

# Gauduchon's form and compactness of the space of divisors (second version)

Daniel Barlet\*

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*En hommage à E. Bishop*

**ABSTRACT.** We show that in a holomorphic family of compact complex connected manifolds parametrized by an irreducible complex space  $S$ , assuming that on a dense Zariski open set  $S^*$  in  $S$  the fibres satisfy the  $\partial\bar{\partial}$ -lemma, the algebraic dimension of each fibre in this family is at least equal to the minimal algebraic dimension of the fibres in  $S^*$ . For instance, if each fibre in  $S^*$  are Moishezon, then all fibres are Moishezon.

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**KEY WORDS.** Family of compact complex manifolds, Algebraic dimension, relative codimension 1 cycle-space,  $\partial\bar{\partial}$ -lemma, strongly Gauduchon manifolds.

## 1 Introduction

In this article we give a rather elementary proof of the following result which answers a “classical” question.

**Theorem 1.0.1** *Let  $\pi : \mathcal{X} \rightarrow D$  be a smooth holomorphic family of compact complex connected manifolds parametrized by the unit disc  $D$  in  $\mathbb{C}$ . Assume that for each  $s \in D^*$  the fiber  $X_s$  of  $\pi$  at  $s$  satisfies the  $\partial\bar{\partial}$ -lemma. Let*

$$a := \inf\{\text{alg}(X_s), s \in D^*\}$$

*where  $\text{alg}(X)$  denotes the algebraic dimension of the compact complex connected manifold  $X$ . Then we have  $\text{alg}(X_0) \geq a$ .*

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\*Institut Elie Cartan, Algèbre et Géométrie,  
Université de Lorraine, CNRS UMR 7502 and Institut Universitaire de France.

As a special case (when  $a = n := \dim X_s$ ) we obtain that, if for any  $s \in D^*$  each  $X_s$  is a Moishezon manifold,  $X_0$  is also a Moishezon manifold.

The previous theorem gives easily the following corollary (see [B.15] for details).

**Corollary 1.0.2** *Let  $\pi : \mathcal{X} \rightarrow S$  be a holomorphic family of compact complex connected manifolds parametrized by a reduced and irreducible complex space  $S$ . Let  $S^*$  be a dense Zariski open set in  $S$ . Assume that for each  $s \in S^*$  the fiber  $X_s$  of  $\pi$  at  $s$  satisfies the  $\partial\bar{\partial}$ -lemma. Let  $a := \inf\{\text{alg}(X_s), s \in S^*\}$ . Then for any  $s \in S$  we have  $\text{alg}(X_s) \geq a$ . ■*

Note that the proof in [B.15] shows that the minimum of the algebraic dimension is obtained at the general point<sup>1</sup> in  $S$ .

In the first section we show that the existence of a Gauduchon metric on a compact complex connected manifold  $X$  implies the compactness of the connected components of the space of divisors in  $X$ . This new proof of this classical result is the key of our proof for the theorem above which appears as a relative version of it.

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## 2 The absolute case

Let me begin by two simple lemmas.

**Lemma 2.0.3** *Let  $M$  be a connected reduced complex space and let  $T : M \rightarrow \mathbb{R}^+$  be a continuous function on  $M$ . Assume that  $T$  is pluri-harmonic on the smooth part  $M'$  of  $M$  and that the function  $T$  achieves its minimum at a point  $x_0 \in M$ . Then  $T$  is constant on  $M$ .*

PROOF. This is an elementary exercise. ■

**Lemma 2.0.4** *Let  $X$  be a compact reduced complex space and let  $\omega$  be a continuous real  $(q, q)$ -form on  $X$  which is strictly positive in the Lelong sense (see section 3.1). Let  $\Gamma$  be a connected component of the reduced complex space of compact  $q$ -cycles in  $X$  and define*

$$\theta : \Gamma \rightarrow \mathbb{R}^+ \quad \text{by} \quad \gamma \mapsto \theta(\gamma) := \int_{\gamma} \omega.$$

*Then the continuous function  $\theta$  achieves its minimum on  $\Gamma$ .*

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<sup>1</sup>this means on the complement of a countable union of closed nowhere dense analytic subsets in  $S$  (see the proposition 3.3.1 below).

