_{n-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n-1} \end{pmatrix}.$$

Then, the above system admits a unique solution  $(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$ .

If  $\limsup \sqrt[n]{|\alpha_n|} = \frac{1}{R} > 0$ , then the function  $\tilde{f}(z) = \sum_{n=0}^{+\infty} \alpha_n z^n$  is holomorphic on the disk  $D(0, R)$ ,

and the exact limit  $\lim_{exact} \sum_{k=1}^n g(\frac{1}{k})$  exists, in addition, we have:

$$\tilde{f}(\delta) = \lim_{exact} \sum_{k=1}^n g(\frac{1}{k}) = \lim_{exact} \sum_{k=1}^n a_k,$$

we get :

$$\left( \lim_{exact} \sum_{k=1}^n a_k \right)^* = \sum_{k=1}^{+\infty} a_k.$$

We note:  $\lim_{exact} \sum a_n = \sum \alpha_n \delta^n$ .

Then:

$$\begin{aligned} \left( \lim_{exact} \sum a_n \right)^* &= \sum_{n=1}^{+\infty} a_n, \\ \left( \sum \alpha_n \delta^n \right)^* &= \sum_{n=1}^{+\infty} a_n. \end{aligned}$$

## 11 The black magic matrix

### 11.1 The calculation of the exact limit with the black magic matrix

Let  $g$  be a metallic function and  $f$  a holomorphic function in a neighborhood of 0.

We assume that the series  $\sum_{k=1}^n g(\frac{1}{k})$  admits the exact limit  $\tilde{f}(\delta)$ .

Let  $(\alpha_n)$  and  $(\beta_n)$  be the real sequences such that  $\tilde{f}(z) = \alpha_0 + \sum_{k=1}^{+\infty} \alpha_k z^k$  and  $g(z) = \sum_{k=0}^{+\infty} \beta_k z^k$ .

We have:

$$\lim_{exact} \sum_{k=1}^n g(\frac{1}{k}) = \tilde{f}(\delta).$$

And:

$$\begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \cdot \\ \cdot \\ \cdot \\ \beta_n \end{pmatrix} = M_{n-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n-1} \end{pmatrix}.$$

**Definition 13.** *The black magic matrix of order  $n$  is defined as  $\psi^{(n)} = M_n^{-1}$ .*

We obtain:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \psi^{(n-1)} \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \vdots \\ \beta_n \end{pmatrix}.$$

The real  $\alpha_0$  is given by  $\alpha_0 = \tilde{f}(0) = \sum_{k=1}^{+\infty} g(\frac{1}{k})$ .

**Remark 14.** *We can verify that:*

$$\sum_{k=1}^m g(\frac{1}{k}) = \alpha_0 + \lim_{n \rightarrow +\infty} \left( \frac{1}{m} \quad \frac{1}{m^2} \quad \frac{1}{m^3} \quad \dots \quad \frac{1}{m^{n-1}} \right) \psi^{(n-1)} \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \beta_n \end{pmatrix},$$

from a certain rank  $m_0$ .

## 11.2 The magical properties of $\psi^{(n)}$

**Property 1.** *The matrix  $\psi^{(n)}$  is given by  $\psi^{(n)} = M_n^{-1}$ , where  $M_n[i, j] = \begin{cases} -(j-1)^i, & \text{if } 1 \leq j-1 \leq i \leq n, \\ 0, & \text{else.} \end{cases}$*

*We deduce that the matrix  $\psi^{(n)}$  is invertible and lower triangular.*

**Property 2.** *We have  $\psi_{i,i}^{(n)} = \frac{-1}{i}$ , then  $\text{tr}(\psi^{(n)}) = -H(n)$  and  $\det(\psi^{(n)}) = \frac{(-1)^n}{n!}$ , where  $(H(n))_{n \geq 1}$*

*is the harmonic series which defined as  $H(n) = \sum_{i=1}^n \frac{1}{i}$ .*

**Proof.** The matrix  $M_n$  is lower triangular and  $Sp(M_n) = \{-i, \text{ for } 1 \leq i \leq n\}$ .  
Then :  $\psi^{(n)}$  is lower triangular and we get  $Sp(\psi^{(n)}) = \{\frac{-1}{i}, \text{ for } 1 \leq i \leq n\}$ .