PROOF. Note first that  $\theta$  is continuous thanks to the prop. IV 3.2.1 of [B-M 1]. Let  $\alpha := \inf\{\theta(\gamma), \gamma \in \Gamma\}$ . Then the subset  $A := \{\gamma \in \Gamma / \theta(\gamma) \leq \alpha + 1\}$  is a compact subset in  $\Gamma$  thanks to Bishop's theorem [Bi.64], because it is a closed subset in  $\Gamma$  such that each cycle in  $A$  has bounded volume (relative to  $\omega$ ) and support in the compact space  $X$  (see th. IV 2.7.23 in [B-M 1]). Then  $\theta$  achieves its minimum on  $A$ . ■

**Proposition 2.0.5** *Let  $X$  be a compact reduced complex space and let  $\omega$  be a continuous real  $(q, q)$ -form on  $X$  which is strictly positive in the Lelong sense (see section 3.1) and satisfies  $\partial\bar{\partial}\omega = 0$  as a current on  $X$ . Then any connected component of the reduced complex space  $\mathcal{C}_q(X)$  of compact  $q$ -cycles in  $X$  is compact.*

PROOF. The function  $\gamma \rightarrow \theta(\gamma) := \int_\gamma \omega$  is continuous on  $\mathcal{C}_q(X)$  and  $\partial\bar{\partial}$ -closed in the sense of currents; as it achieves its minimum on any connected component  $\Gamma$  of  $\mathcal{C}_q(X)$  thanks to the lemma 2.0.4, we conclude that it is constant on each  $\Gamma$ , thanks to lemma 2.0.3, and then any such  $\Gamma$  is compact (see [B-M 1] ch.IV th. 2.7.20). ■

REMARK. As any compact complex manifold  $X$  of dimension  $n$  admits a Gauduchon metric (see [G.77]), the proposition above gives a proof of the fact that the space of divisors of such a  $X$  has always compact connected components, which is a well known classical result (see [C.82] and [Fu.82]).

## 3 The relative case

### 3.1 Relative strong positivity in the Lelong sense

Let  $\pi : X \rightarrow S$  be a surjective holomorphic map between two irreducible complex spaces. Let  $T_{X/D}$  be the tangent Zariski linear space of  $\pi$ , so the kernel of the tangent map

$$T_\pi : T_X \rightarrow \pi^*(T_D).$$

Let  $1 \leq q \leq \dim X - \dim S := n$  and let  $\omega$  a continuous relative  $(q, q)$ -differential form on  $X$ . Then  $\omega$  induces a continuous hermitian form of the fibres on the linear space on  $X$  associated to the coherent sheaf  $\Omega_{X/S}^q := \Omega_X^q / \pi^*(\Omega_S^1) \wedge \Omega_X^{q-1}$ .

Let  $p : Gr_q(X/S) \rightarrow X$  be the grassmannian of  $q$ -planes in  $T_{X/S}$ ,  $U$  the universal  $q$ -vector bundle on  $Gr_q(X/S)$  and  $\theta : \Lambda^q(U) \rightarrow Gr_q(X/S)$  the line bundle which is the determinant of  $U$ . A  $\pi$ -relative continuous  $(q, q)$ -form on  $X$  defines a continuous hermitian form on  $\Lambda^q(U)$ .

**Definition 3.1.1** *We shall say that a  $\pi$ -relative continuous  $(q, q)$ -form  $\omega$  on  $X$  is **strongly Lelong positive** at  $x_0$  if the hermitian form on  $\Lambda^q(U)$  defined by  $\omega$  is a positive hermitian form at each point of  $p^{-1}(x_0)$  (so a continuous hermitian metric on this line bundle).*

In other word that means that for any  $q$ -plane  $P$  in the Zariski tangent space  $T_{X,x_0}$  which is vertical (i.e. contained in the kernel of  $T_{\pi,x_0}$ ) then  $\omega_{x_0}[v_1 \wedge \cdots \wedge v_q] > 0$  when  $v_1, \dots, v_q$  is a basis of  $P$ .

As the map  $p : Gr_q(X/S) \rightarrow X$  is proper, the condition above is open. Moreover, it is an easy exercise to prove the following properties:

1. Assume that the  $\pi$ -relative continuous  $(q, q)$ -form  $\omega$  on  $X$  is strongly Lelong positive at  $x_0$ . Then for any continuous hermitian metric  $h$  on  $X$  there exists a neighbourhood  $V$  of  $x_0$  in  $X$  and a positive constant  $C(V)$  such that on the open set  $p^{-1}(V)$  the metric induced by  $h^{\wedge q}$  and  $\omega$  on the line bundle  $\Lambda^q(U)$  satisfy

$$h^{\wedge q} \leq C(V).\omega.$$

2. An easy consequence of the above estimate is the fact that on any closed (complex) analytic subset  $Z$  of pure dimension  $q$  in a fiber of  $\pi$  over  $V$  we have the inequality of volume forms on  $Z$

$$h|_Z^{\wedge q} \leq C(V).\omega|_Z.$$

3. Assume now that  $\omega$  is strongly Lelong positive on an open set  $V$  in  $X$  containing a compact set  $L$ . Then there exists a positive constant  $C_L$  such that for any compact relative  $q$ -cycle  $\gamma$  contained in  $L$  we have the inequality

$$\int_{\gamma} h^{\wedge q} \leq C_L \cdot \int_{\gamma} \omega.$$

This means that the integral over compact relative  $q$ -cycles of a continuous  $\pi$ -relative  $(q, q)$ -form which is strongly Lelong positive on  $X$  controls the volume of these cycles contained in a given compact set  $L \subset X$  for any given hermitian metric on  $X$ .

The following result follows easily of the description of compact subsets in the space of compact cycles which is a consequence of E. Bishop's theorem.

**Theorem 3.1.2** *Let  $\pi : \mathcal{X} \rightarrow S$  a proper surjective map between irreducible complex spaces. Assume that there exists a smooth  $d$ -closed  $2q$ -differential form  $\tilde{\omega}$  on  $\mathcal{X}$  such that its  $(q, q)$ -part induces a relative strongly Lelong positive form on  $X$ . Then any connected component of  $\mathcal{C}_q(\pi)$ , the space of relative compact  $n$ -cycles of  $\pi$ , is proper over  $S$ .*

PROOF. Take such a connected component  $\Gamma$ . Then the continuous function given by  $\gamma \mapsto \int_{\gamma} \tilde{\omega}$  is constant on  $\Gamma$ . This implies that for any compact subset  $K$  in  $S$  the volume of cycles in  $\Gamma \cap \pi^{-1}(K)$  for any continuous metric is bounded. But then, as  $\pi^{-1}(K)$  is compact in  $X$  this implies that  $\Gamma \cap \pi^{-1}(K)$  is a compact subset in  $\mathcal{C}_q(\pi) \subset \mathcal{C}_q(X)$  (see [B-M 1] IV th.2.7.20), concluding the proof. ■

REMARK. In the case of smooth family of compact connected manifolds on a complex disc (or polydisc)  $\pi : \mathcal{X} \rightarrow D$  each of them satisfying the  $\partial\bar{\partial}$ -lemma, the existence of a smooth  $(q, q)$ -relative differential form  $\omega$  such that  $\partial_{/\pi}\bar{\partial}_{/\pi}\omega = 0$  on  $\mathcal{X}$  which is strongly Lelong positive on the fibres of  $\pi$  allows to prove, in the same way than in the case of relative divisors, the properness over  $D$  of the connected component of the relative cycle's space  $\mathcal{C}_q(\pi)$  :

the first step is to produce for any given  $s \in D$ , using the  $\partial\bar{\partial}$ -lemma on  $X_s := \pi^{-1}(s)$  smooth forms  $\alpha$  and  $\beta$  on  $X_s$  of type  $(q, q-1)$  and  $(q-1, q)$  respectively such that the form  $\omega - \partial\alpha - \bar{\partial}\beta$  is  $d$ -closed on  $X_s$ , but with  $(q, q)$ -part strongly Lelong positive on  $X_s$ . Then, using the local  $\mathcal{C}^\infty$  triviality of  $\pi$  around  $s$  to produce on  $\pi^{-1}(\Delta_s)$ , where  $\Delta_s$  is a small open (poly-)disc around  $s$  in  $D$ , a smooth  $d$ -closed  $2q$ -form  $\Omega_s$  inducing  $\omega - \partial\alpha - \bar{\partial}\beta$  on  $X_s$ . Then, the  $(q, q)$ -part of  $\Omega_s$  is  $\omega|_{X_s}$  so it is strongly Lelong positive on  $X_s$  and then also on  $\pi^{-1}(\Delta_s)$ , if  $\Delta_s$  is small enough around  $s$ , because the strong Lelong positivity is an open property (and in our  $\mathcal{C}^\infty$ -trivialisation used to construct  $\Omega_s$  the complex structure varies smoothly). The theorem above gives then the properness on  $\Delta_s$  of the connected components of the space  $\mathcal{C}_q(\pi|_{\Delta_s})$ . To conclude, consider now a connected component  $\Gamma$  of  $\mathcal{C}_q(\pi)$  and remark that two  $q$ -cycles in  $\Gamma$  which are contained in  $\pi^{-1}(\Delta_s)$  have not only the same image in  $H_{2q}(\mathcal{X}, \mathbb{C})$  but also in  $H_{2q}(\pi^{-1}(\Delta_s), \mathbb{C})$  because the natural map

$$H_{2q}(\pi^{-1}(\Delta_s), \mathbb{C}) \rightarrow H_{2q}(\mathcal{X}, \mathbb{C})$$

is injective (in fact bijective). This implies that the integral of  $\Omega_s$  on any cycle in  $\Gamma \cap \mathcal{C}_q(\pi^{-1}(\Delta_s))$  is constant, so  $\Gamma \cap \mathcal{C}_q(\pi^{-1}(\Delta_s))$  has only finitely many connected components, each of them being proper on  $\Delta_s$ .  $\blacksquare$