**Property 3.** *For every  $1 \leq i \leq n-1$ , we have:*

$$\psi_{i+1,i}^{(n)} = \frac{1}{2}.$$

**Proof.** We have:

$$\delta_{i+1,i} = \sum_{k=1}^n M_n[i+1, k] \psi_{k,i}^{(n)}.$$

Then:

$$\sum_{k=i}^{i+1} M_n[i+1, k] \psi_{k,i}^{(n)} = 0.$$

We deduce that:

$$\psi_{i+1,i}^{(n)} = -\frac{M_n[i+1, i] \psi_{i,i}^{(n)}}{M_n[i+1, i+1]},$$

$$\psi_{i+1,i}^{(n)} = -\frac{\binom{i+1}{i-1}\psi_{i,i}^{(n)}}{\binom{i+1}{i}},$$

finally, we obtain:

$$\psi_{i+1,i}^{(n)} = \frac{1}{2}.$$

**Property 4.** For every  $(m, p) \in \mathbb{N}^2$ , such that:  $2 \leq m$ , and  $2m + p \leq n$ , we have:

$$\psi_{2m+p,1+p}^{(n)} = 0.$$

In particular, for every  $2 \leq m \leq \frac{n}{2}$ , we get:

$$\psi_{n,n-2m+1}^{(n)} = 0.$$

*Proof.* We can see the demonstration in the following pages.

**Property 5.** For every  $1 \leq m \leq n - 1$ , we have:

$$\psi_{m,m}^{(n)} \psi_{m+1,m-1}^{(n)} = \frac{1}{12}.$$

Then:

$$\psi_{m+1,m-1}^{(n)} = \frac{-m}{12}.$$

**Proof.** We have:

$$\psi^{(n)} M_n = I_n.$$

Then:

$$\sum_{k=1}^n \psi_{i,k}^{(n)} M_n[k, j] = \delta_{ij}.$$

In particular:

$$\sum_{k=1}^n \psi_{m+1,k}^{(n)} M_n[k, m-1] = \delta_{m+1,m-1}.$$

Then:

$$\sum_{k=m-1}^{m+1} \psi_{m+1,k}^{(n)} M_n[k, m-1] = 0.$$

Which implies:

$$\psi_{m+1,m-1}^{(n)} M_n[m-1, m-1] + \psi_{m+1,m}^{(n)} M_n[m, m-1] + \psi_{m+1,m+1}^{(n)} M_n[m+1, m-1] = 0.$$

Then:

$$-(m-1)\psi_{m+1,m-1}^{(n)} - \frac{m(m-1)}{4} + \frac{m(m-1)}{6} = 0.$$

Finally, we get:

$$\psi_{m+1,m-1}^{(n)} = \frac{-m}{12}.$$

**Property 6.** For every  $(i, j) \in \mathbb{N}^2$ , such that  $1 \leq i, j \leq n$ , we have:

$$\psi_{i,j}^{(n+1)} = \psi_{i,j}^{(n)}.$$

**Proof.** From the definition of  $M_n$ , we have:

$$M_{n+1} = \begin{pmatrix} & & 0 \\ & M_n & \vdots \\ & & 0 \\ X_n & & -(n+1) \end{pmatrix},$$

where  $X_n = -\binom{n+1}{0}, \binom{n+1}{1}, \dots, \binom{n+1}{n-1}$ .

To prove that  $\psi_{i,j}^{(n+1)} = \psi_{i,j}^{(n)}$ , it is sufficient to show that there exists a row vector  $Y_n$  such that:

$$\psi^{(n+1)} = \begin{pmatrix} & & 0 \\ & \psi^{(n)} & \vdots \\ & & 0 \\ Y_n & & -1/(n+1) \end{pmatrix}.$$

On other hand, we have:

$$M_{n+1}\psi^{(n+1)} = I_{n+1},$$

then:

$$\begin{pmatrix} & & 0 \\ & M_n & \vdots \\ & & 0 \\ X_n & & -(n+1) \end{pmatrix} \begin{pmatrix} & & 0 \\ & \psi^{(n)} & \vdots \\ & & 0 \\ Y_n & & -1/(n+1) \end{pmatrix} = I_{n+1},$$

which implies:

$$\begin{pmatrix} & & 0 \\ & M_n\psi^{(n)} & \vdots \\ & & 0 \\ X_n\psi^{(n)} - (n+1)Y_n & & 1 \end{pmatrix} = I_{n+1}.$$

Finally, we deduce that:  $X_n\psi^{(n)} - (n+1)Y_n = 0_n$ .