TERMINOLOGY. In the case of a compact complex connected manifold  $X$  of dimension  $n$  a **strongly Gauduchon** form in the sense of [B.15] is a smooth  $d$ -closed  $(2n-2)$ -form  $\omega$  on  $X$  such that its  $(n-1, n-1)$ -part is strongly positive in the sense of Lelong (in this case it is the "usual" sense). Assuming that  $X$  satisfies the  $\partial\bar{\partial}$ -lemma, any Gauduchon form<sup>2</sup> gives rise to a strongly Gauduchon form via the method described above. And, thanks to [G.77] a Gauduchon form always exists on a compact complex connected manifold.

### 3.2 Estimation of volumes

Let now consider a holomorphic family  $\pi : \mathcal{X} \rightarrow D$  of compact complex connected manifolds of dimension  $n$  parametrized by the unit disc in  $\mathbb{C}$ . So  $\mathcal{X}$  is a smooth complex manifold of dimension  $n+1$ , and we fix a smooth relative Gauduchon form  $\omega$  on  $\mathcal{X}$ . Then  $\omega$  is a smooth  $\pi$ -relative differential form on  $\mathcal{X}$  on type  $(n-1, n-1)$  which is positive definite in the fibres and satisfies  $\partial_{/\pi}\bar{\partial}_{/\pi}\omega \equiv 0$  on  $\mathcal{X}$ .

<sup>2</sup>so a  $(n-1, n-1)$  smooth differential form positive definite and  $\partial\bar{\partial}$ -closed.

Using Ehresmann's theorem we may assume that we have a  $\mathcal{C}^\infty$  trivialization

$$\begin{array}{ccc} D \times X_0 & \xrightarrow{\Phi} & \mathcal{X} \\ & \searrow pr & \swarrow \pi \\ & & D \end{array}$$

of the map  $\pi$  where  $X_0 := \pi^{-1}(0)$ . Then the form  $\Phi^*(\omega)$  is a smooth differential form on  $X_0$  of degree  $2n - 2$  depending in a smooth way of the real variables  $x$  and  $y$  in  $D$ , and we then identify  $\omega$  with the smooth (absolute)  $(2n - 2)$ -differential form of  $(\Phi^{-1})^*(\Phi^*(\omega))$  on  $\mathcal{X}$ .

We shall define

$$\frac{\partial \omega}{\partial x} := (\Phi^{-1})^*\left(\frac{\partial(\Phi^*(\omega))}{\partial x}\right) \quad \text{and} \quad \frac{\partial \omega}{\partial y} := (\Phi^{-1})^*\left(\frac{\partial(\Phi^*(\omega))}{\partial y}\right)$$

on  $\mathcal{X}$ . Then  $\frac{\partial \omega}{\partial x}$  and  $\frac{\partial \omega}{\partial y}$  are smooth  $(2n - 2)$ -differential forms on  $\mathcal{X}$  and satisfy

$$d\omega = \frac{\partial \omega}{\partial x} \cdot \pi^*(dx) + \frac{\partial \omega}{\partial y} \cdot \pi^*(dy) + d_{/\pi}\omega. \quad (0)$$

where  $d_{/\pi}$  is the  $\pi$ -relative differential.

Now define the  $\pi$ -relative smooth differential forms  $\nabla_x \omega$  and  $\nabla_y \omega$  on  $\mathcal{X}$  as the  $(n - 1, n - 1)$ -parts of the restrictions of  $\frac{\partial \omega}{\partial x}$  and  $\frac{\partial \omega}{\partial y}$  to each fibre of  $\pi$ . We may also consider  $\nabla_x \omega$  and  $\nabla_y \omega$  as smooth hermitian forms on the holomorphic vector bundle  $\Lambda^{n-1}(T_{\mathcal{X}/D})$  on  $\mathcal{X}$ . Then the following lemma is given by an elementary compactness argument, because  $\omega$  is a smooth positive definite hermitian form on this vector bundle.

**Lemma 3.2.1** *For any compact set  $K$  in  $D$  there exists a constant  $C_K > 0$  such on  $\pi^{-1}(K)$  we have the following inequalities*

$$|\nabla_x \omega| \leq C_K \cdot \omega \quad \text{and} \quad |\nabla_y \omega| \leq C_K \cdot \omega \quad (1)$$

between hermitain forms on the vector bundle  $\Lambda^{n-1}(T_{\mathcal{X}/D})$  on  $\pi^{-1}(K)$ . ■

The next lemma will allow us to prove our main estimate of the integral of  $\omega$  over an analytic family of relative divisors in  $\mathcal{X}$ .

**Lemma 3.2.2** *Consider over an open disc  $D' \subset D$  a closed and reduced complex hypersurface  $\mathcal{H}$  in  $\pi^{-1}(D')$  which is  $(n - 1)$ -equidimensional on  $D'$ . Then, for any smooth differential 1-form  $\varphi := \varphi_x \cdot dx + \varphi_y \cdot dy$  on  $D'$  we have at the generic point of  $\mathcal{H}$  the equality*

$$d\omega \wedge \pi^*(\varphi)|_{\mathcal{H}} = (\nabla_x \omega \wedge \pi^*(\varphi_y \cdot dx \wedge dy) - \nabla_y \omega \wedge \pi^*(\varphi_x \cdot dx \wedge dy))|_{\mathcal{H}} \quad (2)$$

PROOF. Near a generic point in  $\mathcal{H}$  we may assume that we have a local holomorphic trivialisation

$$\begin{array}{ccc} D'' \times U \subset D'' \times V & \xrightarrow{\cong} & \mathcal{H} \subset \mathcal{X} \\ & \searrow \scriptstyle{pr} & \swarrow \scriptstyle{\pi} \\ & & D'' \end{array}$$

where  $D''$  is a small open disc in  $D$  and  $V = U \times \Delta$  is the product of an open polydisc in  $\mathbb{C}^{n-1}$  by an open disc  $\Delta$  in  $\mathbb{C}$ . On a fibre of the restriction of  $\pi$  to  $\mathcal{H}$  the form  $d\omega \wedge \pi^*(\varphi)$  is of pure type  $(n, n)$ .

Remark that  $d_{/\pi}\omega \wedge \pi^*(dx)$  and  $d_{/\pi}\omega \wedge \pi^*(dy)$  vanish on  $\mathcal{H} \simeq D'' \times U$  because such forms present only the types  $(n, n-1)$  and  $(n-1, n)$  in the variables  $t_1, \dots, t_{n-1}$  in  $U \subset \mathbb{C}^{n-1}$ . Then, using (0) and the fact that the restrictions to  $\mathcal{H}$  of the forms

$$\frac{\partial \omega}{\partial x} \cdot \pi^*(\varphi_y \cdot dx \wedge dy) \quad \text{and} \quad \frac{\partial \omega}{\partial y} \cdot \pi^*(\varphi_x \cdot dx \wedge dy)$$

come only from the  $(n-1, n-1)$ -parts of the restriction to the fibre of  $\pi$  of  $\frac{\partial \omega}{\partial x}$  and  $\frac{\partial \omega}{\partial y}$  respectively because the restriction to  $\mathcal{H}$  of  $\pi^*(dx \wedge dy)$  is of type  $(1, 1)$ , the conclusion follows, by definition of  $\nabla_x \omega$  and  $\nabla_y \omega$ .  $\blacksquare$

The previous lemma gives the following simple estimate on  $\mathcal{H}$ .

**Corollary 3.2.3** *We keep the notations of the previous lemma. Assuming that the 1-form  $\varphi$  has a compact support in  $D'$  contained in the compact set  $K$  in  $D$  we have on  $\mathcal{H}$  the inequality*

$$|(d\varphi \wedge \pi^*(\varphi))_{\mathcal{H}}| \leq C_K \cdot (|\varphi_x| + |\varphi_y|) \cdot (\omega \wedge \pi^*(dx \wedge dy))|_{\mathcal{H}}. \quad (3)$$

between (smooth) real  $2n$ -differential forms on  $\mathcal{H}$  where  $C_K$  is the constant introduced in the lemma 3.2.1.  $\blacksquare$

The following classical lemma will be useful to conclude our estimation of the volume.

**Lemma 3.2.4** *Let  $U$  a bounded domain in  $\mathbb{C}$ . Assume that there is a point  $s_0 \in U$  such that any point in  $U$  can be joined to  $s_0$  by a  $\mathcal{C}^1$  path in  $U$  with length uniformly bounded by a number  $L$ . For instance, we can restrict ourselves to  $U := D^* \setminus A$  where  $D^*$  is a punctured open disc in  $\mathbb{C}$  and  $A$  a closed discrete subset in  $D^*$ . Let  $\eta : U \rightarrow \mathbb{R}^{+*}$  be a  $\mathcal{C}^1$  function on  $U$  such that*

$$\left| \frac{\partial \eta}{\partial x} \right| \leq C \cdot \eta \quad \text{and} \quad \left| \frac{\partial \eta}{\partial y} \right| \leq C \cdot \eta \quad \text{in } U \quad (@)$$

where  $C > 0$  is a given constant. Then this implies that the function  $\eta$  is uniformly bounded on  $U$  by  $\eta(s_0) \cdot \exp(2C \cdot L)$ .