Then, we can choose  $Y_n$  as a form  $Y_n = \frac{X_n\psi^{(n)}}{n+1}$ , and we get:

$$\psi^{(n+1)} = \begin{pmatrix} & & 0 \\ & \psi^{(n)} & \vdots \\ & & 0 \\ \frac{1}{n+1}X_n\psi_n & & -\frac{1}{n+1} \end{pmatrix}.$$

Finally, we deduce that:  $\psi_{i,j}^{(n+1)} = \psi_{i,j}^{(n)}$ , for every  $1 \leq i, j \leq n$ .

**Remark 15.** From the above result, we deduce that  $\psi_{i,j}^{(i)} = \psi_{i,j}^{(n)}$ , for every  $1 \leq i, j \leq n$ .

We note  $\psi_{i,j}^{(n)} = \psi_{i,j}$ .

**Property 7.** For every  $1 < i \leq n$ , we have:

$$\sum_{k=1}^n \psi_{i,k} = 0, \text{ and } \sum_{k=1}^n \psi_{1,k} = -1.$$

Then:

$$\sum_{i=1}^n C_i = C_1 + C_2 + \dots + C_n = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Where  $C_1, C_2, \dots, C_n$  are the column vectors of the matrix  $\psi^{(n)}$ .

**Proof.** We know that:  $\psi_n M_n = I_n$ , then  $\sum_{k=1}^n \psi_n[i, k] M_n[k, 1] = \delta_{i1}$ .

We deduce that:

$$\begin{cases} \sum_{k=1}^n \psi_{1,k} M_n[k, 1] = 1, \\ \sum_{k=1}^n \psi_{i,k} M_n[k, 1] = 0, \quad \text{if } i \neq 1 \end{cases}$$

Then:

$$\begin{cases} \sum_{k=1}^n \psi_{1,k} = -1, \\ \sum_{k=1}^n \psi_{i,k} = 0, \quad \text{if } i \neq 1. \end{cases}$$

**Property 8.** For every  $1 \leq i \leq n$ , we have:

$$\sum_{k=1}^n (-1)^k \psi_{i,k} = (-1)^{i+1}.$$

Which implies:

$$\sum_{i=1}^n (-1)^{i-1} C_i = C_1 - C_2 + \dots + (-1)^{n-1} C_n = \begin{pmatrix} -1 \\ 1 \\ \vdots \\ (-1)^n \end{pmatrix}.$$

Where  $C_1, C_2, \dots, C_n$  are the column vectors of the matrix  $\psi^{(n)}$ .

**Proof.** From the example 3, we have  $\lim_{\text{exact}} \sum_{k=1}^n \frac{-1}{k(k+1)} = \frac{-1}{1+\delta}$ .

Then :  $\lim_{\text{exact}} \sum_{k=1}^n g\left(\frac{1}{k}\right) = \tilde{f}(\delta)$ , for  $g(z) = \frac{-z^2}{1+z} = \sum_{k=2}^{+\infty} (-1)^k z^k$  and  $\tilde{f}(z) = \frac{-1}{1+z} = \sum_{k=0}^{+\infty} (-1)^{k+1} z^k$ .

By using the Property 9, we deduce that:

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ \vdots \\ \vdots \\ (-1)^{n+1} \end{pmatrix} = \psi^{(n)} \begin{pmatrix} -1 \\ 1 \\ -1 \\ \vdots \\ \vdots \\ (-1)^n \end{pmatrix},$$

finally, we deduce that:

$$\sum_{k=1}^n (-1)^k \psi_{i,k} = (-1)^{i+1}.$$

**Property 9.** Let  $g$  be a metallic function, such that  $g(z) = \sum_{k=0}^{+\infty} \beta_k z^k$  on a neighborhood of 0, we assume that the series  $\sum_{k=1}^n g(\frac{1}{k})$  is convergent and admits a exact limit, then there exists a holomorphic function on a neighborhood of 0  $\tilde{f}$  and a real sequence  $(\alpha_n)$ , such that  $\lim_{\text{exact}} \sum_{k=1}^n g(\frac{1}{k}) = \tilde{f}(\delta)$  and  $\tilde{f}(z) = \sum_{k=0}^{+\infty} \alpha_k z^k$  on a neighborhood of 0, the real sequence  $(\alpha_n)$  is given by:

$$\alpha_0 = \sum_{k=1}^{+\infty} g\left(\frac{1}{k}\right) \quad \text{and} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n-1} \end{pmatrix} = \psi^{(n-1)} \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \cdot \\ \cdot \\ \cdot \\ \beta_n \end{pmatrix},$$

and we have:  $\beta_0 = \beta_1 = 0$ .