PROOF. Firstly consider the case of an open interval  $]a, b[$  in  $\mathbb{R}$  and let  $L := b - a$ . If  $\eta : ]a, b[ \rightarrow \mathbb{R}^{+*}$  is a  $\mathcal{C}^1$  function such that  $|\eta'(x)| \leq C.\eta(x) \quad \forall x \in ]a, b[$ ; we have

$$-C \leq \frac{\eta'(x)}{\eta(x)} \leq C$$

which gives, after integration on  $[x_0, x] \subset ]a, b[$

$$-C.|x - x_0| \leq \text{Log} \frac{\eta(x)}{\eta(x_0)} \leq C.|x - x_0|$$

and then, for any choice of  $x_0 \in ]a, b[$  the estimate  $\eta(x) \leq \eta(x_0). \exp(C.L)$  is valid. In the case of the bounded domain  $U \subset \mathbb{C}$ , choose for any  $s \in U$  a  $\mathcal{C}^1$  path of length strictly less than  $L$  and consider a parametrization of this path extended a little around  $s_0$  and  $s$  :

$$\begin{aligned} \Phi : ]a, b[ \rightarrow U \quad x := \varphi(t), y := \psi(t) \quad \text{with} \quad \varphi'(t)^2 + \psi'(t)^2 \equiv 1 \\ \text{and} \quad \Phi(a + \varepsilon) = s_0 \quad \text{and} \quad \Phi(b - \varepsilon) = s \quad \text{with} \quad 0 < \varepsilon \ll 1. \end{aligned}$$

Then apply the one dimensional case to the function  $F(t) := \eta(\Phi(t))$  which satisfies  $|F'(t)| \leq 2C.F(t)$ . This gives  $\eta(s) \leq \eta(s_0). \exp(2C.L) \quad \forall s \in U$ . ■

### 3.3 proof of the theorem 1.0.1

FIRST STEP. For each  $s \in D^*$  we know that  $X_s$  admits a strongly Gauduchon form and then there exists a strongly Gauduchon form for the family over a small disc around  $s$  (see section 3.1). So we have a smooth  $d$ -closed  $2(n - 1)$ -form such that its  $(n - 1, n - 1)$  part is positive definite on each fibre. This implies that each connected component of the relative cycle space  $\mathcal{C}_{n-1}(\pi^*)$  is proper on  $D^*$ , where  $\pi^*$  is the restriction of  $\pi$  to  $\pi^{-1}(D^*)$ :

in a connected component  $\Gamma$  of the relative cycle space  $\mathcal{C}_{n-1}(\pi^*)$ , for any disc  $\Delta \subset D^*$  all cycles in  $\pi^{-1}(\Delta) \cap \Gamma$  are homologous in  $\pi^{-1}(\Delta)$  because we have a  $\mathcal{C}^\infty$ -relative isomorphism  $\mathcal{X} \simeq D \times X_0$ . So if we dispose of a  $d$ -closed  $2(n - 1)$ -smooth form  $\Omega$  on  $\pi^{-1}(\Delta)$  with a positive definite  $(n - 1, n - 1)$  part on fibres we see that the volume for  $(n - 1)$ -cycles in  $\Gamma \cap \pi^{-1}(\Delta)$  relative to the “volume” defined by  $\Omega$  has to be constant, proving the properness on  $\Delta$  of  $\Gamma \cap \pi^{-1}(\Delta)$ .

SECOND STEP. Applying the theorem 1.0.2 of [B.15] we obtain that the minimum of the algebraic dimension of the fibres over  $D^*$  is obtained on a dense (in fact the complement of a countable subset) subset of  $D^*$ . This implies that it is enough, under the hypothesis of the theorem 1.0.1, to prove that the conclusion holds on a small open disc  $D_0$  with center 0 in  $D$  because this “weak” version of this theorem may be apply then to a small disc centered at any point point in  $D^*$  and this gives the theorem 1.0.1.



Thanks to the second step, we may now assume that there exists a positive constant  $C$  such that the inequalities of the lemma 3.2.1 holds on  $\mathcal{X}$ .

Now, using the proposition 3.3.1 (recalled from [B.15]) it is enough to show that each irreducible component  $\Gamma$  of the space  $\mathcal{C}_{n-1}(\pi)$  of relative  $(n-1)$ -cycles in  $\mathcal{X}$  is proper over  $D$ .

**STEP 3.** Let  $\Gamma$  be an irreducible component of  $\mathcal{C}_{n-1}(\pi^*)$  with reduced generic cycle and such that the projection  $p : \Gamma \rightarrow D^*$  is surjective. As the map  $p$  is proper, let  $\tau : \pi^{-1}(D^*) \rightarrow S$  be a Stein factorization (we may assumed that  $S$  is normal) of  $p$  and let  $\tilde{\tau} : S \rightarrow D^*$  the corresponding proper finite surjective map.

Now let  $\omega$  be a smooth  $\pi$ -relative Gauduchon form on  $\mathcal{X}$  and define on  $\Gamma$  the function

$$\theta : \Gamma \rightarrow \mathbb{R}^{+*} \quad \gamma \mapsto \int_{\gamma} \omega.$$

As the function  $\theta$  is continuous, pluri-harmonic along the fibres of  $p$  as a distribution<sup>3</sup> and proper, it is constant in the fibres of  $\tau$  which are the connected components of fibres of  $p$  over  $D^*$ . So it defines a continuous function  $\tilde{\theta} : S \rightarrow \mathbb{R}^{+*}$ . This situation corresponds to the following diagram:

$$\begin{array}{ccccc}
 & & \Gamma & \xleftarrow{p_1} & G & \xrightarrow{\quad} & \Gamma \times \mathcal{X} \\
 & \theta \swarrow & \downarrow \tau & \searrow p & \searrow p_2 & & \downarrow p_2 \\
 & \mathbb{R}^{+*} & S & \xrightarrow{\tilde{\tau}} & D^* & \longrightarrow & \mathcal{X} \\
 & & & & & & \downarrow \pi \\
 & & & & & & D
 \end{array}$$

**STEP 4.** Let  $\tilde{A}_0 \subset S$  be the set of points  $\sigma \in S$  where the fibre  $\tau^{-1}(\sigma)$  is not contained in the union of the singular set of  $\Gamma$  with the critical set of  $p : \Gamma \rightarrow D^*$ . Then  $\tilde{A}_0$  is a closed analytic subset in  $S$  with no interior point. So  $A_0 := \tilde{\tau}(\tilde{A}_0)$  is a closed discrete subset in  $D^*$ . Let  $A_1 \subset D^*$  be the ramification set of  $\tilde{\tau}$ . It is also closed and discrete in  $D^*$  and finally put  $A := A_0 \cup A_1$ .

We shall show now that the continuous function

$$\eta : D^* \rightarrow \mathbb{R}^{+*}, s \mapsto \eta(s) := \text{Trace}_{\tilde{\tau}}(\tilde{\theta})(s)$$

satisfies the following properties :

- i) The partial derivatives  $\frac{\partial \eta}{\partial x}$  and  $\frac{\bar{\partial} \eta}{\partial y}$  are continuous on  $D^* \setminus A$  (with  $s := x + i.y$ ).
- ii) They satisfy the inequalities  $|\frac{\partial \eta}{\partial x}| \leq C.\eta$  and  $|\frac{\bar{\partial} \eta}{\partial y}| \leq C.\eta$  on  $D^* \setminus A$ , where  $C$  is the constant introduced above (see lemma 3.2.1 and the end of step 2).

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<sup>3</sup>meaning that it is  $\partial\bar{\partial}$ -closed as a distribution along the fibres of  $p : \Gamma \rightarrow D^*$ .

Note that it is enough to prove the properties i) and ii) near each point in  $D^* \setminus A$ . Choose a point  $s_0 \in D^* \setminus A$  and fix an open disc  $\Delta_0 \subset D^* \setminus A$  with center  $s_0$ . Choose now in  $p^{-1}(s_0)$  generic point  $\gamma_0$  and, as  $\gamma_0$  is a smooth point of  $\Gamma$ , such that  $p$  has rank 1 at this point, we can find a smooth locally closed curve  $\Sigma$  through  $\gamma_0$  in  $\Gamma$  such that the restriction of  $p$  to  $\Sigma$  induces an isomorphism of  $\Sigma$  onto  $\Delta_1$  where  $\Delta_1 \subset \Delta_0$  is an open disc with center  $s_0$ .

Define, for  $s \in \Delta_1$  the cycle  $Y_s$  as the cycle corresponding to the point  $p_{|\Sigma}^{-1}(s)$  in  $\Sigma \subset \Gamma$ . This defines an analytic family of  $\pi$ -relative  $(n-1)$ -cycles. So the function  $\eta_1$  which is given by  $s \mapsto \int_{Y_s} \omega$  is continuous on  $\Delta_1$  (see the proposition IV 2.3.1 in [B-M 1]) and coincides with  $\frac{1}{\delta} \cdot \eta$  where  $\delta$  is the degree of the map  $\tilde{\tau}$ . We shall compute now the partial derivatives in the distribution sense of this function  $\eta_1$ .