**Example 9.**

$$(n = 2) \quad \psi^{(2)} = \begin{pmatrix} -1 & 0 \\ 1/2 & -1/2 \end{pmatrix}$$

$$(n = 3) \quad \psi^{(3)} = \begin{pmatrix} -1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -1/6 & 1/2 & -1/3 \end{pmatrix}$$

$$(n = 5) \quad \psi^{(5)} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 \\ -1/6 & 1/2 & -1/3 & 0 & 0 \\ 0 & -1/4 & 1/2 & -1/4 & 0 \\ 1/30 & 0 & -1/3 & 1/2 & -1/5 \end{pmatrix}$$

$$(n = 8) \quad \psi^{(8)} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/6 & 1/2 & -1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/4 & 1/2 & -1/4 & 0 & 0 & 0 & 0 \\ 1/30 & 0 & -1/3 & 1/2 & -1/5 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & -5/12 & 1/2 & -1/6 & 0 & 0 \\ -1/42 & 0 & 1/6 & 0 & -1/2 & 1/2 & -1/7 & 0 \\ 0 & -1/12 & 0 & 7/24 & 0 & -7/12 & 1/2 & -1/8 \end{pmatrix}$$

( $n = 12$ )

$$\psi^{(12)} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/6 & 1/2 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/4 & 1/2 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/30 & 0 & -1/3 & 1/2 & -1/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & -5/12 & 1/2 & -1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/42 & 0 & 1/6 & 0 & -1/2 & 1/2 & -1/7 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/12 & 0 & 7/24 & 0 & -7/12 & 1/2 & -1/8 & 0 & 0 & 0 & 0 \\ 1/30 & 0 & -2/9 & 0 & 7/15 & 0 & -2/3 & 1/2 & -1/9 & 0 & 0 & 0 \\ 0 & 3/20 & 0 & -1/2 & 0 & 7/10 & 0 & -3/4 & 1/2 & -1/10 & 0 & 0 \\ -5/66 & 0 & 1/2 & 0 & -1 & 0 & 1 & 0 & -5/6 & 1/2 & -1/11 & 0 \\ 0 & -5/12 & 0 & 11/8 & 0 & -11/6 & 0 & 11/8 & 0 & -11/12 & 1/2 & -1/12 \end{pmatrix}$$

**Theorem 21.** (To calculate the coefficients of  $(\psi_{i,j})$  by induction)

For every  $1 \leq j \leq n$ , we have:

$$\psi_{n+1,j} = \frac{X_n}{n+1} \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \\ \vdots \\ \psi_{n,j} \end{pmatrix},$$

where  $X_n = -\left(\binom{n+1}{0}, \binom{n+1}{1}, \dots, \binom{n+1}{n-1}\right)$ , and we have  $\psi_{n+1,n+1} = \frac{-1}{n+1}$ .

**Proof.** There exists a row vector  $Y_n = (y_1, y_2, \dots, y_n)$ , such that :

$$\psi^{(n+1)} = \begin{pmatrix} & & 0 \\ & \psi^{(n)} & \vdots \\ & & 0 \\ Y_n & & -\frac{1}{n+1} \end{pmatrix}.$$

On other hand,  $Y_n = \frac{X_n \psi^{(n)}}{n+1}$ , then:

$$y_j = \psi_{n+1,j} = Y_n e_j = \frac{X_n}{n+1} \psi^{(n)} e_j = \frac{X_n}{n+1} \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \\ \vdots \\ \psi_{n,j} \end{pmatrix},$$

where  $(e_1, e_2, \dots, e_n)$  is a canonical base of  $\mathbb{R}^n$ .

**Remark 16.** For every  $n \geq 1$ , we have :

$$X_n = (0, X_{n-1}) + (X_{n-1}, -n).$$



On other hand, we know that  $B_{2k+1} = 0$  for every natural number  $k \geq 1$ , then to prove that  $\psi_{s+2k+1,s} = 0$  it's sufficient to show that the above column has the following form :

$$\begin{pmatrix} 0 \\ \vdots \\ x_s \\ x_{s+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \alpha_s B_0 \\ \alpha_{s+1} B_1 \\ \vdots \\ \alpha_n B_{n-s} \end{pmatrix},$$

where  $\alpha_s, \alpha_{s+1}, \dots, \alpha_n$  are a real numbers. Then:

$$\left( \binom{j}{0}, \binom{j}{1}, \binom{j}{j-1}, 0, \dots, 0 \right) \begin{pmatrix} 0 \\ \vdots \\ \alpha_s B_0 \\ \vdots \\ \alpha_n B_{n-s} \end{pmatrix} = -\delta_{s,j}.$$

If  $s \neq j$ , and  $s > j$ , this product is zero.