Let  $\mathcal{Y} \subset \Delta_1 \times \mathcal{X}$  be the graph of the analytic family  $(Y_s)_{s \in \Delta_1}$  and let  $\varphi$  be in  $\mathcal{C}_c^\infty(\Delta_1)$ . Then we have

$$\begin{aligned} \langle d\eta_1, \varphi \cdot dy \rangle &= -\langle \eta_1, \frac{\partial \varphi}{\partial x} \cdot dx \wedge dy \rangle = -\int_{\mathcal{Y}} \omega \wedge \pi^* \left( \frac{\partial \varphi}{\partial x} \cdot dx \wedge dy \right) \\ &= -\int_{\mathcal{Y}} \omega \wedge \pi^* (d\varphi \wedge dy) = -\int_{\mathcal{Y}} \omega \wedge d(\pi^*(\varphi \cdot dy)) \quad \text{then} \\ \langle d\eta_1, \varphi \cdot dy \rangle &= \int_{\mathcal{Y}} \frac{\partial \omega}{\partial x} \wedge \pi^*(\varphi \cdot dx \wedge dy) = \int_{\mathcal{Y}} \nabla_x \omega \wedge \pi^*(\varphi \cdot dx \wedge dy) \end{aligned}$$

by Stokes formula and using the lemma 3.2.2.

We conclude that the distribution  $\frac{\partial \eta_1}{\partial x}$  is equal to the continuous function

$$s \mapsto \int_{Y_s} \nabla_x \omega$$

with the estimate

$$\left| \frac{\partial \eta_1}{\partial x} \right| \leq C \cdot \eta_1$$

deduced from the inequality of the lemma 3.2.1.

We have the analogous result for  $\frac{\partial \eta_1}{\partial y}$ .

So the lemma 3.2.4 applies and the function  $\theta$  on  $\Gamma$  is uniformly bounded. Then the closure of  $\Gamma$  in  $\mathcal{C}_{n-1}(\pi)$  is proper over  $D$ , using again the characterization of compact subsets in  $\mathcal{C}_{n-1}(\mathcal{X})$  (see [B-M 1] IV th.2.7.20).

**FINAL STEP.** Consider now any irreducible component  $\tilde{\Gamma}$  of  $\mathcal{C}_{n-1}(\pi)$ . Remark that to prove the properness of  $\tilde{\Gamma}$  on  $D$  it is enough to consider the case where the generic cycle of  $\tilde{\Gamma}$  is reduced.

Then either  $\tilde{\Gamma}$  is contained in some  $\mathcal{C}_{n-1}(X_s)$  for some  $s \in D$ , and then it is a connected closed subset in  $\mathcal{C}_{n-1}(X_s)$  and then it is compact, or  $\tilde{\Gamma} \setminus p^{-1}(0)$  is a closed

irreducible subset in  $\mathcal{C}_{n-1}(\pi^*)$  which is proper and surjective on  $D^*$  because we may apply the previous result. In this cases we conclude also that  $\tilde{\Gamma}$  (which is the closure of  $\tilde{\Gamma} \setminus p^{-1}(0)$ ) is proper over  $D$ .

As any irreducible component of  $\mathcal{C}_{n-1}(\pi)$  is proper over  $D$ , the conclusion follows. ■

For the convenience of the reader let me recall the proposition 4.0.8 of [B.15] which is an easy generalization of an old result of F. Campana (see [C.81]).

**Proposition 3.3.1** *Let  $\pi : \mathcal{X} \rightarrow S$  be a proper surjective holomorphic  $n$ -equidimensional map between two irreducible complex spaces. Assume that any irreducible component of the complex space  $\mathcal{C}_{n-1}(\pi)$  is proper over  $S$ . Then there exists a countable union  $\Sigma$  of closed irreducible analytic subsets with no interior points in  $S$  and a non negative integer  $a$  such that:*

1. *For any  $s \in S \setminus \Sigma$  the algebraic dimension of  $X_s$  is equal to  $a$ ;*
2. *For all  $s \in S$  the algebraic dimension of  $X_s$  is at least equal to  $a$ .*

We conclude by noticing that there exists an analytic family of smooth complex compact surfaces of the class VII (not Kähler) parametrized by a disc such that the central fibre has algebraic dimension 0 and all other fibres have algebraic dimension equal to 1, see [F-P.09].

This shows that in our theorem 1.0.1 some “Kähler type” assumption on the general fibre  $X_s$  cannot be avoided in order that the general algebraic dimension gives a lower bound for the algebraic dimensions of all fibres.

Note that our assumption that the generic fibres satisfy the  $\partial\bar{\partial}$ -lemma (in fact for the type  $(n, n-1)$ ) is a rather weak such assumption.

## 4 Bibliography

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