If  $s < j$ , we find :  $\alpha_s B_0 \binom{j}{s-1} + \alpha_{s+1} B_1 \binom{j}{s} + \dots + \alpha_j B_{j-s} \binom{j}{j-1} = 0$ .

From the property 10, we know that :

$$\binom{j-s+1}{0} B_0 + \binom{j-s+1}{1} B_1 + \dots + \binom{j-s+1}{j-s} B_{j-s} = 0.$$

To find the real numbers  $(\alpha_i)$ , it's sufficient to find  $\lambda \in \mathbf{R}$ , such that:

$$\alpha_{s+k} B_k \binom{j}{s+k-1} = \lambda \binom{j-s+1}{k} B_k, \text{ for } s+k \leq j.$$

Then:

$$\alpha_{s+k} = \lambda \frac{\binom{j-s+1}{k}}{\binom{j}{s+k-1}},$$

$$\alpha_{s+k} = \frac{\lambda (s+k-1)! (j-s+1)!}{k! j!}.$$

In the case of  $k = 0$ , we get:

$$\alpha_s = \lambda \frac{(s-1)! (j-s+1)!}{j!}.$$

Then:

$$\lambda = \frac{j! \alpha_s}{(s-1)! (j-s+1)!}.$$

On other hand, we know that  $B_0 = 1$ , then  $\alpha_s = \frac{-1}{s}$ , and we get:

$$\lambda = -\frac{j!}{s! (j-s+1)!}.$$

We replace  $\lambda$  by his value, and we define the real  $(\alpha_i)$  as:

$$\alpha_{s+k} = -\frac{(s+k-1)!}{s! k!} = -\frac{1}{k!} (s+1)(s+2)\dots(s+k-1).$$

Finally, we deduce that :

$$\psi_{k+s,s} = -\frac{B_k}{k!} (s+1)(s+2)\dots(s+k-1).$$

**Corollary 2.** For every  $(l, m) \in \mathbb{N}^2$  such that  $m \geq 2$  et  $2m \leq l$ , we have:

$$\psi_{l, l-2m+1} = 0.$$

**Remark 18.** From the above results, we deduce that the Property 4 is true, and we can show the following theorem.

**Theorem 22.** The matrix of the black magic  $\psi^{(n)} = (\psi_{i,j})_{1 \leq i, j \leq n}$  is given by :

$$\psi_{i,j} = \begin{cases} -\frac{\binom{i}{j} B_{i-j}}{i}, & \text{if } i \geq j \\ 0, & \text{else.} \end{cases}$$

## 11.4 The black magic matrix with the Riemann zeta function

### 11.4.1 Important lemma

**Lemma 4.** The radius of convergence of the series

$$\sum_{k=1}^{+\infty} \psi_{k, s-1} z^k,$$

is zero.

**Proof.** The radius of convergence of the series  $\sum_{k=1}^{+\infty} \psi_{k, s-1} z^k$ , is given by:

$$\frac{1}{R} = \overline{\lim}_{k \rightarrow +\infty} \sqrt[k]{|\psi_{k, s-1}|}.$$

We have:

$$\psi_{k, s-1} = -\frac{B_{k-s+1}}{(k-s+1)!} s(s+1)(s+2)\dots(k-1) = -\frac{B_{k-s+1}}{(k-s+1)!} \frac{(k-1)!}{(s-1)!}.$$

For  $k = 2m + s - 1$ , we get:

$$\psi_{2m+s-1, s-1} = -\frac{B_{2m}}{(s-1)!} \frac{(2m+s-2)!}{(2m)!}.$$

Since  $2m! \sim \sqrt{4\pi m} \left(\frac{2m}{e}\right)^{2m}$  and  $|B_{2m}| \sim 4\sqrt{\pi m} \left(\frac{m}{\pi e}\right)^{2m}$ , then:

$$\frac{|B_{2m}|}{2m!} \sim 2\left(\frac{1}{2\pi}\right)^{2m}.$$

We deduce that:

$${}^{2m+s-1}\sqrt{\frac{|B_{2m}|}{2m!}} \sim \frac{1}{2\pi}.$$

On other hand, we have:

$$(2m+s-1)! \sim \sqrt{2\pi(2m+s-1)} \left(\frac{2m+s-1}{e}\right)^{2m+s-1}.$$

Then:

$$\frac{(2m+s-1)!}{(s-1)!} \sim \frac{\sqrt{2\pi(2m+s-1)}}{(s-1)!} \left(\frac{2m+s-1}{e}\right)^{2m+s-1}.$$

We deduce that:

$${}^{2m+s-1}\sqrt{\frac{(2m+s-1)!}{(s-1)!}} \sim (2m+s-1)^{\frac{1}{4m+2s-2}} \left(\frac{2m+s-1}{e}\right),$$

$${}^{2m+s-1}\sqrt{\frac{(2m+s-1)!}{(s-1)!}} \sim e^{\frac{1}{4m+2s-2} \ln(2m+s-1)} \left(\frac{2m+s-1}{e}\right).$$

Finally, we get:

$${}^{2m+s-1}\sqrt{|\psi_{2m+s-1,s-1}|} \sim \frac{1}{2\pi} e^{\frac{1}{4m+2s-2} \ln(2m+s-1)} \left(\frac{2m+s-1}{e}\right),$$

$${}^{2m+s-1}\sqrt{|\psi_{2m+s-1,s-1}|} \sim \frac{m}{\pi e}.$$

Then:  $\lim_{m \rightarrow +\infty} {}^{2m+s-1}\sqrt{|\psi_{2m+s-1,s-1}|} = +\infty$ .

Which implies that:  $\overline{\lim}_{k \rightarrow +\infty} \sqrt[k]{|\psi_{k,s-1}|} = +\infty$ .

We deduce that the radius of convergence of the series  $\sum_{k=1}^{+\infty} \psi_{k,s-1} z^k$ , is zero.  $\square$

## 11.4.2 The Riemann zeta function

This section will be concerned with the Euler zeta series, which is the function

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s},$$

where  $s$  is a real number greater than 1.

If  $s \in \mathbf{N} \setminus \{0, 1\}$ , consider the real function  $g : x \rightarrow x^s$ , and the series  $s_N = \sum_{k=1}^N g(\frac{1}{k})$ .

**Theorem 23.** *The series  $s_N = \sum_{k=1}^N \frac{1}{k^s}$  does not admits an exact limit.*

**Proof.** We assume that the series  $(s_n)$  admits an exact limit, then there exists a holomorphic function  $\tilde{f}$  on a neighborhood of 0 such that:  $\tilde{f}(\frac{1}{N}) = s_N$ , from a certain rank.

If  $\tilde{f}(z) = \sum_{k=0}^{+\infty} \alpha_k z^k$ , then:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \vdots \\ \alpha_n \end{pmatrix} = \psi^{(n)} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \psi^{(n)} e_{s-1}.$$

Then:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \psi_{1,s-1} \\ \psi_{2,s-1} \\ \cdot \\ \cdot \\ \cdot \\ \vdots \\ \psi_{n,s-1} \end{pmatrix},$$

and  $\alpha_0 = \sum_{k=1}^{+\infty} \frac{1}{k^s} = \zeta(s)$ .

Then:  $\tilde{f}(z) = \zeta(s) + \sum_{k=1}^{+\infty} \psi_{k,s-1} z^k$ .

Since  $\lim_{\text{exact}} \sum_{n=1}^N \frac{1}{n^s} = \tilde{f}(\delta)$ , then:

$$\tilde{f}\left(\frac{1}{N}\right) = \sum_{n=1}^N \frac{1}{n^s} = \zeta(s) + \sum_{k=1}^{+\infty} \frac{\psi_{k,s-1}}{N^k},$$

from a certain rank  $N$ .

Then:

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} - \sum_{k=1}^{+\infty} \frac{\psi_{k,s-1}}{N^k}.$$

**Unfortunately**, this result is not true, because the function  $\tilde{f}$  is not holomorphic on a neighborhood of 0, from the lemma 1, then we obtain a contradiction and we deduce that the series  $(s_n)$  does not admits an exact limit.

### 11.5 The twelfth property of the matrix $\psi^{(n)}$

From the above results, the formula  $\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} - \sum_{k=1}^{+\infty} \frac{\psi_{k,s-1}}{N^k}$  is false, despite this, we can correct this equality by using a new term  $E(M, N, s)$  which is defined as:

$$E(M, N, s) = \zeta(s) - \sum_{n=1}^N \frac{1}{n^s} + \sum_{k=1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k}.$$

Then:

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} - \sum_{k=1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M, N, s).$$

Which implies:

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} - \sum_{k=s-1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M, N, s).$$

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} - \frac{\psi_{s-1,s-1}}{N^{s-1}} - \frac{\psi_{s,s-1}}{N^s} - \frac{\psi_{s+1,s-1}}{N^{s+1}} - \sum_{k=s+2}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M, N, s).$$

We deduce that:

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{1}{s-1} \frac{1}{N^{s-1}} - \frac{1}{2} \frac{1}{N^s} - \sum_{k=s+1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M, N, s).$$

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{s-1} \frac{1}{N^{s-1}} + \frac{1}{2} \frac{1}{N^s} - \sum_{k=s+1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M, N, s).$$

For  $r = k - s$ , we obtain:

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{s-1} \frac{1}{N^{s-1}} + \frac{1}{2} \frac{1}{N^s} - \sum_{r=1}^{2M-1} \frac{\psi_{r+s,s-1}}{N^{r+s}} + E(M, N, s).$$

Then:

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{s-1} \frac{1}{N^{s-1}} + \frac{1}{2} \frac{1}{N^s} + \sum_{r=1}^{2M-1} \frac{\binom{r+s}{s-1} B_{r+1}}{(r+s)N^{r+s}} + E(M, N, s).$$

On other hand, we have  $B_{2k+1} = 0$  for every natural  $k \geq 1$ , then:

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{s-1} \frac{1}{N^{s-1}} + \frac{1}{2} \frac{1}{N^s} + \sum_{m=1}^M \frac{\binom{2m+s-1}{s-1} B_{2m}}{(2m+s-1)N^{2m+s-1}} + E(M, N, s).$$

We get:

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{s-1} \frac{1}{N^{s-1}} + \frac{1}{2} \frac{1}{N^s} + \sum_{m=1}^M \prod_{i=0}^{2m-1} (s+i) \frac{B_{2m}}{(2m)!N^{2m+s-1}} + E(M, N, s).$$

Finally, we find the standard **Euler-Maclaurin** [9]. formula applied to the zeta function  $\zeta(s)$ , where  $s$  is a natural number and  $s \geq 2$ , then we deduce that the matrix of the black magic  $\psi^{(n)}$  has a beautiful twelfth property which given as:

**Property 12.** By using the black magic matrix, we can represent the Euler-maclaurin formula as:

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} - \langle C_{s-1}, \tilde{X}_{M,N,s} \rangle + E(M, N, s),$$

where:

- $\langle \cdot, \cdot \rangle$  is the scalar product  $\langle x, y \rangle = \sum x_i y_i$ .
- $C_{s-1} = \psi^{(2M+s)} e_{s-1}$  is the  $(s-1)$ -th column of the matrix  $\psi^{(2M+s)}$ .

- $\tilde{X}_{M,N,s}$  is the column vector defined as :  $\tilde{X}_{M,N,s} = \begin{pmatrix} \frac{1}{N} \\ \frac{1}{N^2} \\ \vdots \\ \frac{1}{N^{2M+s}} \end{pmatrix}$ .

**Example 10.**

$$\psi^{(10)} = \begin{pmatrix} \zeta(2) & \zeta(3) & \zeta(5) & & \zeta(9) & & \zeta(11) \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/6 & 1/2 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/4 & 1/2 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/30 & 0 & -1/3 & 1/2 & -1/5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & -5/12 & 1/2 & -1/6 & 0 & 0 & 0 & 0 \\ -1/42 & 0 & 1/6 & 0 & -1/2 & 1/2 & -1/7 & 0 & 0 & 0 \\ 0 & -1/12 & 0 & 7/24 & 0 & -7/12 & 1/2 & -1/8 & 0 & 0 \\ 1/30 & 0 & -2/9 & 0 & 7/15 & 0 & -2/3 & 1/2 & -1/9 & 0 \\ 0 & 3/20 & 0 & -1/2 & 0 & 7/10 & 0 & -3/4 & 1/2 & -1/10 \end{pmatrix}.$$

We have:

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} - \sum_{k=1}^{2M+s-1} \frac{\psi_{k,s-1}}{N^k} + E(M, N, s).$$

The coefficients of the first column of  $\psi^{(10)}$  are  $-1, 1/2, -1/6, 0, 1/30, 0, -1/42, 0, 1/30, 0$  then :

$$\zeta(2) = \sum_{n=1}^N \frac{1}{n^2} + \frac{1}{N} - \frac{1}{2} \frac{1}{N^2} + \frac{1}{6} \frac{1}{N^3} - \frac{1}{30} \frac{1}{N^5} + \frac{1}{42} \frac{1}{N^7} - \frac{1}{30} \frac{1}{N^9} + E(4, N, 2).$$

Similarly, we deduce the following formulas :

$$\zeta(3) = \sum_{n=1}^N \frac{1}{n^3} + \frac{1}{2} \frac{1}{N^2} - \frac{1}{2} \frac{1}{N^3} + \frac{1}{4} \frac{1}{N^4} - \frac{1}{12} \frac{1}{N^6} + \frac{1}{12} \frac{1}{N^8} - \frac{3}{20} \frac{1}{N^{10}} + E(4, N, 3).$$

$$\zeta(5) = \sum_{n=1}^N \frac{1}{n^5} + \frac{1}{4} \frac{1}{N^4} - \frac{1}{2} \frac{1}{N^5} + \frac{5}{12} \frac{1}{N^6} - \frac{7}{24} \frac{1}{N^8} + \frac{1}{2} \frac{1}{N^{10}} + E(3, N, 5).$$

$$\zeta(9) = \sum_{n=1}^N \frac{1}{n^9} + \frac{1}{8} \frac{1}{N^8} - \frac{1}{2} \frac{1}{N^9} + \frac{3}{4} \frac{1}{N^{10}} + E(1, N, 9).$$

## 12 The relationship between the hyperreal numbers and the Omicran-reals

Let  $u$  be an element of  $\Delta$ ,  ${}^*\mathbb{R}$  the field of hyperreal numbers and  $u(\delta)$  the Omicran defined by the sequence  $u(\frac{1}{n})_{n \geq 1}$ .  $\langle u(\frac{1}{i}) \rangle$  represents the hyperreal defined by the sequence  $u(\frac{1}{n})_{n \geq 1}$ .

The map defined as:

$$\iota : \begin{array}{ccc} \mathcal{O} & \longrightarrow & {}^*\mathbb{R} \\ u(\delta) & \longrightarrow & \langle u(\frac{1}{i}) \rangle \end{array}$$

is a ring homomorphism. In addition, we prove the following results:

- The map  $\iota$  is injective, in fact: if  $\iota(u(\delta)) = 0$  then  $\langle u(\frac{1}{i}) \rangle = 0$ . We deduce that  $\{i : u(\frac{1}{i}) = 0\} \in \mathcal{F}$ . Then  $u(\frac{1}{i})$  is zero for an infinity of indices  $i$ , from the properties of  $u$  as an element of  $\Delta$ , we deduce that  $\tilde{u} = 0$ . Finally  $u(\delta) = 0$ .
- From the above result, we deduce that the field  $\mathcal{O}$  is isomorphic to a subfield of  ${}^*\mathbb{R}$ . More precisely, we have:

$$\mathcal{O} \approx \iota(\mathcal{O}) \subseteq {}^*\mathbb{R}.$$

- The total order relation defined on  ${}^*\mathbb{R}$  extends the total order defined on  $\mathcal{O}$ . In fact:
  1. If  $u(\delta) \leq v(\delta)$  then there exists  $n_0$  such that  $u(\frac{1}{i}) \leq v(\frac{1}{i})$ , which implies that  $\{i : u(\frac{1}{i}) \leq v(\frac{1}{i})\} \in \mathcal{U}$  ( because, the finite sets are not elements of  $\mathcal{U}$  ). Finally, we deduce that  $\langle u(\frac{1}{i}) \rangle \leq \langle v(\frac{1}{i}) \rangle$ .
  2. Conversely, if  $\langle u(\frac{1}{i}) \rangle \leq \langle v(\frac{1}{i}) \rangle$ , then  $\{i : u(\frac{1}{i}) \leq v(\frac{1}{i})\} \in \mathcal{U}$ . From the properties of the elements of  $\Delta$ , we deduce that there exists  $n_0$  such that  $u(\frac{1}{i}) \leq v(\frac{1}{i})$  for every  $i \geq n_0$ . Finally, we get:

$$u(\delta) \leq v(\delta) \iff \{i : u(\frac{1}{i}) \leq v(\frac{1}{i})\} \in \mathcal{U}.$$

From the above results, we can justify the identification of the field of Omicran-reals  $\mathcal{O}$  by a strict subset of the field of hyperreals, and we deduce that:

”Any property that is true for every hyperreal number is also true for every Omicran.”

## 13 Concluding remark

From the work of Robinson, the construction of the hyperreal numbers is related to the existence of an ultrafilter with special properties. We can find in this ultrafilter the element  $A$  which is very "small" compared to  $A^c$ . Unfortunately, this property is not good for giving a very effective approach in practice. In this work we propose an explicit approach without using the ultrafilters, without adding any axiom, we can find the new notions used to obtain more applications thanks to this new method. Finally, we think that the new method becomes more usable for many researchers in all fields of mathematics not only for the specialists in mathematical logic.

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